The prism over the middle-levels graph is hamiltonian

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Abstract

Let $\mathbf{B}_k$ be the bipartite graph defined by the subsets of $\{1, \ldots, 2k+1\}$ of size k and $k+1$. We prove that the prism over $\mathbf{B}_k$ is hamiltonian. We also show that $\mathbf{B}_k$ has a closed spanning 2-trail.

1 Introduction

Let $[2k+1]$ be the set $\{1, \ldots, 2k+1\}$. Consider the bipartite graph $\mathbf{B}_k$ whose vertices are all subsets of $[2k+1]$ of size k or $k+1$, and whose edges represent the inclusion between two such subsets. The notorious Middle two levels problem is whether $\mathbf{B}_k$ is hamiltonian for all k. Most likely it was first asked by Havel [5] (see the account in [8]).

Many authors attempted to solve this problem. One approach was to prove the assertion for specific values of k. The best result in this direction was obtained by Shields and Savage [10] who proved that $\mathbf{B}_k$ is hamiltonian for $1 \leq k \leq 15$. Another approach aimed at identifying long cycles in $\mathbf{B}_k$. In [10], it was proved that $\mathbf{B}_k$ has a cycle of length $\geq 0.86\,|\mathbf{B}_k|$, where $|\mathbf{B}_k| = 2\binom{2k+1}{k}$ is the number of vertices of $\mathbf{B}_k$. The best lower bound is due to R. Johnson who proved [11] that there is a cycle of length $(1 - o(1))\,|\mathbf{B}_k|$. Yet another direction was to find other structures that hopefully would be useful for finding the elusive

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hamiltonian cycle in $\mathbf{B}_k$. For instance, since a hamiltonian cycle in $\mathbf{B}_k$ is a disjoint union of two 1-factors, one may hope to find a hamiltonian cycle by building a sufficiently large repertoire of 1-factors. Duffus et al. [3] proved that no two 1-factors in the orbit of the lexicographic 1-factor form a hamiltonian cycle. This motivated Kierstead and Trotter [8] to generalize the concept of lexicographic 1-factor to lexical factorizations. Still no hamiltonian cycle was discovered. Another paper, [2], introduced the modular matchings.

In this paper, we use modular matchings to prove that $\mathbf{B}_k$ is *close* to being hamiltonian. The word 'close' can be interpreted in several ways. For instance, one can view a hamiltonian cycle as a spanning closed walk that visits each vertex exactly once. One can also view a hamiltonian path as a spanning tree of maximum degree 2. It is then quite natural to explore the following modifications. Instead of searching for a hamiltonian cycle in a graph, search for a spanning, closed walk in which every vertex is visited at most twice (or, in general, k times). Similarly, instead of searching for a hamiltonian path, one can look for a spanning tree of maximum degree 3 (or k). In accordance with the terminology of [6], we call these spanning structures *k-walks* and *k-trees*, respectively.

It was shown in [6] that any graph with a k-tree has a k-walk, and that the existence of a k-walk guarantees the existence of a $(k+1)$-tree, for any k. This results in the following hierarchy among families of graphs:

$$\begin{aligned} \text{1-walk (hamiltonian cycle)} &\implies \text{2-tree (hamiltonian path)} \\ &\implies \text{2-walk} \implies \text{3-tree} \implies \ldots \end{aligned}$$

Clearly, for every connected graph G, there is a k for which G has a k-walk (just duplicate all edges to obtain an eulerian graph whose Euler trail visits every vertex at most $\Delta(G)$ times). Graphs with a k-walk for a smaller k can be regarded as closer to being hamiltonian. For a nice survey of results on k-walks, k-trees and related topics, we refer the reader to Ellingham [4].

