On factors of 4-connected claw-free graphs

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Abstract

We consider the existence of several different kinds of factors in 4-connected claw-free graphs. This is motivated by the following two conjectures which are in fact equivalent by a recent result of the third author. Conjecture 1 (Thomassen): Every 4-connected line graph is hamiltonian, i.e. has a connected 2-factor. Conjecture 2 (Matthews and Sumner): Every 4-connected claw-free graph is hamiltonian. We first show that Conjecture 2 is true within the class of hourglass-free graphs, i.e. graphs that do not contain an induced subgraph isomorphic to two triangles meeting in exactly one vertex. Next we show that a weaker form of Conjecture 2 is true, in which the conclusion is replaced by the conclusion that there exists a connected spanning subgraph in which each vertex has degree two or four. Finally we show that Conjectures 1 and 2 are equivalent to seemingly weaker conjectures in which the conclusion is replaced by the conclusion that there exists a spanning subgraph consisting of a bounded number of paths.

Keywords: claw-free graph, line graph, Hamilton cycle, Hamilton path, factor

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1 Introduction

We use [1] for terminology and notation not defined here.

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Most of the results in this paper are motivated by the following two conjectures due to THOMASSEN [13] and MATTHEWS AND SUMNER [10], respectively. A graph is *claw-free* if it does not contain an induced subgraph isomorphic to $K_{1,3}$.

Conjecture 1
Every 4-connected line graph is hamiltonian.

Conjecture 2
Every 4-connected claw-free graph is hamiltonian.

Since line graphs are claw-free, Conjecture 2 implies Conjecture 1. A recent result on closures due to the third author [11] (Theorem 3 below) implies that Conjecture 1 and Conjecture 2 are even equivalent.

We first introduce some terminology and notation. All *multigraphs* considered here are finite, undirected, and loopless. We use the term *graph* for a multigraph $G = (V, E)$ in order to indicate that G is *simple*, i.e. there is at most one edge joining two vertices. As usual, $V(G)$ or V denotes the *vertex set* and $E(G)$ or E the *edge set* of a multigraph G. Let $A, B \subseteq V$ and $a, b \in V$. By $[A, B]_G$ we denote the set of edges between vertices of A and B in G, and we let $[a, b]_G := [\{a\}, \{b\}]_G$. If $[a, b]_G = \{e\}$ for some $e \in E$, then we also use ab or $[a, b]_G$ for e.

The *submultigraph* $G[A]$ induced by the set $A \subseteq V(G)$ is defined by $G[A] := (A, [A, A]_G)$, and the *degree* of some vertex $a \in V$ is denoted by $d_G(a) := |[\{a\}, V \setminus \{a\}]_G|$. Let $N_G(A) := \{c \in V \setminus A \mid [A, \{c\}]_G \neq \emptyset\}$, and let $N_G(a) := N_G(\{a\})$. Clearly, $d_G(a) = |N_G(a)|$ provided that G is a graph. The submultigraph $G[N_G(a)]$ is called the *neighborhood* of a in G. By $d_G(a, b)$ we denote the *distance* of a, b in G, i.e. the length of a shortest path between a and b in G. If a, b are not in the same component of G, we simply set $d_G(a, b) := \infty$.

A *claw* in the multigraph G is a set of four distinct vertices a, b, c, y

such that a, b, c are *independent* in G, i.e. pairwise nonadjacent in G, and $a, b, c \in N_G(y)$. G is called *claw-free* if there exists no claw in G. Clearly, a multigraph is *claw-free* if it contains no induced subgraph isomorphic to $K_{1,3}$, but the converse is guaranteed only in graphs.

A spanning submultigraph H of G is called a *factor* of G, and a *2-factor* (of G) if all vertices of H have degree 2 in H. Hence a *Hamilton cycle* is a connected 2-factor. A *circuit* C of G is a closed trail (possibly consisting of a single vertex), and it is said to be *(edge) dominating* if every edge of G is incident with some vertex of C. If, moreover, $V(G) = V(C)$ holds then C is a *spanning circuit.*

The *local completion of a graph G at a vertex v* is the operation of joining all pairs of nonadjacent vertices in $N_G(v)$, i.e. replacing the neighborhood of v by the complete graph on $N_G(v)$.

In [11] the following has been proved.

