On local and global independence numbers of a graph

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Abstract

The local independence number $\alpha_i(G)$ of a graph G at a distance i is the maximum number of independent vertices at distance i from any vertex. We study the impact of restricting $\alpha_i(G)$ on the (global) independence number $\alpha(G)$. Among others, we show that in graphs with bounded diameter, $\alpha(G)$ is bounded if and only if $\alpha_i(G)$ is bounded for at least one i, $2 \leq i \leq (\text{diam}(G) - 1)/4$.

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1 Introduction

All graphs $G = (V(G), E(G))$ considered in this paper are simple, finite and undirected. We assume all graphs G under consideration to be connected (otherwise the results can be applied to the components of G). We follow the most common graph-theoretical notation and terminology. For concepts and notation not defined here we refer the reader to [1].

Specifically, $\text{dist}(x, y)$ denotes the *distance* of vertices $x, y \in V(G)$. For any $x \in V(G)$ we set $\text{dist}(x, x) = 0$. For $e = uv \in E(G)$ and $x \in V(G)$, $\text{dist}(x, e)$ denotes the distance of x from e, i.e. the minimum of $\text{dist}(x, u)$ and $\text{dist}(x, v)$. The *diameter* of G, i.e. the maximum distance between a pair of vertices of G, is denoted by $\text{diam}(G)$. For any $x \in V(G)$ and an integer i, $0 \leq i \leq \text{diam}(G)$, $N_i(x) = \{y \in V(G) |\ \text{dist}(x, y) = i\}$ denotes the *neighborhood of x at distance i*. For a set $S \subset V(G)$, $\langle S \rangle$ denotes the subgraph induced by S, and $d_S(u) = |\{x \in S |\ xu \in E(G)\}|$ denotes the *relative degree* of a vertex $u \in V(G)$ with respect to S.

The *independence number* of a graph G is denoted by $\alpha(G)$. For any i, $0 \leq i \leq \text{diam}(G)$, we set $\alpha_i(G) = \max\{\alpha(\langle N_i(x) \rangle) |\ x \in V(G)\}$. The number $\alpha_i(G)$ is called the

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local independence number of G at a distance i. If $\mathcal{B}$ is a family of graphs, then G is said to be *$\mathcal{B}$-free* if G does not contain an induced subgraph isomorphic to any of the graphs from $\mathcal{B}$. Specifically, the graph $K_{1,3}$ is called the *claw* and for $\mathcal{B} = \{K_{1,3}\}$ we say that G is *claw-free*. By a *clique* we mean a (not necessarily maximal) complete subgraph of a graph G.

There are many results dealing with properties of claw-free graphs. In our notation, it is easy to see that G is claw-free if and only if $\alpha_1(G) \leq 2$ (or, more generally, G is $K_{1,r+1}$-free if and only if $\alpha_1(G) \leq r$). However, the graph G obtained by removing one copy of K_r from the Cartesian product $K_r \times K_r$ shows that $\alpha(G)$ can be arbitrarily large even in claw-free graphs of bounded diameter and arbitrarily large connectivity.

In [3], several upper bounds on $\alpha(G)$ were given in the class of $K_{1,r+1}$-free graphs involving several additional parameters. Shepherd [4] showed that the additional restriction $\alpha_2(G) \leq 2$ on a claw-free graph has many global consequences. In this paper, we follow up in this direction by showing that restricting $\alpha_i(G)$ only at a few distance levels implies a restriction on the global independence number $\alpha(G)$. For more related results on claw-free graphs we refer the reader to survey paper [2].

2 Main results

For any integers $r, t \geq 2$ we set $\mathcal{S}_{r,t} = \{G |\ \alpha_1(G) \leq r, \alpha_2(G) \leq t\}$. Note that all classes $\mathcal{S}_{2,t}$ are subclasses of the class of claw-free graphs, and $\mathcal{S}_{2,2}$ is the family of distance claw-free graphs, introduced in [4].

For any integers k, i, $0 \leq i \leq \lfloor \frac{k}{2} \rfloor$, $B_{k,i}$ denotes the graph obtained by joining all vertices of a disjoint union $K_i \cup K_{k-i+1}$ to a (new) vertex x and by attaching a pendant edge to each vertex except x. For a given $k \geq 2$, $\mathcal{B}_k$ denotes the family of all such graphs $B_{k,i}$, $0 \leq i \leq \lfloor \frac{k}{2} \rfloor$.