The *prism* over a graph G is the Cartesian product $G\Box K_2$ of G with the complete graph K_2 [1, 7, 9]. Thus, it consists of two copies of G and a 1-factor joining the corresponding vertices. It was observed in [7] that the property of having a hamiltonian prism is 'sandwiched' between the existence of a 2-tree and the existence of a 2-walk. That is:

$$\text{2-tree} \implies \text{hamiltonian prism} \implies \text{2-walk}$$

and both implications are sharp. This can be naturally interpreted as saying that graphs with a hamiltonian prism are closer to being hamiltonian than those which only have a 2-walk.

A hamiltonian cycle in a graph is a spanning 2-regular subgraph. In this note, we use the modular factorization to prove that $\mathbf{B}_k$ has a spanning 3-connected cubic subgraph. A direct consequence of this is that $\mathbf{B}_k$ has a hamiltonian prism and also a 2-*trail* (a 2-walk in which each edge is used at most once). As an aside, we note that in case $\mathbf{B}_k$ fails to be hamiltonian, these cubic subgraphs yield a family of cubic, 3-connected bipartite non-hamiltonian graphs.

2 Modular matchings

Our main tool is the concept of a modular matching in $\mathbf{B}_k$, as defined in [2]. We recall the related definitions, generally trying to keep in line with the notation of [2]. The *weight* $\sum B$ of a set $B \subset [2k+1]$ is defined to be the sum of all elements of B. The complement of B is denoted by $\overline{B}$.

Let $A \subset [2k+1]$ be a *k-set* (set of size k). For an integer $i = 1, \dots, k+1$, let $\mathbf{m}_i(A)$ be the set obtained when one adds the j-th largest element of $\overline{A}$ to A, where

$$j \equiv i + \sum A \pmod{k+1}$$

and $1 \leq j \leq k+1$. Let $\mathbf{m}_i$ be the set of edges of $\mathbf{B}_k$ of the form $\{A, \mathbf{m}_i(A)\}$. For easier work with expressions such as $\mathbf{m}_{i+1}$, we set $\mathbf{m}_{k+2} = \mathbf{m}_1$ and $\mathbf{m}_{k+3} = \mathbf{m}_2$.

Theorem 1 ([2]) *For $i = 1, \dots, k+1$, $\mathbf{m}_i$ is a matching in $\mathbf{B}_k$ and the set $\{\mathbf{m}_1, \dots, \mathbf{m}_{k+1}\}$ is a 1-factorization of $\mathbf{B}_k$.* □

An important observation, which is implicit in the proof of [2, Theorem 1], is the following:

Lemma 2 *Define a mapping $\mathbf{b}_i : \mathbf{B}_{k+1} \to \mathbf{B}_k$ by setting $\mathbf{b}_i(B)$ to be the set obtained by removing the j-th smallest element from B, where*

$$j \equiv i + \sum B \pmod{k+1}$$

and the index is based at 1. The composition $\mathbf{b}_i \circ \mathbf{m}_i$ is the identity. □

It will be convenient to view the set $[2k+1]$ as ordered cyclically, with 1 being the successor of $2k+1$. A *segment* in a set $B \subset [2k+1]$ is a maximal contiguous sequence of elements of B. Since the elements 1 and $2k+1$ are considered to be adjacent, a segment may 'wrap around'.

3 A connected spanning subgraph of $\mathbf{B}_k$

In this section, we show that three suitably selected modular matchings in $\mathbf{B}_k$ form a connected spanning cubic subgraph of $\mathbf{B}_k$. To this end, we introduce the following notation. Throughout this section, let A be a k-subset of $[2k+1]$. The elements of $\overline{A}$ can be labeled by numbers $+1, \dots, +(k+1)$ such that adding the element with label $+i$ to A, one obtains the set B such that $\{A, B\} \in \mathbf{m}_i$ (thus, $B = \mathbf{m}_i(A)$). We shall use $A(+i)$ to denote the element of $[2k+1]$ labeled $+i$.

By Lemma 2, the elements $A(+1), \ldots, A(+(k+1))$ form a decreasing sequence (except for at most one increase caused by the wrap-around at 1).

