On weights of induced paths and cycles in claw-free and $K_{1,r}$-free graphs

J. Harant
M. Voigt
DEPARTMENT OF MATHEMATICS
TECHNICAL UNIVERSITY OF ILMENAU
D-98684 ILMENAU, GERMANY

S. Jendroľ [1]
DEPARTMENT OF GEOMETRY AND ALGEBRA
P.J.ŠAFÁRIK UNIVERSITY
SK-04154 KOŠICE, SLOVAKIA

B. Randerath
DEPARTMENT II OF MATHEMATICS
RWTH AACHEN
D-52056 AACHEN, GERMANY

Z. Ryjáček[2]
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF WEST BOHEMIA
CZ-30614 PILSEN, CZECH REPUBLIC

I. Schiermeyer[3]
DEPARTMENT OF MATHEMATICS
TECHNICAL UNIVERSITY OF COTTBUS
D-03013 COTTBUS, GERMANY

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ABSTRACT

Let G be a $K_{1,r}$-free graph ($r \geq 3$) on n vertices. We prove that, for any induced path or induced cycle on k vertices in G ($k \geq 2r-1$ or $k \geq 2r$, respectively), the degree sum of its vertices is at most $(2r-2)(n-\alpha)$ where α is the independence number of G.
As a corollary we obtain an upper bound on the length of a longest induced path and a longest induced cycle in a $K_{1,r}$-free graph.
Stronger bounds are given in the special case of claw-free graphs (i.e. $r = 3$). Sharpness examples are also presented.

1. INTRODUCTION

Claw-free graphs have been a subject of interest of many authors in the last years (see e.g. a recent survey by Faudree et al. [6]). For this class of graphs we investigate problems which have their origin in the theory of planar graphs.

Throughout the paper we use the most common graph theoretical terminology. For the concepts not defined here we refer to [1].

A graph G is called $K_{1,r}$-free if there is no induced subgraph of G isomorphic to the complete bipartite graph $K_{1,r}$. By a *claw* we mean the graph $K_{1,3}$. Thus, a graph G is said to be *claw-free* if it does not contain an induced subgraph that is isomorphic to $K_{1,3}$. For a subgraph H of G the *weight* $w_G(H)$ of H in G is the sum of degrees of vertices of H in G, i.e.

$$w_G(H) := \sum_{v \in V(H)} d_G(v).$$

If no ambiguity can arise we write simply $w(H)$ instead of $w_G(H)$. Grünbaum [7] in connection with a beautiful result of Kotzig, who proved [11] that every planar 3-connected graph contains an edge $e = uv$ of weight $w(e) = d(u) + d(v) \leq 13$, 13 being best possible. This result served as starting point for many investigations mainly in polyhedral graphs.

Ivančo [8] proved that every graph of minimum degree at least 3 and of orientable genus g contains an edge e of weight $w(e) \leq 2g + 13$ if $0 \leq g \leq 2$ and $w(e) \leq 4g + 7$ if $g \geq 3$.

At the Conference at Prachatice (Czechoslovakia) in 1990 P. Erdős posed the question of finding an upper bound for the minimum edge weight in the class $\mathcal{G}(n, m)$ of all graphs having n vertices and m edges. A first step towards a solution was made by Ivančo and Jendroľ [9]. For the precise answer to this question see Jendroľ and Schiermeyer [10].

In planar graphs, results analogous to Kotzig's have been achieved only recently for connected subgraphs of order t with $t \geq 3$. Enomoto and Ota [3] proved that every 3-connected planar graph G of order at least t contains a connected subgraph H of order t such that $w_G(H) \leq 8t - 1$. Furthermore, they found a graph G^* in which each connected subgraph K of order t has weight $w_{G^*}(K) \geq 8t - 5$. Fabrici and Jendroľ[4, 5] proved that every 3-connected planar graph G with $\Delta(G) \geq 5k$ contains a path on k vertices P_k such that $w(P_k) \leq 5k^2$. They constructed a graph G^* in which every path P on k vertices has weight $w_{G^*}(P) \geq k \log_2 k$.

Mohar [12] has proved that each hamiltonian planar graph of order n contains a path P_k on k vertices, $k \leq n$, having weight $w(P_k) \leq 6k-1$. The bound $6k-1$ is tight. Mohar's approach allows to prove that each hamiltonian graph of order n and of size m has a path P_k on k vertices, $k \leq n$, such that $w(P_k) \leq \frac{2km}{n}$.

