Hourglasses and Hamilton cycles in 4-connected claw-free graphs

Tomáš Kaiser[1]
MingChu Li[2]
Zdeněk Ryjáček[1]
Liming Xiong[3]

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Abstract

We show that if G is a 4-connected claw-free graph in which every induced hourglass subgraph S contains two non-adjacent vertices with a common neighbor outside S, then G is hamiltonian. This extends the fact that 4-connected claw-free, hourglass-free graphs are hamiltonian, thus proving a broader special case of a conjecture by Matthews and Sumner.

1 Introduction

A well-known conjecture of Matthews and Sumner (see [7]) states that all 4-connected claw-free graphs are hamiltonian. (A graph is *claw-free* if it contains no *claw*, that is, no induced $K_{1,3}$.) While the conjecture is still wide open, it has been proved in various special cases. One such result concerns *induced hourglasses*, i.e.,

[1]Department of Mathematics, University of West Bohemia, Univerzitní 8, 306 14 Plzeň, Czech Republic, e-mail: {kaisert,ryjacek}@kma.zcu.cz. This research was partly done on a visit to the Institute of Systems Sciences of Academia Sinica (Beijing, China) under the project ME 418 of the Czech Ministery of Education. The first and the third authors are partly supported by the project LN00A056 of the Czech Ministery of Education.

[2]Department of Computer Science and Technology, Tianjin University, Tianjin 300072, P. R. China; and LiuHui Center for Applied Mathematics, Nankai University & Tianjin University, Tianjin 300072, P. R. China, e-mail: mcli@eyou.com. Supported by both the project T23 of LiuHui Applied Mathematics Research of Nankai University & Tianjin University, and the project 2002AA142010 of National 863 Plans of the China Ministery of Science.

[3]Department of Mathematics, Jiangxi Normal University, Nanchang, 330027, P. R. China; and Department of Applied Mathematics, Beijing Institute of Technology, Beijing, 100081, P. R. China, e-mail: lmxiong@eyou.com. Supported by the Fund of Natural Science of Jiangxi Province.

induced subgraphs isomorphic to the graph in Fig. 1a: if a 4-connected claw-free graph is *hourglass-free* (contains no induced hourglass), then it is hamiltonian. This was observed independently by several authors; see, e.g., [1].

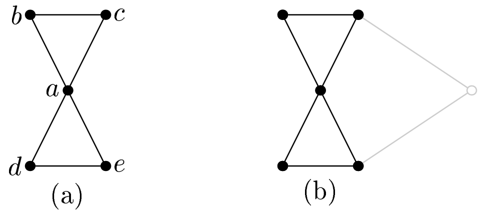

Figure 1: (a) An hourglass. (b) A common neighbor as in the hourglass property.

In the present paper, we generalize this result to graphs which may contain some induced hourglasses. A graph G has the *hourglass property* if in every induced hourglass S, there are two non-adjacent vertices which have a common neighbor in $G - V(S)$ (see Fig. 1b). We prove:

Theorem 1 *Every 4-connected claw-free graph with the hourglass property is hamiltonian.*

2 Notation

Let us fix some notation. The induced subgraph G' of a graph G on vertices $V(G') = \{ a_1, a_2, \ldots, a_k \}$ is denoted by $\langle a_1 a_2 \ldots a_k \rangle_G$. The order of the vertices has a special meaning whenever we speak of an induced hourglass: $\langle abcde \rangle_G$ is an hourglass centered at a and containing edges bc and de (such as the one in Fig. 1). We use a similar convention for claws: by saying that $\langle abcd \rangle_G$ is a claw, we always mean that a is the center.

