Cycles through given vertices and closures

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Abstract

A set $S \subseteq V(G)$ is *cyclable* in G if G contains a cycle C with $S \subseteq V(C)$, and *pancyclable* in G if G contains cycles C_i with $|S \cap V(C_i)| = i$ for all i, $3 \leq i \leq |S|$. We consider stability of the properties of cyclability and pancylability of a given set S under the Bondy-Chvátal closure and under the $*$-closure introduced by the first author. We give refinements of the closure concepts by localizing "close" to the set S and we also derive some sufficient degree conditions for cyclability (of Ore type with $d(x) + d(y) \geq n - 1$ and with $\sigma_3 \geq n - 2$ restricted to S).

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1 Introduction

We consider finite simple undirected graphs $G = (V(G), E(G))$. For concepts and notation not defined here we refer to [4].

We denote by $N(x)$ the *neighborhood* of a vertex $x \in E(G)$, by $N[x] = N(x) \cup \{x\}$ the *closed neighborhood* of a vertex $x \in E(G)$, and by $d(x) = |N(x)|$ the *degree* of x. For a set $A \subseteq V(G)$ we set $N_A(x) = N(x) \cap A$ and $d_A(x) = |N_A(x)|$. By a *clique* we mean a complete subgraph of G (not necessarily maximal). The *distance* of two vertices x and y is denoted by $\mathrm{dist}(x, y)$.

Let $S \subseteq V(G)$. Then we denote $\delta(S) = \min\{d_G(x) \mid x \in S\}$, and $\alpha_G(S) = \max |\{Y \subseteq S \mid Y \text{ is independent in } G\}|$. For $k \leq \alpha_G(S)$ we define $\sigma_k(S) = \min\{\sum_{x \in Y} d_G(x) \mid Y \text{ is an independent set in } G, Y \subseteq S, |Y| = k\}$, otherwise we set $\sigma_k(S) = \infty$.

The *induced subgraph* on A in G is denoted by $\langle A \rangle_G$ (or, if no ambiguity can arise, simply $\langle A \rangle$), and we write $G - A$ for $\langle V(G) \setminus A \rangle_G$. A vertex x is *locally connected* if $\langle N(x) \rangle_G$ is a connected graph; otherwise x is said to be *locally disconnected.* A *claw* in G is an induced subgraph isomorphic to $K_{1,3}$. The (only) vertex of degree 3 of a claw is called its *center.* A vertex $x \in V(G)$ is called a *claw-free vertex* if x is not a center of a claw. A graph G is called *claw-free* if any vertex of G is claw-free. The *circumference* of G, i.e. the length of a longest cycle in G, is denoted by $c(G)$, and the (vertex) connectivity of G is denoted by $\kappa(G)$. We consider cycles to be naturally oriented and, for two vertices x, y of a cycle C, we denote by $x\overrightarrow{C}y$ ($x\overleftarrow{C}y$) the subpath of C with endvertices x, y and with the same (opposite) orientation with respect to the orientation of C. A similar notation is used for subpaths of a path.

Let S be a subset of $V(G)$. A vertex v is called an S-*vertex* if $v \in S$. Many results were published about cycles containing given subsets of vertices, see for example [16]. Following [12], [13], [16], the set S of vertices is called *cyclable* in G if all vertices of S belong to a common cycle in G. The graph G is also said to be S-cyclable and we speak of the cyclability of S in G. The S-*length* of a cycle in G is defined as the number of S-vertices that it contains and the graph G is said S-*pancyclable* if it contains cycles of all S-lengths from 3 to $|S|$. We also say that S is pancyclable in G and speak about pancyclability of S in G. The S-*circumference* of G, denoted by $c_S(G)$, is the S-length of a cycle that contains as many S-vertices as possible.

Conditions implying cyclability and pancyclability of a subset S of vertices have been investigated (see for example [12]) and there is a local version of the Ore condition that implies both cyclability and pancyclability of a set S as can be seen in Theorems A and B below.

Theorem A [18]. *Let G be a 2-connected graph of order n and S a subset of $V(G)$ with $|S| \geq 3$. If $d(x) + d(y) \geq n$ for every pair of nonadjacent vertices x and y in S, then S is cyclable in G.*

Theorem B [13]. *Let G be a graph of order n and S a subset of $V(G)$. If $d(x)+d(y) \geq n$ for every pair of nonadjacent vertices x and y of S, then either G is S-pancyclable or else n is even, $S = V(G)$ and $G = K_{\frac{n}{2},\frac{n}{2}}$, or $G[S] = K_{2,2} = C_4 := x_1x_2x_3x_4x_1$ and the structure of G is as follows: $V(G)$ is partitioned into $S \cup V_1 \cup V_2 \cup V_3 \cup V_4$; for any i, $1 \leq i \leq 4$, $G[V_i]$ is any graph on $|V_i|$ vertices with $|V_i| \geq 0$, and each vertex x_i is adjacent to all the vertices of V_{i+1} and V_i where the index i is taken modulo 4.*

In the same vein, Fournier [14] proved the following cyclability version of the well-known theorem by Chvátal and Erdös [10].

Theorem C [14]. *Let G be a 2-connected graph and let $S \subseteq V(G)$. If $\alpha_G(S) \leq \kappa(G)$, then S is cyclable in G.*

It is interesting to notice that the behaviour of both those properties is quite analogous to those of hamiltonicity and pancyclicity if considering this local Ore condition. It is well known that the general Ore condition, where the minimum degree sum is taken over all pairs of nonadjacent vertices in G instead of S, is closely linked to the definition of the Bondy-Chvátal closure. From this we have been motivated to study how the closure (and which closure) is related to the properties of cyclability and pancyclability.

Let us first recall the concept of k-closure of a graph G introduced by Bondy and Chvátal in [3]. The *k-closure* of G, denoted $C_k(G)$, is the graph obtained from G by recursively joining pairs of nonadjacent vertices satisfying $d(x) + d(y) \geq k$ until no such pair remains. The following was proved in [3].

Theorem D [3]. *Let G be a graph of order n and let k be an integer, $1 \leq k \leq 2n - 3$. Then*

(i) *the closure $C_k(G)$ is uniquely determined,*

(ii) *G is hamiltonian if and only if $C_n(G)$ is hamiltonian,*

(iii) *G is pancyclic if and only if $C_{2n-3}(G)$ is pancyclic.*

A property $\mathcal{P}$ is said to be *k-stable* [3], if G has $\mathcal{P}$ if and only if $C_k(G)$ has $\mathcal{P}$. Thus, (ii) says that hamiltonicity is n-stable, and (iii) says that pancyclicity is $(2n - 3)$-stable.

In Section 2, we easily derive results for S-cyclability and S-pancyclability analogous to those of Theorem D and we observe that in fact instead of considering the general Bondy-Chvátal closure we only need to consider a restricted form dealing with the vertices of S.

We also notice that for a graph G of order n and a subset $S \subseteq V(G)$, the property of S-cyclability is not $(n-1)$-stable, but we however get in Theorem 8 a good characterization of those graphs G such that $d_G(x) + d_G(y) \geq n - 1$ for any $x, y \in S$ with $x, y \notin E(G)$ that are not S-cyclable.