During the C^5-workshop at Burg near Cottbus in November 1997 the authors started investigations of subgraphs having restricted weights in the family of claw-free graphs on n vertices. Since complete graphs are claw-free, there is no interesting upper bound for $w_G(H)$ if H is considered to be a subgraph of G. However, the situation changes if H is required to be an induced subgraph of G. The present paper gives first results concerning claw-free graphs which contain induced paths or cycles.

We consider a connected simple claw-free graph G with vertex set $V(G)$, edge set $E(G)$ and of order $n = |V(G)|$.

Assume G has an induced path on k vertices denoted by P_k. Analogously, by C_k denote an induced cycle on k vertices. We are interested in an upper bound for $w(P_k)$ and $w(C_k)$ of such a path or cycle, respectively.

Theorems 1, 2 and 3 give our main results which will be proved in Section 2.

Theorem 1. Let G be a connected claw-free graph of order n. Let I^* be a maximum independent set of G with $\alpha = |I^*|$.
Then for every induced path P_k in G ($k \geq 5$),

$$w(P_k) \leq 4n - 6\alpha + 2k - 2t + \min(t, 2) - (r_3 + r_1 + 2r_0),$$

where t denotes the number of vertices of I^* belonging to P_k and r_i is the number of vertices belonging neither to I^* nor to P_k and having exactly i neighbors on P_k.
Furthermore, for every induced cycle C_k in G ($k \geq 6$),

$$w(C_k) \leq 4n - 6\alpha + 2k - 2\bar{t} - (\bar{r_3} + 2\bar{r_0}),$$

where $\bar{t}$ denotes the number of vertices of I^* belonging to C_k and $\bar{r_i}$ is the number of vertices belonging neither to I^* nor to C_k and having exactly i neighbors on C_k.

Note that the bound is computable in polynomial time since the determination of a maximum independent set is known to be polynomial in the class of claw-free graphs [**?**].

The following theorem gives a more transparent result.

Theorem 2. Let G be a connected claw-free graph of order n and independence number α. If H is an induced path of length at least 5 or an induced cycle of length at least 6, then

$$w(H) \leq 4n - 4\alpha.$$

In fact, we can prove a little bit more. Let I^* be a maximum independent set of G, P_k an induced path of G ($k \geq 5$) and t the number of vertices of I^* belonging to P_k. Then, $w(P_k) \leq 4n - 4\alpha - 2 + \min(t, 2)$.

It is interesting to ask about the sharpness of these bounds. In Section 3 we shall give examples which attain equality in Theorems 1 and 2. Furthermore we shall show that the coefficients in Theorem 1 are best possible.

Theorem 2 can be generalized to $K_{1,r}$-free graphs.

Theorem 3. Let G be a connected $K_{1,r}$-free graph ($r \geq 3$) of order n and let I^* be a maximum independent set of G with $\alpha = |I^*|$. If H is an induced path of length at least $2r-1$ or an induced cycle of length at least $2r$ in G, then

$$w(H) \leq (2r-2)(n-\alpha) - (r-3)(k-t) \leq (2r-2)(n-\alpha),$$

where t denotes the number of vertices of I^* belonging to H.

Again, for every induced path P_k in G ($k \geq 2r-1$) we have a refinement of the inequality: $w(P_k) \leq (2r-2)(n-\alpha) - 2 + \min(t, 2) - (r-3)(k-t)$.

Note that the argument used in the proof of Theorem 3 admits proving an analogous result for any independent set I. Replacing α by $|I|$ we obtain a result that is slightly weaker but easily computable.

Theorem 3 and further information on $\alpha(G)$ yield immediately an upper bound on the number of vertices of an induced path or induced cycle in a $K_{1,r}$-free graph. An example is the following

Corollary 4. Let G be a $K_{1,r}$-free graph having n vertices, m edges and minimum degree δ. If G contains an induced path P_k or an induced cycle C_k on k vertices, then $k \leq (2r-2)\frac{2mn}{\delta(2m+n)}$.

