Short disjoint paths in locally connected graphs

Chuanping Chen*
Roman Čada†
Tomáš Kaiser†
Zdeněk Ryjáček†

November 28, 2003

1 Introduction

A well-known observation due to Chartrand and Pippert [2] says that a connected, locally k-connected graph is $k+1$-connected. If we take into account the lengths of the paths involved in the definition of connectivity, we may ask, for instance, the following: Is there a function $f(d)$ such that in any locally k-connected graph G of diameter d, any two vertices can be joined by $k+1$ vertex-disjoint paths of length at most $f(d)$?

We discuss several related questions, usually trying to find disjoint paths that are (in some sense) as short as possible, as in the following theorems (proved in Section 3). The relevant definitions are reviewed in the following section. The proofs of our results, together with some sharpness examples, are given in Sections 3, 4 and 5.

Theorem 1 *Let G be a connected, locally k-edge-connected graph, and $x, y \in V(G)$ with $dist(x, y) = d$. Then there are $k+1$ edge-disjoint xy-paths $P^0, \ldots, P^k$ such that*

$$|E(P^0)| = d \text{ and } |E(P^i)| \leq 2d \text{ for } 1 \leq i \leq k.$$

*Institute of Systems Science, Academy of Mathematics and Systems Science, Chinese Acadmy of Sciences, Beijing 100080, P.R. China

†Department of Mathematics, Univesity of West Bohemia, Univerzitní 8, 306 14 Plzeň, Czech Republic, e-mail: `cadar, kaisert, ryjacek@kma.zcu.cz`. This research was partly done on a visit to the Institute of Systems Science of Academia Sinica (Beijing, China) under the project ME 418 of the Czech Ministery of Education. These authors are partly supported by the project LN00A056 of the Czech Ministery of Education.

Theorem 2 *Let G be a connected, locally k-vertex-connected graph, and $x, y \in V(G)$ with $dist(x, y) = d$. Then there are $k+1$ edge-disjoint xy-paths $P^0, \ldots, P^k$ such that*

$$|E(P^0)| = d \text{ and } \frac{3}{2}d - 1 \leq |E(P^i)| \leq 2d \text{ for } 1 \leq i \leq k.$$

Theorem 3 *Let G be a connected, locally k-vertex-connected graph, and $x, y \in V(G)$ with $dist(x, y) = d$. Then there are $k+1$ edge-disjoint xy-paths $P^0, \ldots, P^k$ such that*

$$|E(P^0)| = d, \ \frac{3}{2}d - 1 \leq |E(P^i)| \leq 2d \text{ and}$$
$$||E(P^i)| - |E(P^j)|| \leq 2 \text{ for } 1 \leq i, j \leq k, \ i \neq j.$$

Theorem 4 *Let G be a connected, locally k-vertex-connected graph, and $x, y \in V(G)$ with $dist(x, y) = d$. Then there are $k+1$ edge-disjoint xy-paths $P^0, \ldots, P^k$ such that*

$$\frac{5}{4}d - 1 \leq |E(P^i)| \leq 2d \text{ for } i = 0, 1 \text{ and } \frac{3}{2}d - 1 \leq |E(P^i)| \leq 2d \text{ for } i = 2, \ldots, k.$$

The following result says that in a connected locally k-connected graph, one can find, between given two vertices, k vertex-disjoint paths, one of which is a distance path. There are examples to show that little can be said about the lengths of the other paths.

Theorem 5 *Let G be a locally k-connected graph, and let $x, y \in V(G)$ with $dist(x, y) = d$, where $0 < d < \infty$. Then there are k vertex-disjoint xy-paths $P^0, \ldots, P^{k-1}$ such that $|E(P^0)| = d$.*

It seems natural in this setting to introduce the following graph parameters.

Definition 6 Let G be a graph and $k \geq 1$ an integer. The *k-diameter* of G, $\mathrm{diam}^k(G)$, is the smallest r such that any two vertices of G can be joined by k vertex-disjoint paths of length at most r. If there is no such r, we set $\mathrm{diam}^k(G) = \infty$. Note that the 1-diameter coincides with the ordinary diameter.