Theorem 3
Let G be a claw-free graph, v a vertex of G whose neighborhood is connected, and G' the graph obtained from G by local completion at v. Then

(i) G' *is claw-free, and*

(ii) for every cycle C' *of* G' *there exists a cycle* C *of* G *such that* $V(C') \subseteq V(C)$.

For a claw-free graph G, we define the *closure* $\mathrm{cl}(G)$ of G as the graph obtained from G by iteratively performing local completions at vertices with connected neighborhoods until no more edges can be added. As shown in [11], $\mathrm{cl}(G)$ is uniquely determined by G, and $\mathrm{cl}(G)$ is the line graph of a triangle-free graph. Moreover, in [11] it is shown that Theorem 3 has the following consequences. Let $c(G)$ denote the *circumference* of G, i.e. the length of a longest cycle of G.

Theorem 4
Let G *be a claw-free graph. Then*

(i) $c(cl(G)) = c(G)$.

(ii) If $cl(G)$ *is complete and* $|V(G)| \geq 3$, *then* G *is hamiltonian.*

(iii) Every nonhamiltonian claw-free graph is a factor of a nonhamiltonian line graph.

Theorem 4(iii) together with a result of ZHAN [15] and, independently, JACKSON [5] implies that every 7-connected claw-free graph is hamiltonian. Moreover it yields the mentioned equivalence of Conjecture 1 and Conjecture 2.

Here we prove several results concerning the existence of certain factors in 4-connected claw-free graphs or multigraphs.

In the next section we give a short proof of Conjecture 2 within the subclass of *hourglass-free* graphs, i.e. graphs that do not contain an induced subgraph isomorphic to the *hourglass*, a graph consisting of two triangles meeting in exactly one vertex. This result also follows from a recent result due to the second author [7].

In Section 3 we prove the validity of a weaker form of Conjecture 2 in which we replace the conclusion by the conclusion that there exists a connected factor in which each vertex has degree 2 or 4.

Finally, in Section 4 we show that Conjecture 1 and 2 are equivalent to seemingly weaker conjectures in which we replace the conclusion by the conclusion that there exists a factor consisting of a bounded number of paths.

2 Hourglass-free graphs

Our aim in this section is to prove that all 4-connected claw-free hourglass-free graphs are hamiltonian. For this purpose we need the fact that all 4-connected *inflations* are hamiltonian.

We start this section by introducing some additional terminology. A multigraph G is called *essentially k-edge connected* if it is connected and if every edge cut E' of G such that $G - E'$ has at least two components containing an edge, has at least k edges. It is well-known and easy to check that a line graph $L(G)$ of a multigraph G is k-connected if and only if G is essentially k-edge connected. The *inflation* $I(G)$ of a graph G is the graph obtained from G by replacing all vertices $v_1, v_2, \ldots, v_n$ of G by disjoint complete graphs on $d(v_i)$ vertices $v_{i,1}, v_{i,2}, \ldots v_{i,d(v_i)}$, and all edges v_iv_j by disjoint edges $v_{i,p}v_{j,q}$ $(i, j \in \{1, \ldots, n\}; p \in \{1, \ldots, d(v_i)\}; q \in \{1, \ldots, d(v_j)\})$. Alternatively, as shown in [10, Lemma 2], $I(G)$ is the line graph of the *subdivision graph* $S(G)$, i.e. the graph obtained from G by subdividing each edge of G once. We use the term *inflation* for a graph that is isomorphic to the inflation of some graph. It is obvious that inflations are claw-free and hourglass-free.

The following result has been observed by several graph theorists, but we have not found it in the literature (therefore, we include its proof).

Lemma 5
Every 4-connected inflation is hamiltonian.

Proof Let G be a 4-connected inflation. Then $G = L(S(H))$ for some essentially 4-edge connected subdivision $S(H)$ of a 4-edge connected graph H. As shown in [13], using the result of KUNDU [8] that H has two edge-disjoint spanning trees, it is easy to show that H contains a spanning circuit, hence $S(H)$ contains a dominating circuit. By a result of HARARY AND NASH-WILLIAMS [3] this implies $G = L(S(H))$ is hamiltonian. ∎

The connectivity bound in Lemma 5 cannot be decreased, since there are nonhamiltonian 3-connected inflations, e.g. the inflation of the Petersen graph. These graphs also show that the connectivity bound in the next result is best possible.

Theorem 6
Every 4-connected claw-free hourglass-free graph is hamiltonian.