Proof. Y. Caro [2] and V.K. Wei [13] independently proved that $\alpha(G) \geq \sum_{x \in V(G)} \frac{1}{1+d_G(x)}$ for an arbitrary graph G. Using Jensen's inequality $\phi(\sum \lambda_i x_i) \leq \sum \lambda_i \phi(x_i)$ for any convex function ϕ and $\sum \lambda_i = 1, \lambda_i \geq 0$ we have $\alpha(G) \geq \frac{n^2}{2m+n}$. With $w(P_k) \geq k\delta$, $w(C_k) \geq k\delta$ and Theorem 3 the Corollary follows. □

2. PROOFS OF THE MAIN RESULTS

In the sequel we usually consider the case where H is an induced path P_k and discuss the situation for the induced cycle C_k only if the result differs from the first one.

The vertices of P_k (C_k) will be denoted by $v_1, \ldots, v_k$. We consider a maximum independent set I^* of G with $\alpha = |I^*|$. Let $T = I^* \cap P_k$ ($\bar{T} = I^* \cap C_k$) be the set of vertices of I^* which belong to P_k (C_k) and $I = I^* \setminus P_k$ ($\bar{I} = I^* \setminus C_k$) the set of vertices of I^* which do not belong to P_k (C_k). The cardinality of T ($\bar{T}$) is denoted by t ($\bar{t}$). The set $R = V(G) \setminus (P_k \cup I)$ ($\bar{R} = V(G) \setminus (C_k \cup I)$) is the set of remaining vertices belonging neither to I nor to P_k (C_k). Let $N(v)$ be the set of neighbors of v and $R_i = \{v \in R\ ;\ |N(v) \cap P_k| = i\}$ $(\bar{R_i} = \{v \in \bar{R}\ ;\ |N(v) \cap C_k| = i\})$ be the set of vertices of R $(\bar{R})$ which are adjacent to exactly i vertices of P_k (C_k). The number of elements of R_i $(\bar{R_i})$ is denoted by r_i $(\bar{r_i})$. Note that $R_i = \emptyset$ for all $i \geq 5$ since otherwise $v \in R_i$ $(i \geq 5)$ and three of its neighbors on the path build a claw in G which is forbidden. The same statement is true for $\bar{r_i}$ if $|C_k| \geq 6$. Furthermore for the induced cycle we have immediately $\bar{R_1} = \emptyset$ because otherwise a claw occurs. Thus $\bar{r_1}$ does not occur in the inequalities.

Now let us consider the cardinality of the following edge sets.

(1) $E_I = \{uv \mid u \in P_k \textit{ and } v \in I\}$ is the set of edges between P_k and I,
(2) $E_R = \{uv \mid u \in P_k \textit{ and } v \in R\}$ is the set of edges between P_k and R.

Denote the corresponding edge sets for the cycle by $\bar{E_I}$ and $\bar{E_R}$.
Obviously we have

$$w(P_k) = 2k - 2 + |E_I| + |E_R|, \quad w(C_k) = 2k + |\bar{E_I}| + |\bar{E_R}|. \tag{1}$$

Lemma 5.

$$|E_I| \leq 2k - 4t + \min(t, 2) \leq 2k - 4t + 2. \tag{2}$$

Proof. As G is claw-free, every vertex has at most two neighbors in I^* and no vertex of T has a neighbor in I. Hence, $|E_I| \leq 2(k-t)$. This estimation is sharp if every vertex of P_k except the vertices of T has exactly two neighbors in I.

First assume that $t \geq 2$ and consider an interval $v_i, \ldots, v_j$ $(i < j)$ of vertices of the path where v_i and v_j belong to $T = I^* \cap P_k$ and no vertex v_s with $i < s < j$ is an element of T. We want to show that every such interval contains either a vertex which has no neighbor in I or two vertices which have at most one neighbor in I. Compared with the above estimation we lose in both cases two edges for each of the $t-1$ intervals.

First note that $j \neq i+1$. If $j = i+2$ then v_{i+1} has no neighbor in I. If $j > i+2$ then both v_{i+1} and v_{j-1} have at most one neighbor in I. It follows that $|E_I| \leq 2(k-t) - 2(t-1) = 2k - 4t + 2$.

If $t = 0$, it follows immediately that $|E_I| \leq 2k = 2k - 4t$. For $t = 1$ we have $|E_I| \leq 2k - 3 = 2k - 4t + 1$. □

By the same argument we obtain immediately the following result.