The *local k-diameter* of G, $\mathrm{diam}_L^k(G)$, is the maximum k-diameter taken over all neighborhoods $\langle N(v) \rangle$, $v \in V(G)$. The *local diameter* is defined to be the local 1-diameter.

In Section 5, we prove the following bound on the $k+1$-diameter in terms of the usual diameter and the local k-diameter. It extends the result of Chartrand and Pippert mentioned in the beginning of this section.

Theorem 7 *For any graph G with $\mathrm{diam}_L^k(G) \geq 2$ and any integer $k \geq 1$,*

$$\mathrm{diam}^{k+1}(G) \leq k^2 \,\mathrm{diam}(G)\Big(\mathrm{diam}_L^k(G) - 1\Big).$$

We remark that the existence of k disjoint paths of bounded length has been studied, from a different perspective, by Lovász et al. [4]. They proved the following Menger-type theorem:

Theorem 8 *Let x, y be vertices of a graph G. If there are at most k pairwise vertex-disjoint xy-paths of length at most ℓ, then there is a set $X \subset V$ $(x, y \notin X)$ of size at most $k\ell/2$ such that $G - X$ has no xy-path of length at most ℓ.*

An even stronger result of this type holds if we replace 'paths of length $\leq \ell$' by 'shortest paths'. Consult [3] for the details.

2 Definitions

The purpose of this section is to fix terminology and notation in cases where ambiguity might arise. For a background in graph theory, we refer the reader e.g. to [1].

All the graphs we consider are without loops and multiple edges. Let $G = (V, E)$ be a graph. The *neighborhood* of a vertex $v \in V$ is defined as $N(x) = \{\, y \,|\, xy \in E \,\}$. For $X \subset V$, we set $N(X) = \bigcup_{x \in X} N(x)$. If H is a subgraph of G, we write $N(H)$ for the neighborhood of its vertex set.

The induced subgraph of G on a set $X \subset V$ is denoted by $\langle X \rangle$.

G is *locally k-connected* if the neighborhood of every vertex is k-connected. Locally k-edge-connected graphs are defined in an analogous way.

We use the following notation for paths. If P is a path in G passing through vertices x and y, then we let xPy stand for the portion of P which has x and y as endpoints. If Q is another path passing through y and z, then $xPyQz$ is the walk arising from the concatenation of xPy and yQz. This definition can easily be extended to the situation involving more than 2 paths.

The distance of vertices x, y of G is denoted by $\mathrm{dist}\,(x, y)$. The length of a path is the number of edges it contains. If $\mathrm{dist}\,(x, y) = d$, then any xy-path of length d is called a *distance* xy-path or a *shortest* xy-path.

A basic result concerning higher connectivity is the theorem of Menger [5] which says that there are k pairwise disjoint xy-paths in G if and only if the removal of no $k - 1$ vertices from G disconnects x from y. In particular, if G is k-connected, then there are k pairwise disjoint xy-paths for any $x, y \in V$. We shall occasionally use the following easy consequence of this theorem:

Theorem 9 *If G is k-connected, then for any $x \in V$ and $Y = \{\, y_1, \ldots, y_k \,\} \subset V$, there are k vertex-disjoint xy_i-paths in G $(i = 1, \ldots, k)$.*

A similar theorem holds for the edge-connectivity version where G is assumed k-edge-connected and the resulting paths are edge-disjoint.

3 Edge-disjoint paths

Proof of Theorem 1. Fix a shortest xy-path $P^0 = x_0x_1\dots x_d$, where $x_0 = x$ and $y_d = y$.

We shall prove the stronger assertion that the paths $P^1, \dots, P^k$ can be chosen to satisfy

(1) $V(P^i) \subset V(P^0) \cup N(P^0)$ for all i=1,..., k,

(2) the predecessors of y on P^i and on P^0 are adjacent.

in addition to the properties specified in the theorem.

The proof is by induction on the length d of the distance path.