Symmetrically, if B is a $(k+1)$-subset of $[2k+1]$, then the elements of B can be labeled by $-1, \ldots, -(k+1)$ in such a way that removing the element labeled $-i$ from B (we shall write $B(-i)$ for the element), one obtains the set $\mathbf{b}_i(B)$. Again by Lemma 2, the sequence $B(-1), \ldots, B(-(k+1))$ is increasing (with a possible wrap-around at $2k+1$).

We need to be able to describe a sequence of additions and removals of elements of the above type. First, let $i, j \in [k+1]$. We write A^{+i} for the set obtained by adding $A(+i)$ to A (i.e., the set $\mathbf{m}_i(A)$). The symbol $A^{+i,-j}$ denotes the outcome of the removal of $A^{+i}(-j)$ from A^{+i}. The definition is extended to sequences like $A^{+i_1,-i_2,\ldots,\pm i_r}$ (in which the signs must alternate) in a natural way. Expressions like $B^{-i_1,+i_2,\ldots,\pm i_s}$, where B is a $(k+1)$-set, are defined symmetrically.

To help the reader, we introduce a graphical notation for the above operations, used in Figure 1. A sequence of additions and deletions is represented using a rectangular grid, each of whose rows corresponds to a set involved in the sequence. For brevity, we identify the rows with such sets. Columns correspond to (and are identified with) elements of $[2k+1]$. A square in row S and column x is marked gray iff $x \in S$. A label like $+i$ in row S denotes the element $S(+i)$. Labels on the left and on the top of a diagram mark special sets and elements. Finally, a square marked in bold represents the element whose addition/removal leads to the next set in the sequence. Observe that this is always a square with a label ($+i$ or $-i$).

Lemma 3 *For any k-set $A \neq \{1, \ldots, k\}$ and $i \in \{1, \ldots, k+1\}$, the spanning subgraph of $\mathbf{B}_k$ formed by the edges in $\mathbf{m}_i \cup \mathbf{m}_{i+1} \cup \mathbf{m}_{i+2}$ contains a path P that starts in A and ends in a k-set of smaller weight.*

Proof. To keep the notation simple, we prove the assertion for $i = 1$ (the proof in the general case is a trivial modification). Assume first that some element of A is larger than $a = A(+2)$ (see Figure 1a), and set $C = A^{+2,-3}$. Since

$$C(-3) = A^{+2}(-3) > A^{+2}(-2) = a,$$

the weight of C is less than that of A. Thus, the path that starts at A and follows first the edge of $\mathbf{m}_2$ and then the edge of $\mathbf{m}_3$ has the required property.

We may therefore assume that a is the largest element of A^{+2} (as in Figures 1b and c). Let s and z be the first and the last element of the last segment σ of A preceding a, respectively. Clearly, $s < a$ (although our definition allows a segment to wrap around). Furthermore, $s > 1$ since $A \neq \{1, \ldots, k\}$.

Note that $A^{+2}(-1) = z$. Set $D = A^{+2,-1}$, observing that $D(+2) = s - 1$. Furthermore, set $E = D^{+2,-3} = A^{+2,-1,+2,-3}$.

To interpret E, we distinguish two cases based on the length of σ. If σ has length 1 (i.e., $s+1 \notin A$, see Figure 1b), then $D^{+2}(-3) = a$. Consequently, E

