Hamiltonian decompositions of prisms over cubic graphs

Roman Čada*
Tomáš Kaiser*
Moshe Rosenfeld†
Zdeněk Ryjáček*

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Abstract

A conjecture of Alspach and Rosenfeld states that the prism $G \square K_2$ over any 3-connected cubic graph G has a decomposition into two Hamilton cycles. Using a method based on colored diagrams, we show this conjecture to hold for 3-connected planar bipartite cubic graphs and for one other class of planar cubic graphs known as 'kleetope duals'. We also give a new proof of the fact that $G \square K_2$ is hamiltonian for any 3-connected cubic graph G.

1 Introduction

The prism over a graph G, denoted by $G \square K_2$, is obtained by taking two copies of G and joining the two clones of each vertex by an edge. The motivation for studying Hamilton cycles in prisms over cubic 3-connected graphs goes back to the (still open) conjecture of Barnette that all simple 4-polytopes are hamiltonian. Initally, Rosenfeld and Barnette [7] proved that the prism over simple 3-polytopes (which are the 1-skeleton of some simple 4-polytopes) are hamiltonian if the 4-color conjecture is true (at that time it was still a conjecture). Later, this result was extended by various authors using techniques that avoided use of the 4-color theorem. Finally, in 1993 Paulraja [6] proved the most general possible result:

*Department of Mathematics, University of West Bohemia, Univerzitní 8, 306 14 Plzeň, Czech Republic, and the Institute of Theoretical Computer Science (ITI), Charles University, Praha, Czech Republic. E-mail: {cadar,kaisert,ryjacek}@kma.zcu.cz. Supported by project LN00A056 of the Czech Ministery of Education.

†Computing & software systems program, University of Washington, Box 354350, Tacoma, Washington 98402-3100, U.S.A. E-mail: moishe@u.washington.edu. Partial support from NSF grant INT-9802416, the Czech grants KONTAKT 338/99 and GAČR 201/99/0242 (DIMATIA) is gratefully acknowledged.

Theorem 1.1 *The prism over any 3-connected cubic graph is hamiltonian.*

A simple proof of this result, which lends itself to estimating the complexity of constructing the Hamilton cycle in the prism, is included in Section 3 of the present paper.

Alspach and Rosenfeld [1] conjectured that the prism over any 3-connected cubic graph actually admits a hamiltonian decomposition, and they proved the conjecture for some infinite families of graphs. We shall show that the conjecture holds for two classes of graphs.

A *kleetope* is any planar triangulation which can be obtained from the complete graph K_4 by repeatedly adding a vertex and joining it to the 3 vertices of a face. We prove the following theorem in Section 8.

Theorem 1.2 *The prism over the dual of any kleetope admits a hamiltonian decomposition.*

We remark that by [4], kleetopes are precisely the planar graphs with a unique 4-coloring of the vertices.

Using a characterization of 3-connected bipartite cubic planar graphs due to Batagelj [2], we obtain the following result.

Theorem 1.3 *The prism over any 3-connected bipartite planar cubic graph admits a hamiltonian decomposition.*

In Section 4, we shall see that the connectivity requirement in Alspach and Rosenfeld's conjecture cannot be relaxed since there are 2-connected planar cubic graphs whose prisms have no hamiltonian decomposition. In fact, one can make the examples bipartite, thereby showing that 3-connectivity is also essential in Theorem 1.3.

2 Notation

Let G be a graph. We generally write $V(G)$ for the set of vertices and $E(G)$ for the set of edges of G. Only simple graphs are considered. We refer the reader to [3] for any graph-theoretic concepts we use without definition.

The prism over G was defined in the Introduction; note that it can be viewed as the Cartesian (or box) product $G\Box K_2$ of G with K_2 as well. We identify G with one of its copies in $G\Box K_2$, and write v^* for the clone of a vertex $v \in V(G)$ in the other copy of G.

If $w = v^*$, we set $w^* = v$; in other words, $(v^*)^* = v$.

This notation is extended, in the obvious way, to edges, sets of vertices, and sets of edges in $G\Box K_2$. For instance, if $F \subset E(G\Box K_2)$, then $F^* = \{ u^*v^* \mid uv \in F \}$.

3 The hamiltonicity of prisms

We shall now present a simplified proof of Theorem 1.1. The original proof appears in [6].

We begin with the following lemma.

Lemma 3.1 *A 3-connected cubic graph contains a 2-connected bipartite spanning subgraph.*

Proof. Let H be a maximal 2-connected, bipartite subgraph of G. If H does not span G, there is a vertex $g \in V(G) - V(H)$. Assume further that the vertices of H are properly colored in black and white. Since G is 3-connected, it contains 3 vertex-disjoint paths P_1, P_2, P_3, each connecting g to a vertex in H. Let h_i be the first vertices of H on each of the paths. Without loss of generality, we may assume that h_1 and h_2 are colored white. Then the lengths of P_1 and P_2 have distinct parities, for otherwise P_1, P_2 can be added to H, contradicting its maximality. If h_3 is also colored white, then among the 3 paths there are two with the same parity, and we can add these two paths to H, obtaining a larger spanning 2-connected bipartite subgraph. Hence h_3 is colored black. Since the lengths of P_1 and P_2 have distinct parities, one of these lengths will have distinct parity from the length of the path P_3, we can add these two paths to H and again increase the size of H. Hence if H is maximal it must be a spanning subgraph. □

A *cactus* is a connected graph C such that any two cycles in C are vertex-disjoint, every vertex of degree at least 3 lies on a cycle, and C has at least two vertices. The cycles of C are called its *leaves*. A cactus is *even* if all of its leaves are cycles of even length. See Fig. 1 for an example of an even cactus.