Since 1972, when Bondy and Chvátal introduced the closure concept, various closure concepts appeared, in relations with some special properties or some special families of graphs (see for example the survey [8]). Let us recall the closure concept introduced in [17] for claw-free graphs. Let G be a claw-free graph, let $x \in V(G)$ be locally connected with noncomplete neighborhood (such a vertex is called *eligible*) and let G'_x be the graph obtained from G by adding to $\langle N(x)\rangle_G$ all missing edges (i.e., $\langle N(x)\rangle_{G'_x}$ is a clique). The graph G'_x is called the *local completion* of G at x, and the graph, obtained from G by recursively repeating the local completion operation, as long as this is possible, is called the (*claw-free*) *closure* of G and denoted by $\mathrm{cl}(G)$. The following was proved in [17].

Theorem E [17]. *Let G be a claw-free graph. Then*

(i) *the closure* $\mathrm{cl}(G)$ *is uniquely determined,*

(ii) *there is a triangle-free graph H such that G is the line graph of H,*

(iii) $c(G) = c(\mathrm{cl}(G))$,

(iv) *G is hamiltonian if and only if* $\mathrm{cl}(G)$ *is hamiltonian.*

Similarly to the Bondy-Chvátal closure, there is a stability concept related to this closure. A subclass $\mathcal{C}$ of the class of claw-free graphs is *stable* if $G \in \mathcal{C}$ implies $\mathrm{cl}(G) \in \mathcal{C}$, and a property $\mathcal{P}$ is *stable in* $\mathcal{C}$ if, for every $G \in \mathcal{C}$, G has $\mathcal{P}$ if and only if $\mathrm{cl}(G)$ has $\mathcal{P}$. Thus, by Theorem E(iv), hamiltonicity is stable in claw-free graphs.

Motivated by this closure concept which turned out to be a powerfull tool in claw-free graphs, and by the closure concepts introduced by Broersma and Trommel in [9], the following strenghthening was introduced in [11]. For a tree T denote $S(T) = \{s \in V(T) \mid d_T(s) \geq 2\}$. Let $N^2(x) = \{y \in V(G) \mid 1 \leq \mathrm{dist}(x, y) \leq 2\}$.

Following [11] we say that x is an $*$-*eligible* vertex of G if

(i) x is a claw-free vertex (not necessarily locally connected),

(ii) $\langle N(x)\rangle$ is not a complete graph,

(iii) there is a tree T such that

α) $N(x) \subseteq V(T) \subseteq N^2(x)$,

β) for any $s \in S(T)$ the set $N(s) \setminus N[x]$ induces a clique (possibly empty),

γ) $V(T) \setminus S(T) \subseteq N(x)$.

Let G be a (general) graph, let $x \in V(G)$ be an $*$-eligible vertex of G and let G_x^* be the graph obtained from G by adding to $\langle N(x) \rangle_G$ all missing edges (i.e., $\langle N(x) \rangle_{G_x^*}$ is a clique). The graph G_x^* is called the *local completion* of G at x.

The following results summarize basic properties of the local completion operation at $*$-eligible vertices.

Proposition F [11]. *Let $x \in V(G)$ be a $*$-eligible vertex in G and let $y \in V(G)$*

(i) If y is claw-free in G, then y is claw-free in G_x^*.

(ii) If y is $*$-eligible in G, then y is $*$-eligible in G_x^*.

Proposition G [11]. *Let G be a graph, let $x \in V(G)$ be $*$-eligible in G and let G_x^* be the local completion of G at x. Then for any cycle C' in G_x^* there is a cycle C in G such that $V(C') \subseteq V(C)$.*

Analogously to the closure in claw-free graphs, we can define the $*$-*closure* of a graph G, denoted by $\mathrm{cl}^*(G)$, as a graph, obtained from G by recursively repeating the local completion operation at $*$-eligible vertices, as long as this is possible.

Proposition F(ii) implies the first part and Proposition G implies the second part of the following theorem.

Theorem H [11]. *Let G be a graph. Then*

(i) $\mathrm{cl}^*(G)$ is well-defined (i.e., uniquely determined),

(ii) $c(\mathrm{cl}^*(G)) = c(G)$.

We will also need the following result.

Proposition I [11]. *Let x be a locally connected claw-free vertex of G such that every vertex in $N(x)$ is also claw-free. Then x is $*$-eligible in G.*

Specifically, Proposition I implies that if G is a claw-free graph, then $\mathrm{cl}(G)$ is contained in $\mathrm{cl}^*(G)$.

In Section 3, we reformulate the $*$-closure for cyclability of a set $S \subseteq V(G)$. However, in the class of claw-free graphs it was proved in [5] that for every integer $k \geq 2$ there is a k-connected claw-free graph G such that G is not pancyclic but $\mathrm{cl}(G)$ is pancyclic. Thus, we cannot hope to obtain a pancyclability result for the $*$-closure as well. As an application of the $*$-closure, we obtain in Theorem 12 a σ_3-condition for cyclability, assuming the claw-free property only locally, “close” to the set S.

2 The Bondy-Chvátal closure

We are first interested in the stability of S-cyclability and S-pancyclability for a given subset S of vertices in a graph of order n. We get results generalizing those for hamiltonicity and pancyclicity that now appear as corollaries.

Theorem 1. *Let G be a graph of order n, let $S \subseteq V(G)$, $S \neq \emptyset$, and let k be an integer, $1 \leq k \leq |S|$. Let $u, v \in V(G)$ be such that $uv \notin E(G)$ and $d(u) + d(v) \geq n$. Then G contains a cycle C with $|V(C) \cap S| \geq k$ if and only if $G' = G + uv$ contains a cycle C' with $|V(C') \cap S| \geq k$.*

Proof. The "only if" part is trivial since every cycle in G is also a cycle in G'. Let, conversely, C' be a cycle in G' such that $|V(C') \cap S| \geq k$ and let $uv \in E(C')$ (otherwise there is nothing to do). Denote by P the uv-path $P = C' - uv$ in G and set $t = |V(P)|$ and $R = V(G) \setminus V(P)$.

Suppose first that $d_P(u) + d_P(v) \geq t$. Then for the sets $M = \{x \in V(P) \mid x^+u \in E(G)\}$ and $N = \{x \in V(P) \mid xv \in E(G)\}$ we have $|M| + |N| = d_P(u) + d_P(v) \geq t$. Since $v \notin M \cup N$, there is a vertex $x \in M \cap N$. But then $C = u\overrightarrow{P}xv\overleftarrow{P}x^+u$ is the required cycle.

Let next $d_P(u) + d_P(v) \leq t - 1$. Then $d_R(u) + d_R(v) = d(u) + d(v) - (d_P(u) + d_P(v)) \geq n - (t - 1)$. Since $|R| = n - t$, there is a vertex $y \in N_R(u) \cap N_R(v)$ and then $C = u\overrightarrow{P}vyu$ is a cycle in G with $|V(C) \cap S| \geq k$. ■

Remark 2. It is easy to construct an example of a graph G_0 showing that Theorem 1 fails if the conditions $|V(C) \cap S| \geq k$ and $|V(C') \cap S| \geq k$ are replaced by $|V(C) \cap S| = k$ and $|V(C') \cap S| = k$. The graph G_0 consists of an odd path $u = x_0x_1x_2 \ldots x_f \ldots x_{2f-2}x_{2f-1}x_{2f} = v$ and an additional vertex w; u is adjacent to $x_1, x_2, \ldots, x_f$ and w; symetrically, v is adjacent to $x_f, \ldots, x_{2f-2}, x_{2f-1}$ and w. The set S is equal to $V(G_0) \setminus \{u, v\}$. There is a cycle of S-length 1 in G' but not in G.

