Clique covering and degree conditions for hamiltonicity in claw-free graphs

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Abstract

By using the closure concept introduced by the last author, we prove that for any sufficiently large nonhamiltonian claw-free graph G satisfying a degree condition of type $\sigma_k(G) > n + k^2 - 4k + 7$ (where k is a constant), the closure of G can be covered by at most $k-1$ cliques. Using structural properties of the closure concept, we show a method for characterizing all such nonhamiltonian exceptional graphs with limited clique covering number. The method is explicitly carried out for $k \leq 6$ and illustrated by proving that every 2-connected claw-free graph G of order $n \geq 77$ with $\delta(G) \geq 14$ and $\sigma_6(G) > n + 19$ is either hamiltonian or belongs to a family of easily described exceptions.

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1 Introduction

In this paper we consider only finite undirected graphs $G = (V(G), E(G))$ without loops and multiple edges. For any set $A \subset V(G)$, $\langle A \rangle$ denotes the subgraph of G induced on A, $G - A$ stands for $\langle V(G) \setminus A \rangle$. If $A, B \subset V(G)$, then we denote $N_A(B) = \{x \in A | xy \in E(G)$ for some $y \in B\}$. If $x \in V(G)$, then we simply denote $N(x) = N_{V(G)}(\{x\})$. A vertex $x \in V(G)$ is said to be *locally connected* if $\langle N(x) \rangle$ is connected. The graph G is said to be *claw-free* if it does not contain any induced subgraph isomorphic to the claw $K_{1,3}$. The *independence number* of a graph G is denoted by $\alpha(G)$ and its *clique covering number* (i.e. the minimum number of cliques necessary for covering $V(G)$) by $\theta(G)$. The notation $\delta(G)$ stands for the minimum degree of G and $\sigma_k(G)$ ($k \geq 1$) for the minimum degree sum of any k independent vertices in G (for $k > \alpha(G)$ we set $\sigma_k(G) = \infty$). The (vertex) *connectivity* of G is denoted by $\kappa(G)$, the *matching number* of G (i.e. the maximum number of edges in a matching of G) is denoted by $\nu(G)$, and the *vertex covering number* of G (the minimum cardinality of a vertex covering, i.e. is of a set T of vertices such that each edge of G has at least one vertex in T) is denoted by $\tau(G)$. The line graph of a graph G is denoted by $L(G)$. For other notation and terminology not defined here we refer e.g. to [1].

Claw-free graphs have been intensively studied during the last decade, and particularly sufficient conditions for a 2-connected claw-free graph to be hamiltonian have been subject of many papers (see for example the survey [5]). Recently, a closure concept for claw-free graphs was introduced by Ryjáček [13] as follows: the *closure* $\mathrm{cl}(G)$ of a claw-free graph G is obtained by recursively completing the neighborhood of any locally connected vertex of G, as long as this is possible. The closure $\mathrm{cl}(G)$ is well-defined (i.e. unique), remains a claw-free graph and its connectivity is at least equal to the connectivity of G. The following basic properties of the closure $\mathrm{cl}(G)$ were proved in [13].

Proposition A [13]. *Let G be a claw-free graph and $\mathrm{cl}(G)$ its closure. Then*

(*i*) *there is a triangle-free graph H_G such that $\mathrm{cl}(G)$ is the line graph of H_G,*

(*ii*) *both graphs G and $\mathrm{cl}(G)$ have the same circumference.*

Consequently, G is hamiltonian if and only if $\mathrm{cl}(G)$ is hamiltonian.

If G is a claw-free graph such that $G = \mathrm{cl}(G)$, then we say that G is *closed.* It is apparent that a claw-free graph G is closed if and only if every vertex $x \in V(G)$ is either *simplicial* (i.e. $\langle N(x) \rangle$ is a clique), or is *locally disconnected* (i.e. $\langle N(x) \rangle$ consists of two vertex disjoint cliques).

In [13], the closure concept was used to answer an old question by showing that every 7-connected claw-free graph is hamiltonian. H. Li [8] extended this result as follows.

Theorem B [8]. *Every 6-connected claw-free graph with at most 34 vertices of degree 6 is hamiltonian.*

Several other results linked to the closure concept can be mentioned. For example, Brandt, Favaron and Ryjáček [2], Ryjáček, Saito and Schelp [14] and Ishizuka [6] studied the behavior of some other properties dealing with cycles and paths under the closure operation for claw-free graphs. Brousek [3] gave a characterization of nonhamiltonian 2-connected claw-free graphs that are minimal, i.e. that contain no nonhamiltonian 2-connected claw-free graph as a proper induced subgraph.

2 Nonhamiltonian closed claw-free graphs with small clique covering number

Let G be a 2-connected closed claw-free graph and $\mathcal{P}$ be an arbitrary set of maximal cliques in G. We will often use the following properties of $\mathcal{P}$.

1. Two distinct cliques in $\mathcal{P}$ never share more than one vertex. Assume otherwise that the distinct cliques C_1 and C_2 of $\mathcal{P}$ have common vertices x and y. Then $C_1 \cup C_2 - \{x\}$ is a connected part of $N(x)$ and thus, by the claw-freedom and by the definition of the closure, $C_1 = C_2$, which contradicts our choice of two distinct cliques. Analogously, if C_1 and C_2 are two disjoint cliques of $\mathcal{P}$, any vertex of C_1 has at most one neighbor in C_2 and symmetrically.

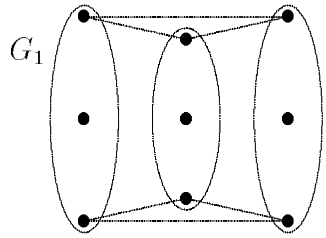
2. By the claw-freedom, three distinct cliques of $\mathcal{P}$ cannot share a common vertex and if there are three cliques such that one of them shares one vertex with the two others, then the last two cliques are disjoint.

In the following theorem we show that all 2-connected nonhamiltonian closed claw-free graphs with small clique covering number can be described as spanning subgraphs of several easily described graphs. The classes $\mathcal{F}_3$, $\mathcal{F}_4$, $\mathcal{F}_5$ are shown in Figures 1, 2, 3 (where the circular and elliptical parts represent cliques containing at least one simplicial vertex).

Theorem 1. *Let G be a 2-connected closed claw-free graph.*

(i) If $\theta(G) \leq 2$, then G is hamiltonian.

(ii) If $3 \leq \theta(G) \leq 5$, then either $G \in \cup_{i=3}^{\theta(G)} \mathcal{F}_i$, or G is hamiltonian.