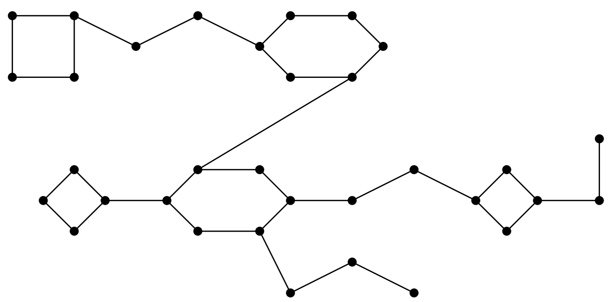

Figure 1: An even cactus.

Lemma 3.2 *Any 2-connected graph H of maximum degree $\Delta(H) \leq 3$ has a spanning subgraph C such that C is a cactus (not necessarily even) and its leaves contain all vertices v with $\deg_H(v) = 3$.*

Proof. The proof proceeds by induction on the number of vertices of H. The lemma is trivially true for graphs H of order at most 4.

If H is cubic, then since it is 2-connected, it contains a 1-factor F by the Petersen theorem. The complement $\bar{F}$ of F is a 2-factor. If we contract, in H, each component of $\bar{F}$ to a single vertex, we obtain a connected graph H'. Adding the edge set of any spanning tree of H' to $\bar{F}$ (with the obvious identifications), we get a spanning cactus of H.

Thus, we may assume that H contains vertices of degree 2. Let h be such a vertex and let a, b be its two neighbors.

If $ab \in E(H)$, then 2-connectedness (and the fact that H has more than 4 vertices) implies that a, b each have one additional neighbor a', b' (respectively), where $a' \neq b'$. We now shrink the triangle ahb to a single vertex c of degree 2 as in Fig. 2.

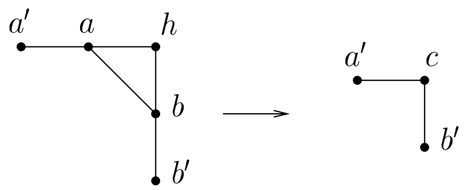

Figure 2: The contraction of a triangle.

The new graph clearly remains 2-connected. By the induction hypothesis, it contains a spanning cactus C. If the path $a'cb'$ is part of a leaf, we modify it to the path $a'ahbb'$ (and get a spanning cactus in G). If $a'cb'$ is not part of a leaf, then without loss of generality we may assume that $a'c$ is contained in $E(C)$. We can now modify C by removing c, adding the triangle ahb and the edge aa'; if $b'c \in E(C)$, we also add bb'. In either case, the resulting cactus has the required properties (note that the triangle is a leaf).

If $ab \notin E(H)$, we remove the vertex h and add the edge ab. Again, by the induction hypothesis, the resulting graph contains a spanning cactus C. We note that the degrees of a or b have not changed in this graph. If $ab \in E(C)$, we just replace it by the path ahb. Assume $ab \notin E(C)$. If one of the vertices, say a, has degree 3 (in H), then by the induction hypothesis it is contained in a leaf and we may extend C by adding the edge ah. If both a and b have degree 2, then since $ab \notin E(C)$, a has degree 1 in C. Again, we can add the edge ah to C. In either case we obtain the desired spanning cactus. □

The relevance of even cacti to Hamilton cycles is shown by the following simple lemma.

Lemma 3.3 *The prism over any even cactus C with $\Delta(C) \leq 3$ is hamiltonian.*

Proof. We prove, by induction on the number of vertices of C, that $C\Box K_2$ has a Hamilton cycle F such that:

$$\begin{array}{c} F \text{ contains the edge } xx^* \\ \text{for each degree 2 vertex } x \text{ belonging to a leaf of } C. \end{array} \tag{1}$$

The assertion is trivial if $|V(C)| = 2$. We may also assume that C is not a cycle. Let T be the tree obtained by contracting each cycle Q of C to a vertex v_Q and discarding loops. Since T contains at least two vertices, we may choose a vertex t of degree 1 in T.

Assume first that $t \in V(C)$, i.e. t is of degree one in C and does not lie on a leaf. Let u be the unique neighbor of t in C. If u belongs to a leaf of the cactus $C - t$, then by induction (using (1) and the bound on the maximum degree), $(C - t)\Box K_2$ has a Hamilton cycle containing the edge uu^*. But if u does not belong to a leaf of $C - t$, then its degree in $C - t$ is 1, and so any Hamilton cycle of $(C - t)\Box K_2$ must contain uu^*. In either case, we can replace the edge uu^* by the path utt^*u^*, obtaining a Hamilton cycle in $C\Box K_2$, moreover one for which (1) holds.

It remains to consider the case where $t = v_Q$ corresponds to a cycle Q of C. The fact that t has degree one in T implies that only one vertex $w \in V(Q)$ has degree 3 in C. Let C' be obtained by removing all the vertices of Q, except for w, from C. By induction, $C'\Box K_2$ has a Hamilton cycle F' satisfying (1), which must necessarily contain ww^* as the degree of w in C' is one. The cycle F' is easily modified to a Hamilton cycle of C that uses all the edges between Q and Q^* except for ww^*, and agrees with F' outside Q. Thus, (1) is preserved. The proof is complete. □

We are now in a position to prove the main result of this section.

Proof of Theorem 1.1. We first show that G contains a spanning even cactus. By Lemma 3.1, G contains a spanning 2-connected bipartite subgraph H with $\Delta(H) \leq 3$. By Lemma 3.2, H contains a spanning cactus C. Since H is bipartite, C is even. Finally, Lemma 3.3 implies that $C\Box K_2$ (and hence $G\Box K_2$) contains a Hamilton cycle. □

Remark 3.4 It is easy to see that the complexity of constructing a Hamilton cycle in $H\Box K_2$ is dominated by the complexity of finding a 1-factor in the cubic graph H.