Proposition 1. *Let G be a graph. Then $G \in \mathcal{S}_{2,k}$ if and only if G is claw-free and $\mathcal{B}_k$-free.*

Proof. If G contains a claw, then $\alpha_1(G) \geq 3$, and if G contains an induced subgraph $B \in \mathcal{B}_k$, then $\alpha_2(G) \geq k+1$. In both cases, $G \notin \mathcal{S}_{2,k}$.

Conversely, let $G \notin \mathcal{S}_{2,k}$. If $\alpha_1(G) \geq 3$, then clearly G contains a claw; hence suppose $\alpha_1(G) \leq 2$ (implying G is claw-free) and $\alpha_2(G) \geq k+1$. Let x be a vertex such that $N_2(x)$ contains an independent set I with $|I| \geq k+1$. For every $y_i \in I$ choose a $z_i \in N_1(y_i) \cap N_1(x)$, $i = 1, \ldots, k+1$. Since G is claw-free, $z_i \neq z_j$ for $i \neq j$. Thus, if $z_i z_{j_1}, z_i z_{j_2} \in E(G)$ for some i, j_1, j_2, then $z_{j_1} z_{j_2} \in E(G)$, for otherwise $\langle \{z_i, z_{j_1}, z_{j_2}, y_i\} \rangle$ is a claw. This implies that $\langle \{z_1, \ldots, z_{k+1}\} \rangle$ is a disjoint union of cliques. Since G is claw-free, $\langle \{z_1, \ldots, z_{k+1}\} \rangle$ consists of at most two cliques, implying $\langle \{x, z_1, \ldots, z_{k+1}, y_1, \ldots, y_{k+1}\} \rangle \in \mathcal{B}_k$. ■

The following theorem shows that the restriction on independence number at distances 1 and 2, given in the definition of the class $\mathcal{S}_{2,k}$, implies an upper bound on $\alpha_\ell(G)$ at all distances ℓ.

Theorem 2. *Let $G \in \mathcal{S}_{2,k}$ and let $\ell \geq 3$. Then*

$$\alpha_\ell(G) \leq \begin{cases} k(\lfloor \frac{k}{2} \rfloor \lceil \frac{k}{2} \rceil)^{\frac{\ell}{2}-1} & \text{for } \ell \text{ even,} \\ 2(\lfloor \frac{k}{2} \rfloor \lceil \frac{k}{2} \rceil)^{\frac{\ell-1}{2}} & \text{for } \ell \text{ odd,} \end{cases}$$

and this bound is sharp.

Before proving Theorem 2, we first prove one auxiliary statement on trees.

Proposition 3. *Let k be a positive integer and let T be a tree rooted at edge e such that $d(x) + d(y) \leq k + 2$ for every edge xy of T. Let $A_i = \{x \in V(T) |\ \mathrm{dist}(x, e) = i\}$, $i = 1, \ldots, \mathrm{diam}(T) - 1$. Then, for any fixed $i \geq 2$, $|A_i|$ is maximum if $d(x) + d(y) = k + 2$ for every non-end edge xy of T and $d(x) = \lceil \frac{k}{2} \rceil + 1$ or $d(x) = \lfloor \frac{k}{2} \rfloor + 1$ for every non-end vertex x of T. In this case,*

$$|A_i| = \begin{cases} 2(\lfloor \frac{k}{2} \rfloor \lceil \frac{k}{2} \rceil)^{\frac{i}{2}} & \text{for } i \text{ even,} \\ k(\lfloor \frac{k}{2} \rfloor \lceil \frac{k}{2} \rceil)^{\frac{i-1}{2}} & \text{for } i \text{ odd.} \end{cases}$$

(Equivalently, $|A_i|$ is maximum if and only if T is a balanced or a nearly balanced tree rooted at e.)

Proof. Let $e = uv \in E(T)$, and set $A_j^u = \{x \in V(T) |\ \mathrm{dist}(x, u) = j,\ \mathrm{dist}(x, v) = j + 1\}$ and $A_j^v = \{x \in V(T) |\ \mathrm{dist}(x, v) = j,\ \mathrm{dist}(x, u) = j + 1\}$, $j = 0, 1, \ldots, i$.