$\mathcal{F}_3$ is the set of spanning subgraphs of G_1

Figure 1

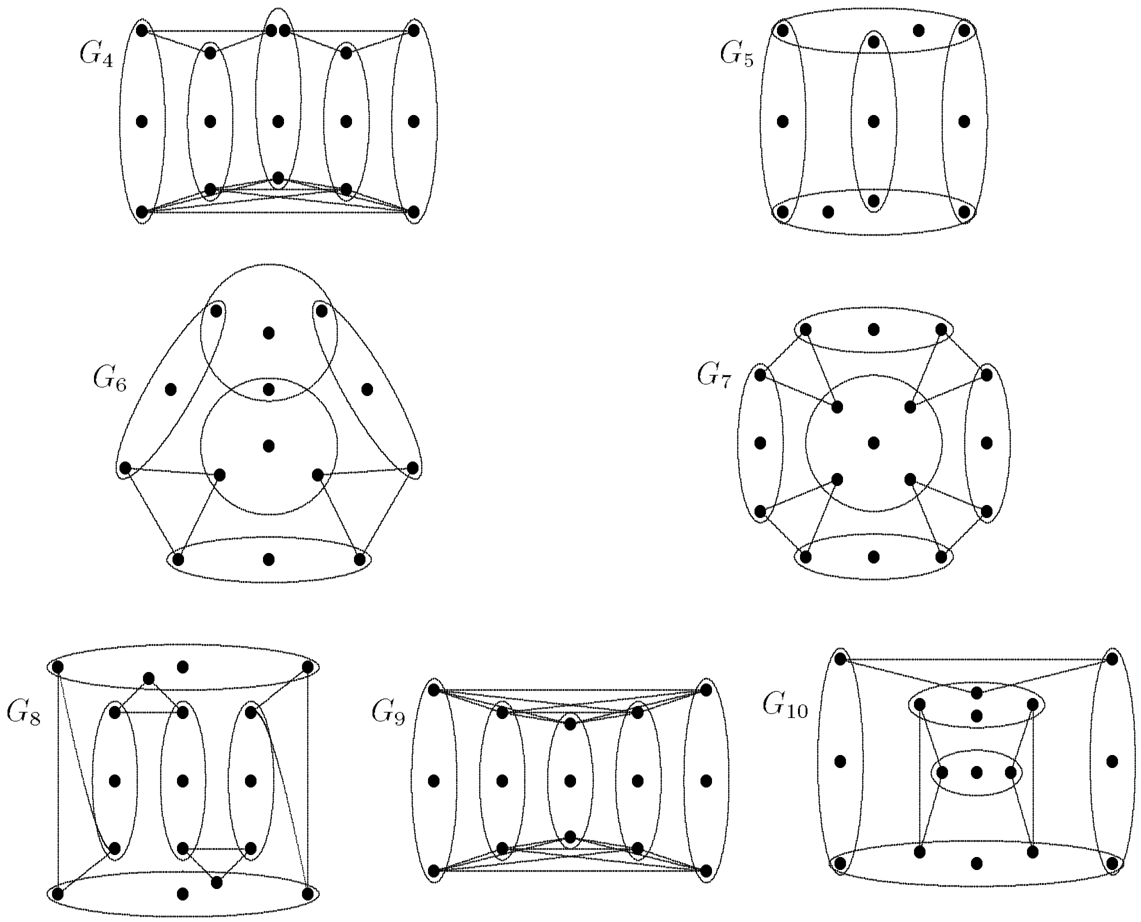
$\mathcal{F}_4$ is the set of spanning subgraphs of G_2 and G_3

Figure 2

Remark. The method of finding the classes $\mathcal{F}_i$ is illustrated by proving the cases $\theta(G) \leq 4$. The proof for $\theta(G) = 5$ is lengthy and somewhat tedious and is thus postponed to Section 4. The authors nevertheless believe that the general method could be applicable even for larger values of $\theta(G)$, e.g. with help of a computer.

Proof. Part (i) of the theorem can be seen immediately and we thus concentrate on the case $\theta(G) \geq 3$.

Let G be a 2-connected nonhamiltonian closed claw-free graph. Then G is the line graph of a unique triangle-free graph H. Let D_1 be the set of degree 1 vertices of H and $H' = H - D_1$. The graphs G and H' are respectively 2-connected and 2-edge-connected. Let θ be the clique covering number of G. We choose a minimum clique covering $\mathcal{P} = \{B_1, B_2, \cdots, B_\theta\}$ of G such that each clique B_i is maximal. Since $\mathcal{P}$ is minimum, each B_i contains at least one *proper vertex*, i.e. a vertex belonging to no other clique of $\mathcal{P}$. The cliques B_i correspond to stars of H which are centered at distinct vertices b_1, b_2, $\ldots$, b_θ of H', called the *black vertices* of H. The other vertices of H will be called *white*. Since the set $B = \{b_1, b_2, \cdots, b_\theta\}$ is a vertex covering of H (i.e., every edge of H has at least one vertex in B), the set $W = V(H) \setminus B$ of the white vertices is independent. We call $\mathcal{S}$ any complete bipartite induced subgraph of H', one class of which is formed by two black vertices b and b', and the second one by white vertices.



$\mathcal{F}_5$ is the set of spanning subgraphs of G_4, G_5, G_6, G_7, G_8, G_9 and G_{10}

Figure 3

Harary and Nash-Williams [12] proved that, for any graph H, the line graph $L(H)$ is hamiltonian if and only if H contains a dominating closed trail, i.e. a closed trail T such that the graph $G - V(T)$ has no edge. Specifically, since the graph G is nonhamiltonian, H contains no dominating closed trail (shortly, DCT), and thus H' contains no closed trail (shortly, CT) containing all its black vertices.

Note that if an endblock of H' (i.e., a block of H' with exactly one cutvertex) is a subgraph $\mathcal{S}$ with a black cutvertex, say b, and if we call J the graph obtained from H by deleting b' and the neighbors of b' in D_1, then H has a DCT if and only if J has a DCT. Moreover, the line graph of H spans a graph obtained from the line graph of J by enlarging a clique containing simplicial vertices. Since the clique covering number of the line graph of J is equal to $\theta - 1$, this case can be reduced to the case $\theta := \theta - 1$.

Two vertices of H are said to be *related* if they are adjacent or if they are both black and there exists between them a path of length 2, the inner vertex of which is white. If T is a CT in H, v is a vertex on T and b is a black vertex outside T, then we say that b, v are $\bar{T}$*-related* if $bv \in E(H)$ or v and b have a white common neighbor outside T.