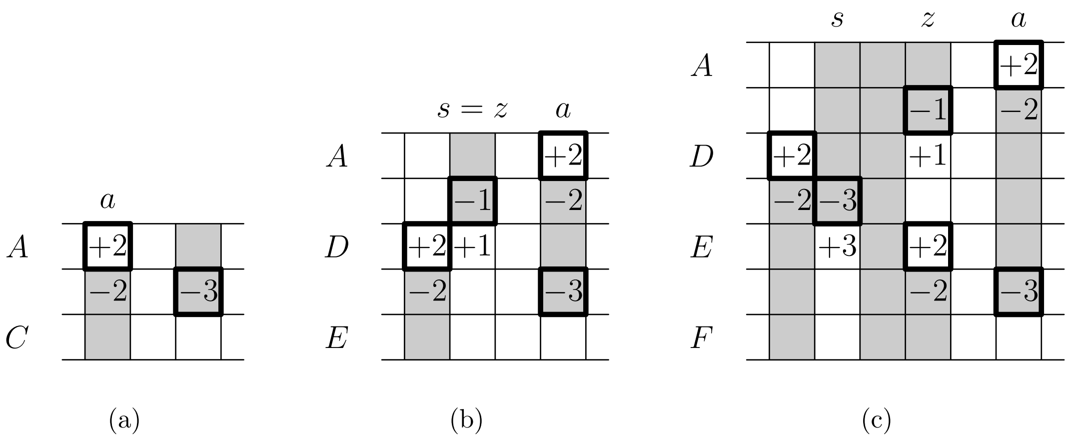

Figure 1: An illustration to the proof of Lemma 3.

differs from A in that it has $s-1$ in place of s. We infer that $\sum E < \sum A$. The desired path follows the matchings in the order $\mathbf{m}_2, \mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3$ starting from A.

It remains to consider the case that the length of σ is more than 1 (Figure 1c). The element $D^{+2}(-3)$ is now s, so $E = A \cup \{a\} \setminus \{s\}$. Since $E^{+2} = z$, one has $E^{+2}(-3) = a$. Setting $F = E^{+2,-3}$, one has $F = A \cup \{s-1\} \setminus \{s\}$, and hence $\sum F < \sum A$. Recalling that $F = A^{+2,-1,+2,-3,+2,-3}$, one sees that a path from A to F uses edges of $\mathbf{m}_2, \mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3, \mathbf{m}_2$ and $\mathbf{m}_3$ in order. The proof is finished. □

Theorem 4 *For $i \in \{1, \dots, k+1\}$, the union M_i of the matchings $\mathbf{m}_i$, $\mathbf{m}_{i+1}$ and $\mathbf{m}_{i+2}$ is a connected spanning cubic subgraph of $\mathbf{B}_k$.*

Proof. Lemma 3 implies that every vertex different from $A_0 = \{1, \dots, k\}$ is joined to A_0 by a path in M_i. Therefore, M_i is connected. It is cubic since $\mathbf{m}_i, \mathbf{m}_{i+1}, \mathbf{m}_{i+2}$ are pairwise disjoint by Theorem 1. □

4 The subgraph M_i is 3-connected

We now strengthen the result of Section 3 by showing that the spanning cubic subgraph M_i of $\mathbf{B}_k$ is actually 3-connected. When working with elements of $[2k+1]$, we perform all our computations modulo $2k+1$, using $2k+1$ in place of 0. Thus, for instance, $(2k+1)+1$ is 1.

Let $A \subset [2k+1]$. The *shift* $\mathrm{sh}(A)$ of A is the set

$$\mathrm{sh}(A) = \{x+1 : x \in A\}.$$

Thus, as we consider the elements 1 and $2k+1$ to be adjacent, the shift of A is obtained from A by a translation by one to the right. Set $\mathrm{sh}^0(A) = A$ and, for $n > 0$, $\mathrm{sh}^n(A) = \mathrm{sh}(\mathrm{sh}^{n-1}(A))$. Clearly, $\mathrm{sh}^{2k+1}(A) = A$.

The following lemma is implicit in [2]. For convenience, we include a short proof based on an idea suggested by a referee.