Since $\langle N(x_1)\rangle$ is k-edge-connected, there are k edge-disjoint x_0x_2-paths $\bar{P}_1^1, \dots, \bar{P}_1^k$ in $\langle N(x_1)\rangle$. Let y_0^i be the successor of x_0 on $\bar{P}_1^i$ and y_1^i the predecessor of x_2 on $\bar{P}_1^i$ (not excluding the possibility $y_0^i = y_1^i$). Since $V(\bar{P}_1^i) \subset N(x_1)$, we have $y_0^i \in N(x_0) \cap N(x_1)$ and $y_1^i \in N(x_1) \cap N(x_2)$ for $i = 1, \dots, k$. Since all the $\bar{P}_1^i$ are edge-disjoint, we have $y_0^i \neq y_0^j$ and $y_1^i \neq y_1^j$ for distinct i, j between 1 and k. Thus we can set, for all $i = 1, \dots, k$,

$$\begin{aligned} P_1^i &= x_0y_0^ix_1, \\ P_2^i &= \begin{cases} x_0y_0^ix_1y_1^ix_2 & \text{if } y_0^i \neq y_1^i, \\ x_0y_1^ix_1 & \text{if } y_0^i = y_1^i. \end{cases} \end{aligned}$$

It is easy to see that fixing $j = 1$ or $j = 2$, the paths P_j^i $(i = 1, \dots, k)$ are edge-disjoint, satisfy (1) and (2), and their length is at most $2j$. Note that since the P_j^i are disjoint, the vertices y_j^i are distinct as i ranges over $1, \dots, k$.

For the induction step, let $3 \leq j \leq d-1$ and assume we have already constructed edge-disjoint paths $P_j^1, \dots, P_j^k$ of length at most $2j$, satisfying (1) and (2). Denoting the predecessor of x_j on P_j^i by y_{j-1}^i, the disjointness of the paths again implies that the y_{j-1}^i are distinct. By our assumptions, $\langle N(x_j)\rangle$ is k-edge-connected, and so Theorem 9 implies that there are k edge-disjoint paths $\bar{P}_j^i$ in $\langle N(x_j)\rangle$ joining y_{j-1}^i to x_{j+1} for $i = 1, \dots, k$.

Let y_j^i be the predecessor of x_{j+1} on $\bar{P}_j^i$, $i = 1, \dots, k$. Then $y_j^i \in N(x_j) \cap N(x_{j+1})$ and $y_j^{i_1} \neq y_j^{i_2}$ for $1 \leq i_1 < i_2 \leq k$ (since the paths $\bar{P}_j^i$ are edge-disjoint). Now for $i = 1, \dots, k$, set

$$P_{j+1}^i = \begin{cases} x_0P_j^iy_j^ix_{j+1} & \text{if } y_{j-1}^i = y_j^i, \\ x_0P_j^ix_{j-1}y_j^ix_{j+1} & \text{otherwise.} \end{cases}$$

The paths P_{j+1}^i satisfy (1) and (2), they are edge-disjoint, and clearly $|E(P_{j+1}^i)| \leq |E(P_j^i)| + 2 \leq 2(j+1)$.

For $j = d$, we get the required paths $P^i = P_d^i$, $i = 1, \dots, k$.

□

In fact we have shown in the proof of Theorem 1 that a pair of paths P^i and P^0 (for some fixed $i = 1, \ldots, k$) can be constructed as a sequence of two figures A, B depicted in Fig. 1. In such a drawing the path P^0 forms the bottom contour whereas the path P^i the upper one.

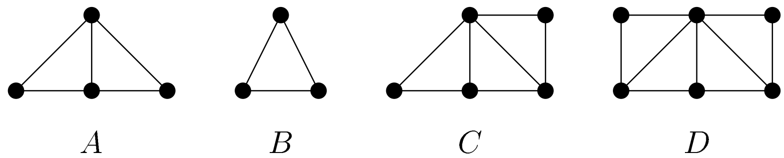

Figure 1: Four basic structures from the proof of Theorems 1, 2, 3 and 4.