The *black length* of a CT T of H is the number of its black vertices. We choose T of maximum black length and denote this length by $\mathrm{bla}(T)$. Since T does not contain all the θ black vertices of H, we have $\mathrm{bla}(T) \leq \theta - 1$. We also denote by $\mathrm{blo}(H')$ the number of blocks of H', and by $\mathrm{blo}(T)$ the number of blocks of T. Since H' is 2-edge-connected and triangle-free, each block of H' contains a cycle of length at least 4, and thus, by the independence of W, at least two black vertices. The same argument holds for T. Therefore, $1 \leq \mathrm{blo}(H') \leq \theta - 1$ and $1 \leq \mathrm{blo}(T) \leq \mathrm{bla}(T) - 1 \leq \theta - 2$ (in particular, $\theta \geq 3$). Moreover, if $2 \leq \mathrm{blo}(H') = \theta - 1$ or $\theta - 2$, then at least one of the endblocks of H' exactly contains two black vertices and thus has the structure $\mathcal{S}$ with a black cutvertex. We know that this case can be reduced to the case $\theta := \theta - 1$. Hence we can suppose $\mathrm{blo}(H') = 1$ or $2 \leq \mathrm{blo}(H') \leq \theta - 3$, where the second case can happen only for $\theta \geq 5$.

Case $\theta = 3$. Let $B = \{b_1, b_2, b_3\}$.

By the above, the graph H contains a DCT except possibly if $\mathrm{blo}(H') = 1$, $\mathrm{bla}(T) = 2$ and $\mathrm{blo}(T) = 1$. Then T has the structure $\mathcal{S}$ with, say, b_1 and b_2 as black vertices. Since $\mathrm{blo}(H') = 1$, the third black vertex b_3 of H is related to at least two vertices of T. It is not $\bar{T}$-related to any of b_1, b_2 for otherwise we could find a CT of H through b_1, b_2 and b_3. Hence b_3 is adjacent to two white vertices w_1 and w_2 of T. These two vertices are the only white vertices of T for otherwise we can again find a CT of H of black length 3. Since H is triangle-free, H' is a complete bipartite graph of classes $\{b_1, b_2, b_3\}$ and $\{w_1, w_2\}$. Moreover, each vertex b_i has at least one neighbor in D_1 for otherwise if, say, b_1 has no neighbor in D_1, then $w_1 b_2 w_2 b_3 w_1$ is a DCT of H.

Therefore, the line graph G of H is the graph G_1 of Figure 1.

Case $\theta = 4$. Let $B = \{b_1, b_2, b_3, b_4\}$.

The graph H contains a DCT except possibly if $\mathrm{blo}(H') = 1$ and either $\mathrm{bla}(T) = 2$ and $\mathrm{blo}(T) = 1$, or $\mathrm{bla}(T) = 3$ and $\mathrm{blo}(T) = 2$ or 1.

(i) If $\mathrm{bla}(T) = 2$ and $\mathrm{blo}(T) = 1$, then T has the structure $\mathcal{S}$ with, say, b_1 and b_2 as black vertices. Since $\mathrm{blo}(H') = 1$ and by the choice of T, each of the two other black vertices b_3 and b_4 of H is adjacent to two white vertices of T. Moreover, if T contains more than two white vertices, then we can find a CT of black length larger than 2. Hence, as in the case $\theta = 3$, H' is isomorphic to a complete bipartite graph of classes $\{b_1, b_2, b_3, b_4\}$ and

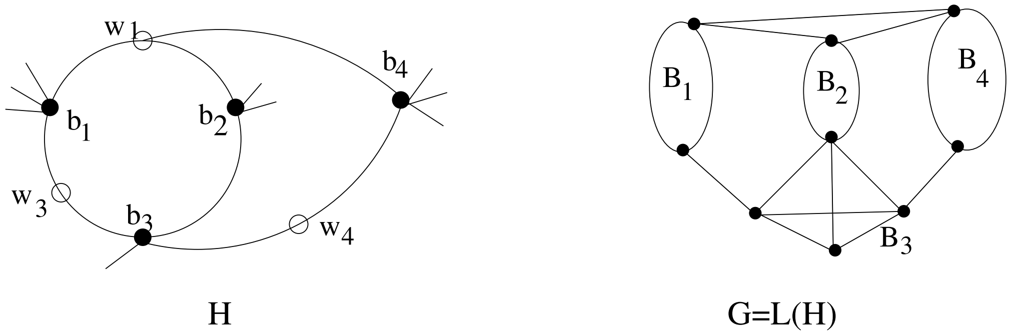


Figure 4

$\{w_1, w_2\}$. But $b_1w_1b_2w_2b_3w_1b_4w_2b_1$ is then a DCT of H, which proves the impossibility of this case.

(*ii*) If $\text{bla}(T) = 3$ and $\text{blo}(T) = 2$, then the two blocks of T have the structure $\mathcal{S}$, say $\mathcal{S}_1$ of black vertices b_1 and b_2, and $\mathcal{S}_2$ of black vertices b_2 and b_3. Since $\text{blo}(H') = 1$, the fourth black vertex b_4 of H is $\bar{T}$-related to some vertex in $\mathcal{S}_1 \setminus \{b_2\}$ and some vertex in $\mathcal{S}_2 \setminus \{b_2\}$. Whichever these vertices are, we get a CT of H containing the four black vertices, which proves the impossibility of this case.

(*iii*) Hence the case $\theta = 4$ reduces to $\text{blo}(H') = 1$, $\text{bla}(T) = 3$ and $\text{blo}(T) = 1$. Then T necessarily contains a cycle C through, say, b_1, b_2 and b_3. Since H is triangle-free, C also contains at least one white vertex. Let $C = b_1w_1b_2w_2b_3w_3b_1$, where w_2 and w_3 possibly do not exist. Since $\text{blo}(H') = 1$, the fourth black vertex b_4 of H is $\bar{T}$-related to at least two vertices of C. By the choice of T, two such vertices cannot be adjacent on C, neither they can be both black. We can distinguish three situations.

- The vertex b_4 is adjacent to exactly one white vertex of C, say w_1, and is $\bar{T}$-related to b_3. By the choice of T, there are no other white vertices in H' (relating two of the four black vertices) than w_1 and possibly w_2, w_3 and a vertex w_4 that $\bar{T}$-relates b_4 and b_3.

  The line graph G of H is then a spanning subgraph of the graph G_1 if b_3 has no neighbor in D_1, of the graph G_2 otherwise. For instance, Figure 4 shows G if w_3 and w_4 exist, w_2 does not exist, and b_3 has one neighbor in D_1.