Lemma 5 *Let A be a subset of $[2k+1]$ with $|A| \in \{k, k+1\}$. Then $\mathrm{sh}^n(A) \neq A$ for all $n = 1, ..., 2k$.*

Proof. The proof relies on the fact that shifting changes the weight by $|A|$ modulo $2k+1$; in symbols,

$$\sum \mathrm{sh}(A) \equiv \left(\sum A\right) + |A| \pmod{2k+1}.$$

Let $n \leq 2k+1$ be the smallest positive integer such that $\mathrm{sh}^n(A) = A$. By the above, $n \cdot |A|$ is divisible by $2k+1$. Since $|A|$ and $2k+1$ are relatively prime, n is divisible by $2k+1$, whence $n = 2k+1$. □

We shall make use of a result that appears as Theorem 3 in [2]:

Lemma 6 *For any $i \in \{1, \ldots, k+1\}$, if $\{A, B\} \in \mathbf{m}_i$, then $\{\mathrm{sh}(A), \mathrm{sh}(B)\} \in \mathbf{m}_i$ as well.* □

The edges of $\mathbf{m}_i \cup \mathbf{m}_j$, $1 \leq i \neq j \leq k+1$, form a 2-factor of $\mathbf{B}_k$. We now describe some properties of the cycles of the 2-factors $\mathbf{m}_i \cup \mathbf{m}_{i+1}$:

Lemma 7 *Let C be a cycle in the 2-factor $\mathbf{m}_i \cup \mathbf{m}_{i+1}$, where $i \in \{1, \ldots, k+1\}$. Let $A \subset [2k+1]$ be a set on C.*

(i) For all t, $\mathrm{sh}^t(A)$ is on C.

(ii) If A has t segments, then the length of C is $2(2k+1)(t+\delta)$, where $\delta \in \{0, 1\}$. In particular, if a set B is also on C, then the numbers of segments of A and B differ by at most 1.

(iii) The sets A, $\mathrm{sh}(A)$, …, $\mathrm{sh}^{2k}(A)$ are uniformly distributed on C, i.e.,

$$d_C(A, \mathrm{sh}(A)) = d_C(\mathrm{sh}^t(A), \mathrm{sh}^{t+1}(A)),$$

where $t \in \{1, \ldots, 2k\}$ and d_C denotes the distance on C.

(iv) If a set $B \subset [2k+1]$ is on the cycle C and $\{A, B\} \in E(\mathbf{B}_k)$, then either $d_C(A, B) = 1$ or $d_C(A, B) > d_C(A, \mathrm{sh}(A))$.

Proof. (i) A segment of a set A will be denoted by $[a, b]$, where a and b are the smallest and the largest numbers in the segment, respectively. Let A be a k-set and $[a_j, b_j]$, $j = 1, \ldots, n$, be its segments. We label the segments in such a way that $[a_1, b_1]$ is the first segment to the right of $A(+i)$, and the segment $[a_j, b_j]$ is the first segment to the right of the segment $[a_{j-1}, b_{j-1}]$, with a possible wrap-around. It is easy to see that the path P through the vertices given below is a part of the cycle C (see Figure 2 for an illustration).

$$
\begin{aligned}
&A = \{[a_1, b_1], [a_2, b_2], \ldots, [a_{n-1}, b_{n-1}], [a_n, b_n]\},\\
&\{A(+i), [a_1, b_1], [a_2, b_2], \ldots, [a_{n-1}, b_{n-1}], [a_n, b_n]\},\\
&\{A(+i), [a_1 + 1, b_1], [a_2, b_2], \ldots, [a_{n-1}, b_{n-1}], [a_n, b_n]\},\\
&\{A(+i), [a_1 + 1, b_1 + 1], [a_2, b_2], \ldots, [a_{n-1}, b_{n-1}], [a_n, b_n]\},\\
&\{A(+i), [a_1 + 1, b_1 + 1], [a_2 + 1, b_2], \ldots, [a_{n-1}, b_{n-1}], [a_n, b_n]\},\\
&\{A(+i), [a_1 + 1, b_1 + 1], [a_2 + 1, b_2 + 1], \ldots, [a_{n-1}, b_{n-1}], [a_n, b_n]\},\\
&\qquad\vdots\\
&\{A(+i), [a_1 + 1, b_1 + 1], [a_2 + 1, b_2 + 1], \ldots, [a_{n-1} + 1, b_{n-1} + 1], [a_n, b_n]\},\\
&\{A(+i), [a_1 + 1, b_1 + 1], [a_2 + 1, b_2 + 1], \ldots, [a_{n-1} + 1, b_{n-1} + 1], [a_n + 1, b_n]\}\\
&\qquad = B.
\end{aligned}
$$