4 Non-3-connected graphs

As mentioned in the Introduction, prisms over 2-connected cubic planar graphs do not necessarily possess a hamiltonian decomposition. The following class of examples was found by W. McCuaig [5]. Join two vertices u and v by three internally disjoint paths of at least 3 edges each, and replace every degree 2 vertex by a 'diamond' (K_4 minus an edge) so as to obtain a cubic graph G (shown in Fig. 3). It is easy to see that G is 2-connected and planar, and we shall show that $G\Box K_2$ has no hamiltonian decomposition.

Suppose that $G\Box K_2$ can be decomposed into Hamilton cycles C_1, C_2. One of the cycles (say, C_1) does not contain the edge uu^*. Since both u and u^* are of degree two in C_1, there is an edge of G adjacent to u (say, e_1 as shown in Fig. 3) such that both e_1 and $e_1{}^*$ are contained in C_1.

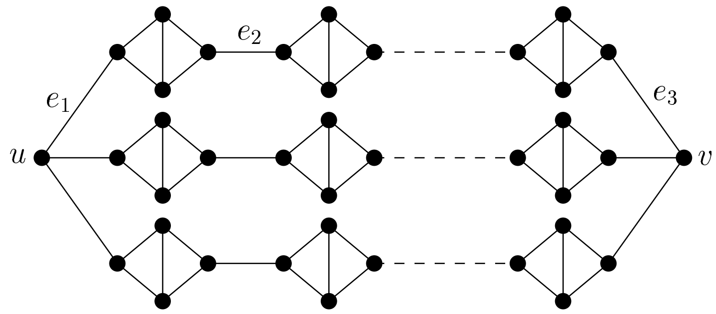

Figure 3: Cubic, 2-connected planar graphs whose prisms do not admit hamiltonian decompositions.

Consider now the edges e_2 and e_3 as in Fig. 3. Since $\{ e_1, e_1{}^*, e_2, e_2{}^* \}$ is an edge cut in $G\Box K_2$, it must be intersected by both C_1 and C_2. It follows that both e_2 and $e_2{}^*$ are contained in C_2. By the same token, e_3 and $e_3{}^*$ are contained in C_2. But then C_2 contains all of the edge cut $\{ e_2, e_2{}^*, e_3, e_3{}^* \}$, which means that C_1 cannot be a Hamilton cycle, a contradiction.

Observe also that it is easy to modify this example to obtain a *bipartite* cubic planar 2-connected graph (simply replacing each diamond by the 3-cube minus an edge). This shows that the assumption of 3-connectivity in Theorem 1.3 cannot be removed.

5 Colorings and 2-factors

Let G be a cubic graph and let F be a 2-factor of the prism $G\Box K_2$. The factor F induces a coloring of the edges of G in 4 colors by the following rule.

We color an edge $e \in E(G)$ (with respect to F) blue if $e \in E(F)$ and $e^* \notin E(F)$; yellow if $e \notin E(F)$ and $e^* \in E(F)$; and green if $e, e^* \in E(F)$. The remaining edges of G will be colored red. (The green color was chosen to represent 'both blue and yellow'. Admittedly, using black in place of red might be more in the spirit of this analogy.)

Note that this coloring need not be a proper edge-coloring of G. However, only 4 combinations of colors can appear at any given vertex. We abbreviate colors by the initial letter (e.g. blue is B) and define the *type* of a vertex to be the unordered collection of colors of the adjacent edges. Then it is easy to see that

$$\text{any vertex of } G \text{ is of type BYG, BYR, GGR or RRG.} \tag{2}$$

Indeed, consider a vertex $v \in V(G)$ and distinguish two cases. Firstly, if $vv^* \in E(F)$, denote the edge in $E(G) \cap E(F)$ by e. In case $e^* \in E(F)$, the vertex v is RRG, otherwise v is BYR. Secondly, if $vv^* \notin E(F)$, denote the two edges in $E(G) \cap E(F)$ by e_1 and e_2. The vertex v is GGR or BYG according to whether both of ${e_1}^*$ and ${e_2}^*$ are contained in $E(F)$ or they are not.

Note the effect of passing to the complement $\overline{F}$ of F: the coloring induced by $\overline{F}$ has blue interchanged with yellow, and green with red.

The above correspondence can be reversed. That is, any edge-coloring of G with property (2) determines a 2-factor in the prism, which can be obtained as follows. If $x, y \in V(G)$, then the edges of F will include:

- xy if it is colored blue or green,
- x^*y^* if xy is colored yellow or green,
- xx^* if x is of type BYR or RRG.

It is straightforward to check, for the 4 possible types of x, that x and x^* are of degree two in F. Thus F is a 2-factor. Moreover, we have obtained a bijection between 2-factors of $G \square K_2$ and edge-colorings of G satisfying (2). We shall call any such coloring *admissible*.

A more dynamic view of the above correspondence may be helpful. Starting with an admissible coloring, we can trace the associated 2-factor as follows. Imagine a robot with two possible states (labeled blue and yellow) walking through G. The green edges of G are taken to be colored both yellow and blue, while the red edges have none of these colors. To each of the two states, we associate one of the two copies of G in $G \square K_2$ and refer to them as the blue copy and the yellow copy. If an edge is used during the walk in a particular state, we include its clone from the corresponding copy in the 2-factor.