Lemma 6. If $v_1 \notin T$ or $v_k \notin T$, then $|E_I| \leq 2k - 4t + 1$. If both $v_1 \notin T$ and $v_k \notin T$, then $|E_I| \leq 2k - 4t$. □

If we consider induced cycles then there are no end-vertices which play a special role and so we obtain the following upper bound.

Lemma 7.

$$|\bar{E_I}| \leq 2k - 4\bar{t}. \tag{3}$$

Since every vertex of R has at most four neighbors in P_k $(k \geq 5)$ we obtain a first estimation for $|E_R|$:

$$|E_R| \leq 4(n - \alpha - (k - t)). \tag{4}$$

The analogous inequality is true for $|\bar{E_R}|$ of an induced cycle C_k with at least 6 vertices. From the equalities and inequalities (1) - (4) it follows

$$w(P_k) \leq 2(k-1) + 2k - 4t + \min(t,2) + 4(n - \alpha - (k-t)) = 4n - 4\alpha - 2 + \min(t,2)$$

and

$$w(C_k) \leq 2k + 2k - 4\bar{t} + 4(n - \alpha - (k - \bar{t})) = 4n - 4\alpha,$$

proving Theorem 2. ■

If we consider $K_{1,r}$-free graphs with $r \geq 3$ we obtain by analogous arguments

$$|E_I| \leq (r-1)k - (r+1)t + \min(t,2) \leq (r-1)k - (r+1)t + 2,$$

$$|\bar{E_I}| \leq (r-1)k - (r+1)\bar{t},$$

$$|E_R|, |\bar{E_R}| \leq (2r-2)(n - \alpha - (k-t)),$$

proving Theorem 3. ■

In the sequel we consider again the special case $r = 3$, i.e. G is assumed to be claw-free. Now we can estimate $|E_R|$ more carefully.

Lemma 8.

$$|E_R| \leq 4n - 6\alpha - 2k + 2t + 2 - (r_3 + r_1 + 2r_0) \tag{5}$$

and

$$|\bar{E_R}| \leq 4n - 6\alpha - 2k + 2\bar{t} - (\bar{r_3} + 2\bar{r_0}). \tag{6}$$

Proof. Let $I_1 = \{u \in I \mid N(u) \cap P_k \neq \emptyset\}$ be the set of vertices of I which have at least one neighbor on P_k whereas $I_0 = \{u \in I \mid N(u) \cap P_k = \emptyset\}$ is the set of vertices of I which have no neighbor on P_k. Denote the cardinality of I_1 by y_1 and the cardinality of I_0 by y_0.

(a) $y_0 \le r_0 + r_1 + r_2$
Notice first that every vertex of I_0 has a neighbor in R since G is connected.

Assume $a \in I_0$ and let $a_1 \in R$ be a neighbor of a. Then $a_1 \notin R_i$ for $i \ge 3$ since otherwise two of the neighbors of a_1 on the path together with a_1 and a build a claw. Thus, if $|I_0| \in \{0, 1\}$ we are done.

Now assume $a, b \in I_0$ $(a \ne b)$ and consider shortest paths $(a, a_1, \ldots, a_s)$ and $(b, b_1, \ldots, b_q)$ such that $a_s, b_q \in P_k$ and $a_1, \ldots, a_{s-1}, b_1, \ldots, b_{q-1} \notin P_k$.

Consequently, $a_1, b_1 \in R_0 \cup R_1 \cup R_2$ because a and b have no neighbors on $P_k \cup I$ and vertices of R_i, $i \ge 3$ have no neighbors in I_0.

If $a_1 = b_1$, then a, b, a_1 and a_2 induce a claw, a contradiction.

It follows that $a_1 \ne b_1$ for every pair of vertices $a, b \in I_0$. Thus every vertex of I_0 has its private neighbor in $R_0 \cup R_1 \cup R_2$. Hence $y_0 = |I_0| \le r_0 + r_1 + r_2$.

(b) $y_1 \le k - 2t + 1$ for P_k and $\bar{y_1} \le k - 2\bar{t}$ for the cycle C_k

If a vertex $w \in I$ has exactly one neighbor v_i on P_k, then $i = 1$ or $i = k$, otherwise v_{i-1}, v_i, v_{i+1} and w build a claw.