Proof Let G be a 4-connected claw-free hourglass-free graph. Then by a result in [2] $\mathrm{cl}(G)$ is also claw-free and hourglass-free. Hence by Theorem 4 we can assume that $G = \mathrm{cl}(G)$. This implies that the neighborhood of each vertex of G induces either a complete graph or a disjoint union of two complete graphs. Since G is hourglass-free, in the latter case one of the complete graphs is a K_1. Hence G contains two types of edges, namely edges that are contained in a complete subgraph on more than 2 vertices, and edges that are contained in a K_2 only. Moreover, all maximal complete subgraphs on more than two vertices contain two types of vertices, namely vertices with a complete neighborhood (contained in the subgraph) which are called *simplicial* vertices, and vertices with precisely one neighbor outside the subgraph. It is not difficult to check that the graph G' obtained from G by deleting all simplicial vertices is a 4-connected inflation. Hence G' is hamiltonian by Lemma 5. Clearly, a Hamilton cycle in G' contains at least one edge of each maximal complete subgraph on more than 2 vertices, and all the maximal complete subgraphs of G containing simplicial vertices correspond to such subgraphs. Hence a Hamilton cycle of G' can easily be extended to a Hamilton cycle in G. ∎

3 Connected factors with degree restrictions

By Theorem 3.1 in [6], every connected claw-free graph has a 2-*walk*, i.e. a (closed) walk which passes every vertex at most twice. Clearly, the edges of a 2-walk induce a connected factor of maximum degree at most 4.

The aim of this section is to prove that every 4-connected claw-free graph contains a connected factor with vertices of degree 2 or 4. We start with a series of lemmas on *congruent* factors of multigraphs, i.e. factors of a multigraph G which have the same parity of degrees at every vertex. Lemma 7 will allow us to apply the closure introduced in Section 1 later on. (Note that $\mathrm{cl}(G)$ can be constructed from G by iteratively adding the missing edge in a subgraph $K_4 - e$.)

Lemma 7
Let F be a connected factor of a multigraph G and let e be an edge contained in some complete subgraph K_4 of G. Then $G - e$ has a connected factor F' such that $d_{F'}(x) \equiv d_F(x) \bmod 2$ for all $x \in V(G)$.

Proof For two multigraphs G_1, G_2 we define $G_1 \cup G_2 := (V(G_1) \cup V(G_2), E(G_1) \cup E(G_2))$, $G_1 \cap G_2 := (V(G_1) \cap V(G_2), E(G_1) \cap E(G_2))$, and $G_1 \Delta G_2 := (G_1 \cup G_2) - E(G_1 \cap G_2)$. ($G_1 \Delta G_2$ is the *symmetric difference* of G_1 and G_2.)

Let w, x, y, z be the vertices of the subgraph $H \cong K_4$ which contains e, say $e \in [w, x]$. The conclusion of the lemma is obviously true if $e \notin E(F)$. So we may assume $e \in E(F)$. We define the following four w, x-subpaths of H: $Q := w, y, x$, $R := w, z, x$, $S := w, y, z, x$, and $T := w, z, y, x$. It is easy to see that if F' is the symmetric difference of $F - e$ and any of these paths, then $d_{F'}(u) \equiv d_F(u) \bmod 2$ holds for all $u \in V(H)$. Hence it suffices to prove that the symmetric difference F' of one of these paths and $F - e$ contains a connected spanning subgraph of H. We denote $(F - e) \cap H$ by H'.

If $d_{H'}(y) = 3$, then $F' := (F - e)\Delta R$ will serve, if $d_{H'}(y) = 0$ and $d_{H'}(z) \neq 0$ then $F' := (F - e)\Delta Q$ will do, and if $d_{H'}(y) = d_{H'}(z) = 0$ then $D' := (F - e)\Delta T$ will. So we may assume that y and, by symmetry, z have degree 1 or 2 in H'.

Without loss of generality, we may assume that $d_{H'}(w) \geq d_{H'}(x)$. We distinguish three cases.

Case 1. $d_{H'}(w) = 2$ and $d_{H'}(x) \geq 1$. Without loss of generality, x is adjacent to y in H'. Since $d_{H'}(y) \neq 3$, there is no edge between y and z in H'. It follows that $F' := (F - e)\Delta S$ is an appropriate factor.

Case 2. $d_{H'}(w) = 2$ and $d_{H'}(x) = 0$. If y is adjacent to z in H', then $F' := (F - e)\Delta Q$ will do; otherwise $F' := (F - e)\Delta S$ will.