Proof of Theorem 2. We consider a pair of paths P^i and P^0 for some fixed $i = 1, \ldots, k$. We will speak here about a sequence of figures A and B.

Our aim is to use some modifications of such a sequence (i.e. of paths P^i constructed in the proof of Theorem 1) in order to get a lower bound for the length of paths P^i ($i = 1, \ldots, k$).

We make use of the following

Claim. Suppose we have a sequence of figures A and B. Then a subsequence AA can be replaced by AC (see Fig. 1).

Proof. Consider two neighboring figures A as in Fig. 2.

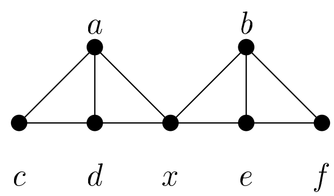

Figure 2: Illustration to the proof of Theorem 2.

Let x be a common vertex of two A's. Since x is a locally k-vertex-connected there is a path Q in $\langle N(x) \rangle$ joining vertices a and e. We will consider now the same path used in the proof of Theorem 1, i.e. b is the predeccessor of e. This choice of such a path Q for every path P^i ($i = 1, \ldots, k$) ensure that the resulting modified paths will be still edge-disjoint.

Note that since P^0 is a distance path we have $ae, ab, bd, cx, fx \notin E(G)$).

Denote the successor of a on Q by a' and the predeccesor of b by b' (the orientation of Q is taken from a to e).

If $a' = d$, then $b' \neq a'$ and $\langle \{x, b', b, e, f\} \rangle$ induces the figure C. Assume now $a' \neq d$. Then possibly $b' = a'$ but $\langle \{x, b', b, e, f\} \rangle$ induces also the figure C. $\square$

Note that modifications of paths introduced in Claim do not change the validity of the upper bound from the Theorem 1. Application of Claim to the sequence

$AA\dots A$ gives the sequence $AC\dots C$. (Note that it is also possible, if needed, to construct in some circumstances sequences containing figure D but we will not use this fact.)

The important fact is that it is possible to get the new modified paths P^i edge disjoint since now G is locally k-vertex-connected.

At first we modify the sequences (for every $i = 1,\dots,k$) from the proof of Theorem 1 replacing subsequences $AA\dots A$ with help of Claim by $AC\dots C$. Recall that these resulting paths will be edge-disjoint.

The proof of the lower bound for the length of paths P^i is now by induction on the length of the modified sequence (number of letters in the sequence). Firstly, for all figures A, B, C the lower bound holds.

Assume now that the lower bound holds for (modified) sequences of all lengths between 1 and some n. Consider a sequence S_{n+1} of length $n+1$. If the last element is B, then $|E(P^i_{n+1})| = |E(P^i_n)| + 2 \geq \frac{3}{2}(d+1) - 1$ obviously holds (P^i_n denotes the subpath of P^i created by the sequence S_n). Similarly for the last element being C.

Thus let the last element be A. The previous element must be now B, i.e. we have a sequence $S_{n-1}BA$.

If S_{n-1} is empty, then it is not difficult to check the validity of the lower bound for BA.

Suppose that S_{n-1} does not contain B. By Claim S_{n-1} is a sequence $ACCC\dots C$. Simple counting (for the whole sequence $S_{n-1}BA$) gives the lower bound.

It remains to deal with the case in which S_{n-1} contains at least one B. Take the last such B in S_{n-1}. We have now the sequence S_pBS_qCA (for some $0 \leq p, q \leq n-1$). Also here S_q is the sequence $ACCC\dots C$. For S_p holds the induction hypothesis and counting the edges in the rest of the sequences gives the lower bound. $\square$

Proof of Theorem 3. We consider the paths $P^1,\dots,P^k$ constructed in the proof of Theorem 2.

By a *common point* of two paths P^i and P^j we mean a vertex in $V(P^i) \cap V(P^j) \cap V(P^0)$. Two common points of paths P^i and P^j are *neighboring*, if there is no other common point on P^0 between them.

Firstly we introduce one useful claim. Its proof is obvious.