The *neighborhood* of a vertex v in G is denoted by $N_G(v)$. (Note that $v \notin N_G(v)$.) With a slight abuse of notation, we also write $N_G(v)$ for the induced subgraph $\langle N_G(v) \rangle_G$. The vertex v is *locally connected* if $N_G(v)$ is connected. Otherwise, v is *locally disconnected.*

3 Stability

We need the concept of the *claw-free closure* as defined in [9]. Let G be a claw-free graph. If x is a locally connected vertex of G, then the *local completion at* x is the operation of adding all possible edges between vertices in $N_G(x)$. The resulting graph, denoted by G'_x, is easily shown to be claw-free again. Iterating local completions, we finally arrive at a graph in which all locally connected vertices have complete neighborhoods. This graph does not depend on the order of local completions (which is not quite obvious); it is called the *closure* of G and denoted by $cl(G)$.

Theorem 2 (Ryjáček [9]) *For a claw-free graph G,*

(i) the (well-defined) closure $cl(G)$ of G is the line graph of some triangle-free graph,

(ii) the closure is idempotent, i.e. $cl(cl(G)) = cl(G)$,

(iii) G is hamiltonian if and only if $cl(G)$ is hamiltonian.

The following proposition will enable us to restrict our attention to line graphs.

Proposition 3 *Let G be a claw-free graph having the hourglass property. Then its closure $cl(G)$ has the hourglass property, too.*

The proof of Proposition 3 is given below. First note that a local completion of an hourglass-free graph G may contain induced hourglasses. However, as shown in [2], all such hourglasses are destroyed by subsequent local completions, so the closure $cl(G)$ is hourglass-free. This motivates the following definition.

An induced hourglass in G is *permanent* if its vertex set induces an hourglass in $cl(G)$. The graph G has the *permanent hourglass property* if for every permanent induced hourglass $S \subset G$, some two non-adjacent vertices of S have a common neighbor in $G - V(S)$. Thus the permanent hourglass property is just the hourglass property restricted to permanent hourglasses. We shall need two lemmas about the closure.

Lemma 4 *Let v be a vertex of a claw-free graph G. If u is any internal vertex of an induced path in $N_G(v)$, then $N_{cl(G)}(u)$ is complete.*

Proof. Follows easily from the observations that u must be a locally connected vertex in G, and that local completions cannot make any locally connected vertex locally disconnected. □

Lemma 5 *An induced hourglass $H = \langle ab_1b_2c_1c_2\rangle_G$ in a claw-free graph G is permanent if and only if $N_{cl(G)}(a)$ contains no b_ic_j-path.*

Proof. Assume that H is permanent. If $N_{cl(G)}(a)$ contains a b_ic_j-path, then the edges b_1b_2 and c_1c_2 are in the same component of $N_{cl(G)}(a)$. Since each component of $N_{cl(G)}(a)$ is clearly a complete subgraph, H cannot be an induced hourglass.

For the converse, note that an edge joining b_i and c_j is a particular example of a b_ic_j-path. If there are no such edges, then H is an induced subgraph of $cl(G)$ and is therefore permanent. □

Proposition 6 *If a claw-free graph G has the permanent hourglass property, then so does its local completion G'_x at any locally connected vertex x.*

Proof. Suppose that $H = \langle ab_1b_2c_1c_2 \rangle_{G'_x}$ is a permanent hourglass in G'_x such that no b_i has a common neighbor with any c_j except a. Trivially, $H^- = \langle ab_1b_2c_1c_2 \rangle_G$ cannot be an induced hourglass, for it would be permanent and the above common neighbor would have to exist.

Thus, some edges of H are missing in H^-. At least one edge adjacent to a must be missing. To see this, note that if $b_1b_2 \notin E(H^-)$ and H^- contains all the edges adjacent to a, then $\langle ab_1b_2c_1 \rangle_G$ is a claw. By symmetry, we may henceforth assume that $ab_1 \notin E(H^-)$. Since the local completion at x in G adds the edge ab_1, we have $xa, xb_1 \in E(G)$. Since b_1 is non-adjacent in G'_x to c_i (for $i = 1, 2$), and $ac_i \in E(G'_x)$, we can conclude that $ac_i \in E(G)$. The same argument proves that $c_1c_2 \in E(G)$.