- The vertex b_4 is adjacent to exactly two white vertices of C. Say, b_4 is adjacent to w_1 and w_2, but not to w_3 (if w_3 exists). By the choice of T, there are no relations between two of the b_i's except those which are shown in Figure 5, and both b_2 and b_4 have some neighbor in D_1.

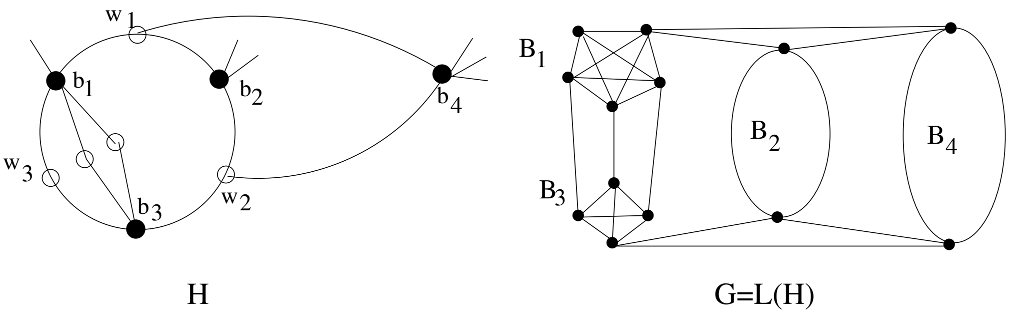


Figure 5

The line graph G of H is then a spanning subgraph of the graph G_1. For instance, Figure 5 shows G when b_1 and b_3 are related by three paths of length 2, b_1 has one neighbor in D_1, and b_3 has none.

– The three vertices w_1, w_2, w_3 exist and the vertex b_4 is adjacent to all of them. By the choice of T, there is no relation between two of the four b_i's, except the relations created by w_1, w_2, w_3. Moreover, each of the four b_i's has some neighbor in D_1.

The line graph G of H is then the graph G_3. ■

3 Degree conditions for hamiltonicity

In the main result of this section, Theorem 8, we show that for any integer $k \geq 4$, every sufficiently large graph G with minimum degree sum $\sigma_k(G) > n+(k-2)^2$ is either hamiltonian or its closure has small clique covering number (and in this case the method of Section 2 is applicable for finding the classes of exceptions).

Before formulating the main result, we first prove several auxiliary statements.

Lemma 2. *Let G be a closed claw-free graph of order n and $\{a_1, a_2, \ldots, a_t\} \subset V(G)$ an independent set. Then*

(i) $|N(a_i) \cap N(a_j)| \leq 2, \quad 1 \leq i < j \leq t,$

(ii) $\sum_{i=1}^{t} d(a_i) \leq n + t^2 - 2t.$

Proof. (i) Suppose that e.g. $b_1, b_2, b_3 \in N(a_1) \cap N(a_2)$. If $\{b_1, b_2, b_3\}$ is independent, then $\langle\{a_1, b_1, b_2, b_3\}\rangle$ is an induced claw. Hence we can suppose that e.g. $b_1 b_2 \in E(G)$, but then, since G is closed, $a_1 a_2 \in E(G)$, a contradiction.

(ii) By part (i), $\sum_{i=1}^{t} d(a_i) \leq n - t + 2\frac{t(t-1)}{2} = n + t^2 - 2t$. ■

Lemma 3. *(i) Any triangle-free graph H whose matching number $\nu(H)$ and vertex covering number $\tau(H)$ satisfy $\nu(H) < \tau(H)$, contains an edge xy such that $d(x)+d(y) \leq \nu(H)+\tau(H)$.*

(ii) Let G be a closed claw-free graph. If $\alpha(G) < \theta(G)$, then

$$\delta(G) \leq \alpha(G) + \theta(G) - 2.$$

Proof. (i) Let T be a minimum vertex covering of H and choose a maximum matching M of H such that M saturates as few vertices of T as possible. Note that, since T is a vertex covering, $V(H) \setminus T$ is an independent set, and since the matching M is maximum, the set of insaturated vertices is independent.

If T contains an insaturated vertex x, then all the neighbors of x are saturated. If all the vertices of T are saturated by M then, since $\nu(H) < \tau(G)$, $\langle T\rangle$ contains some edge xx' of M. If x has some insaturated neighbor $w \in V(H)\setminus T$, then the matching $M' = (M\setminus\{xx'\})\cup\{xw\}$ contradicts the choice of M. Since all the vertices of $\langle T\rangle$ are saturated, again all the neighbors of x are saturated. Therefore in both cases, since x is adjacent to at most one endvertex of each edge of M by the triangle-freeness hypothesis, $d(x) \leq |M| = \nu(H)$.

The vertex x has at least one neighbor y in $V(H) \setminus T$ for otherwise $T' = T \setminus \{x\}$ is a vertex covering contradicting the minimality of T. Since $V(H) \setminus T$ is independent, $N(y) \subset T$ and thus $d(y) \leq |T| = \tau(H)$, which achieves the proof of Part (i).

(ii) Let H be the triangle-free graph such that $G = L(H)$. Then $\alpha(G) = \nu(H)$, $\theta(G) = \tau(H)$ and the degree of a vertex u of G corresponding to an edge xy of H is equal to $d_H(x) + d_H(y) - 2$. The result is thus a direct consequence of Part (i). ■

Lemma 4. *Let G be a closed claw-free graph. Then*

$$\theta(G) \leq 2\alpha(G).$$

Proof. Let $D = \{d_1, d_2, ...d_t\}$ be a maximal independent set in G. Then we have $V(G) = \bigcup_{i=1}^{t} N(d_i) \bigcup D$. Since $N(d_i) \cup \{d_i\}$, $1 \leq i \leq t$, can be covered by one or two cliques, G can be covered by at most $2t \leq 2\alpha(G)$ cliques. ■

The following proposition shows that a lower bound on degrees of a claw-free graph G implies an upper bound on the clique covering number of its closure $\mathrm{cl}(G)$.

Proposition 5. *Let $k \geq 2$ be an integer and let G be a claw-free graph of order n such that $\delta(G) > 3k-5$ and*

$$\sigma_k(G) > n + k^2 - 2k.$$

Then $\theta(\mathrm{cl}(G)) \leq k-1$.

Proof. If G satisfies the assumptions of the theorem, then clearly so does its closure $\mathrm{cl}(G)$. Hence we can suppose that G is closed.