From Theorem 1, we easily obtain the following result concerning, for graphs G of order n and a (given) set $S \subseteq V(G)$, the value of S-circumference of G and the cyclability of S in G.

Corollary 3. *Let G be a graph of order n and let $S \subseteq V(G)$, $S \neq \emptyset$. Then*

(*i*) $c_S(G) = c_S(C_n(G))$,

(*ii*) *S is cyclable in G if and only if S is cyclable in $C_n(G)$.*

Proof. The proof follows immediately from Theorem 1. ■

Corollary 3 says in fact that for graphs G of order n, the property of cyclability of a (given) set $S \subseteq V(G)$ is n-stable. The following example shows that the value n for stability is sharp.

Example 4. Let G and $S \subseteq V(G)$ satisfy the following conditions.

(*i*) $S = \{s_1, s_2, s_3, s_4, s_5\}$,

(*ii*) $V(G) = V_1 \cup V_2 \cup \{s_3\}$, where $V_1 \cap V_2 = \emptyset$, $\{s_1, s_2\} \subseteq V_1$, $\{s_4, s_5\} \subseteq V_2$,

(*iii*) $N(s_1) = V_1 \cup \{s_4\}$, $N(s_5) = V_2 \cup \{s_2\}$, $N(s_3) = \{s_2, s_4\}$.

Then $s_1 s_5 \notin E(G)$, $d(s_1) + d(s_5) = n - 1$ and S is cyclable in $G + s_1 s_5$ but not in G. Thus, the property of cyclability of a (given) set $S \subseteq V(G)$ is not $(n-1)$-stable.

The following result is an analogue of Theorem 1 for S-pancyclability.

Theorem 5. *Let G be a graph of order n, let $S \subseteq V(G)$, $|S| \geq 3$, and let $u, v \in V(G)$ be such that $uv \notin E(G)$ and*

$$d(u) + d(v) \geq n + |S| - 3.$$

Then S is pancyclable in G if and only if S is pancyclable in $G + uv$.

Proof. Suppose that for some k, $3 \leq k \leq |S|$, there is a cycle C' in $G' = G + uv$ with $|V(C') \cap S| = k$ and no cycle C in G with $|V(C) \cap S| = k$. Clearly $uv \in E(C')$. Let P be the uv path $P = C' - uv$ in G (with an orientation from u to v) and set $t = |V(P)|$ and $R = V(G) \setminus V(P)$. By our assumption, $|V(P) \cap S| = k$.

If $d_P(u) + d_P(v) \geq t$, then, similarly as in the proof of Theorem 1, there is a vertex $x \in V(P)$ with $ux^+, vx \in E(G)$ and then $C = u\overrightarrow{P}xv\overleftarrow{P}x^+u$ gives a contradiction. Hence $d_P(u) + d_P(v) \leq t - 1$. This implies $d_R(u) + d_R(v) = d(u) + d(v) - (d_P(u) + d_P(v)) \geq n + |S| - 3 - (t - 1) = n - t + |S| - 2$. Since $k \geq 3$, we further have $d_R(u) + d_R(v) \geq n - t + |S| - 2 - k + 3 = n - t + |S| - k + 1$. Since $|R| = n - t$ and $|R \cap S| = |S| - k$, the vertices u and v have a common neighbor $y \in R \setminus S$. Then $C = u\overrightarrow{P}vyu$ is a cycle in G with $|V(C) \cap S| = k$, a contradiction. ■

Corollary 6. *Let G be a graph of order n and let $S \subseteq V(G)$, $|S| \geq 3$. Then S is pancyclable in G if and only if S is pancyclable in $C_{n+|S|-3}(G)$.*

Clearly Theorems 1 and 5 are true if assuming that the nonadjacent vertices u and v are both in the subset S of $V(G)$. We now localize the Bondy Chvátal closure as follows. Define for the subset $S \subseteq V(G)$ and any integer k the (k, S)-*closure* of G (denoted $C_k^S(G)$) as the graph obtained by recursively adding all missing edges uv with $d(u) + d(v) \geq k$, $u, v \in S$. The closure $C_k^S(G)$ is uniquely determined. Moreover, if G is large and S is small, considering $C_k^S(G)$ instead of $C_k(G)$ can reduce the complexity. We in fact have proved the following statement which is an easy consequence of Theorems 1 and 5 and of the definition of $C_k^S(G)$.

Proposition 7. *Let G be a graph of order n and let $S \subseteq V(G)$, $|S| \geq 3$. Then*

(*i*) $c_S(G) = c_S(C_n^S(G))$,

(*ii*) *S is cyclable in G if and only if G is cyclable in $C_n^S(G)$,*

(*iii*) *S is pancyclable in G if and only if S is pancyclable in $C_{n+|S|-3}^S(G)$.*

Note that Proposition 7 has Theorem D as a corollary if setting $S = V(G)$.

Coming back to S-cyclability for a subset of vertices of a graph of order n, we have shown that the property was n stable but not $(n-1)$-stable. However, in the next result we show that it is still possible to prove an Ore-type condition with $n-1$ instead of n if accepting some exception graphs. Theorem 8 generalizes analogous results for hamiltonicity and pancyclicity which can be found in [1] and [2], respectively.

Theorem 8. *Let G be a 2-connected graph of order n and let $S \subseteq V(G)$ be such that*

$$d_G(x) + d_G(y) \geq n - 1$$

for any $x, y \in S$ with $x, y \notin E(G)$. Then either S is cyclable in G, or n is odd and G contains an independent set $S_1 \subseteq S$ such that $|S_1| = \frac{n+1}{2}$ and every vertex of S_1 is adjacent to all vertices in $G - S_1$.

The **proof** of Theorem 8 is lengthy and therefore it is postponed to Section 4.

Theorem 8 has as an easy consequence the following.

Corollary 9. *Let G be a 1-tough graph of order n and let $S \subseteq V(G)$ be such that $|S| \geq 3$ and*

$$d(x) + d(y) \geq n - 1$$

for any nonadjacent $x, y \in S$. Then S is cyclable in G.

Proof. Assume S is not cyclable in G and let S_1 be defined as in Theorem 8. Set $R = G - S_1$. We then have $S_1 = G - R$ and S_1 has $|S_1| = |R| + 1$ components, contradicting the assumption that G is 1-tough. ■

3 The $*_H$-closure

Similarly as with the Bondy-Chvátal closure, we localize the $*$-closure as follows. Let $H \subseteq V(G)$ be an arbitrary set of vertices, and let $\mathrm{cl}_H^*(G)$ be the graph obtained from G by recursively performing the local completion operation at those $*$-eligible vertices that belong to H. Using Proposition F(ii), it is not difficult to show that, for any graph G, its $*_H$-closure $\mathrm{cl}^*_H(G)$ is uniquely determined (if G', G'' are two $*_H$-closures of G and xy is the first edge that occurs in $E(G') \setminus E(G'')$, then Proposition F(ii) immediately implies $xy \in E(G'')$, a contradiction). Then we have the following result.