The walk begins by choosing a blue edge arbitrarily and traversing it to one of its endvertices in the blue state. At each vertex, the robot determines the edge to take next, after a possible change of state. Let us say that it has arrived to a vertex v along an edge e in the blue state. If there is a blue edge $e' \neq e$ adjacent to v, then the walk will continue on e'. If not, the state will change to yellow (which corresponds to including the vertical edge vv^* in the 2-factor) and a yellow edge $e' \neq e$ will be chosen. If there is none, then it must be that e is a green edge which is adjacent to 2 red edges at v, and the robot will return along e. Once it visits a vertex in the same state for the second time, a new starting vertex is chosen among those which have not been visited in both states, and the process is repeated (possibly starting in the yellow state). If there are no such vertices, we have a collection of closed walks which represents the associated 2-factor.

For an example, consider the coloring of the 3-cube given in Fig. 4. The figure also shows the convention used to represent colored edges. Starting at x in the blue state, we continue in the same state to w, z and y, where we are forced to switch to yellow and go on to y', where we switch back to blue, etc. The whole walk is $xwzy/y'/x'w'z'/zww'x'x/$, where the slashes indicate a change of state. In fact, one can see that each vertex was visited precisely once in each state. Thus the 2-factor in the prism corresponding to this walk is connected, hence a Hamilton cycle in the prism.

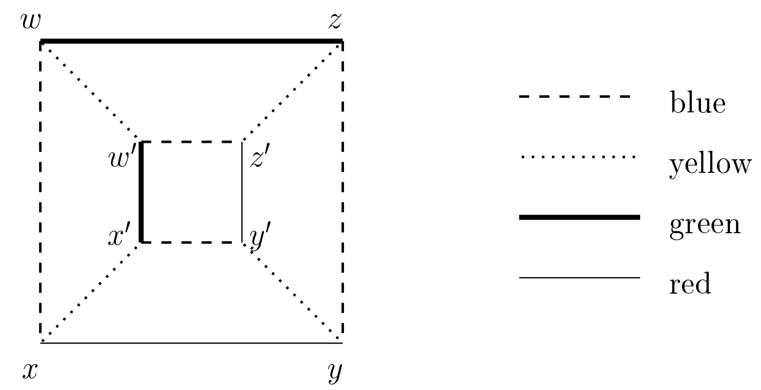

Figure 4: A coloring of the 3-cube inducing a Hamilton cycle in the prism.

In some of the subsequent sections, we shall be interested in colorings whose associated 2-factors, as well as their complements, are Hamilton cycles — in other words, colorings which induce a hamiltonian decomposition of $G\Box K_2$.

6 Matchings

We shall now describe how a 2-factor in the prism over a graph G induces a matching on a set of vertices of any subgraph H of G. The idea is simple, and if the description seems to be a bit technical, it is because in Section 10, we shall need a fairly rigorous treatment of this correspondence.

Let G be a cubic graph and F a 2-factor in the prism over G. For any subgraph H of G, let $F|H$ be the restriction of F to $V(H\Box K_2)$ minus all edges xx^*, where x is a vertex of H whose degree (in H) is even.

The set of *terminals* of H (with respect to F), $T_F(H) \subset V(H\Box K_2)$, consists of all vertices of $H\Box K_2$ whose degree in $F|H$ is 1. Note that some terminals may correspond to vertices of H with degree larger than 1 (that is, 2), and, on the other hand, there need not be any corresponding terminal for a vertex of degree 1 in H.

Let $G - H$ be the graph arising from G by removing all vertices of degree 3 in H, along with all edges of H. Note that if H has no isolated vertices, then $G - (G - H) = H$, and in any case, $E(G - H) = E(G) - E(H)$.

Observe also that the edge sets of $F|H$ and $F|(G - H)$ form a partition of $E(F)$. It follows that the set of terminals of $G - H$ coincides with $T_F(H)$.

There is a naturally defined perfect matching $M_F(H)$ in the complete graph on the vertex set $T_F(H)$: since all degrees in $F|H$ are at most 2, $F|H$ is a disjoint union of paths and cycles (and isolated vertices), and the endvertices of the paths are precisely the terminals. We shall match two terminals by an edge in $M_F(H)$ if they are distinct endvertices of the same path. (Matchings like $M_F(H)$ will be simply referred to as matchings on $T_F(H)$, without explicitly mentioning the complete graph they are contained in. Such matchings are never assumed to exist in G or H.)

Applying the same procedure to the 2-factor $\overline{F}$, we obtain another matching on $T_F(H)$, namely $M_{\overline{F}}(H)$. The following simple observation will often be useful.

Proposition 6.1 *For any subgraph H of G, the factor F is a Hamilton cycle if and only if*

(a) $M_F(H) \cup M_F(G - H)$ is the edge set of a cycle on $T_F(H)$, and

(b) neither $F|H$ nor $F|(G - H)$ contain cycles. □

Two perfect matchings M_1, M_2 on the same vertex set will be called *compatible* if, as in (a) above, their union is a single cycle.

7 Local modifications

In this section, we develop tools allowing us to alter a given coloring after a local modification of the underlying graph while preserving the existence of a hamiltonian decomposition of the prism.

Let G, G' be cubic graphs with admissible colorings inducing 2-factors F and F' in the prisms over G and G', respectively. Let $H \subset G$ and $H' \subset G'$ be subgraphs such that $G - H$ equals $G' - H'$ as a colored graph. This is the setting for all the assertions of the present section.