Claim. Let c_1 and c_2 be two neighboring common points of two paths P^i and P^j. Then their subpaths $c_1P^ic_2$ and $c_1P^ic_2$ are of equal length or have the following form (up to symmetry): BB and A; $BACC\dots CB$ (C can appear here also zero times) and $ACCC\dots C$ or $BCCC\dots CB$ (C can appear zero times) and $CCCC\dots C$.

The proof of the theorem is now by induction on the length of the distance paths.

If $d = 1$ then all paths P^i are of the same length. Assume that the Theorem holds for all lengths between 1 and some d.

Let P^i and P^j be two paths such that $|E(P^i)| - |E(P^j)| > 2$. Let c be the last common point of P^i and P^j on P^0 and let $P^i = P^i_1 c P^i_2$ and $P^j = P^j_1 c P^j_2$. Assume $|P^i_2| = |P^j_2|$. Then obviously by the induction hypothesis (applied to P^i_1 and P^j_1) $|E(P^i)| - |E(P^j)| \leq 2$, a contradiction. Then we have by the previous claim (and up to symmetry) the following cases: $P^i_2 = BB$, $P^j_2 = A$ or $P^i_2 = BACC \ldots CB$, $P^j_2 = ACCC \ldots C$ or $P^i_2 = BCCC \ldots CB$, $P^j_2 = CCCC \ldots C$. In all these cases we can if necessary exchange P^i_1 and P^j_1. Since always $||E(P^i_2)| - |E(P^j_2)|| \leq 2$ the new resulting paths satisfy the statement of the theorem. $\square$

Proof of Theorem 4. The main idea here is to 'lend' the path P^0 some longer intervals of a path P^i. Obviously, the worst case is, when all paths P^i ($i = 1, \ldots, k$) are vertex-disjoint and consist of A and C, i.e. they are of the form $ACCC \ldots C$. In this case take one of them, say P^1, and modify P^0 to $ICICIC \ldots$ and P^1 to $AICICIC \ldots$. By I we mean a subpath of P^0 of length 2. Counting gives then the lower bound $\frac{5}{4}d - 1$ for P^0 and P^1. $\square$

Note that for $k = 1$ the lower bounds given in Theorem 4 are sharp. See Fig. 3.

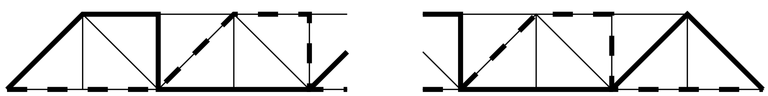

Figure 3: Sharpness example to Theorems 2 and 4.

4 Vertex-disjoint paths, one of them shortest

We now prove Theorem 5 which says that local k-connectivity guarantees the existence of k vertex-disjoint xy-paths, one of which is a shortest xy-path. Note the difference in comparison to Theorem 1 where there are $k + 1$ edge-disjoint paths.

We actually prove the stronger statement that the paths can be chosen such that in addition,

$$V(P^i) \subset N\Big(V(P^0) - \{\, y \,\}\Big) \tag{1}$$

for all $i \geq 1$.

We proceed by induction on $d = \mathrm{dist}\,(x, y)$. If $d = 1$, then y must have a neighbor $x' \neq x$ (unless $k = 1$, in which case the assertion is trivial). There are k vertex-disjoint xx'-paths $T^1, \ldots, T^k$ in $\langle N(y) \rangle$. Letting z^i be the neighbor of x on T^i, we can define $P^i = xz^iy$ (for $1 \leq i \leq k$) and $P^0 = xy$. Thus, in this case, we get $k + 1$ paths with the required properties, which is even more than is necessary.

For the induction step, assume that the assertion is true for all pairs of vertices at distance $d' < d$. Let y' be the neighbor of y on any distance path from x to y. Since $\mathrm{dist}\,(x, y') = d-1$, we can find (by the induction hypothesis) vertex-disjoint xy'-paths $Q^0, \dots, Q^{k-1}$ satisfying (1) and such that $|E(Q^0)| = d - 1$. We may assume them to be chordless.