Let, to the contrary, $\theta(G) \geq k$. For $\alpha(G) \geq k$ we have an immediate contradiction with Lemma 2. Hence $\alpha(G) \leq k-1$, implying $\alpha(G) < \theta(G)$. By Lemma 3 and Lemma 4 then $\delta(G) \leq \alpha(G) + \theta(G) - 2 \leq k - 1 + 2(k-1) - 2 = 3k - 5$, a contradiction. ■

Corollary 6. *Let $k \geq 2$ be an integer and let G be a claw-free graph of order $n \geq 2k^2 - 3k$ and minimum degree*

$$\delta(G) > \frac{n}{k} + k - 2.$$

Then $\theta(\mathrm{cl}(G)) \leq k-1$.

Proof. For $n \geq 2k^2 - 3k$ and $\delta(G) > \frac{n}{k} + k - 2$ clearly $\delta(G) > 3k-5$. The rest of the proof follows immediately from Proposition 5. ■

Example. Let t, k be integers, $k \geq 2$, $t \geq 2k-2$, and let $G = K_k \times K_t$ be the Cartesian product of two cliques K_k, K_t. Then $|V(G)| = n = kt$, $\delta(G) = k + t - 2 \geq 3k - 4 > 3k - 5$ and $\sigma_k(G) = k\delta(G) = k(k+t-2) = n + k^2 - 2k$, but $\theta(G) = k$. This example shows that the lower bounds on $\sigma(G)$ and $\delta(G)$ in Proposition 5 and Corollary 6 are sharp.

However, in the following we show that these lower bounds on $\sigma(G)$ and $\delta(G)$ can be improved under the additional assumption that G is nonhamiltonian.

We again begin with an auxiliary statement.

Lemma 7. *Let G be a closed claw-free graph of order n and connectivity $\kappa(G)$ such that $1 \leq \kappa(G) < \alpha(G)$ and let $A = \{a_1, \ldots, a_\alpha\}$ be a maximum independent set in G. Then*

$$\sum_{i=1}^{\alpha} d(a_i) \leq n + \alpha^2 - 4\alpha + 2 + \kappa(G).$$

Remark. The well-known theorem by Chvátal and Erdős [4] states that every graph G with $\alpha(G) \leq \kappa(G)$ is hamiltonian. Thus, the assumption $\kappa(G) < \alpha(G)$ of Lemma 7 is satisfied by any nonhamiltonian graph G.

Proof. Let $S \subset V(G)$ be a minimum vertex cutset in G (i.e., $|S| = \kappa = \kappa(G)$), and let G_1, G_2 be the components of $G - S$. (Note that, by the minimality of S, each of the vertices of S has adjacencies in all components of $G - S$, and hence $G - S$ has two components since G is claw-free). Let $r = |V(G_1) \cap A|$, $s = |S \cap A|$ and $t = |V(G_2) \cap A|$. Suppose that A is chosen such that s is minimum and the notation is chosen such that $r \leq t$. Since $s \leq \kappa < \alpha$, $t \geq 1$.

By part (i) of Lemma 2, any two vertices $x, y \in A$ can have in G at most two common neighbors. In addition to this fact, we make the following observations.

- If $x \in S \cap A$ and $y \in V(G_i)$ $(i = 1, 2)$, then x and y can have at most one common neighbor outside S, since if e.g. $N(x) \cap N(y) = \{z_1, z_2\} \subset V(G_1)$, then $z_1 z_2 \notin E(G)$ (since G is closed and x, y are independent), but then, for any $v \in N(x) \cap V(G_2)$, $\langle \{x, v, z_1, z_2\} \rangle$ is an induced claw.
- For any vertex $z \in S \setminus A$ there is at most one pair $x, y \in A$ such that $z \in N(x) \cap N(y)$ (since if z is a common neighbor for two different pairs, then z has at least three independent neighbors and hence z is a center of an induced claw).
- If $x \in V(G_1) \cap A$ and $y \in V(G_2) \cap A$, then $N(x) \cap N(y) \subset S \setminus A$ (since S is a cutset and A is independent).

Thus, out of the total $\binom{\alpha}{2}$ pairs of vertices of A, $rs + ts$ pairs have at most one common neighbor outside S, rt pairs have no common neighbor outside S, and $\kappa - s$ vertices in $S \setminus A$ can play the role of common neighbors for at most $\kappa - s$ additional pairs. This gives $\sum_{i=1}^{\alpha} d(a_i) \leq n - \alpha + rs + ts + 2(\frac{\alpha(\alpha-1)}{2} - rs - ts - rt) + \kappa - s = n + \alpha^2 - 2\alpha - 2rt - rs - ts - s + \kappa$, from which, using $t = \alpha - r - s$,

$$\sum_{i=1}^{\alpha} d(a_i) \leq n + \alpha^2 - 2\alpha + \kappa - f(r, s),$$

where

$$f(r, s) = \alpha(2r + s) - 2r^2 - 2rs - s^2 + s.$$

If $r \geq 1$, then, by the definition of r, s and t, $1 \leq r \leq \dfrac{\alpha - s}{2}$ and $0 \leq s \leq \kappa$. A straightforward calculation then shows that, for these values of r, s and under the assumption that $\kappa < \alpha$, the function $f(r, s)$ achieves for $r = 1$ and $s = 0$ its minimum value $f_{min} = 2\alpha - 2$.

If $r = 0$, then necessarily $s \geq 2$, since otherwise adding a vertex of G_1 to A (if $s = 0$) or replacing in A the only vertex of $S \cap A$ by a vertex of G_1 (if $s = 1$) we get a contradiction with the choice of A. Hence we have in this case $f(0, s) = \alpha s - s^2 + s$ for $2 \leq s \leq \kappa$ and again a straightforward checking shows that the minimum value of $f(0, s)$ for $2 \leq s \leq \kappa$ and $2 \leq \kappa \leq \alpha - 1$ is equal to $f_{min} = f(0, 2) = 2\alpha - 2$.

Hence in both cases we have

$$\sum_{i=1}^{\alpha} d(a_i) \leq n + \alpha^2 - 2\alpha + \kappa - f_{min} = n + \alpha^2 - 4\alpha + 2 + \kappa.$$

■

Now we can prove the main result of this section.

Theorem 8. *Let $k \geq 4$ be an integer and let G be a 2-connected claw-free graph with $|V(G)| = n$ such that $n \geq 3k^2 - 4k - 7$, $\delta(G) \geq 3k - 4$ and*

$$\sigma_k(G) > n + k^2 - 4k + 7.$$

Then either $\theta(\mathrm{cl}(G)) \leq k - 1$, or G is hamiltonian.

Remark. In the first case, i.e. if $\theta(\mathrm{cl}(G)) \leq k-1$, then G is hamiltonian or belongs to some of the classes of nonhamiltonian exceptions that can be found by using the method indicated in Section 2.