Proposition 7.1 *The restrictions $F'|(G' - H')$ and $F|(G - H)$ are equal. In particular, $T_{F'}(G' - H') = T_F(G - H)$ and $M_{F'}(G' - H') = M_F(G - H)$.*

Proof. By definition, an edge e belongs to $E(F|(G-H))$ if and only if $e \in E(F)$, $e \in E(G - H)$ and $e \neq xx^*$ for all $x \in V(G - H)$ of even degree in $G - H$. But since $G - H = G' - H'$, the edge sets and vertex degrees in these graphs are the same. The fact that the colorings of these graphs are identical implies that an edge of $G - H$ is in $E(F)$ iff it is in $E(F')$. The rest of the proposition is trivial. □

Proposition 7.2 *Assume that F is a Hamilton cycle. If $F'|H'$ contains no cycles and $M_{F'}(H')$ is compatible with $M_F(G-H)$, then F' is a Hamilton cycle.*

Proof. If F is a Hamilton cycle, then Propositions 6.1 and 7.1 imply that $F'|(G'-H')$ contains no cycles and that $M_{F'}(H')$ is compatible with $M_{F'}(G'-H')$. Since we are assuming that $F'|H'$ contains no cycles, it follows from Proposition 6.1 that F' is a Hamilton cycle. $\square$

In most situations we shall deal with, $M_{F'}(H')$ happens to be equal to $M_F(H)$, so by Proposition 6.1, it is automatically compatible with $M_F(G-H)$.

Corollary 7.3 *If F is a Hamilton cycle, $F'|H'$ contains no cycles, and $M_F(H) = M_{F'}(H')$, then F' is a Hamilton cycle.* $\square$

Turning to hamiltonian decompositions, we have to ensure that $\overline{F}$ is also a Hamilton cycle. For this, Proposition 7.2 can be used twice:

Corollary 7.4 *Assume that F and $\overline{F}$ are both Hamilton cycles. If*

(a) $F'|H'$ and $\overline{F'}|H'$ contain no cycles, and

(b) $M_F(H) = M_{F'}(H')$ and $M_{\overline{F}}(H) = M_{\overline{F'}}(H')$,

then $F' \cup \overline{F'}$ is a hamiltonian decomposition of $G' \square K_2$. $\square$

8 Kleetope duals

From the definition of kleetopes in the introduction, it is easy to see that their duals are precisely the graphs which can be obtained from the complete graph K_4 by repeated *triangle inflations* as shown in Fig. 5. We now show that the prisms over all kleetope duals possess a hamiltonian decomposition (Theorem 1.2).

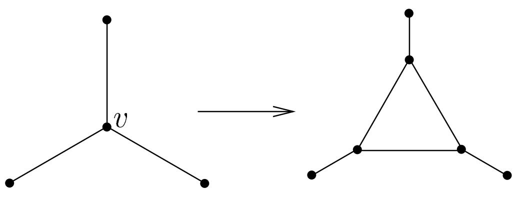

Figure 5: The triangle inflation at a vertex v.

Let G be a cubic graph with an edge-coloring that induces a hamiltonian decomposition of the prism over G (the Hamilton cycle corresponding to the coloring will be denoted by F), and let G' arise from G by a triangle inflation at v. Assume that v is a BYG vertex.

Denoting the new added triangle by T, we identify $E(G') - E(T)$ with $E(G)$ in the obvious way, and use the color of any edge of G to color the corresponding edge of G'. It remains to color T.

Here and in the following sections, if $X \subset V$, we write $A_G(X)$ for the subgraph of G formed by the edges with at least one end in X, together with all their endvertices. For $v \in V(G)$, $A_G(v)$ stands for $A_G(\{ v \})$.

Let $H = A_G(v)$ and $H' = A_{G'}(V(T))$, and use the coloring of Fig. 7 to color H'. The coloring corresponds to a 2-factor F' in $G' \Box K_2$. Although $M_F(H)$ and $M_{\bar{F}}(H)$ cannot be explicitly determined from the type of v alone, it is not hard to check that Corollary 7.4 applies, ensuring that the new coloring induces a hamiltonian decomposition of $G' \Box K_2$.

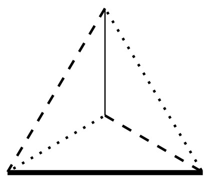

Figure 6: A coloring of K_4.

The case of v being a BYR vertex is similar. In the remaining two cases, however, it is not clear how to color H'. Nevertheless, this case will never occur: all vertices in the coloring for K_4 in Fig. 6 are BYG or BYR, and this is preserved by each triangle inflation. This proves Theorem 1.2.

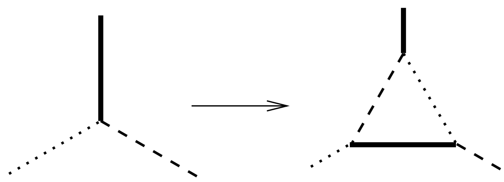

Figure 7: A coloring for the triangle inflation.

9 Bipartite planar graphs

The objective of this and the following sections is to show that the prism over any 3-connected cubic planar bipartite graph has a hamiltonian decomposition (Theorem 1.3).

Batagelj [2] proved that all 3-connected cubic bipartite planar graphs can be obtained from the cube by a succession of the two operations depicted in Fig. 8: the *diamond inflation* of any vertex, and the A_1 *subdivision*. The latter operation, applied to a pair of non-adjacent edges uv, wz, adds 2 new vertices to each of uv, wz, and also adds 2 independent edges on these 4 vertices. (The operations can be chosen such that all the intermediate graphs are planar bipartite, but this is not important for our needs.)

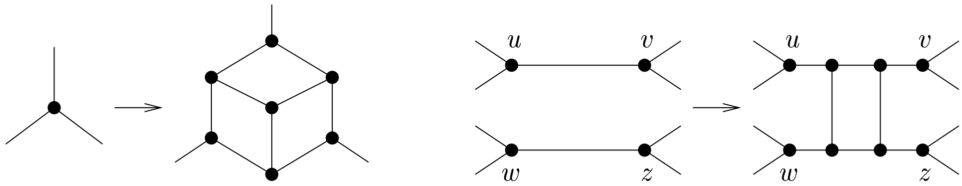

Figure 8: Transformations generating the 3-connected bipartite planar cubic graphs: (a) diamond inflation, (b) A_1 subdivision.