Note that Q^0 does not pass through y as $\mathrm{dist}\,(x, y) = d$, and by (1), y does not lie on the other paths Q^i for the same reason.

Denote the predecessor of y' on Q^i $(0 \le i \le k-1)$ by y^i. By Theorem 9, there are k vertex-disjoint paths $\bar{P}^i$ joining y^i to y $(0 \le i \le k-1)$ in $\langle N(y') \rangle$. Define P^0 to be $xQ^0y'y$ and set, for $1 \le i \le k-1$,

$$P^i = xQ^iy^i\bar{P}^iy.$$

We claim that $P^0, \dots, P^{k-1}$ are vertex-disjoint paths. To see this, observe that $V(Q^i) \cap V(\bar{P}^j)$ is empty if $i \neq j$, and equals $\{ y^i \}$ if $i = j$. Indeed, the $\bar{P}^j$ are paths in $\langle N(y') \rangle$, so that any other intersection would imply a chord in the xy'-path Q^i, which is however assumed chordless.

Furthermore, the length of P^0 is d and condition (1) is clearly satisfied. This concludes the proof.

Example 10 The following example shows that in general, we cannot expect to find, under the hypotheses of Theorem 5, $k+1$ vertex-disjoint xy-paths, one of which is of length at most αd (where α is any fixed constant).

Fix integers k and ℓ. Take a path P_ℓ of length ℓ on vertices $v_0, \dots, v_\ell$ and a complete graph K_{k+1} on vertices $w_0, \dots, w_k$. Let H be the composition $P_\ell[K_k]$ in which $V(P_\ell) \times \{ w_0 \}$ is contracted to a vertex w, and $\{ v_0 \} \times \{ w_1, \dots, w_k \}$ is contracted to a vertex v. (Multiple edges and loops are suppressed.) Take another copy H' of H (denoting a copy of $v \in V(H)$ by v') and form a graph G by identifying, in the disjoint union $H \cup H'$, w with $(v_\ell, w_1)'$, w' with (v_ℓ, w_1), and (v_ℓ, w_i) with $(v_\ell, w_i)'$ for $i \ge 2$. (See Fig. 4 for an illustration with $k = 2$ and $\ell = 3$.)

The vertices $x = v$ and $y = v'$ are at distance 3, but it is easy to see that the length of the shortest of any $k+1$ vertex-disjoint xy-paths can be made arbitrarily large by choosing large ℓ.

Also note that the same example (with ℓ large) shows that in Theorem 5, we cannot upper-bound the lengths of the paths $P^1, \dots, P^{k-1}$ if the length of P^0 is (a constant times) d.

5 The k-diameter

The k-diameter and local k-diameter $(k \ge 1)$ were defined in the Introduction. In this section, we prove Theorem 7. We begin with an easy observation on the diameter, which implies an upper bound on the local diameter of $K_{1,r}$-free graphs (that is, graphs containing no induced copy of the complete bipartite graph $K_{1,r}$).

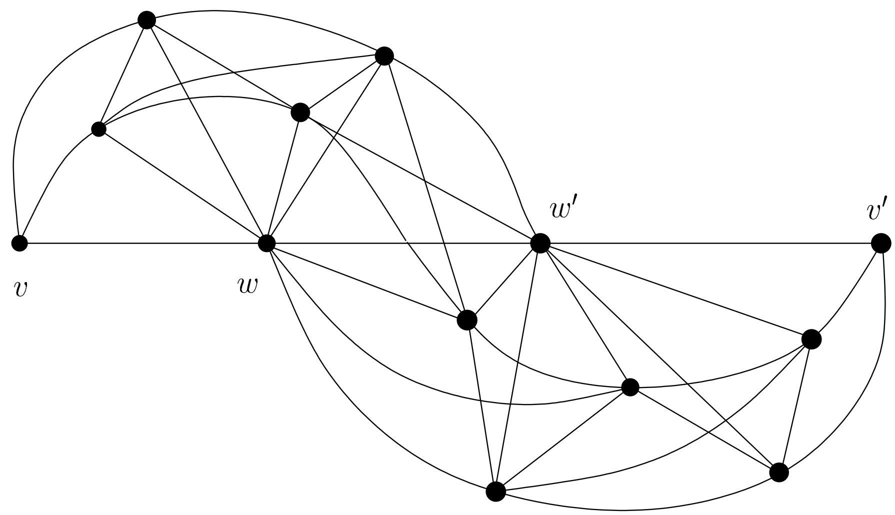

Figure 4: A sharpness example for Theorem 5.