If $v_1 \in T$ and $v_k \in T$, then none of them has a neighbor in I because they belong to the independent set. Thus all vertices of I_1 have at least 2 neighbors on P_k. Using inequality (2) it follows $2y_1 \le |E_I| \le 2k - 4t + 2$ and $y_1 \le k - 2t + 1$.

If $v_1 \notin T$ and $v_k \in T$, then there is at most one vertex $w \in I_1$ which has only one neighbor (namely v_1) on P_k, otherwise there would be a claw. Using Lemma 6 it follows that $2y_1 - 1 \le |E_I| \le 2k - 4t + 1$ and $y_1 \le k - 2t + 1$.

The same arguments can be applied for the case $v_1 \in T$ and $v_k \notin T$.

If $v_1 \notin T$ and $v_k \notin T$, then two of the vertices of I_1 can possibly have exactly one neighbor on P_k. Using Lemma 6 it follows $2y_1 - 2 \le |E_I| \le 2k - 4t$ and $y_1 \le k - 2t + 1$.

For the cycle C_k we have always $2\bar{y_1} \le |\bar{E_I}|$ and together with inequality (3) we obtain $\bar{y_1} \le k - 2\bar{t}$.

(c) $\alpha - k + t - 1 \le r_0 + r_1 + r_2$ for P_k and $\alpha - k + t \le r_0 + r_1 + r_2$ for C_k

Notice that $\alpha = t + y_1 + y_0$. Thus $\alpha - y_0 - t = y_1 \le k - 2t + 1$ and $\alpha - k + t - 1 \le y_0 \le r_0 + r_1 + r_2$. The result for the cycle can be obtained in an analogous way.

Obviously we have $|E_R| = 4r_4 + 3r_3 + 2r_2 + r_1 = 4(r_4 + r_3 + r_2 + r_1 + r_0) - (r_3 + 2r_2 + 3r_1 + 4r_0) = 4(n - \alpha - (k - t)) - 2(r_2 + r_1 + r_0) - (r_3 + r_1 + 2r_0)$. The application of inequality (c) gives $|E_R| \le 4(n - \alpha - (k - t)) - 2(\alpha - k + t - 1) - (r_3 + r_1 + 2r_0) = 4n - 6\alpha - 2k + 2t + 2 - (r_3 + r_1 + 2r_0)$.

Analogous arguments give the result for C_k. □

Using the equalities and inequalities (1, 2) and (5) we obtain

$$w(P_k) \le 4n - 6\alpha + 2k - 2t + \min(t, 2) - (r_3 + r_1 + 2r_0)$$

and using (1, 3) and (6) we have

$$w(C_k) \leq 4n - 6\alpha + 2k - 2\bar{t} - (\bar{r_3} + 2\bar{r_0})$$

which proves Theorem 1. ■

3. SHARPNESS OF THE UPPER BOUNDS

In this section we prove the following results concerning the sharpness of the proved bounds.

Theorem 9. For every n, α, k, there exists a graph G such that $r_0 + r_1 + r_2 + r_3 = 0$ ($\bar{r_0} + \bar{r_2} + \bar{r_3} = 0$) and equality holds in both Theorem 1 and Theorem 2.

Theorem 10. For every n, α, k, there exists a graph G such that $r_0 + r_1 + r_2 + r_3 > 0$ ($\bar{r_0} + \bar{r_2} + \bar{r_3} > 0$) and equality holds in Theorem 1. The coefficients of r_0, r_1, r_2 and r_3 ($\bar{r_0}, \bar{r_2}, \bar{r_3}$) of the bounds in Theorem 1 are best possible.

First, we consider induced paths on k vertices where k is odd.

Notice that the independence number α of G defined as the number of vertices in a maximum independent set is at least $\frac{k+1}{2}$ because $\frac{k+1}{2}$ vertices of the induced path P_k build an independent set.

Proof of Theorem 9
Since $r_0 + r_1 + r_2 + r_3 = 0$ and by the inequalities (a) and (b) of the proof of Lemma 8, we have $y_0 = 0$ and $|I| = y_1 + y_0 = y_1 = \alpha - t \leq k - 2t + 1$. Thus, it follows $\alpha \leq k - t + 1 \leq k + 1$. For $\alpha = k - t + 1$ we have

$$4(n - \alpha) - 2 + \min(t, 2) = 4n - 6\alpha + 2k - 2t + \min(t, 2) - (r_1 + r_3 + 2r_0).$$

Thus Theorem 1 and Theorem 2 give the same bound for $\alpha = k - t + 1$ and it is sufficient to find graphs such that one of the bounds is sharp.