Choose a shortest ab_1-path P in $N_G(x)$ and let p be the neighbor of a on P. (Refer to Fig. 2 for an illustration.) We claim that $J = \langle axpc_1c_2 \rangle_G$ is an hourglass. If not, then either xc_i or pc_i is an edge for some i. In either case, we get a b_1c_i-path in $N_{G'_x}(a)$ and hence in $N_{cl(G)}(a)$; these paths are b_1xc_i and b_1xpc_i, respectively. By Lemma 5, H is not permanent. This contradiction implies that J is an hourglass. In fact, J is a permanent hourglass: if a subsequent local completion destroys J, the added edge (xc_i or pc_i) creates a b_1c_i-path in $N_{cl(G)}(a)$.

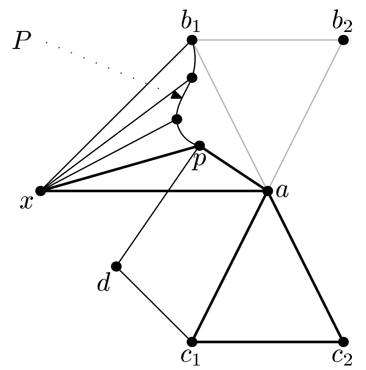

Figure 2: An illustration to the proof of Proposition 6. 'Guaranteed' edges of G are shown black, those of G'_x grey, the hourglass J bold.

By the permanent hourglass property of G, either x or v has a common neighbor d with some c_i. If d is a common neighbor of x and c_i, then $b_1d \in E(G'_x)$ and d is a common neighbor of b_1 and c_i in H, a contradiction.

Hence we may assume that d is a common neighbor of p and c_i in G. By Lemma 4, $N_{cl(G)}(p)$ is complete, and thus ad is an edge in $cl(G)$. But then b_1xpdc_i is a path in $N_{cl(G)}(a)$, so H is not permanent by Lemma 5. This finishes the proof. □

Proof of Proposition 3. If G has the hourglass property, then in particular, it has the permanent hourglass property. By Proposition 6 (used once for each local completion), so does $cl(G)$. However, as $cl(cl(G)) = cl(G)$, all induced hourglasses in $cl(G)$ are permanent, so $cl(G)$ in fact has the hourglass property as required. □

4 Collapsible graphs

We utilize the concept of a collapsible graph, introduced by Catlin [3] (see also [4]). A graph G is *collapsible* if for any subset $X \subset V(G)$ of even size, one can find a connected spanning subgraph $H \subset G$ such that the set of vertices v with odd degree $d_H(v)$ is precisely X. If H is any subgraph of G, then the graph G/H is obtained by contracting H to a single vertex, discarding any loops but keeping all multiple edges.

Theorem 7 (Catlin [3]) *Let H be a collapsible subgraph of G. Then G has a spanning closed trail if and only if G/H does.*

A large supply of collapsible graphs is provided by the following theorem.

Theorem 8 (Catlin [3]) *Any 4-edge-connected graph is collapsible.*

A different class of collapsible graphs is obtained from the following remarkable result of Lai [6] (conjectured by P. Catlin as a strengthening of a conjecture due to Paulraja [8]).

Theorem 9 (Lai [6]) *Let G be a 2-connected graph with minimum degree $\delta(G) \geq 3$. If every edge of G is contained in a cycle of length at most 4, then G is collapsible.*

5 Line graphs

By Section 3, we may restrict our attention to the class of line graphs of triangle-free graphs. This offers us the advantage of passing to the preimage G of the line graph $L(G)$. We first need to interpret the hourglass condition in this setting.

An *I-tree* in G is any subgraph of G isomorphic to the (unique) tree J on 6 vertices, 2 of which have degree 3 in J. (Note that the tree is shaped like the letter I, see Fig. 3.) To describe an I-tree, we only give its edges, listing the edge joining the degree 3 vertices as the first one. A graph G has the *I-tree property* if in any I-tree $J \subset G$, there are two vertices of distance 3 in J that are adjacent in G.