Theorem 10. *Let G be a graph, let $S \subseteq V(G)$, $S \neq \emptyset$, let k be an integer, $1 \leq k \leq |S|$. Let $H \subseteq V(G)$ be an arbitrary set of vertices. Then G contains a cycle C with $|V(C) \cap S| \geq k$ if and only if $\mathrm{cl}^*_H(G)$ contains a cycle C' with $|V(C') \cap S| \geq k$.*

Proof. Clearly any cycle in G is a cycle in $\mathrm{cl}^*_H(G)$. Let, conversely, C' be a cycle in $\mathrm{cl}^*_H(G)$ with $|V(C') \cap S| \geq k$. Then the existence of a required cycle C in G follows immediately from Proposition G. ■

Corollary 11. *Let G be a graph and let $H, S \subseteq V(G)$ be arbitrary sets of vertices. Then*

(i) $c_S(G) = c_S(\mathrm{cl}^*_H(G))$,

(ii) *S is cyclable in G if and only if S is cyclable in $\mathrm{cl}^*_H(G)$.*

Proof. The proof follows immediately from Theorem 10. ■

Broersma and Lu [7] have localized some well-known sufficient conditions for hamiltonicity in claw-free graphs to cyclability, but under the assumption that the whole graph G is claw-free. As an application of the $*_H$-closure concept, we show that the well known σ_3-condition for hamiltonicity in claw-free graphs can be extended to cyclability, assuming that G is claw-free only "locally", i.e. under the assumption that no vertex in $S \cup N(S)$ is a claw center. Note that a proper choice of the set H of the $*_H$-closure plays a crucial role in the proof: e.g. with the choice $H = V(G)$ even the statement (i) of Claim 2 could not be proved, since then, for a vertex in S, its neighbors in $\mathrm{cl}^*_H(G)$ are not necessarily claw-free.

Theorem 12. *Let G be a 2-connected graph of order $n \geq 33$ and let $S \subseteq V(G)$, $S \neq \emptyset$, be such that*

(i) *no vertex in $S \cup N(S)$ is a claw center,*

(ii) $\sigma_3(S) \geq n - 2$.

Then S is cyclable in G.

Theorem 12 obviously implies the following minimum degree condition.

Corollary 13. *Let G be a 2-connected graph of order $n \geq 33$ and let $S \subseteq V(G)$, $S \neq \emptyset$, be such that*

(i) *no vertex in $S \cup N(S)$ is a claw center,*

(ii) $\delta(S) \geq \frac{n-2}{3}$.

Then S is cyclable in G.

Note that Theorem 12 also implies for $n \geq 33$ as immediate corollaries the following well-known results by Broersma [6] and Zhang [19] and by Matthews and Sumner [15].

Corollary J [6], [19]. *Every 2-connected claw-free graph of order $n \geq 3$ satisfying $\sigma_3(G) \geq n-2$ is hamiltonian.*

Corollary K [15]. *Every 2-connected claw-free graph of order $n \geq 3$ satisfying $\delta(G) \geq \frac{n-2}{3}$ is hamiltonian.*

Proof of Theorem 12. Suppose G and S satisfy the assumptions of the theorem but S is not cyclable in G, and set $H = S \cup \{x \in V(G) \mid$ there are $u_1, u_2 \in S$ such that $u_1u_2 \notin E(G)$ and $x \in N(u_1) \cap N(u_2)\}$. Then by Corollary 11, the same holds for $\mathrm{cl}^*_H(G)$. Thus, in the following we suppose that $G = \mathrm{cl}^*_H(G)$.

We first prove four auxiliary statements describing the structure of G "close" to S.

Claim 1. For any $u \in S$, $\langle N(u)\rangle_G$ either is a clique or consists of two vertex-disjoint cliques.

Proof of Claim 1. If u is $*$-eligible, then $\langle N(u)\rangle_G$ is a clique since G is $*_H$-closed and $S \subseteq H$. If u is not $*$-eligible, then, by Proposition I, u is locally disconnected and $\langle N(u)\rangle_G$ consists of two vertex-disjoint cliques since u is not a claw center. $\square$

Claim 2. Let $u, v \in S$, $uv \notin E(G)$. Then

(i) for any component K_u of $\langle N(u)\rangle$ and K_v of $\langle N(v)\rangle$, $|K_u \cap K_v| \leq 1$,

(ii) $|N(u) \cap N(v)| \leq 2$,

(iii) if at least one $\langle N(u)\rangle$, $\langle N(v)\rangle$ is a clique, then $|N(u) \cap N(v)| \leq 1$.

Proof of Claim 2. (i) Let, to the contrary, $x, y \in V(K_u) \cap V(K_v)$. Then $x, y \in H$ and $xy \in E(G)$. We show that x is $*$-eligible. By the choice of the set H, x is a claw-free vertex. Since $u, v \in N(x)$ and $uv \notin E(G)$, we have $N(x) \subseteq N[u] \cup N[v]$ (otherwise x is a claw center). This implies that all vertices in $N(x)$ are claw-free (see Proposition F(i)) and, since $y \in N(x)$, $\langle N(x)\rangle$ is connected. By Proposition I, x is $*$-eligible. Since $x \in H$, $\langle N(x)\rangle_G$ is a clique, implying $uv \in E(G)$, a contradiction.

(ii) If $|N(u) \cap N(v)| \geq 3$, then for some two components K_u, K_v of $\langle N(u)\rangle$ and $\langle N(v)\rangle$ we have $|V(K_u) \cap V(K_v)| \geq 2$, contradicting part ($i$).

(iii) Suppose that $\langle N(u)\rangle_G$ is a clique. If $N(u) \cap N(v) = \{x, y\}$, then, similarly as above, x is $*$-eligible, implying $uv \in E(G)$, a contradiction. $\square$

Claim 3. If $\{u_1, u_2, u_3\} \subseteq S$ is an independent set, then

(i) $\sum_{1 \leq i < j \leq 3} |N(u_i) \cap N(u_j)| \geq 1$,

(ii) $|N(u_1) \cap N(u_2) \cap N(u_3)| = 0$.

Proof of Claim 3. (*i*) If $N(u_i) \cap N(u_j) = \emptyset$ for all i, j, $1 \leq i < j \leq 3$, then $n \geq d(u_1) + d(u_2) + d(u_3) + 3 \geq \sigma_3(S) + 3 \geq n - 2 + 3 = n + 1$, a contradiction.

(*ii*) If $x \in N(u_1) \cap N(u_2) \cap N(u_3)$, then $\langle \{x, u_1, u_2, u_3\} \rangle$ is a claw, a contradiction. □

Claim 4. Let $M = \{s_1, s_2, s_3\} \subseteq S$ be an independent set. Then $S \subseteq N(M)$.