If $A(+i) = b_n + 1$, then $B = \mathrm{sh}(A)$. Otherwise, the two vertices on P that immediately follow B are

$$
\begin{aligned}
&\{A(+i), [a_1 + 1, b_1 + 1], [a_2 + 1, b_2 + 1], \ldots, [a_{n-1} + 1, b_{n-1} + 1],\\
&\quad [a_n + 1, b_n + 1]\},\\
&\{[a_1 + 1, b_1 + 1], [a_2 + 1, b_2 + 1], \ldots, [a_{n-1} + 1, b_{n-1} + 1], [a_n + 1, b_n + 1]\}\\
&\quad = \mathrm{sh}(A).
\end{aligned}
$$

This shows that if a k-set A is on C then its shift is on C as well. Applying the same argument repeatedly, we get that the vertex $\mathrm{sh}^t(A)$ is on C for all $t > 1$. By Lemma 6, the same applies to each $(k + 1)$-set on C.

(ii) Immediate from (i) and Lemma 5.

(iii) Follows from Lemma 6 and the proof of (i).

(iv) We assume that $|A| = k$ and note that the argument for the case $|A| = k + 1$ is analogous. We retain the notation of the proof of part (i). This proof shows that only one $(k + 1)$-set on the path P from A to $\mathrm{sh}(A)$, namely $\mathbf{m}_i(A)$, contains the number a_1. Similarly, the only $(k + 1)$-set on the shorter subpath of C from $\mathrm{sh}^{2k}(A)$ to A containing the element b_n is $\mathbf{m}_{i+1}(A)$. Since $A \subset B$, the

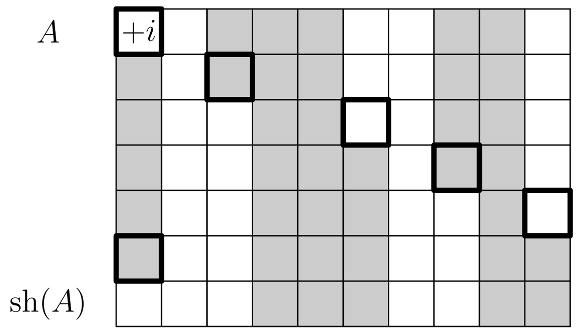

Figure 2: The path P in the proof of Lemma 7 (i).

set B contains both a_1 and b_n. It follows that either B is a neighbor of A, or $d_C(A,B) > d_C(A,\mathrm{sh}(A))$. □

Theorem 8 *For $i \in \{1,\dots,k+1\}$, the union M_i of the matchings $\mathbf{m}_i$, $\mathbf{m}_{i+1}$ and $\mathbf{m}_{i+2}$ is a 3-connected graph.*

Proof. By Theorem 4, the cubic graph M_i is connected. To establish the theorem, it is enough to show that M_i is 3-edge-connected. For the sake of contradiction, let F be any edge-cut of size 1 or 2 in M_i. The matchings $\mathbf{m}_j$ $(i \le j \le i+2)$ induce a 3-edge-coloring of M_i, and the well-known Parity Lemma of Blanuša (see, e.g., [12, Lemma 3.7.2]) implies that the parity of $|\mathbf{m}_j \cap F|$ is the same for all $j \in \{i, i+1, i+2\}$. It follows that $|F| = 2$ and both edges of F come from the same matching $\mathbf{m}_p$.