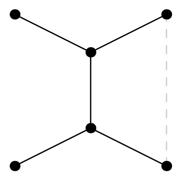

Figure 3: An I-tree (black) for which the I-tree property is satisfied.

Lemma 10 *A triangle-free graph G has the I-tree property if and only if its line graph $L(G)$ has the hourglass property.*

Proof. To prove the 'if' part, let $L(G)$ have the hourglass property and let $J = \{ xy, xx_1, xx_2, yy_1, yy_2 \}$ be an I-tree in G. The edges of J constitute an induced hourglass in $L(G)$, and so there is an edge $e \neq xy$ adjacent to, say, both xx_1 and yy_1. Since G is triangle-free, $e \notin \{ xy_1, yx_1 \}$, and so $e = x_1y_1$. Since J was arbitrary, we have established the I-tree property for G. The 'only if' implication is even more straightforward. □

The characterization of the preimages of hamiltonian line graphs is well-known. Recall that a closed trail T in a graph G is *dominating* if $G - V(T)$ is an edgeless graph.

Theorem 11 (Harary–Nash-Williams [5]) *The line graph $L(G)$ of a graph G is hamiltonian if and only if G has a dominating closed trail.*

A graph is *essentially k-edge-connected* if every edge cut of size less than k is trivial (no more than one component contains any edges). It is easy to see that G is essentially k-edge-connected if and only if its line graph $L(G)$ is k-connected.

We shall derive Theorem 1 directly from the following proposition. One definition: to *suppress* a degree 2 vertex means to contract one of the edges incident with it (discarding the loop).

Proposition 12 *Any essentially 4-edge-connected, triangle-free graph with the I-tree property has a dominating closed trail.*

Proof. Let G be a graph with the stated properties. Let A be the set of vertices of degree 1 in G. The graph $G - A$ has no degree one vertices, for otherwise we could find an essential 1-cut in G. Similarly, every vertex of degree 2 in $G - A$ has degree 2 in G.

Let $B_\diamond \subset V(G - A)$ be the set of vertices of degree 2 contained in some 4-cycle, and denote the set of all other degree 2 vertices of $G - A$ by B. (The proof is illustrated in Fig. 4.) Define $G_\diamond$ to be the graph obtained from $G - A$ by suppressing all vertices in $B_\diamond$. We aim to show that each component of $G_\diamond - B$ is collapsible.

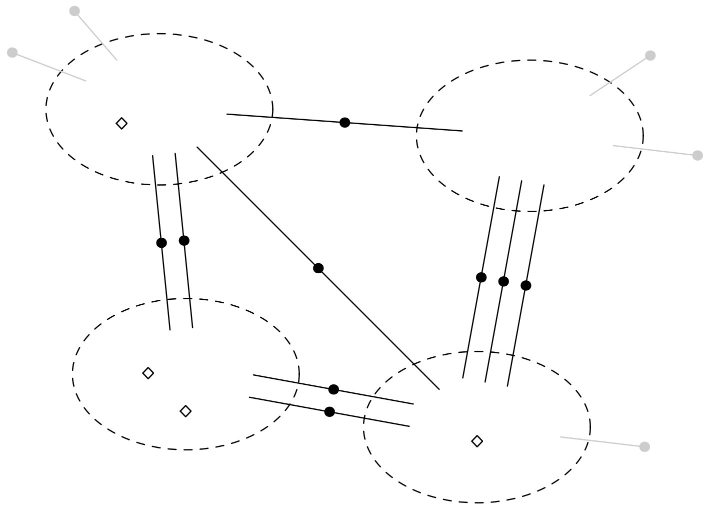

Figure 4: The structure of the graph G. Vertices in A are represented by grey dots, those in B by black ones, and those in $B_\diamond$ by $\diamond$. The ovals correspond to the components of $G_\diamond - B$.

Thus let C be a component of $G_\diamond - B$. If C is *trivial* (i.e. it consists of a single vertex), there is nothing to prove. We thus assume that C contains at least two vertices. It is our aim to verify the hypotheses of Theorem 9 for C.