Proof of Claim 4. Let, to the contrary, $a \in S \setminus N(M)$. Then $M' = M \cup \{a\}$ is an independent set and $M' \subseteq S$. Since $\sigma_3 \geq n-2$, we have $d(x_1) + d(x_2) + d(x_3) \geq n - 2$ for every 3-element subset $\{x_1, x_2, x_3\} \subset M'$, from which

$$3(d(s_1) + d(s_2) + d(s_3) + d(a)) \geq 4(n-2).$$

By Claim 2(*ii*) and Claim 3(*ii*), there are at most 12 vertices that are common neighbors for some pair of vertices of M'. This implies

$$n \geq |N(M')| \geq d(s_1) + d(s_2) + d(s_3) + d(a) + 4 - 12,$$

from which

$$3(n+8) \geq 3(d(s_1) + d(s_2) + d(s_3) + d(a)) \geq 4(n-2),$$

implying $n \leq 32$, a contradiction. □

Now, if $\alpha(S) \leq 2$, then S is cyclable by Theorem C. Thus, for the rest of the proof suppose that $\alpha(S) \geq 3$. Let $\{s_1, s_2, s_3\} \subseteq S$ be an independent set. By Claim 2(*ii*) and Claim 3(*i*), we can suppose the notation is chosen such that $1 \leq |N(s_1) \cap N(s_2)| \leq 2$. Set $F = \langle \{s_1, s_2\} \cup N(s_1) \cup N(s_2) \rangle$.

Since G is 2-connected, there are two vertex-disjoint $a_i b_i$-paths P_i (possibly trivial) such that $a_i \in V(F) \setminus \{s_1, s_2\}$ and $b_i \in N(s_3)$, $i = 1, 2$.

Claim 5.

(*i*) *There is an $a_1 a_2$-path P such that P is internally vertex-disjoint from $V(F)$ and $N[s_3] \subseteq V(P)$.*

(*ii*) *Moreover, if there is another ab-path Q with $a \in V(F)$ and $b \in V(P)$, then there is also a path P' from a to some of a_1, a_2 such that $N[s_3] \subseteq V(P')$.*

Proof of Claim 5. (*i*) If $\langle N(s_3) \rangle$ is a clique, we can set $P = a_1 \overrightarrow{P_1} b_1 \langle N(s_3) \rangle b_2 \overleftarrow{P_2} a_2$. Thus, let K_3^1, K_3^2 be the components of $\langle N(s_3) \rangle$. If $b_i \in V(K_3^i)$, $i = 1, 2$, then clearly $P = a_1 \overrightarrow{P_1} b_1 K_3^1 s_3 K_3^2 b_2 \overleftarrow{P_2} a_2$. Thus, up to a symmetry, it remains to consider the case when $b_1, b_2 \in V(K_3^1)$.

Since s_3 cannot be a cutvertex, there is an $a_3 b_3$-path P_3 avoiding s_3 such that $a_3 \in V(K_3^1)$ and $b_3 \in V(K_3^2)$. If P_3 has a common vertex with P_1, P_2 or F, then we are in the previous case, hence P_3 is vertex-disjoint from P_1, P_2 and F. Then the path $P = a_1 \overrightarrow{P_1} b_1 a_3 \overrightarrow{P_3} b_3 K_3^2 s_3 (K_3^1 - \{b_1, a_3\}) b_2 \overleftarrow{P_2} a_2$ has the required properties.

(ii) If Q is vertex-disjoint from both P_1 and P_2, then $b \in N[s_3]$ and we apply the construction from the proof of part (i) to Q and to any of P_1, P_2. Thus, Q intersects at least one of P_1, P_2. Let c be the first (along Q from a to b) vertex in $V(Q) \cap (V(P_1) \cup V(P_2))$. If $c \in V(P_1)$, then we apply the proof of part (i) to the paths $a\overrightarrow{Q}c\overrightarrow{P_1}b_1$ and P_2; if $c \in V(P_2)$, then we use $a\overrightarrow{Q}c\overrightarrow{P_2}b$ and P_1. $\square$

We now distinguish two cases.

Case 1: $|N(s_1) \cap N(s_2)| = 2$.

Denote $\{z_1, z_2\} = N(s_1) \cap N(s_2)$. By Claim 2$(iii)$, s_1 and s_2 are locally disconnected. Denote by K_i^j that of the two components of $\langle N(s_i)\rangle$, which contains z_j, and set $\bar{K}_i^j = \langle V(K_i^j) \cup \{s_i\}\rangle_G$ for $i, j = 1, 2$. If a_1, a_2 are in the same clique of F, say, $a_1, a_2 \in V(K_1^1)$, then the cycle $C = a_1\overrightarrow{P}a_2z_1K_2^1s_2K_2^2z_2K_1^2s_1(K_1^1 - \{a_2, z_1\})a_1$ contains all vertices of S. Similarly, if a_1, a_2 are in two consecutive cliques of F, say, $a_1 \in V(K_1^1)$ and $a_2 \in V(K_1^2)$, then $C = a_1\overrightarrow{P}a_2K_1^2z_2K_2^2s_2K_2^1z_1\bar{K}_1^1a_1$ contains all vertices of S. Thus, up to a symmetry, we can suppose that $a_1 \in V(K_1^2)$ and $a_2 \in V(K_2^1)$. We show that then either $V(K_1^1) \cap S = \emptyset$ or $V(K_2^2) \cap S = \emptyset$. Let, to the contrary, $u_i \in V(K_i^i) \cap S$, $i = 1, 2$. By Claim 3(i), some two of the vertices u_1, u_2, s_3 have a common neighbor.

Suppose first that $v \in N(u_1) \cap N(u_2)$. If $v \in V(G) \setminus (V(F) \cup V(P))$, then the cycle $C = a_1\overrightarrow{P}a_2\bar{K}_2^1z_1\bar{K}_1^1u_1vu_2K_2^2z_2K_1^2a_1$ contains all vertices of S. Hence $v \in V(F) \cup V(P)$. If $v \in (V(K_1^1) \setminus \{z_1\}) \cup (V(K_2^2) \setminus \{z_2\})$, then we analogously have a cycle C with $S \subseteq V(C)$, and the possibility $v \in V(K_1^2) \cup V(K_2^1) \cup \{s_1, s_2\}$ contradicts the fact that G is closed. Hence $v \in V(P) \setminus \{a_1, a_2\}$, but then, by Claim 5$(ii)$ we have a path P' joining u_1 to some of a_1, a_2 and containing all vertices of $N[s_3]$ and we are in the previous subcase. Hence $N(u_1) \cap N(u_2) = \emptyset$. The remaining (up to a symmetry) case $v \in N(u_1) \cap N(s_3)$ yields a contradiction in an analogous way.

Hence one of K_1^1, K_2^2 (say K_1^1) contains no vertex of S. But then all vertices of S are on the cycle $C = a_1\overrightarrow{P}a_2K_2^1s_2K_2^2z_2\bar{K}_1^2a_1$.

Case 2: $|N(s_1) \cap N(s_2)| = 1$.

Denote $\{z\} = N(s_1) \cap N(s_2)$. Recall that, by Claim 4, $\{s_1, s_2, s_3\}$ dominates S and since G is closed, each $\langle N(s_i)\rangle$ is a clique or consists of two vertex-disjoint cliques. We want to show that there is a cycle C containing all vertices of S.