Observation 11 *For any graph G,*

$$\operatorname{diam}(G) < 2\alpha(G),$$

where $\alpha(G)$ is the independence number of G.

Proof. Let $P = x_0x_1 \dots x_d$ be a path in G of length $d = \operatorname{diam}(G)$ joining vertices x_0, x_d whose distance is exactly d. Then the set $A = \{ x_{2i} \mid 0 \le i \le d/2 \}$ must be independent, for otherwise we could join x_0 to x_d by a shorter path. Since $|A| \ge (d+1)/2$, the claim follows. $\square$

Corollary 12 *Let $r \ge 2$. If G does not contain $K_{1,r}$ as an induced subgraph, then* $\operatorname{diam}_L(G) \le 2r - 3$. $\square$

It is easy to see that $\operatorname{diam}^{k+1}(G) \ge \operatorname{diam}^k(G)$ for any k and G. Thus $\operatorname{diam}^k(G) \ge \operatorname{diam}(G)$. In the opposite direction, the following theorem bounds the $k+1$-diameter of G in terms of its diameter and local k-diameter.

For the proof of Theorem 7, we shall need the following lemma.

Lemma 13 *Let G be a graph with* $\operatorname{diam}^k(G) \le d$ *and let $S = \{ x_1, \dots, x_k \}$ be a set of vertices of G. Then for any vertex $z \notin S$, there are k vertex-disjoint paths $P_1, \dots, P_k$ such that each P_i is an zx_i-path of length at most k^2d.*

Proof. Since $\operatorname{diam}^k(G) \le d$, we have, for any $x_i \in S$, k vertex-disjoint zx_i-paths Q_{ij} $(1 \le j \le k)$ of length at most d each. Let H be the subgraph of G with vertices $\bigcup_{i,j=1}^{k} V(Q_{ij})$ and edges $\bigcup_{i,j=1}^{k} E(Q_{ij})$. Let H' be the graph obtained from H by adding a new vertex w, together with edges wx_i $(1 \le i \le k)$. H' is

k-vertex-connected between z and w. Indeed, no set Y of $k-1$ vertices of H' can separate z from w as Y must miss at least one vertex $x_i \in S$, and there are k disjoint paths joining x_i to z in H'. Thus by Menger's theorem, there are k vertex-disjoint wz-paths $P'_1, \ldots, P'_k$ in H'. Restricting to H, these give us the zx_i-paths P_i as desired. Since $|E(H)| \leq k^2 d$, no P_i can have more edges than this. $\square$

Proof of Theorem 7. Let two vertices x, y at distance d be given. We claim that there are $k+1$ vertex-disjoint xy-paths of length at most $k^2 d \cdot \mathrm{diam}_L^k(G)$. The proof is by induction on d. We may assume that $\mathrm{diam}_L^k(G)$ is finite, i.e. that G is locally k-connected.

If $d = 1$ then we can find k vertex-disjoint xy-paths of length 2 just as in the beginning of the proof of Theorem 5; together with the path of length 1, they have the desired properties. (Note the assumption that $\mathrm{diam}_L^k(G) \geq 2$.)