A hamiltonian decomposition for the prism over the 3-cube is given by the coloring in Fig 4. In the following, we shall prove that hamiltonian-decomposeability of the prism is preserved by diamond inflations and A_1 subdivisions.

We consider the diamond inflation first. Let a graph G have a hamiltonian-decomposeable prism, and let G' arise from G by the diamond inflation of a vertex $v \in V(G)$.

Let $H = A_G(v)$ and $H' = A_{G'}(X)$, where X is the set of the 7 new added vertices in G'. There are essentially two cases to distinguish: v is either a BYG vertex or a GGR vertex (the other possibilities are covered by symmetry). In both of these cases, it is easy to extend the existing coloring to the diamond. Explicitly, the colorings in Fig. 9 show how to color H', keeping the old coloring on the rest of G', so that the hypotheses of Corollary 7.4 are satisfied. The check is straightforward.

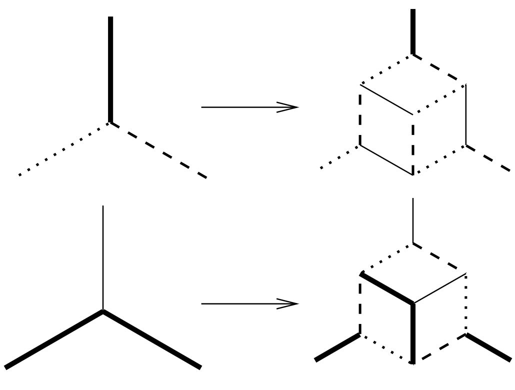

Figure 9: Colorings for the diamond inflation.

Most cases of the A_1 subdivision are no harder. Assume we subdivide edges uv and wz in G to obtain G'. (No restrictions are placed on uv and wz except that they are independent.) Let H be the subgraph of G formed by uv, wz and their endvertices. Let $H' = A_{G'}(V(T))$, where T is the 4-cycle on the new added vertices.

There are, up to symmetry, five possible combinations of colors of uv and wz. Four of them are covered by the colorings of H' in Fig. 10. (The coloring of the rest of G' is as in G.) Corollary 7.4 implies that these colorings correspond to hamiltonian decompositions. Note that in these cases, each outward edge of H' is given the same color as the edge of G which gave rise to it by subdivision.

In the fifth (and last) case, both uv and wz are colored green. We cannot hope to find a suitable coloring of H' with all the outward edges colored green, for the green edges would separate the 4-cycle in H' from the rest of G', and the complement of the associated 2-factor would necessarily be disconnected. Thus we cannot get a hamiltonian decomposition in this manner. We discuss the A_1 subdivision of two green edges in the following section.

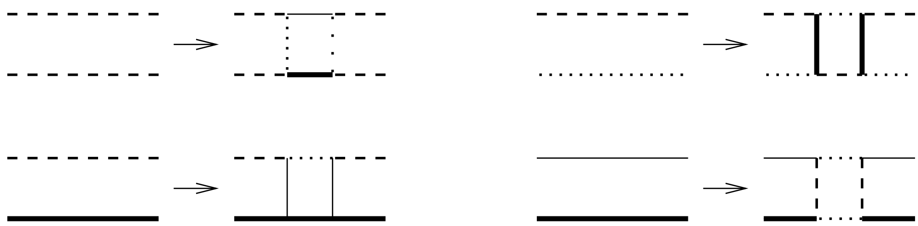

Figure 10: Colorings for the easy cases of A_1 subdivision.

10 The green-green case

In this section, we show that the A_1 subdivision preserves the existence of a hamiltonian decomposition of the prism in the last remaining case: when both edges being subdivided are green (or both red) in the coloring induced by the original hamiltonian decomposition.

Let G be a cubic graph and let G' be the result of the A_1 subdivision performed on edges $uv, wz \in E(G)$. Assume that $G \square K_2$ has a Hamilton cycle F whose complement $\overline{F}$ is also hamiltonian, and consider the corresponding coloring of G. It is sufficient to discuss the case where uv and wz are colored green: if they are both red, we may interchange the red and green colors. All the other color combinations have been dealt with in the preceding section.

Set $X = \{u, v, w, z\} \subset V(G)$ and let $T = T_F(H)$ be the set of terminals, as defined in Section 6. The assumption that uv and wz are green means that $\{uv, u^*v^*, wz, w^*z^*\} \subset E(F)$. We let $X' = \{u', v', w', z'\} \subset V(G')$ be the four vertices that were added to G in the construction of G', where for each $x \in X$, x' is adjacent to x in G'. Let H' be the subgraph of G' on $X \cup X'$ formed by all edges with at least one endvertex in X'. We shall denote the vertex set of $H \square K_2$ by X^+.

As we shall see, the only really interesting case is when no vertex from X is RRG (i.e. adjacent to two red edges in G). Let us treat the opposite case first.

Case I: some vertices from X are RRG. Note that u and v cannot be RRG at the same time. Otherwise, the edge uv would be separated from the rest of the graph by a cut consisting of red edges, contradicting the fact that F is a Hamilton cycle. The same applies to w and z.

Thus, there are (up to symmetry) only 3 possible placements of RRG vertices: (a) z is the only RRG vertex, (b) v and z are RRG, or (c) u and z are RRG.