We first prove that $|A_i|$ is maximum if $d(x) + d(y) = k + 2$ for any non-end edge xy and $d(x) = \lceil \frac{k}{2} \rceil + 1$ or $d(x) = \lfloor \frac{k}{2} \rfloor + 1$ for every non-end vertex x of T. We prove this statement by induction on i. We will anchor the induction for $i = 1$ and $i = 2$, and show when the result holds for i, it also holds for $i + 2$.

1. Let first $i = 2$ and let $d(u) = r + 1$, $d(v) = s + 1$, $r + s \leq k$. Every vertex in A_1^u has, under the degree constraint, at most $k - r$ neighbors in A_2^u and, similarly, any vertex in A_1^v has at most $k - s$ neighbors in A_2^v. Since $|A_1^u| = r$ and $|A_1^v| = s$, we have

$$|A_2| = |A_2^u| + |A_2^v| \leq r(k - r) + s(k - s).$$

Under the asumption $r + s \leq k$, this function is maximized when $r = \lceil \frac{k}{2} \rceil$ and $s = \lfloor \frac{k}{2} \rfloor$, or $r = \lfloor \frac{k}{2} \rfloor$ and $s = \lceil \frac{k}{2} \rceil$.

2. Assume for any tree T' rooted at e a maximum number of vertices in level i, $i \geq 1$, is attained if T' is a balanced or a nearly balanced tree with $d(x) + d(y) = k + 2$ for any non-end edge xy. Let T be a tree rooted at e and having maximum number of vertices in level $i + 2$, $i \geq 1$. Consider an arbitrary vertex $x \in A_i^u$. Assuming no degree constraint on x, if x has r neighbors in A_{i+1}^u, then x has a maximum number of descendants in A_{i+2}^u if $r(k - r)$ is maximum, i.e. when $r = \lfloor \frac{k}{2} \rfloor$ or $r = \lceil \frac{k}{2} \rceil$. Let $T^* = T - (A_{i+1} \cup A_{i+2})$. If T^* does not have maximum number of vertices at level i, then, by the induction assumption, it can be maximized by replacing T^* by a balanced or nearly balanced tree T^{**}. Replacing T^* by T^{**} in T, we can enlarge the number of vertices at level $i + 2$. Consequently, we can assume that T^* is the required (nearly) balanced tree. By the first part of the proof, A_{i+2} is maximized if the subtrees at levels $i, i + 1, i + 2$ are also (nearly) balanced. This gives the required statement.

By symmetry, we can assume $d(u) = r + 1 = \lfloor \frac{k}{2} \rfloor + 1$ and $d(v) = k - r + 1 = \lceil \frac{k}{2} \rceil + 1$. A simple counting argument then gives $|A_j^u| = |A_j^v| = (r(k-r))^{\frac{j}{2}}$ for j even, and $|A_j^u| = r(r(k-r))^{\frac{j-1}{2}}$ and $|A_j^v| = (k-r)(r(k-r))^{\frac{j-1}{2}}$ for j odd, $1 \le j \le i$. Hence $|A_i| = 2(r(k-r))^{\frac{i}{2}}$ for i even, and $|A_i| = (k(r(k-r))^{\frac{i-1}{2}}$ for i odd, which gives the required result. ■