Let the set of vertices of a component of $M_i - F$ be denoted by R, and set $S = V(M_i) - R$. Suppose first that $p = i$. Let C be the cycle of $\mathbf{m}_i \cup \mathbf{m}_{i+1}$ containing the edge x. By parts (i) and (iii) of Lemma 7, there is a vertex A on C such that $A \in R$ and $\mathrm{sh}^t(A) \in S$ for some $t \le 2k$. Let C' be a cycle of $\mathbf{m}_{i+1} \cup \mathbf{m}_{i+2}$ passing through A. Since F only contains edges from $\mathbf{m}_i$, all the vertices of C' are in R. Thus, by Lemma 6, $\mathrm{sh}^t(A) \in R$, a contradiction. An analogous argument applies if $p = i+2$. Thus, we are left with the case $p = i+1$.

Let C be a cycle of $\mathbf{m}_i \cup \mathbf{m}_{i+1}$ that contains F. Write P_1 and P_2 for the two paths of $C - F$, assuming $|P_1| \le |P_2|$. Let A_x denote the vertex of P_1 incident with an edge $x \in F$. Without loss of generality, we assume that $P_1 \subset R$. Suppose first that the vertex $B = \mathbf{m}_{i+2}(A_x)$ is on C as well. Then, by Lemma 6 and Lemma 7 (iv), there is an r with the property that $\mathrm{sh}^r(A_x) \in P_1$, but $B' \in P_2$, where $z = \{\mathrm{sh}^r(A), B'\} \in \mathbf{m}_{i+2}$. This would mean that $z \in F$ and $|F| > 2$. Thus, $B = \mathbf{m}_{i+2}(A_x) \notin C$. We infer that B is on the cycle C' of $\mathbf{m}_{i+1} \cup \mathbf{m}_{i+2}$ that passes through x. Clearly, $B \in R$, for otherwise $|F| > 2$. By the same token as above, y is on C' as well. Assume that, for some t, $\mathrm{sh}^t(B) \in S$. Consider a cycle C'' of $\mathbf{m}_i \cup \mathbf{m}_{i+1}$ passing through B. As B is not on C, all vertices of C'' are in

R. By Lemma 7 (i), all the shifts of B are in C'', which contradicts the fact that $\mathrm{sh}^t(B)$ is in S, and we get $|F| > 2$. We need to consider now the case that for all t, $\mathrm{sh}^t(B) \in R$. Let T_1 and T_2 denote the two paths of $C' - F$, and suppose that B is on T_1. As $\mathrm{sh}^t(B)$ is on T_1 for all $t \geq 0$, then, by Lemma 7 (iii), for any E on C', there is at most one $t_e < 2k+1$ so that $\mathrm{sh}^{t_e}(E)$ is on T_2. However, A_x is on both P_1 and C', and $|P_1| \leq |P_2|$ leads to a contradiction with the previous statement as $|P_1| \leq |P_2|$ implies (Lemma 7 (iii)) that at least two distinct shifts of A_x have to be on $P_2 \subset S$, hence on T_2. The proof is complete. □

Corollary 9 *The prism over the graph* $\mathbf{B}_k$ *is hamiltonian.*

Proof. By [9] (see also [1]), any 3-connected cubic graph has a hamiltonian prism. Thus, the assertion follows from Theorem 8. □

We remark that Corollary 9 can also be directly derived from Theorem 4, by showing that a connected cubic bipartite graph has a hamiltonian prism. We conclude the paper with the following observation on 2-trails (defined in Section 1):

Corollary 10 *The graph* $\mathbf{B}_k$ *has a* 2*-trail.*

Proof. Adding any matching $\mathbf{m}_j$ ($j \notin \{i, i+1, i+2\}$) to M_i, we obtain a connected spanning 4-regular subgraph of $\mathbf{B}_k$. Any Euler trail of this subgraph is a 2-trail in $\mathbf{B}_k$. □

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