Case 3. $d_{H'}(w) = 1$. Without loss of generality, w is adjacent to y in H'. If x is not adjacent to z in H', then $F' := (F - e)\Delta R$ will do; in the other case, $d_{H'}(x) = 1$ as well, and $F' := (F - e)\Delta T$ contains a connected spanning subgraph of H, since it contains all edges of $H - e$ except possibly an edge between y, z. ■

Lemma 8 guarantees the existence of a connected low degree factor in a claw-free multigraph which is congruent to a given one.

Lemma 8
Let F be a connected factor of a claw-free multigraph G. Then there exists a connected factor F' of G without vertices of degree exceeding 4 such that $d_{F'}(x) \equiv d_F(x)$ mod 2 for all $x \in V(G)$.

Proof Throughout the proof, we call a connected factor F' with $d_{F'}(x) \equiv d_F(x)$ mod 2 for all $x \in V(G)$ a *good* factor. Among all good factors we choose one, say F', with a minimum number of edges. We claim that F' contains no vertex of degree exceeding 4.

Suppose to the contrary that $x \in V(G)$ had degree at least 5 in F'. We distinguish two cases.

Case 1. $F' - x$ is connected. First note that there is no pair of distinct edges $e, f \in E(F')$ between x and some $y \in V(G)$, for otherwise $F' - e - f$ would be a good factor, contradicting the choice of F. So $|N_{F'}(x)| \geq 5$. Let $e \in [y, z]_G$ be an edge in $G[N_{F'}(x)]$. Then $e \in E(F')$, too, for otherwise $(F' - [x, y] - [x, z]) + e$ would be a good factor, a contradiction. Furthermore, e is a bridge of $F' - x$, for otherwise $F' - [x, y] - [x, z] - e$ is a good factor, which is absurd again. So every edge in $G[N_{F'}(x)]$ is a bridge of $F' - x$, and in particular, $G[N_{F'}(x)]$ contains no cycle. But then $N_{F'}(x)$ must contain three independent vertices (since $|N_{F'}(x)| \geq 5$), which form a claw together with x, a contradiction.

Case 2. $F' - x$ is not connected. First note that there is no triple $e, f, h \in E(F')$ between x and some $y \in V(G)$, for otherwise $F' - e - f$ would be a good factor. Let C, D be distinct components of $F' - x$, and let $Y := N_{F'}(x) \cap V(C)$ and $Z := N_{F'}(x) \cap V(D)$. There is no edge in G between a vertex of Y and one of Z, for otherwise there were edges $e \in [x, y]_{F'}$, $f \in [x, z]_{F'}$, $h \in [y, z]_G \setminus E(F')$ for some $y \in Y$, $z \in Z$, and $(F' - e - f) + h$ would be a good factor, a contradiction. In particular, C and D are the *only* components of $F' - x$. Since G is claw-free, Y and Z are complete in G. Without loss of generality, we may assume that there are at least three edges between x and vertices of Y (otherwise we swap the roles of Y and Z). Then Y must be complete in F' as well, for otherwise there would be edges $e \in [x, y]_{F'}$, $f \in [x, z]_{F'}$, $h \in [y, z]_G \setminus E(F')$, and so $(F' - e - f) + h$ would be a good factor, a contradiction. It follows that there cannot be a pair e, f of distinct edges between x and $y \in Y$, for otherwise $F' - e - f$ would be a good factor, a contradiction. So $|Y| \geq 3$. But then $F' - [x, y] - [x, z] - e$ is a good factor for arbitrary $e \in [y, z]_{F'} \neq \emptyset$, $y, z \in Y$, our final contradiction. ■

Lemma 9 deals with the existence of a connected even factor in 4-connected line graphs of multigraphs.

Lemma 9
Every 4-connected line graph of a multigraph contains a connected factor which has degree two or four at each vertex.

Proof Let G be a multigraph such that $L(G)$ is 4-connected. Suppose that x is a vertex of degree 3 in G. If a neighbor y of x has degree less than 3, then the vertex in $L(G)$ corresponding to some edge in $[x, y]_G$ had degree less than four, which is impossible. So *doubling* an edge e incident with x, i.e. adding a further, new edge e^+ with the same endvertices as e, will not produce a vertex of degree less than four at one of its ends. So there exists a set $E' \subseteq E(G)$ such that doubling each edge of E' (once) produces a graph G' without vertices of degree 3, with $E(G') = E(G) \cup \{e^+ \mid e \in E'\}$, and with $V(G') = V(G)$. Furthermore, no edge $e \in E'$ has an endvertex of degree one or two in G.