Proof of Theorem 2. Let $x \in V(G)$ be such that $N_\ell(x) \neq \emptyset$ for $\ell \geq 2$ and let $A = \{x_1^\ell, \ldots, x_r^\ell\}$ be a maximum independent set in $\langle N_\ell(x) \rangle$. For each vertex x_i^ℓ, choose its neighbor $x_i^{\ell-1} \in N_{\ell-1}(x)$. Then the vertices $x_1^{\ell-1}, \ldots, x_r^{\ell-1}$ are distinct, for otherwise, if $x_{i_1}^{\ell-1} = x_{i_2}^{\ell-1}$ for some $i_1 \neq i_2$, then, for a neighbor y of $x_{i_1}^{\ell-1}$ in $N_{\ell-2}(x)$, $\langle \{x_{i_1}^{\ell-1}, x_{i_1}^\ell, x_{i_2}^\ell, y\} \rangle$ is a claw. Next observe that $\langle N_{\ell-1}(x) \rangle$ consists of a collection of vertex disjoint cliques, since if $x_{i_1}^{\ell-1} x_{i_2}^{\ell-1} \in E(G)$ and $x_{i_1}^{\ell-1} x_{i_3}^{\ell-1} \in E(G)$, but $x_{i_2}^{\ell-1} x_{i_3}^{\ell-1} \notin E(G)$, then $\langle \{x_{i_1}^{\ell-1}, x_{i_2}^{\ell-1}, x_{i_3}^{\ell-1}, x_{i_1}^\ell\} \rangle$ is a claw. Finally, if B is a clique in $\langle N_{\ell-1}(x) \rangle$, then all vertices of B are adjacent to the same vertex in $N_{\ell-2}(x)$, for if $x_{i_1}^{\ell-1}, x_{i_2}^{\ell-1} \in V(B)$ are such that $x_{i_1}^{\ell-1} y_1 \in E(G)$ and $x_{i_2}^{\ell-1} y_2 \in E(G)$ but $x_{i_1}^{\ell-1} y_2 \notin E(G)$ for some $y_1, y_2 \in N_{\ell-2}(x)$, then $\langle \{x_{i_2}^{\ell-1}, x_{i_2}^\ell, x_{i_1}^{\ell-1}, y_2\} \rangle$ is a claw.

By induction, we obtain that the vertices of the system of distance paths from the vertices of A to the vertex x induce in G a tree-like subgraph H with the following properties:

- $\langle N_j(x) \cap V(H) \rangle$ is a disjoint union of cliques,
- for each clique in $\langle N_j(x) \cap V(H) \rangle$, all its vertices have the same neighbor in $N_{j-1}(x) \cap V(H)$,

$j = 1, \ldots, \ell - 1$. Moreover, by Proposition 1, for any two cliques in H sharing a vertex the sum of their orders is at most k for otherwise we have a forbidden subgraph from $\mathcal{B}_k$. This implies that the graph $H - A$ is the line graph of a tree in which $d(u) + d(v) \leq k + 2$ for any its edge uv. Proposition 3 (for $i = \ell - 1$) then gives the required bound on $N_{\ell-1}(x) \cap V(H)$ and hence also on $|A| = \alpha(\langle N_\ell(x) \rangle)$. Since x is arbitrary, the result follows. ■

Our next result shows that arbitrary fixed upper bounds on $\alpha_1(G)$ and $\alpha_2(G)$ (not necessarily $\alpha_1(G) \leq 2$) also imply an upper bound on $\alpha_\ell(G)$ for any ℓ.

Theorem 4. *Let $r, s \geq 2$ be fixed integers and let $G \in \mathcal{S}_{r,s}$. Then*

$$\alpha_\ell(G) \leq s[r(r+s+1)]^{\ell-2}$$

for any $\ell = 3, \ldots, \mathrm{diam}(G)$.

Corollary 5. *Let $r, s, d \geq 2$ be fixed integers and let $\mathcal{S}_{r,s}^d$ be the class of all graphs $G \in \mathcal{S}_{r,s}$ with $\mathrm{diam}(G) \leq d$. Then there is a constant K such that $\alpha(G) \leq K$ for any $G \in \mathcal{S}_{r,s}^d$.*

(Equivalently, in graphs with bounded diameter, an upper bound on $\alpha_1(G)$ and $\alpha_2(G)$ implies an upper bound on $\alpha(G)$.)

Proof of Theorem 4. Let $x \in V(G)$ and let A be a maximum independent set in $\langle N_i(x) \rangle$ for an arbitrary fixed i, $3 \leq i \leq \ell$. For each vertex $a \in A$ choose exactly one neighbor $b \in N_{i-1}(x)$ and let $S \subset N_{i-1}(x)$ be the set of these neighbors. Since $\alpha_1(G) \leq r$, every vertex in S has at most r neighbors in A, implying $|A| = \alpha(\langle N_i(x) \rangle) \leq r|S|$, from which