We claim that the minimum degree in C is at least 3. Suppose, to the contrary, that $v \in V(C)$ is a vertex of degree less than 3. Clearly $d_{G-A}(v) \geq 3$ for otherwise it would have been either deleted or suppressed. Hence v has a neighbor $w_1 \in B$ in the graph $G - A$. We claim that in fact, it has at least two neighbors in B^-. If not, then the edges of C incident with v, together with the edge $e \in E(G - A)$ incident with w_1 but not with v, constitute an essential cut in G of size at most 3. This is impossible, so v has another neighbor $w_2 \in B$.

By our assumption that C is non-trivial, v is adjacent to a vertex $z \in V(C)$. Similarly as for v, we have $d_{G-A}(z) \geq 3$, so we may choose two neighbors $y_1, y_2 \in V(G-A)$ of z. Since G contains no triangles, $w_i \notin \{ y_1, y_2 \}$ for $i = 1, 2$. Thus the edge set $\{ vz, vw_1, vw_2, zy_1, zy_2 \}$ induces an I-tree. By the I-tree property, there is an edge between some w_i and some y_j, which yields a 4-cycle vzy_jw_i containing w_i, again a contradiction. We have shown that $\delta(C) \geq 3$. The same argument shows that every edge of C is contained in a 4-cycle of C.

Finally, we need to show that C is 2-connected. To begin with, C cannot be just a single edge, since $\delta(C) \geq 3$. Assume thus that u is a cut-vertex of C, and choose its neighbors u_1, u_2 in different components of $C - u$. Since $\delta(C) \geq 3$, u has a third neighbor u_3. Similarly, u_1 has at least two neighbors a, b besides u. If $u_3 \notin \{ a, b \}$, we may consider the I-tree with edges $\{ uu_1, u_1a, u_1b, uu_2, uu_3 \}$. Since an edge between u_2 and either of a or b is ruled out (u is a cut-vertex), we must have an edge between u_3 and a or b. To sum up, u_3 is adjacent either to u_1 or to one of its neighbors other than u. A symmetric argument shows the same for u_2 in place of u_1. But then u_3 shows that u is not a cut-vertex.

We have proved that every non-trivial component C of $G_\diamond - B$ satisfies the requirements of Theorem 9 and is therefore collapsible. Let H be the graph obtained from $G_\diamond$ by suppressing all vertices in B. Consider all non-trivial components C of $G_\diamond - B$ as subgraphs of H, contract them and suppress all vertices in B. Each vertex of the resulting graph H' corresponds to a component of $G' - B$, each edge of H' corresponds to a vertex in B. It is easy to see that H' is 4-edge-connected, for every edge cut in H' gives rise to an *essential* edge cut in G of the same size. By Theorem 8, H' has a spanning closed trail, and hence Theorem 7 implies that H has a spanning closed trail T. It is routine to check that the corresponding closed trail in G dominates every edge adjacent to a vertex in A, B or $B_\diamond$, and hence it is a dominating closed trail in G. The proof is complete. □

We now prove our main theorem.

Proof of Theorem 1. Let H be a 4-connected claw-free graph satisfying the hourglass condition. By Proposition 3 and Theorem 2, we may assume that H is a line graph, say $H = L(G)$, where G is essentially 4-connected and triangle-free. By Lemma 10, G has the I-tree property. By Proposition 12, G has a dominating closed trail, and hence, by Theorem 11, H is hamiltonian. □

Remark 13 The assumption of Theorem 1 that G is 4-connected cannot be relaxed to include 3-connected graphs, even with a lower bound on the minimum degree. This is demonstrated by the following example. Let k be an integer. Subdivide each edge of the Petersen graph by one vertex, attach k pendant edges to each vertex of degree 3, and denote the line graph of the resulting graph by H. Then H is 3-connected with minimum degree $\delta(H) = k + 2$. Since H contains no induced hourglass, it trivially has the hourglass property. It is, however, non-hamiltonian.

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