By [7], there exists a dominating circuit of G which contains all vertices of degree at least 4 in G', and here we can specify that if it contains exactly one of e and e^+, then it contains e. Among all dominating circuits with these properties we choose one, say F, with as few edges as possible. It follows by the choice of F, that if F contains both edges e and e^+ for some $e \in E'$, then $F - e - e^+$ is disconnected. We orient the edges of F according to one way of traversing the circuit, starting at an arbitrary vertex. If $f = (x, y)$ is an arc of the orientation, we call x the *inneighbour* and y the *outneighbour* of f. Hence the orientation of F corresponds to a sequence T of edges such that the outneighbour of e is equal to the inneighbour of f whenever e and f are consecutive in T or e is the last and f is the first element of the sequence. Since $F - e - e^+$ is disconnected whenever e and e^+ are in F for some $e \in E'$, e and e^+ are oriented oppositely (if they are both in F).

Now we produce a sequence T' of edges of G by inserting some of the edges not in $E(F)$ (not necessarily once) at some position into the sequence of edges corresponding to T, according to the following rules:

1) If e and f with $f = e^+$ or $e = f^+$ are consecutive on T, then we insert two edges of $E(G) \setminus E(F)$ incident with the outvertex of e (i.e. the invertex of f) at the position in between e and f (such edges exist).

2) If e and f, and f^+ and e^+ are both consecutive on T, then we insert an edge incident with the outvertex of f^+ at the position in between f^+ and e^+ (such an edge exists).

The sequence T' need not be a circuit. Note that every inserted edge occurs at most twice in T' and all others occur once in T'; those which have been inserted twice never occur consecutively in T'. Neither e and e^+ nor e^+ and e are consecutive in T', and if e and f are consecutive in T', then f^+ and e^+ are not.

Now we construct T'' from T' by inserting sequentially the remaining edges: If there is an edge e in $E(G)$ not inserted so far into T'', then we insert it at a position between consecutive f and g, whenever e, f and g have a common endvertex. If this is not possible, then e has a common endvertex with the first and the last edge of T'', and we add e at the end of T''. All edges inserted in this way into T' occur only once.

Finally, we construct T''' from T'' by replacing each doubled edge e^+, $e \in E'$, by the original edge e.

T''' is a sequence of edges of G with the following properties:

1) Any two consecutive edges have a common vertex, and the first and the last one have a common vertex.

2) Two consecutive edges of T''' are distinct.

3) If $e, f \in E'$ are consecutive in T''', then f and e are not.

4) Every edge of G occurs in T''' at least once, at most $3 \cdot |E'|$ edges occur twice, and no edge of G occurs more than twice.

Therefore, the edges of T''' form a connected factor of $L(G)$ with vertices of degree 2 or

4, and with at most $3 \cdot |E'|$ vertices of degree 4. ■

In general, one cannot expect an upper bound for $|E'|$ better than the number $v_3(G)$ of vertices of degree 3 in G, which leads, according to the proof of Lemma 9, to an upper bound of $3 \cdot v_3(G)$ for the number of vertices of degree 4 in the factor. Unfortunately, this bound may equal $|V(L(G))|$, for example if G is an essentially 4-edge-connected bipartite graph where one color class consists of vertices of degree 3.

If one provides more structure on G, then one can improve this bound. For example, if in G the vertices of degree 3 are independent, then one gets $|E'| \leq v_3(G)$ by similar arguments as above. This implies, for example, that a 4-connected line graph with minimum degree 5 contains a connected factor with more than 2/3 of its vertices having degree 2 and all others having degree 4.

Now we are able to establish the main result of this section.

Theorem 10
Every 4-connected claw-free graph contains a connected factor which has degree two or four at each vertex.

Proof Let G be a 4-connected claw-free graph. Then $\mathrm{cl}(G)$ is a 4-connected line graph. By Lemma 9, $\mathrm{cl}(G)$ contains a connected factor which has degree two or four at each vertex. By Lemma 7, G contains a connected factor which has even degree at each vertex. Finally, by Lemma 8, the assertion follows. ■

By the results of [7] it is also possible to prove the stronger result that between every pair of distinct vertices in a 4-connected line graph there exists a spanning trail which passes every vertex at most twice.