In order to reduce the number of cases to be considered, we define a graph G' by the following construction: if $\langle N(s_i)\rangle_G$ is a clique of order t (note that necessarily $t \geq 2$), we partition its vertices into two subsets N_1, N_2 of order $\lfloor \frac{t}{2} \rfloor$ and $\lceil \frac{t}{2} \rceil$, respectively, and remove all edges xy with $x \in N_1$ and $y \in N_2$, $i = 1, 2$. Clearly, both $\langle N(s_1)\rangle_{G'}$ and $\langle N(s_2)\rangle_{G'}$ consist of two vertex-disjoint cliques. Denote by K_i^1, K_i^2 the components of $\langle N(s_i)\rangle_{G'}$, $i = 1, 2$, choose the notation such that $z \in V(K_1^2) \cap V(K_2^1)$, and denote $\bar{K}_i^j = \langle V(K_i^j) \cup \{s_i\}\rangle_G$,

$i, j = 1, 2$. Set $F' = \langle \bar{K}_1^1 \cup \bar{K}_1^2 \cup \bar{K}_2^1 \cup \bar{K}_2^2 \rangle_{G'}$. Recall that by Claim 5$(i)$, there is an a_1a_2-path P such that $a_1, a_2 \in V(F')$, $N[s_3] \subseteq V(P)$ and P is internally vertex-disjoint from F'.

If the path P can be chosen such that $a_1 \in V(K_1^1)$ and $a_2 \in V(K_2^2)$, then $C = a_1\overrightarrow{P}a_2K_2^2s_2K_2^1zK_1^2s_1K_1^1a_1$ is the required cycle. Hence, by the symmetry, we can suppose that for every a_1a_2-path with $a_1, a_2 \in V(F')$ and $N[s_3] \subseteq V(P)$ we have $a_1, a_2 \in V(K_1^2) \cup V(K_2^1) \cup V(K_2^2)$. This implies by Claim 5(ii) that there is no path from $V(K_1^1)$ to $N[s_3]$ which is internally vertex-disjoint from F'. Since s_1 cannot be a cutvertex of G, there is an a_3b_3-path P_3 in G such that $a_3 \in V(K_1^1)$, $b_3 \in V(K_1^2) \cup V(K_2^1) \cup V(K_2^2)$ and P_3 is internally disjoint with F' and P (note that P_3 can be one of the edges removed during the construction of G').

Suppose first that K_2^2 has the same property as K_1^1, i.e. that there is no path from K_2^2 to $N[s_3]$ internally vertex-disjoint from F'. Then, symmetrically, there is a_4b_4-path P_4 such that $a_4 \in V(K_2^2)$, $b_4 \in V(K_1^1) \cup V(K_1^2) \cup V(K_2^1)$ and P_4 is internally vertex-disjoint with F' and P. Moreover, P_4 is internally vertex disjoint also with P_3 since otherwise we are in some of the previous subcases. Then we have, up to a symmetry, the following possibilities.

Case $a_1 \in V(K_1^2)$, $a_2 \in V(K_2^1)$

Subcase	Cycle C with $S \subseteq V(C)$
$b_3 \in V(K_2^2)$	$a_1\overrightarrow{P}a_2K_2^1s_2K_2^2b_3\overleftarrow{P_3}a_3K_1^1s_1(K_1^2 - z)a_1$
$b_3 \in V(K_2^1)$, $b_4 \in V(K_2^1)$	$a_1\overrightarrow{P}a_2(K_2^1 - \{b_3, b_4\})s_2K_2^2a_4P_4b_4b_3\overleftarrow{P_3}a_3K_1^1s_1(K_1^2 - z)a_1$
$b_3 \in V(K_2^1)$, $b_4 \in V(K_1^2)$	$a_1\overrightarrow{P}a_2s_2K_2^2a_4\overrightarrow{P_4}b_4z(K_2^1 - a_2)b_3\overleftarrow{P_3}a_3K_1^1s_1(K_1^2 - \{b_4, z\})a_1$
$b_3 \in V(K_1^2)$, $b_4 \in V(K_2^1)$	$a_1\overrightarrow{P}a_2s_2K_2^2a_4\overrightarrow{P_4}b_4(K_2^1 - a_2)zb_3\overleftarrow{P_3}a_3K_1^1s_1(K_1^2 - \{z, b_3\})a_1$

Case $a_1, a_2 \in V(K_1^2)$

Subcase	Cycle C with $S \subseteq V(C)$
$b_3 \in V(K_2^2)$	$a_1\overrightarrow{P}a_2zK_2^1s_2K_2^2b_3\overleftarrow{P_3}a_3K_1^1s_1(K_1^2 - \{a_2, z\})a_1$
$b_3 \in V(K_2^1)$, $b_4 \in V(K_2^1)$	$a_1\overrightarrow{P}a_2z(K_2^1 - \{b_3, b_4\})s_2K_2^2a_4\overrightarrow{P_4}b_4b_3\overleftarrow{P_3}a_3K_1^1s_1(K_1^2 - \{a_2, z\})a_1$
$b_3 \in V(K_2^1)$, $b_4 \in V(K_1^2)$	$a_1\overrightarrow{P}a_2b_4\overleftarrow{P_4}a_4K_2^2s_2K_2^1b_3\overleftarrow{P_3}a_3K_1^1s_1(K_1^2 - \{a_2, b_4, z\})a_1$
$b_3 \in V(K_1^2)$, $b_4 \in V(K_1^2)$	$a_1\overrightarrow{P}a_2zK_2^1s_2K_2^2a_4\overrightarrow{P_4}b_4b_3\overleftarrow{P_3}a_3K_1^1s_1(K_1^2 - \{a_2, z, b_3, b_4\})a_1$

In the remaining subcase in this case, i.e. $b_3 \in V(K_1^2)$ and $b_4 \in V(K_2^1)$, we observe that, by the 2-connectedness, there is an a_5b_5-path P_5 with $a_5 \in K_1^2$ and $b_5 \in K_2^1$. Since the path P_5 is internally vertex-disjoint from P_3, P_4, P and F' (otherwise we can transform the situation to some of the previous cases), $C = a_1\overrightarrow{P}a_2z(K_2^1 - \{a_4, b_5\})s_2K_2^2a_4\overrightarrow{P_4}b_4b_5\overleftarrow{P_5}a_5b_3\overleftarrow{P_3}a_3K_1^1s_1(K_2^1 - \{z, a_5, b_3\})a_1$, a contradiction.

Hence for the rest of the proof we can suppose that $a_2 \in K_2^2$ and $a_1 \in V(K_1^2) \cup V(K_2^1) \cup V(K_2^2)$. Then we have $b_3 \in V(K_1^2) \cup V(K_2^1) \cup V(K_2^2)$. We distinguish three cases.