$$|S| \geq \frac{\alpha(\langle N_i(x) \rangle)}{r}. \tag{1}$$

If a vertex $u \in S$ is adjacent to $v_1, \ldots, v_t \in S$, then from the choice of S the set of vertices $\{v_1, v_2, \ldots, v_t\}$ has at least t neighbors in A. Let this set of neighbors of $v_1, v_2, \ldots, v_t$ in A be $\{w_1, w_2, \ldots, w_t\}$. Since $\alpha_1(G) \leq r$, vertex u is adjacent to at most r of $w_1, w_2, \ldots, w_t$, making at least $t - r$ of them at distance 2 from u. Since $\alpha_2(G) \leq s$, this implies $d_S(u) \leq r + s$ for any $u \in S$. Since $|V(H)| \leq (\Delta(H)+1) \cdot \alpha(H)$ for any graph H, we have $|S| \leq (r + s + 1) \cdot \alpha(\langle S \rangle)$, implying

$$\alpha(\langle S \rangle) \geq \frac{|S|}{r + s + 1}. \tag{2}$$

From (1) and (2) we then have

$$\alpha(\langle N_{i-1}(x) \rangle) \geq \alpha(\langle S \rangle) \geq \frac{|S|}{r + s + 1} \geq \frac{\alpha(\langle N_i(x) \rangle)}{r(r + s + 1)},$$

from which

$$\alpha(\langle N_i(x) \rangle) \leq r(r + s + 1) \cdot \alpha(\langle N_{i-1}(x) \rangle).$$

Hence

$$\alpha(\langle N_\ell(x) \rangle) \leq s[r(r + s + 1)]^{\ell - 2}.$$

Since x is arbitrary, the result follows. ■

The next theorem shows that a bound on the independence number at a certain distance implies bounds at all smaller distances.

Theorem 6. *Let k be a positive integer and let G be a graph of diameter $d \geq 4k + 1$. Then*

$$\alpha_k(G) \leq (2k + 1) \cdot \alpha_{k+1}(G).$$

Proof. We show that $\alpha_k(G) = s$ implies $\alpha_{k+1}(G) \geq \frac{s}{2k+1}$. Let $x \in V(G)$ be such that $\alpha(\langle N_k(x) \rangle) = s$ and let S be a maximum independent set in $\langle N_k(x) \rangle$. Let y be a vertex at distance $2k + 1$ from x and let $P : x = x_0, x_1, \ldots, x_{2k+1} = y$ be a shortest x,y-path. Set $S_1 = \{u \in S |\ \mathrm{dist}(u, x_1) = k + 1\}$ and $S_i = \{u \in S \setminus (S_1 \cup \ldots \cup S_{i-1}) |\ \mathrm{dist}(u, x_i) = k + 1\}$, $i = 2, \ldots, 2k + 1$. Then $\{S_1, \ldots, S_{2k+1}\}$ is a partition of S. Thus $|S_i| \geq \frac{|S|}{2k+1}$ for some i, $1 \leq i \leq 2k + 1$. Since all vertices in S_i are at distance $k + 1$ from x_i, this implies

$$\alpha_{k+1}(G) \geq \alpha(\langle N_{k+1}(x_i) \rangle) \geq |S_i| \geq \frac{|S|}{2k + 1} = \frac{\alpha_k(G)}{2k + 1},$$

as requested. ■

Combining Theorems 4 and 6, we obtain the following result.

Theorem 7. *Let $d \geq 9$ be an integer, let $\mathcal{C}^d = \{G|\ \mathrm{diam}(G) = d\}$ and let $\mathcal{C} \subset \mathcal{C}^d$. Then $\alpha(G)$ is bounded in $\mathcal{C}$ if and only if $\alpha_i(G)$ is bounded in $\mathcal{C}$ for at least one i, $2 \leq i \leq \frac{d-1}{4}$.*

Proof. Clearly, any bound on $\alpha(G)$ is a bound on $\alpha_i(G)$ as well. Conversely, suppose $\alpha_i(G)$ is bounded for some i, $2 \leq i \leq \frac{d-1}{4}$. Then both $\alpha_1(G)$ and $\alpha_2(G)$ are bounded by Theorem 6, implying $\alpha_\ell(G)$ is bounded for all ℓ, $1 \leq \ell \leq d$, by Theorem 4. But then $\alpha(G) \leq \sum_{\ell=1}^{d} \alpha_\ell(G)$ is also bounded. ■

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