4 Factors consisting of a bounded number of paths

In this section we prove that Conjecture 1 and 2 are equivalent to seemingly weaker conjectures in which the conclusion that G is hamiltonian is replaced by the conclusion that G contains a factor consisting of a number of paths bounded by a constant, or, more generally, by a function which is sublinear in the number of vertices of the graph. In particular we show that every k-connected claw-free graph is hamiltonian if and only if every k-connected claw-free graph is traceable, i.e. contains a Hamilton path. For convenience we use the term *r-path-factor* for a factor consisting of at most r paths. A *path-factor* is an r-path factor for some r, and its *endvertices* are the vertices of degree less than 2 of its components.

We start with an auxiliary result. Here a *k-clique* of a graph G is a subset of k vertices of G inducing a complete subgraph in G.

Lemma 11
Let $k \geq 2$ be an integer. If there exists a k-connected nonhamiltonian claw-free graph on n vertices, then there exists a k-connected nonhamiltonian claw-free graph on at most $2n-2$ vertices containing a k-clique.

Proof Let G be a k-connected nonhamiltonian claw-free graph on n vertices, and assume that $G = \text{cl}(G)$. Hence G is the line graph of some triangle-free (simple) graph H. We may assume $k \geq 4$, since for $k \leq 3$ the claw-freeness clearly implies that there is a k-clique in G. If all vertices of H have degree at least 4, then it is easy to see that H is 4-edge connected; by the result of [14] G is hamiltonian. If there is a vertex in H with precisely one neighbor u, then the edges incident with u induce a clique in G with at least k vertices. Hence we may assume there is a vertex x of degree 2 or 3 in H. Therefore, assuming G does not contain a k-clique, G contains a vertex whose neighborhood consists of disjoint cliques R and Q with $|R| \geq |Q| \in \{1, 2\}$. If some vertex of G is contained in a k-clique, then we are done. Hence we may assume that $|R| = k - 2$ and $|Q| = 2$. Now consider two copies G_1 and G_2 of G with the same fixed vertex x called x_i in G_i $(i = 1, 2)$ and the same partition of $N(x)$ into two cliques Q_i, R_i in G_i with $|Q_i| = 2$ and $|R_i| = k - 2$ for $i = 1, 2$, respectively. Define the graph G' on $2n - 2$ vertices obtained from G_1 and G_2 by deleting x_1 and x_2, and joining all vertices of Q_1 to all vertices of Q_2, and joining all vertices of R_1 to all vertices of R_2. Denote by E' the set of edges joining vertices of $G_1 - x_1$ and $G_2 - x_2$. Then one easily checks that G' is claw-free and k-connected, and that G' contains a k-clique. We complete the proof by showing that G' is nonhamiltonian.

Suppose to the contrary that G' has a Hamilton cycle C. Then $F_i := C \cap (G_i - x_i)$ is a path-factor of $G_i - x_i$ with all endvertices in $Q_i \cup R_i$. Either F_1 contains no path between the vertices of Q_1, or F_2 contains no path between the endvertices, for otherwise these two paths, together with two edges of E', would form a proper subcycle of C, which is absurd. Without loss of generality, F_1 contains no path between the endvertices of Q_1.

Case 1. Q_1 contains no endvertex of F_1. Then $F_1 \cup \{x_1\}$ is a path-factor of G_1 with all endvertices in the clique $R_1 \cup \{x_1\}$.

Case 2. Q_1 contains endvertices of exactly one component P of F_1. Then Q_1 contains precisely one endvertex of P, and hence $(F_1 - P) \cup (P + x_1)$ is a path-factor of G_1 with all endvertices in the clique $R_1 \cup \{x_1\}$.

Case 3. Q_1 contains endvertices of two distinct components $P \neq P'$ of F_1. Then $(F_1 - P - P') \cup (P + x_1 + P')$ is a path-factor of G_1 with all endvertices in the clique R_1.

Since a graph on at least 3 vertices is hamiltonian if and only if it has a path-factor with all endvertices being contained in the same clique, it follows in either case that G_1 is hamiltonian, a contradiction. ∎

We use the above lemma to prove the following result.

Theorem 12
Let $k \geq 2$ and $r \geq 1$ be two integers. Then the following statements are equivalent.

(1) There is a k-connected claw-free nonhamiltonian graph.

(2) There is a k-connected claw-free graph without an r-path-factor.