In the case $\alpha < k - t + 1$ and $R_0 \cup R_1 \cup R_2 \cup R_3 = \emptyset$, Theorem 2 gives a better bound than Theorem 1 whereas Theorem 1 gives a better bound for $\alpha > k - t + 1$. Consequently, we investigate the case $\alpha = k - t + 1$ where $0 \leq t \leq \frac{k+1}{2}$ by definition. We shall construct graphs G_i with independence number $\alpha_i = \lfloor \frac{k+1+i}{2} \rfloor$ for $0 \leq i \leq \frac{k+1}{2}$.

Lemma 11. Let k, t, n and α be positive integers such that k is odd, $0 \leq t \leq \frac{k+1}{2}$, $n \geq \alpha + k - t$ and $\alpha = k - t + 1$. Then there is a graph G of order n with independence number α which has an induced path on k vertices such that $w(P_k) = 4n - 4\alpha - 2 + \min(t, 2)$.

Note that for the case $t \geq 2$ even the bound $4n - 4\alpha$ is sharp.

Proof. First assume $t \geq 2$. Thus $\alpha \leq k-1$. Define graphs G_i where $0 \leq i \leq k-3$ in the following way:

- G_0 corresponds to an induced path $P_k = (v_1, v_2, \ldots, v_k)$.
- For $i = 1$ to $k-3$ add a vertex y_i to the graph G_{i-1} and add the edges $y_i v_{i+1}$ and $y_i v_{i+2}$.

Let $w_i(P_k) = \sum_{v \in P_k \subseteq V(G_i)} d(v)$ be the weight of the path P_k in G_i, α_i be the independence number of G_i, n_i be the order of G_i and t_i be the number of vertices of a maximum independent set which belong to P_k in G_i.

Consider the case that i is even. Obviously, we have $\alpha_i = \frac{k+1+i}{2}$, $t_i = \frac{k+1-i}{2}$, $n_i = k+i$ and $w_i(P_k) = 2(k-1)+2i$. Thus it follows $w_i(P_k) = 4(n_i - \alpha_i)$. Notice that α runs from $\frac{k+1}{2}$ to $k-1$.

Now let $\alpha = k$. Thus $t = 1$.

Construct a graph G_{k-1} by adding vertices z_1 and z_2 to the graph G_{k-3} and joining z_1 with v_1 and v_2 and z_2 with v_1 (see Fig. 1).

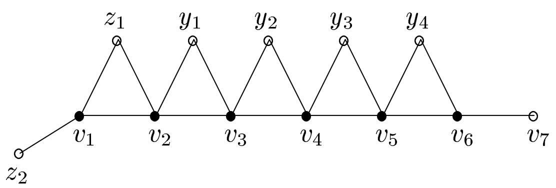

Figure 1. Graph G_6 for $k = 7$

We have $\alpha_i = \frac{k+1+i}{2} = k$, $t_i = \frac{k+1-i}{2} = 1$, $n_i = k+i = 2k-1$ and $w_i(P_k) = 2(k-1)+2i = 4k-5$. It follows that $w_i(P_k) = 4(n_i - \alpha_i) + 2 - \min(t, 2)$.

For $\alpha = k+1$ we have $t = 0$.

Construct a graph G_{k+1} by adding vertices x_1 and x_2 to the graph G_{k-1} and joining x_1 with v_{k-1} and v_k and x_2 with v_k. It is easy to see that $\alpha_i = \frac{k+1+i}{2} = k+1$, $t_i = \frac{k+1-i}{2} = 0$, $n_i = k+i = 2k+1$ and $w_i(P_k) = 2(k-1)+2k = 4k-2$. It follows that $w_i(P_k) = 4(n_i - \alpha_i) + 2 - \min(t, 2)$.

So far, all constructed graphs G_i have the minimum possible number of vertices $n_i = k + \alpha_i - t_i$. Now, we shall construct graphs G_i^s which have $n_i^s = k + \alpha_i - t_i + s$ vertices.