Assume that z is the only RRG vertex. It is easy to see that in this case, $T_F(H)$ equals $\{ u, v, w, u^*, v^*, w^* \}$, and $M_F(H)$, $M_{\overline{F}}(H)$ are as given in Fig. 11. We color H' as in Fig. 12a, and use the coloring of G in the rest of G'. This gives an admissible coloring; let F' be the corresponding 2-factor. Looking at Fig. 12b, one can check directly that $F'|H'$ and $\overline{F'}|H'$ contain no cycles, $M_{F'}(H') = M_F(H)$, and $M_{\overline{F'}}(H') = M_{\overline{F}}(H)$. By Proposition 7.4, the coloring of G' induces a hamiltonian decomposition.

Figure 11: The matchings induced by F and $\overline{F}$ if z is the only RRG vertex.

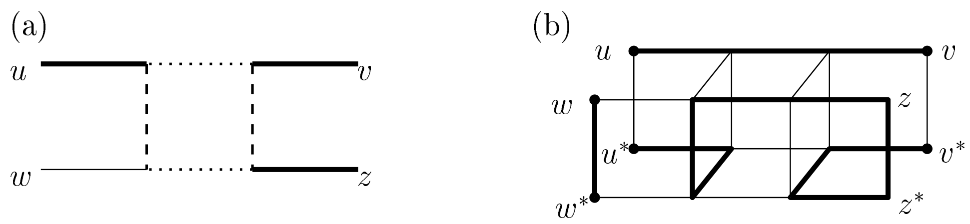

Figure 12: (a) a coloring of H', (b) the restriction $F'|H'$ if z is the only RRG vertex.

Next, assume that there are two RRG vertices v and z. The set of terminals of H is then $\{ u, w, u^*, w^* \}$. $M_F(H)$ matches u to u^* and w to w^*, and so does $M_{\overline{F}}(H)$. As before, one can check that the coloring of H' given in Fig. 12a extends the hamiltonian decomposition.

The same coloring works in the third subcase also, namely when u and z are the RRG vertices.

Case II: no vertex from X is RRG. With this assumption, one can see that the set of terminals of H is X^+, and $M_F(H)$, $M_{\overline{F}}(H)$ are as in Fig. 13. The difficulty with this case is that it is the only one where Corollary 7.4 cannot be applied, as there is no coloring of H' which would have the associated matchings on X^+ identical to $M_F(H)$ and $M_{\overline{F}}(H)$. Instead, we prove that for at least one out of a certain set of colorings of H', $M_{F'}(H')$ is compatible with $M_F(G-H)$, which makes Proposition 7.2 applicable.

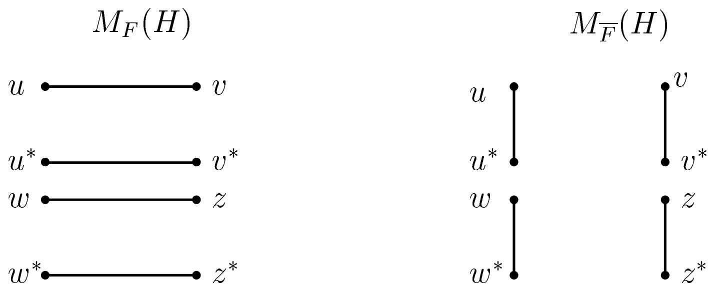

Figure 13: The matchings $M_F(H)$ and $M_{\overline{F}}(H)$ in Case II.

Identify X^+ with the set of vertices of the combinatorial 3-cube $\mathbb{Z}_2^3$ (viewed as a vector space over $GF(2)$) as follows. Make u, v, w and z correspond to vectors $(001), (011), (101)$ and (111), respectively; for each $x \in X$, make x^* correspond to (001) plus the vector for x. See Fig. 14 in which a vector is represented by a point with the corresponding coordinates in the 3-space. Note that the cube is not necessarily a subgraph of the prism of G or G'.

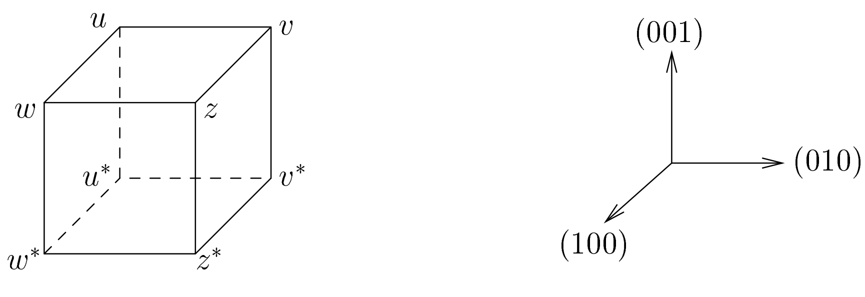

Figure 14: The 3-cube and the coordinate system used to draw it. The point representing u^* is at the origin (000).

Let $\tau = (010)$. Since the matching $O = M_F(G-H)$ is compatible with $M_F(H)$, we get that

$$\text{if } \{ x, y \} \text{ is in } O, \text{ then } \{ x+\tau, y+\tau \} \text{ is not,} \tag{3}$$

and in fact this characterizes the perfect matchings compatible with $M_F(H)$. We call any matching satisfying (3) *asymmetric*.

Unless otherwise specified, all matchings considered in this section are perfect matchings on X^+.

For $x \in X^+$, set $|\, x\,|$ to be the scalar product $x \cdot x$ modulo 2. This corresponds to a linear form on $\mathbb{Z}_2^3$. By an *edge* on $\mathbb{Z}_2^3$, we simply mean any pair of distinct vertices. The *direction* of an edge xy is the vector $y - x = x + y$. We call the edge *odd* if $|\, x + y\,| = 1$ and *even* otherwise. A τ*-edge* is an edge of type $\{\, x, x + \tau\,\}$; that is, an edge parallel to the y axis in Fig. 14. Note that an asymmetric matching cannot contain any τ-edge.