Moreover, if there is an example for (1) on n vertices, then there is an example for (2) with at most $(2r + 1)(2n - 2)$ vertices.

Proof It is clear that we only have to show that the existence of a k-connected claw-free nonhamiltonian graph on n vertices implies the existence of a k-connected claw-free graph without an r-path-factor on at most $(2r+1)(2n-2)$ vertices.

Let G be a k-connected claw-free nonhamiltonian graph on n vertices. Then by Lemma 11 there is a k-connected claw-free nonhamiltonian graph H on at most $2n-2$ vertices containing a k-clique Q. We may assume that $H = \mathrm{cl}(H)$. Let G_r be the graph obtained from $2r+1$ disjoint copies of H by joining all vertices corresponding to the k-clique Q in all copies, forming a $(2r+1)k$-clique. Clearly, G_r is claw-free and k-connected and has at most $(2r+1)(2n-2)$ vertices. We complete the proof by showing that G_r admits no r-path-factor. Suppose to the contrary that P is an r-path-factor of G_r. Then P has at most $2r$ vertices of degree zero or one. Since G_r contains $2r+1$ disjoint copies of H, this implies that for at least one copy of H, $V(H) \setminus Q$ contains no endvertices of P. It is obvious that we can construct a Hamilton cycle in this copy of H, contradicting the assumption that H is nonhamiltonian. ∎

Theorem 12 has a number of interesting consequences, the first of which is obvious and given without proof.

Corollary 13
Let $k \geq 2$ be an integer. Then the following statements are equivalent.

(1) Every k-connected claw-free graph is hamiltonian.

(2) Every k-connected claw-free graph is traceable.

In particular Corollary 13 shows that Conjecture 2 is equivalent to the conjecture that every 4-connected claw-free graph is traceable. We can weaken the conclusion a little further. The next consequences of Theorem 12 can be obtained by examining the order of the graph G_r in the proof of the theorem.

Corollary 14
Let $k \geq 2$ be an integer, and let $f(n)$ be a function of n with the property that $\lim_{n\to\infty} \frac{f(n)}{n} = 0$. Then the following statements are equivalent.

(1) Every k-connected claw-free graph is hamiltonian.

(2) Every k-connected claw-free graph on n vertices has an $f(n)$-path-factor.

(3) Every k-connected claw-free graph on n vertices has a 2-factor with at most $f(n)$ components.

(4) Every k-connected claw-free graph on n vertices has a spanning tree with at most $f(n)$ vertices of degree one.

(5) Every k-connected claw-free graph on n vertices has a path of length at least $n - f(n)$.

Proof We only prove that (2) implies (1). The other cases are similar and left to the reader.

Suppose (2) is true and suppose there exists a k-connected claw-free nonhamiltonian graph on m vertices. Then by Theorem 12 there is a k-connected claw-free graph G_r without an r-path-factor on $n_r \le (2r+1)(2m-2)$ vertices. If we let r tend to infinity, then G_r is a graph on n_r vertices without an r-path-factor, while $\lim_{r\to\infty} \frac{r}{n_r} \ge \frac{1}{4m-4}$ for a fixed integer $m > 1$. This contradicts the assumption that (2) is true. ■

In particular Corollary 14 shows that Conjecture 2 is true if one could show that, e.g., every 4-connected claw-free graph on n vertices admits a factor consisting of a number of paths which is sublinear in n.

Recently, in [4] it has been shown that a claw-free graph G has an r-path-factor if and only if $\mathrm{cl}(G)$ has an r-path-factor. Similarly, in [12] it has been shown that a claw-free graph G has a 2-factor with at most r components if and only if $\mathrm{cl}(G)$ has such a 2-factor. These results immediately imply the equivalence of the following statements related to Conjecture 1.

Corollary 15
Let $k \ge 2$ be an integer, and let $f(n)$ be a function of n with the property that $\lim_{n\to\infty} \frac{f(n)}{n} = 0$. Then the following statements are equivalent.

(1) Every k-connected line graph is hamiltonian.

(2) Every k-connected line graph on n vertices has an $f(n)$-path-factor.

(3) Every k-connected line graph on n vertices has a 2-factor with at most $f(n)$ components.

In particular Corollary 15 shows that Conjecture 1 is true if one could show that, e.g., every 4-connected line graph on n vertices admits a 2-factor consisting of a number of components which is sublinear in n. The equivalences between (1) and (2) of Corollary 14 and of Corollary 15 appear also in a sequence of equivalences in [9].

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