The graph G_i^s can be obtained from the graph G_i by adding a clique on s vertices $u_1, \ldots, u_s$, joining every vertex u_j with four consecutive vertices $v_\ell, \ldots, v_{\ell+3}$ ($\ell \geq 2$) of the path P_k, say v_2, v_3, v_4, v_5, and with two additional vertices $y_{\ell-1}$ (y_1) and $y_{\ell+1}$ (y_3) if they occur in G_i (see Fig 2).

Observe that G_i^s is connected, claw-free and except for the equality for the number of vertices it fulfils the same equalities as G_i considered above. $\square$

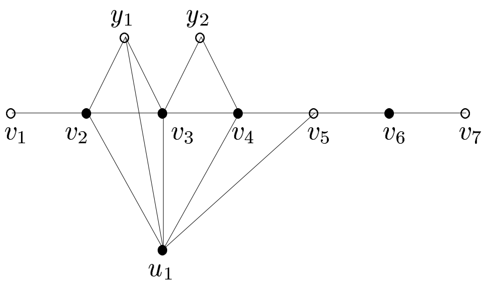

Figure 2. Graph G_2^1 for $k = 7$

If we consider induced paths P_k on even number of vertices, then the analogous graphs for odd i give sharp examples. The corresponding constructions lead to sharp examples for cycles.

Thus the proof of Theorem 9 is complete. ■

By adding a suitable number of isolated vertices the above constructed graphs can be extended to obtain sharp examples for the case $\alpha > k$ or $\alpha > k + 1$, respectively.

Proof of Theorem 10

First, we again consider induced paths on k vertices where k is odd.

Sharp example for Theorem 1 if $r_3 \neq 0$.

Construct a graph H_3^s in the following way (see Fig. 3). Take an induced path on k vertices $v_1, \ldots, v_k$, add a clique with s vertices $u_1, \ldots, u_s$ and join every vertex u_i with three consecutive vertices of the path, say v_2, v_3 and v_4. Denote the corresponding parameters by n_3^s, α_3^s, t_3^s and $w_3^s(P_k)$.

We have $n_3^s = k+s$, $\alpha_3^s = t_3^s = \frac{k+1}{2}$ (for a suitable choice of the maximum independent set), $r_3 = s$ and $w_3^s(P_k) = 2(k-1) + 3s$.

Furthermore we have $4n_3^s - 6\alpha_3^s + 2k - 2t_3^s + \min(t_3^s, 2) = 2(k-1) + 4s$. Thus, in Theorem 1 equality holds for these graphs and the coefficient "-1" corresponding to r_3 in the bound is best possible.

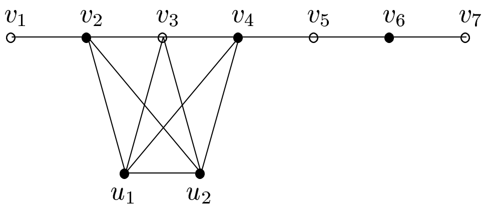

Figure 3. Graph H_3^2 for $k = 7$

We can combine this example with one of the following to get sharp examples for $r_3 \neq 0$ and α arbitrarily large.

Sharp example for Theorem 1 if $r_2 \neq 0$.

Construct a graph H_2^s in the following way (see Fig. 4). Take an induced path on k vertices $v_1, \ldots, v_k$, add a clique with s vertices $u_1, \ldots, u_s$ and join every vertex u_i with two consecutive vertices of the path, say v_2 and v_3. Furthermore add s vertices $z_1, \ldots, z_s$ and join z_i with u_i for every i. Denote the corresponding parameters by n_2^s, α_2^s, t_2^s and $w_2^s(P_k)$.

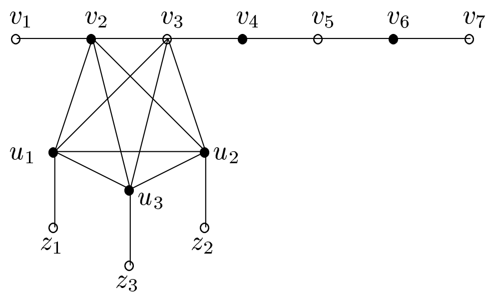

Figure 4. Graph H_2^3 for $k = 7$

We have $n_2^s = k + 2s$, $\alpha_2^s = \frac{k+1}{2} + s$, $t_2^s = \frac{k+1}{2}$, $r_2 = s$ and $w_2^s(P_k) = 2(k-1) + 2s$.