Proof. If G is a nonhamiltonian graph satisfying the assumptions of the theorem, then clearly so does its closure $\mathrm{cl}(G)$, and hence we can suppose that G is closed. It remains to show that $\theta(G) \leq k - 1$. Let, to the contrary, $\theta(G) \geq k$.

If $\alpha(G) \geq k+1$, then by Lemma 2, we have $\sigma_{k+1}(G) \leq n+(k+1)^2-2(k+1) = n+k^2-1$, implying $\sigma_k(G) \leq \frac{k}{k+1}(n + k^2 - 1) \leq n + k^2 - 4k + 7$ for $n \geq 3k^2 - 4k - 7$, a contradiction. Hence $\alpha(G) \leq k$.

If $\alpha(G) \leq k - 1$, then $\alpha(G) < \theta(G)$ and, by Lemma 3 and Lemma 4, $\delta(G) \leq \alpha(G) + \theta(G) - 2 \leq (k - 1) + 2(k - 1) - 2 = 3k - 5$, a contradiction.

Hence we have $\alpha(G) = k$. By Theorem B, $\kappa(G) \leq 5$ (since $\delta(G) \geq 3k - 4 \geq 8$ for $k \geq 4$). By the Chvátal-Erdős Theorem [4] (see also the remark after Lemma 7), $\kappa(G) < \alpha(G)$. Lemma 7 then gives $\sigma_k(G) \leq n + k^2 - 4k + 7$, a contradiction. ■

From Theorem 8 we obtain the following minimum degree result.

Theorem 9. *Let $k \geq 4$ be an integer and let G be a 2-connected claw-free graph with $|V(G)| = n$ such that $n \geq 3k^2 - 4k - 7$ and*

$$\delta(G) > \frac{n + k^2 - 4k + 7}{k}.$$

Then either $\theta(\mathrm{cl}(G)) \leq k - 1$, or G is hamiltonian.

Proof. For $n \geq 3k^2 - 4k - 7$ and $k \geq 4$ obviously $\delta(G) > \dfrac{n + k^2 - 4k + 7}{k} \geq 3k - 4$. The rest of the proof follows immediately from Theorem 8. ∎

From Theorems 8 and 9 we obtain the following corollaries, in which $\mathcal{F}_3$, $\mathcal{F}_4$, $\mathcal{F}_5$ are the classes introduced in Theorem 1 (see Figures 1, 2, 3).

Corollary 10. *Let G be a 2-connected claw-free graph with $n \geq 77$ vertices such that $\delta(G) \geq 14$ and*

$$\sigma_6(G) > n + 19.$$

Then either $G \in \mathcal{F}_3 \cup \mathcal{F}_4 \cup \mathcal{F}_5$, or G is hamiltonian.

Proof follows immediately from Theorems 8 and 1 by setting $k = 6$. ∎

It is easy to see that Corollary 10 yields in a straightforward way a corresponding minimum degree result. We show that the additive constant in this condition can be slightly inproved.

Corollary 11. *Let G be a claw-free graph of connectivity $\kappa(G) = 2$ with $n \geq 78$ vertices satisfying*

$$\delta(G) > \frac{n + 16}{6}.$$

Then either $G \in \mathcal{F}_3 \cup \mathcal{F}_4 \cup \mathcal{F}_5$, or G is hamiltonian.

Proof. We can again suppose that G is closed. Let $\theta(G) \geq 6$. Similarly as in the proof of Theorem 8, for $\alpha(G) \geq 9$ we have $\sigma_6(G) \leq \frac{6}{9}(n + 63) \leq n + 16$ for $n \geq 78$, and for $\alpha(G) \leq 5$ we have $\delta(G) \leq \alpha(G) + \theta(G) - 2 \leq 13$, both contradicting the assumptions. Hence $6 \leq \alpha(G) \leq 8$.

If $\alpha(G) = 6$, then, by Lemma 7, $\sigma_6(G) \leq n + 16$, implying $\delta(G) \leq \frac{n+16}{6}$, a contradiction.

If $\alpha(G) = 7$, then similarly Lemma 7 gives $\sigma_7(G) \leq n + 25$, implying $\delta(G) \leq \frac{n+25}{7} \leq \frac{n+16}{6}$ for $n \geq 78$, and for $\alpha(G) = 8$ analogously $\sigma_8(G) \leq n + 36$, implying $\delta(G) \leq \frac{n+36}{8} \leq \frac{n+16}{6}$

for $n \geq 78$. This contradiction proves that $\theta(G) \leq 5$. The rest of the proof follows from Theorem 1. ■

Remarks. 1. M.C. Li [10], [11] and later on G. Li, M. Lu and Z. Liu [7] proved that every 3-connected claw-free graph satisfying $\delta(G) \geq \frac{n+9}{6}$ ([10], [11]) or $\delta(G) \geq \frac{n+7}{6}$ ([7]), respectively, is hamiltonian. Hence Corollary 11 remains true if we replace the assumption "$\kappa(G) = 2$" by "2-connected".

2. Corollary 10 improves the strongest known result in this direction by Li, Lu, Tian and Wei [9].

3. Using Corollary 6 instead of Lemma 7, we can get the result of Corollary 11 with $\delta(G) > \frac{n}{6} + 4$ for $n \geq 54$.

4. Trommel, Veldman and Verschut [15] proved that every claw-free graph G of order n and minimum degree $\delta(G) > \sqrt{3n+1} - 2$ contains cycles of all lengths from 3 to the circumference of G. This result immediately implies that all graphs that are hamiltonian by Corollary 11 (and Remark 1) are pancyclic.

4 Appendix

Proof of Theorem 1, case $\theta = 5$.

We follow the notation and terminology introduced in the first part of the proof. Let $B = \{b_1, b_2, b_3, b_4, b_5\}$, and let T be a CT in H with maximum black length. Under this assumption, we assume that $\text{blo}(T)$ is minimum, and under both assumptions that T has also minimum length (length meaning now the number of edges as usual). Then clearly $2 \leq \text{bla}(T) \leq 4$ and $1 \leq \text{blo}(T) \leq 3$. We consider each of these cases separately. The subcase for $\text{bla}(T) = k$ and $\text{blo}(T) = \ell$ will be referred to as Subcase k/ℓ. We assume that the notation is chosen such that T contains b_1, b_2, ..., b_k. For a vertex v outside T, we denote by $R(v)$ the set of vertices of T that are $\bar{T}$-related to v. We also denote by $w_{i,j}$ the white vertex on T between b_i and b_j when there is no ambiguity.