(i) If a_1, b_3 are in the same clique, we have the following possibilities:

Case	Cycle C with $S \subseteq V(C)$
$a_1, b_3 \in V(K_1^2)$	$a_1 \overrightarrow{P} a_2 K_2^2 s_2 K_2^1 z b_3 \overleftarrow{P_3} a_3 K_1^1 s_1 (K_1^2 - \{z, b_3\}) a_1$
$a_1, b_3 \in V(K_2^1)$	$a_1 \overrightarrow{P} a_2 K_2^2 s_2 b_3 \overleftarrow{P_3} a_3 K_1^1 s_1 K_1^2 z (K_2^1 - b_3) a_1$
$a_1, b_3 \in V(K_2^2)$	$a_1 \overrightarrow{P} a_2 b_3 \overleftarrow{P_3} a_3 K_1^1 s_1 K_1^2 z K_2^1 s_2 (K_2^2 - \{a_2, b_3\}) a_1$

(ii) If P and P_3 are non-overlapping and a_1, b_3 are in different cliques, then the following possibilities can occur:

$\alpha)$ $b_3 \in V(K_1^2)$, $a_1 \in V(K_2^1)$,

$\beta)$ $b_3 \in V(K_1^2)$, $a_1 \in V(K_2^2)$,

$\gamma)$ $b_3 \in V(K_2^1)$, $a_1 \in V(K_2^2)$.

In the subcase $\alpha)$ the vertex z cannot be a cutvertex, implying there is a path P' from $V(K_1^1) \cup V(K_1^2)$ to $V(K_2^1) \cup V(K_2^2)$. In the subcase $\beta)$, there is either a path P' from $V(K_1^1) \cup V(K_1^2)$ to $V(K_2^2)$, or a pair of paths P' from $V(K_1^1) \cup V(K_1^2)$ to $V(K_2^1)$ and P'' from $V(K_2^1)$ to $V(K_2^2)$.

In the subcase $\gamma)$, there is a path P' from $V(K_2^2)$ to some of $V(K_2^1)$, $V(K_1^2)$ or $V(K_1^1)$.

In each of these subcases, it is straightforward to check that there is a cycle containing all vertices of S.

(iii) Thus, P and P_3 are overlapping. Then we have the following possibilities.

$\alpha)$ $b_3 \in V(K_2^1)$, $a_1 \in V(K_1^2)$,

$\beta)$ $b_3 \in V(K_2^2)$, $a_1 \in V(K_2^1)$,

$\gamma)$ $b_3 \in V(K_2^2)$, $a_1 \in V(K_1^2)$.

In the first two subcases $\alpha)$ and $\beta)$, the cycles $a_1 \overrightarrow{P} a_2 K_2^2 s_2 K_2^1 b_3 \overleftarrow{P_3} a_3 K_1^1 s_1 (K_1^2 - z) a_1$ and $a_1 \overrightarrow{P} a_2 K_2^2 b_3 \overleftarrow{P_3} a_3 K_1^1 s_1 K_1^2 z \bar{K}_2^1 a_1$, respectively, contain all vertices of S. Thus, to complete the proof, it remains to consider subcase $\gamma)$. If s_1 is eligible in G, then $\bar{K}_1 = \langle V(\bar{K}_1^1) \cup V(\bar{K}_1^2) \rangle_G$ is a clique in G and the cycle (in G) $a_1 \overrightarrow{P} a_2 K_2^2 s_2 K_2^1 z \bar{K}_1 a_1$ contains all vertices of S. Similarly, if s_2 is $*$-eligible in G, then $\bar{K}_2 = \langle V(\bar{K}_2^1) \cup V(\bar{K}_2^2) \rangle_G$ is a clique in G and the cycle $a_1 \overrightarrow{P} a_2 \bar{K}_2 b_3 \overleftarrow{P_3} a_3 K_1^1 s_1 (K_1^2 - z) a_1$ contains all vertices of S. Hence both s_1 and s_2 are locally disconnected in G, implying that $\bar{K}_1^1$ and $\bar{K}_2^1$ are maximal cliques in G.

We show that at most one of K_1^1, K_2^1 can contain a vertex of S. Let, to the contrary, $u_i \in V(K_i^1) \cap S$, $i = 1, 2$, $u_i \neq s_1, s_2$. Then clearly $\{u_1, u_2, s_3\}$ is an independent set and, u_1 and u_2 cannot have a common neighbor (otherwise there is a cycle containing S). Since there is no path from K_1^1 to $N[s_3]$, also u_1 and s_3 have no common neighbor. Hence by Claim 3(i), there is a vertex $y \in N(u_2) \cap N(s_3)$. But then, by Claim 5(ii), we get a $u_2 a_2$-path ($a_1 u_2$-path) P' with $N[s_3] \subseteq V(P')$, and we transform this subcase to some of the previous subcases. Thus, we have proved that either K_1^1 or K_2^1 contains no vertex of S. But this implies that in the first case $a_1 \overrightarrow{P} a_2 K_2^2 s_2 K_2^1 z \bar{K}_1^2 a_1$ and in the second case $a_1 \overrightarrow{P} a_2 \bar{K}_2^2 b_3 \overleftarrow{P_3} a_3 K_1^1 s_1 K_1^2 a_1$ is a cycle containing all vertices of S. ■

4 Proof of Theorem 8

We first prove the following lemma.

Lemma 14. *Let G be a 2-connected graph of order n and $S \subseteq V(G)$ such that $d_G(x) + d_G(y) \geq n-1$ for every pair x, y of vertices of S with $xy \notin E(G)$. Let $u, v \in S$ be nonadjacent. If S is cyclable in $G' = G + uv$, then either S is cyclable in G or n is odd and S contains an independent set $S_1 \subseteq S$ such that $|S_1| = \frac{n+1}{2}$ and every vertex of S_1 is adjacent to all vertices in $G - S_1$.*

Proof. Let S be cyclable in G' and let C' be a cycle in G' containing S. Suppose S is not cyclable in G. Clearly $uv \in E(C')$ and we get an uv-path P in G such that $S \subseteq V(P)$. Denote $|V(P)| = t$ and $R = G - P$.

If $d_P(u) + d_P(v) \geq t$, similarly as in the proof of Theorem 1, there is an $x \in V(P)$ with $ux^+, vx \in E(G)$ and the cycle $u\overrightarrow{P}xv\overleftarrow{P}x^+u$ yields a contradiction.

If $d_P(u) + d_P(v) \leq t-2$, then $d_R(u) + d_R(v) \geq n-1-(t-2) = n-t+1$, implying u and v have a common neighbor in R and so we also get a cycle in G that contains S.

We then necessarily have $d_P(u) + d_P(v) = t-1$. Set

$$A = \{x \in V(P) \mid u, v \in N(x)\}, \quad B^u = \{x \in V(P) \mid u \in N(x), v \notin N(x)\}$$
$$B^v = \{x \in V(P) \mid u \notin N(x), v \in N(x)\}, \quad C = \{x \in V(P) \mid u, v \notin N(x)\},$$

and denote $A = a_1, a_2, \ldots, a_k$ (with the indices increasing along P from u to v). For any two vertices $x, y \in V(P)$, $x \neq y$, $x \in u\overrightarrow{P}y$, denote $P[x,y] = x\overrightarrow{P}y$, $P(x,y] = x^+\overrightarrow{P}y$, $P[x,y) = x\overrightarrow{P}y^-$ and if $x^+ \neq y$ then also $P(x,y) = x^+\overrightarrow{P}y^-$.

Now, if $A = \emptyset$, then $d_P(u) + d_P(v) \leq t-2$, a contradiction. Hence $A \neq \emptyset$, implying $k \geq 1$.

Claim 1. If $k \geq 2$, then every $P(a_i, a_{i+1})$ $(i = 1, \ldots, k-1)$ contains at least one vertex from the set S.