Furthermore we have $4n_2^s - 6\alpha_2^s + 2k - 2t_2^s + \min(t_2^s, 2) = 2(k-1) + 2s$. Thus, equality holds in Theorem 1 for these graphs and the coefficient "0" corresponding to r_2 in the bound is best possible.

Sharp example for Theorem 1 if $r_1 \neq 0$.

Notice first that every vertex of R_1 has to be adjacent to v_1 or v_k of the path (otherwise there is a claw).

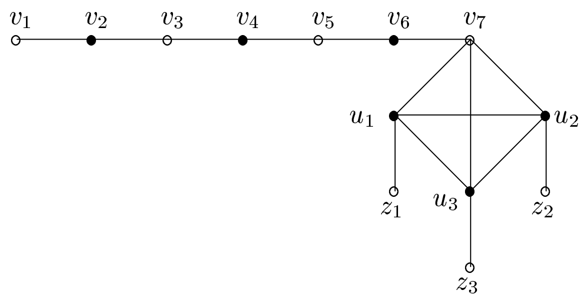

Figure 5. Graph H_1^3 for $k = 7$

Construct a graph H_1^s in the following way (see Fig. 5). Take an induced path on k vertices $v_1, \ldots, v_k$, add a clique with s vertices $u_1, \ldots, u_s$ and join every vertex u_i with

v_1 or v_k, say v_k. Furthermore add s vertices $z_1, \ldots, z_s$ and join z_i with u_i for every i. Denote the corresponding parameters by n_1^s, α_1^s, t_1^s and $w_1^s(P_k)$.

We have $n_1^s = k + 2s$, $\alpha_1^s = \frac{k+1}{2} + s$, $t_1^s = \frac{k+1}{2}$, $r_1 = s$ and $w_1^s(P_k) = 2(k-1) + s$. Furthermore we have $4n_1^s - 6\alpha_1^s + 2k - 2t_1^s + \min(t_1^s, 2) = 2(k-1) + 2s$. Thus, equality holds in Theorem 1 for these graphs and the coefficient "-1" corresponding to r_1 in the bound is best possible.

Sharp example for Theorem 1 if $r_0 \neq 0$.

Construct a graph H_0^s in the following way (see Fig. 6). Take a graph H_2^s and add s vertices $a_1, \ldots, a_s$ and s vertices $b_1, \ldots, b_s$. Join every vertex a_i with z_i and b_i.

Denote the corresponding parameters by n_0^s, α_0^s, t_0^s and $w_0^s(P_k)$.

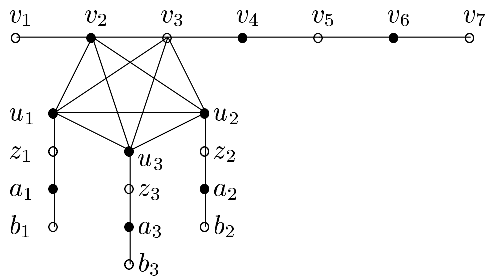

Figure 6. Graph H_0^3 for $k = 7$

We have $n_0^s = k + 4s$, $\alpha_0^s = \frac{k+1}{2} + 2s$, $t_0^s = \frac{k+1}{2}$, $r_0 = s$ and $w_0^s(P_k) = 2(k-1) + 2s$.

Furthermore we have $4n_0^s - 6\alpha_0^s + 2k - 2t_0^s + \min(t_0^s, 2) = 2(k-1) + 4s$. Thus, equality holds in Theorem 1 for these graphs and the coefficient "-2" corresponding to r_0 in the bound is best possible.

For all constructed graphs we may add a clique on x vertices each of them joined in an appropriate way with the same four consecutive vertices of the path obtaining sharp examples for every possible n. Sometimes additional edges like in Fig. 2 will be necessary to avoid a claw.

If we consider induced paths P_k where k is even then we may start the constructions e.g. with a path on k vertices and an additional vertex y joined with v_2 and v_3. Sometimes additional edges like in Fig. 2 will be necessary to avoid a claw. In this way we obtain sharp examples for that case.

Corresponding constructions provide sharp examples for induced cycles on k vertices.

This completes the proof of Theorem 10. ■

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