A black vertex $b \notin V(T)$ will be said to be *insertible* if b is $\bar{T}$-related to two vertices of T, r_b^1 and r_b^2, in such a way that if we replace the part of T between r_b^1, r_b^2 by the two paths relating b to r_b^1 and r_b^2, we get a CT containing b and all black vertices of T. We also analogously speak of insertibility for several black vertices $b_{i_1}, \ldots, b_{i_k}$ outside T if they are connected by a b_{i_1}, b_{i_k}-path P and b_{i_j} is $\bar{T}$-related to a vertex r_b^j on T ($j = 1, k$) in such a way that P together with the paths relating b_{i_j} to r_b^j ($j = 1, k$) and with the rest of T yield a CT containing $b_{i_1}, \dots, b_{i_k}$ and all black vertices of T. Clearly no insertible vertex can exist outside T if T is supposed to have maximum black length.

Let F be a subgraph of H' containing $s+1$ $(1 \leq s \leq \theta - 2)$ black vertices $b_1, \dots, b_{s+1}$ and let H'_F be the graph obtained from H' by contracting F to one black vertex (i.e., by replacing F by a new black vetex b, adjacent in H'_F to all vertices of $H' - F$ that were adjacent in H' to come vertex of F). Then clearly $L(H'_F)$ can be covered by $\theta - s$ cliques. In some subcases, we will often meet a situation when it is straightforward to check that, for a certain subgraph F of H', H has a DCT if and only if H'_F has a CT containing all its black vertices. We will then say that the subcase reduces to $\theta - s$ by contracting $b_1, \dots, b_{s+1}$ to a clique. This occurs e.g. if F is the structure $\mathcal{S}$ with a black cutvertex. Another example can be seen in Figure 5, where the edge cutset formed by the matching $\{b_1w_1, b_3w_2\}$ of H separates the subgraph F containing the black vertices b_1 and b_3 from the rest of H. Clearly, H has a DCT if and only if H'_F has a CT containing all black vertices. In $G = L(H)$, the two cliques B_1 and B_3 form a spanning subgraph of one clique with vertex set $V(B_1) \cup V(B_3)$.

Subcase 4/3. There are - up to a symmetry - 2 possibilities:

Subcase 4/3-1. There is a unique black cutvertex, say b_1 common to the three blocks. The black vertices b_2, b_3 and b_4 are respectively inner vertices of each block and related to b_1 using two white vertices. Vertex b_5 is outside T and the only case when it is not insertible corresponds to b_5 being adjacent to the two white vertices of the same block (say for example to the white neighbors of b_2). Then, contracting b_3, b_4, b_1 to a clique, we reduce to $\theta = 3$.

Subcase 4/3-2. There are two black cutvertices, say b_2 and b_3, b_1 is in the same endblock as b_2, b_4 in the same endblock as b_3, every block consists exactly of a C_4 with alternate black and white vertices and b_5 is out of T. If b_5 is not insertible, then necessarily it is adjacent to two white vertices in the same block, but then this case can be reduced to $\theta = 3$ or $\theta = 4$.

Subcase 4/2. Here also, there are two possible structures for T.

Subcase 4/2-1. The cutvertex is white and the two blocks consist exactly of two C_4's with alternate black and white vertices and a common white cutvertex. We also assume that b_1 and b_2 are in, say, block 1, b_3 and b_4 in block 2. Vertex b_5 is out of T and the only case when b_5 is not insertible and blo(T) is minimum corresponds to b_5 being only related to white vertices on T.

Subcase 4/2-1-1: $|R(b_5) \cap T|$ is included in block 1. Then contracting b_3 and b_4 into a clique, we reduce to $\theta = 4$.

Subcase 4/2-1-2: $|R(b_5) \cap T| = 2$ and both white vertices related to b_5 are in different blocks. Then also contracting b_3 and b_4 into a clique reduces this case to $\theta = 4$.

Subcase 4/2-1-3: $|R(b_5) \cap T| = 3$. We then get the exception graph G_4.

Subcase 4/2-2. The cutvertex is black, one block, say block 1, consists exactly of a C_4 with alternate black and white vertices, we also assume that b_1 is in block 1, b_2 in both blocks, b_3 and b_4 in block 2 (with at least one additional white vertex) and b_5 is outside T. The vertex b_5 is not insertible and $\text{blo}(T)$ is minimum only in some cases when b_5 is $\bar{T}$-related to at most one black vertex of T.

Subcase 4/2-2-1: $|R(b_5) \cap B| = 1$. The only two possible cases are $R(b_5) \cap T = \{b_2, w_{3,4}\}$ where $w_{3,4}$ is the (possible) white vertex on T between b_3 and b_4, or $R(b_5) \cap T = \{b_4, w_{2,3}\}$ where $w_{2,3}$ is the (possible) white vertex on T between b_2 and b_3. Both cases are reducible to $\theta = 4$.

Subcase 4/2-2-2: $|R(b_5) \cap B| = 0$. By the non-insertibility and since $\text{blo}(T)$ is minimum, the white vertices in $R(b_5)$ are necessarily in the same block and then the situation is reducible to $\theta = 3$ or 4.

Subcase 4/1. In this subcase we first show that we can suppose that T is a cycle containing b_1, b_2, b_3, b_4 (with possible diagonal edges not on T). Let, to the contrary, $A = \{v_1, \ldots, v_k\} \subset V(T)$ be the set of vertices of T having degree in T at least 4. Since $\text{blo}(T) = 1$, $k \geq 2$. The trail T then consists of at least $2k$ paths $P_1, \ldots, P_\ell$ $(\ell \geq 2k)$ with endvertices in A.

Suppose that $k \geq 3$. Since B is a covering and $|B \cap V(T)| = 4$, at least one vertex of A is black (otherwise some edge of some of the paths P_i remains uncovered by B). But then at least $2k - 3$ of the paths P_i have all interior vertices white, implying that some of these paths yield a cycle C such that all vertices of C outside A are white. Removing the edges of C from T we get a contradiction with the choice of T. Hence $k = 2$.

Let first $A = \{b_3, b_4\} \subset B$. Then similarly at least 2 of the paths P_i have all interior vertices white, contradicting the choice of T.

Next suppose that $A = \{b_4, w\}$ for some white vertex w. If at least two of the remaining black vertices are on the same P_i, we have a cycle containing all black vertices of T. Hence we can suppose P_i contains b_i, $i = 1, 2, 3$, as an interior vertex. Since w is white, $\ell = 4$ and P_4 is the edge wb_4 (otherwise we have and edge with no black vertex). Since H is triangle-free, there are white vertices w_1, w_2, w_3 such that $P_i = wb_iw_ib_4$, $i = 1,2,3$. But then, whichever the vertices in $R(b_5)$ are, we always get a DCT in H.