Proof of Claim 1. If $P(a_i, a_{i+1})$ contains no vertex of S, then the cycle $u\overrightarrow{P}a_iv\overleftarrow{P}a_{i+1}u$ yields a contradiction. □

Claim 2. If $k \geq 2$, then every $P(a_i, a_{i+1})$ $(i = 1, \ldots, k-1)$ contains at least one vertex from the set C.

Proof of Claim 2. If not, then all the interior vertices of $P(a_i, a_{i+1})$ belong to $B^u \cup B^v$. Let z be the first vertex in $P(a_i, a_{i+1}]$ such that $zu \in E(G)$ (such a vertex exists since $a_{i+1}u \in E(G)$). Then $z^-v \in E(G)$ (with possibly $z^- = a_i$) and the cycle $u\overrightarrow{P}z^-v\overleftarrow{P}zu$ yields a contradiction. □

Claim 3. If $k \geq 2$, then every $P(a_i, a_{i+1})$ $(i = 1, \ldots, k-1)$ contains exactly one vertex from the set C.

Proof of Claim 3. Otherwise we have $|C| \geq |A|$, implying $d_P(u)+d_P(v) = 2|A|+|B^u|+|B^v| \leq |A|+|B^u|+|B^v|+|C| \leq t-2$, a contradiction (recall that $t = |A|+|B^u|+|B^v|+|C|+2$). $\square$

Denote by c_i the only vertex in $P(a_i, a_{i+1}) \cap C$ $(i = 1, \ldots, k-1)$.

Claim 4.

(*i*) *$P(u, a_1) \subseteq B^u$ and $P(a_k, v) \subseteq B^v$.*

(*ii*) *If $k \geq 2$, then moreover $P(a_i, c_i) \subseteq B^v$ and $P(c_i, a_{i+1}) \subseteq B^u$, $i = 1, \ldots, k-1$.*

Proof of Claim 4. It is a direct consequence of Claims 3. $\square$

Now, if $k = 1$ then for any $x_2 \in P[u, a_1)$ and $x_2 \in P(a_1, v]$ there is no x_1x_2-path Q in G with interior vertices outside P (otherwise the cycle $u\overrightarrow{P}x_1Qx_2\overrightarrow{P}vx_2^-\overleftarrow{P}x_1^+u$ yields a contradiction), but then a_1 is a cutvertex of G contradicting the assumption that G is 2-connected. Hence $k \geq 2$.

We distinguish three cases.

Case 1: $P(c_i, a_{i+1}) \neq \emptyset$ for some i, $1 \leq i \leq k-1$.

Then $P(u, a_1) \cap S \neq \emptyset$, since otherwise the cycle $ua_{i+1}\overrightarrow{P}va_1\overrightarrow{P}a_{i+1}^-u$ contains all vertices of the set S.

If $sa_{i+1} \in E(G)$, where s is the last vertex of $P(u, a_1) \cap S$, then we are also done since $P(u, a_1) \cap S$ can be inserted into $a_{i+1}^-a_{i+1}$ (i.e., we have a cycle $u\overrightarrow{P}sa_{i+1}\overrightarrow{P}va_1\overrightarrow{P}a_{i+1}^-u$ containing S).

Thus, let z be the first vertex in $P(u, a_1) \cap S$ nonadjacent to a_{i+1}, and let w be its predecessor in S. Set $P' = zPa_{i+1}^-uPwa_{i+1}Pv$. Then P' is a z,v-path with $V(P') \subseteq V(P)$ and $S \subseteq V(P')$. Clearly, z and v are nonadjacent. By the assumption, $d(u) + d(v) \geq n-1$. Repeating the previous argument for P' instead of P, we get (let $t' = |V(P')|$) $d_{P'}(z) + d_{P'}(v) = t'-1$. Hence the structure of $N_{P'}(z)$ and $N'_P(v)$ is described by Claims 1 to 4, too.

Now it is straightforward to check that z cannot be adjacent to any of the c_i's (otherwise we are done). Hence z has at least k common neighbors with v, since otherwise a counting argument gives that we cannot have $d_{P'}(z) + d_{P'}(v) = t'-1$.

By Claim 4(*ii*), no vertex in any $P(c_i, a_{i+1})$ can be a common neighbor of z and v. If a vertex in some $P(a_i, c_i)$ is a common neighbor of z and v, we have a contradiction (using Claim 4(*ii*)). Hence $N_{P'}(z) \cap N_{P'}(v) = A$, implying that $za_{i+1} \in E(G)$, which is also a contradiction.

Case 2: $P(a_i, c_i) \neq \emptyset$ for some $i, 1 \leq i \leq k-1$.

This case is symmetric to Case 1.

Case 3: $P(a_i, c_i) = P(c_i, a_{i+1}) = \emptyset$ for all $i, 1 \leq i \leq k-1$.

Then, by Claim 1, $c_i \in S$ and it is straightforward to check that $N_P(c_i) \subseteq A$ for $i = 1, \ldots, k-1$. Hence $d_P(c_i) \leq k$, $i = 1, \ldots, k-1$. Suppose that $P(u, a_1) \neq \emptyset$ and let $d = |P(u, a_1)|$. Then $t = |V(P)| \geq (k+1) + k + d = 2k + d + 1$, implying $|V(R)| \leq n - 2k - d - 1$. Since v and c_1 are independent and both in S, using Claim 4(i) we have $d_R(v) + d_R(c_1) = d_G(v) + d_G(c_1) - d_P(z) + d_P(c_1) \geq n - 1 - [(t - (k+1) - 1) + k] = n - t + 1$. This implies that v and c_1 have a common neighbor $r \in R$ and the cycle $u\overrightarrow{P}c_1 r v \overleftarrow{P} a_2 u$ yields a contradiction.

Hence $P(u, a_1) = \emptyset$. By symmetry, $P(a_k, v) = \emptyset$. This implies that $S_1 = \{u, c_1, \ldots c_k, v\} \subseteq S$, $|S_1| = k+1$, S_1 is independent and $d_P(x) \leq k$ for any $x \in S_1$.

We show now that $V(P) = V(G)$. Suppose, to the contrary, that some $x \in S_1$ has a neighbor $z \in R$, and let $y_1, y_2 \in S_1 - \{x\}$. Since no two vertices in S_1 can have a common neighbor in R, we have $d_G(y_1) + d_G(y_2) = d_R(y_1) + d_R(y_2) + d_P(y_1) + d_P(y_2) \leq n - t - 1 + k + k$ $= (n - (2k+1) - 1) + 2k = n - 2$, a contradiction. Hence no vertex of S_1 has a neighbor outside P. This implies $n - 1 \leq d_G(y_1) + d_G(y_2) = d_P(y_1) + d_P(y_2) \leq 2k = t - 1$, i.e. $t \geq n$. Since obviously $t \leq n$, we have $t = n$, from which $V(P) = V(G)$.

Now the only way to satisfy the assumption $d(x) + d(y) \geq n - 1$ for every $x, y \in S_1$ is that every $x \in S_1$ is adjacent to all vertices of $A = V(G) \setminus S_1$. This implies that G has the required structure. ■

Proof of Theorem 8. The set S is obviously cyclable in the graph G' obtained from G by adding all edges with $x, y \in S$. By Lemma 14, S is cyclable in G. ■

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