We call a (perfect) matching on X^+ *special* if it consists of odd edges with pairwise distinct directions. It is easy to see that any pair of odd edges on X^+ with distinct directions can be completed to a unique special matching. It follows that there are exactly 8 special matchings.

Observe that if we extend the coloring of G to G' by coloring H' as in Fig. 12a, then $M_{F'}(H')$ is special and $F'|H'$ contains no cycles. (See Fig. 15.) Both of these assertions remain true if we interchange (in H') the colors of the red edge and any green edge, and/or interchange all the blue and yellow colors. There are 8 ways to make these changes, and in fact they yield all the special matchings.

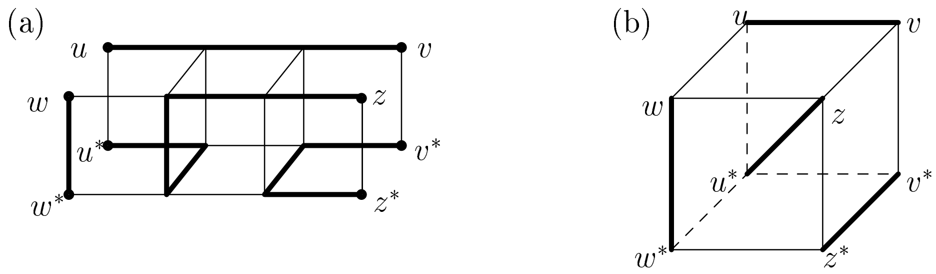

Figure 15: $F'|H'$ and $M_{F'}(H')$ (shown in the cube) for the diagram of Fig. 12a in Case II.

We do not have to worry about $M_{\overline{F'}}(H')$: for all of the above colorings, it equals $M_{\overline{F}}(H)$ as shown in Fig. 13, and obviously induces no cycles in $\overline{F'}|H'$. Roughly speaking, this is because these colorings of H' only include 1 red edge; $\overline{F'}$ is just like $\overline{F}$, except that it enters $H' \square K_2$ through this edge, traverses it and exits through the red edge again.

In view of Proposition 7.2, our only concern is to find, for a given asymmetric matching O, a compatible special matching. To get one, it is sufficient to find a partial matching S consisting of 3 odd edges with distinct directions, in such a way that there is an induced path in $(X^+, S \cup O)$, consisting of 3 S-edges alternating with 2 O-edges. (Here, an S-edge is simply an edge from S.) Indeed, since the path is induced, its endvertices are not connected by an O-edge, so the remaining 2 O-edges extend the path to an alternating path of length 7, and it is easy to see that we may add the edge connecting its endvertices to S. This observation will be useful in the proof of the following proposition.

Proposition 10.1 *For any asymmetric matching O on X^+, there is a compatible special matching.*

Proof. Any τ-edge is adjacent to exactly 2 edges from O. Assume first that there is a τ-edge, say $\{ a, a + \tau \}$, such that both of the adjacent edges from O are of the same parity.

Since O does not include any τ-edge, we may put $\{ a, a + \tau \}$ in S. Let $\{ a, x \}, \{ a + \tau, y \}$ be the adjacent O-edges. We may assume that $|\, y \,| = |\, a \,|$, using $a + \tau$ instead of a if necessary.

Clearly $x, y \notin \{ a, a + \tau \}$, $x \neq y$, and by (3), $x \neq y + \tau$.

Set

$$\begin{aligned} \alpha(x) &= a + x + y + \tau, \\ \alpha(y) &= a + x + y. \end{aligned}$$

Define the two remaining edges of S by joining x to $\alpha(x)$ and y to $\alpha(y)$. First, it is easy to check algebraically that $\alpha(x), \alpha(y)$ do not fall in $\{ x, y, a, a + \tau \}$. For instance, if $\alpha(x) = y$, then $x = a + \tau$, which we know to be false. The other inequalities are no harder.

Second, $\alpha(x)$ and $\alpha(y)$ form a τ-edge, which cannot appear in O. It follows that $(\alpha(x), x, a, a + \tau, y, \alpha(y))$ is an induced alternating path in $S \cup O$.

Finally, note that $\{ x, \alpha(x) \}$ and $\{ y, \alpha(y) \}$ are odd edges whose directions are distinct and both different from τ. Thus we are free to add them to S. This finishes the first case.

For the second case, assume that all τ-edges are adjacent to one odd and one even edge of O. Take the τ-edge $\{ (000), \tau \}$; let a be the end-vertex adjacent to an even O-edge ax and let y be the end-point of the (odd) edge adjacent to $a + \tau$. Clearly $|\, x \,| = |\, y \,| = |\, a \,|$.

Set

$$\begin{aligned} \alpha(x) &= a + x + y + \tau, \\ \alpha(y) &= x + \tau. \end{aligned}$$

As before, it is straightforward to check that the sequence $(\alpha(x), x, a, a + \tau, y, \alpha(y))$ defines an alternating path, the edges $\{ x, \alpha(x) \}$ and $\{ y, \alpha(y) \}$ are odd, and their directions are distinct and different from τ.

We need to show, however, that O does not join $\alpha(x)$ to $\alpha(y)$. Since

$$|\, \alpha(x) + \alpha(y) \,| = |\, a + y \,| = 0,$$

$\{ \alpha(x), \alpha(y) \}$ is an even edge. If it were contained in O, then the τ-edge $\{ x, x+\tau \} = \{ x, \alpha(y) \}$ would be adjacent to two even O-edges, contrary to our assumption. Hence we may complete S as necessary. □

Using Proposition 7.2, we can easily finish the argument for the green-green case, thus establishing Theorem 1.3.

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