For the inductive step, let y' be the neighbor of y on any distance path from x to y. We may apply induction to x and y' since $\mathrm{dist}\,(x, y) = d - 1$. This yields xy'-paths $Q^0, \ldots, Q^k$ as in the claim. We assume them (as we may) to be chordless. Denote by y^i $(0 \leq i \leq k)$ the predecessor of y' on Q^i. Use Lemma 13 on $\langle N(y') \rangle$, setting $S = \{ y^1, \ldots, y^k \}$ and $z = y$. We get k vertex-disjoint $y^i y$-paths $\bar{P}^i$ $(1 \leq i \leq k)$ of length at most $k^2 \cdot \mathrm{diam}^k(\langle N(y') \rangle) \leq k^2 \cdot \mathrm{diam}_L^k(G)$. Since P^0 is chordless, the $\bar{P}^i$ cannot intersect it in any vertex except y and possibly y^0. For any $\bar{P}^i$ not passing through y^0, we set $P^i = xQ^i y^i \bar{P}^i y$. If some one of the paths, say $\bar{P}^s$, contains y^0, then we let $P^s = xQ^s y^s \bar{P}^s y^0 y' y$ and $P^0 = xQ^0 y^0 \bar{P}^s y$. Otherwise, we set $P^0 = xQ^0 y' y$. The lack of chords in the paths Q^i implies that we obtain vertex-disjoint paths by this construction. The lengths of the paths are clearly as desired, and so the proof is complete. $\square$

Corollary 14 *Let G be a connected, locally k-connected $K_{1,r}$-free graph, where $k \geq 1$ and $r \geq 3$. If $\mathrm{diam}(G) = d$, then*

$$\mathrm{diam}^{k+1}(G) \leq 2k^2 d(r-2).$$

Example 15 Theorem 7 is probably not sharp, especially if k is not fixed. However, we shall give an example of a graph G with $\mathrm{diam}(G) \leq d+1$, $\mathrm{diam}_L^k(G) \leq \ell + 2$, and $\mathrm{diam}_L^{k+1}(G) > d \cdot \ell$, where d and ℓ are any given integers. Take the Cartesian product $H = P_{d\ell} \otimes K_{k+1}$ of a path on $d \cdot \ell + 1$ vertices $\{ v_0, \ldots, v_{d\ell} \}$ with the complete graph on vertices $\{ w_0, \ldots, w_k \}$. For $0 \leq i \leq d-1$, let $S_i \subset V(H)$ be defined as

$$S_i = \{ v_{i\ell+1}, v_{i\ell+2}, \ldots, v_{i\ell+\ell-1} \} \times \{ w_0 \}.$$

To form G, first contract each S_i to a vertex s_i, suppressing multiple edges and loops, and then remove all vertices $(v_{i\ell}, w_0)$ where $0 < i < d$. (See Fig. 5 for an illustration with $d = 3, \ell = 3$ and $k = 2$.) It is straightforward to check that G has the required properties (to see that $\mathrm{diam}^{k+1}(G) > d\ell$, consider disjoint paths between the vertices (v_0, w_0) and $(v_{d\ell}, w_0)$.)

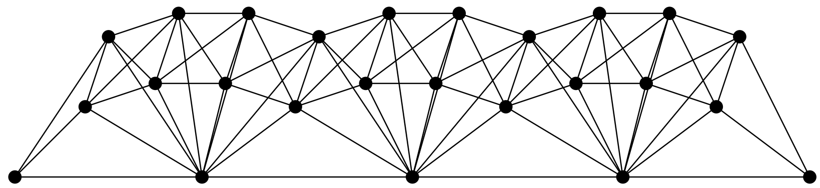

Figure 5: An example for Theorem 7.

References

[1] J.A. Bondy and U.S.R. Murty: Graph Theory with Applications. Macmillan, London, 1976.

[2] G. Chartrand and R.E. Pippert: Locally connected graphs. *Časopis pro pěstování matematiky*, **99** (1974), 158–163.

[3] A. Frank: Connectivity and network flows. *Handbook of Combinatorics* (R.L. Graham et al. eds.), Elsevier, Amsterdam, 1995, pp. 111–177.

[4] L. Lovász, V. Neumann-Lara and M.D. Plummer: Mengerian theorems for paths of bounded length, *Period Math. Hungar.* **9** (1978), 269–276.

[5] K. Menger: Zur allgemeinen Kurventheorie, *Fund. Math.* **10** (1927), 96–115.