Finally, let $A = \{w_1, w_2\} \subset V(T) \setminus B$. Then $\ell = 4$, and $P_i = w_1b_iw_2$ for $i = 1,2,3$. For $R(b_5) = \{w_1, w_2\}$ we get the exception class G_9, all other possibilities yield a DCT in H.

Hence we can suppose for the rest of Subcase 4/1 that T is the cycle $b_1b_2b_3b_4b_1$ with possibly a white vertex $w_{i,i+1}$ added between the two consecutive black vertices b_i and b_{i+1} (indices are considered modulo 4) for some values of i from 1 to 4. Some diagonals can also exist except between two white vertices and only if they do not create triangles. Vertex b_5 is out of T and supposed not to be insertible, whence if it is related to b_i it will not be related to $w_{i,i+1}$ nor to b_{i+1} (and symmetrically with $i-1$ instead of $i+1$). We then distinguish two cases.

<u>Subcase 4/1-1:</u> $|R(b_5) \cap B| \geq 1$. By symmetry, we only have to consider three subcases.

<u>Subcase 4/1-1-1:</u> $R(b_5) = \{b_1, b_3\}$. Straightforward checking shows that there are no additional edges (otherwise we have a DCT), and then we have the exception graph G_5 or G_1 or G_2.

<u>Subcase 4/1-1-2:</u> $R(b_5) = \{b_1, w_{2,3}, w_{3,4}\}$. Analogously, there are no additional edges and this graph yields the exception graph G_6.

<u>Subcase 4/1-1-3:</u> $R(b_5) = \{b_1, w_{2,3}\}$. Since we are not in the previous case, no edge from b_5 to T can be added. However, there is no DCT if $b_1w_{3,4} \in E(H)$ or $b_3w_{1,4} \in E(H)$. If $b_1w_{3,4} \notin E(H)$, then, contracting b_3, b_4 and $w_{4,1}$ (if any) into a clique reduces the case to $\theta = 4$, if $b_1w_{3,4} \in E(H)$ and $b_3w_{1,4} \notin E(H)$, then we reduce the situation to $\theta = 4$ by contracting b_1, b_4 and $w_{3,4}$ into a clique. If both $b_1w_{3,4} \in E(H)$ and $b_3w_{1,4} \in E(H)$, we have the exception graph G_{10}.

<u>Subcase 4/1-2:</u> $|R(b_5) \cap B| = 0$. Here we need to distinguish four possibilities.

<u>Subcase 4/1-2-1:</u> $R(b_5) = \{w_{1,2}, w_{2,3}, w_{3,4}, w_{4,1}\}$. There are no additional edges and this graph yields the exception graph G_7.

<u>Subcase 4/1-2-2:</u> $R(b_5) = \{w_{1,2}, w_{2,3}, w_{3,4}\}$. Straightforward checking shows that the only possible additional relation not to get a DCT is (up to symmetry) between b_2 and $w_{4,1}$. If $b_2w_{4,1} \in E(H)$, then $L(H) = G_8$, if $b_2w_{4,1} \notin E(H)$, contracting b_1, $w_{4,1}$, b_4 into one clique reduces the situation to the case $\theta = 4$.

<u>Subcase 4/1-2-3:</u> $R(b_5) = \{w_{1,2}, w_{2,3}\}$. If b_2 has some additional neighbor in T, we can exchange b_2 and b_5 and we then are in Case 4/1-2-1 or in Case 4/1-2-2.

Otherwise (i.e. if b_2 has no other neighbor), contracting $b_1,\ b_3,\ b_4$ into a clique reduces the case to $\theta = 3$.

Subcase 4/1-2-4: $R(b_5) = \{w_{1,2}, w_{3,4}\}$. If there is no black-white edge between any of $b_1,\ b_4,\ w_{4,1}$ and any of $b_2,\ b_3,\ w_{2,3}$, we can reduce the case to $\theta = 3$ by contracting $b_1,\ b_4,\ w_{4,1}$ into a clique and $b_2,\ b_3,\ w_{2,3}$ into another one. So we can suppose $b_1 w_{2,3} \in E(H)$. Then $b_4 w_{2,3} \notin E(H)$ (otherwise $b_1 w_{2,3} b_2 w_{1,2} b_5 w_{3,4} b_3 w_{2,3} b_4 (w_{4,1}) b_1$ is a DCT). If $b_3 w_{4,1} \in E(H)$, then checking that any additional relation yields a DCT, we have $L(H) = G_8$. Hence also $b_3 w_{4,1} \notin E(H)$. This implies that there is no relation between $b_1,\ b_2,\ w_{1,2},\ w_{2,3},\ w_{4,1}$ and the rest of $V(H)$, but then contracting these vertices into one clique reduces the situation to $\theta = 4$.

Subcase 3/2. Assume that b_2 is the cutvertex of T, b_1 is in block 1, b_3 in block 2, with both blocks consisting of a C_4 with alternate black and white vertices. Vertices b_4 and b_5 are out of T. If b_4 and b_5 are related, then they both also have a relation on T, otherwise the case can be reduced to $\theta = 3$ or 4. If at least one relation is a black vertex on T, or both are white but not in the same block, then in any case b_4 and b_5 constitute an insertible path. If both relations on T are white in the same block, we can reduce to $\theta = 4$. So we assume that b_4 and b_5 are not related. If b_4 or b_5 is related to a black vertex on T or to two white vertices in different blocks of T, then it is insertible. Hence each of b_4 and b_5 is related to the two white vertices of a block. If both are related to the white vertices of the same block, we get a DCT, otherwise, contracting for example b_2, b_3 and b_5 into a clique, we reduce to $\theta = 3$.

Subcase 3/1. T is the cycle $b_1 b_2 b_3 b_1$ with at least one and at most three white vertices added between two black vertices, and with no additional edges. As in Subcase 3/2, b_4 and b_5 cannot be related (otherwise they either constitute an insertible path or yield a CT of black length 4). If b_4 or b_5 has two consecutive relations on T (one of them black), then we also have the insertibility property. It remains to consider the case when b_4 and b_5 are related either to two white vertices or to a black vertex and a non-consecutive white vertex. Checking all the different possible combinations, we always get a closed trail of black length 4 or 5.

Subcase 2/1. We immediately see that this case never occurs since we can always obtain a closed trail of black length at least 3. ■

Remark. The authors were recently informed that a result analogous to Corollary 11 was independently obtained by E.J. Kuipers and H.J. Veldman.

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