Contractible Subgraphs, Thomassen's Conjecture and the Dominating Cycle Conjecture for Snarks

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Abstract

We show that the conjectures by Matthews and Sumner (every 4-connected claw-free graph is hamiltonian), by Thomassen (every 4-connected line graph is hamiltonian) and by Fleischner (every cyclically 4-edge-connected cubic graph has either a 3-edge-coloring or a dominating cycle), which are known to be equivalent, are equivalent with the statement that every snark (i.e. a cyclically 4-edge-connected cubic graph of girth at least five that is not 3-edge-colorable) has a dominating cycle.

We use a refinement of the contractibility technique which was introduced by Ryjáček and Schelp in 2003 as a common generalization and strengthening of the reduction techniques by Catlin and Veldman and of the closure concept introduced by Ryjáček in 1997.

Keywords: dominating cycle, contractible graph, cubic graph, snark, line graph, hamiltonian graph

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1 Introduction

In this paper we consider finite undirected graphs. All the graphs we consider are loopless (with one exception in Section 3), however we allow the graphs to have multiple edges. We follow the most common graph-theoretic terminology and notation and for concepts and notation not defined here we refer the reader to [2]. If F, G are graphs then $G - F$ denotes the graph $G - V(F)$ and by an *a, b-path* we mean a path with end vertices a, b. A graph G is *claw-free* if G does not contain an induced subgraph isomorphic to the claw $K_{1,3}$.

In 1984, Matthews and Sumner [8] posed the following conjecture.

Conjecture A [8]. *Every 4-connected claw-free graph is hamiltonian.*

Since every line graph is claw-free (see [1]), the following conjecture by Thomassen is a special case of Conjecture A.

Conjecture B [12]. *Every 4-connected line graph is hamiltonian.*

A closed trail T in a graph G is said to be *dominating*, if every edge of G has at least one vertex on T, i.e., the graph $G-T$ is edgeless (a closed trail is defined as usual, except that we allow a single vertex to be such a trail). The following result by Harary and Nash-Williams [6] shows the relation between the existence of a dominating closed trail (abbreviated DCT) in a graph G and hamiltonicity of its line graph $L(G)$.

Theorem 1 [6]. *Let G be a graph with at least three edges. Then $L(G)$ is hamiltonian if and only if G contains a DCT.*

Let k be an integer and let G be a graph with $|E(G)| > k$. The graph G is said to be *essentially k-edge-connected* if G contains no edge cut R such that $|R| < k$ and at least two components of $G - R$ are nontrivial (i.e. containing at least one edge). If G contains no edge cut R such that $|R| < k$ and at least two components of $G - R$ contain a cycle, G is said to be *cyclically k-edge-connected*.

It is well-known that G is essentially k-edge-connected if and only if its line graph $L(G)$ is k-connected. Thus, the following statement is an equivalent formulation of Conjecture B.

Conjecture C. *Every essentially 4-edge-connected graph contains a DCT.*

By a *cubic* graph we will always mean a regular graph of degree 3 without multiple edges. It is easy to observe that if G is cubic, then a DCT in G becomes a dominating cycle (abbreviated DC), and that every essentially 4-edge-connected cubic graph must be triangle-free, with a single exception of the graph K_4. To avoid this exceptional case, we will always consider only essentially 4-edge-connected cubic graphs on at least 5 vertices.

Since a cubic graph is essentially 4-edge-connected if and only if it is cyclically 4-edge-connected (see [5], Corollary 1), the following statement, known as the Dominating Cycle Conjecture, is a special case of Conjecture C.

Conjecture D. *Every cyclically 4-edge-connected cubic graph has a DC.*

Restricting to cyclically 4-edge-connected cubic graphs that are not 3-edge-colorable, we obtain the following conjecture posed by Fleischner [4].

Conjecture E [4]. *Every cyclically 4-edge-connected cubic graph that is not 3-edge-colorable has a DC.*

In [10], a closure technique was used to prove that Conjectures A and B are equivalent. Fleischner and Jackson [5] showed that Conjectures B, C and D are equivalent. Finally, Kochol [7] established the equivalence of these conjectures with Conjecture E. Thus, we have the following result.

Theorem 2 [5], [7], [10]. *Conjectures A, B, C, D and E are equivalent.*

A cyclically 4-edge-connected cubic graph G of girth $g(G) \geq 5$ that is not 3-edge-colorable is called a *snark*. Snarks have turned out to be an important class of graphs for example in the context of nowhere zero flows. For more information about snarks see the paper [9]. Restricting our considerations to snarks, we obtain the following special case of Conjecture E.

Conjecture F. *Every snark has a DC.*

The following theorem, which is the main result of this paper, shows that Conjecture F is equivalent with the previous ones.

Theorem 3. *Conjecture F is equivalent with Conjectures A, B, C, D and E.*

The proof of Theorem 3 is postponed to Section 4.

As already noted, every cyclically 4-edge-connected cubic graph other than K_4 must be triangle-free. Thus, the difference between Conjectures E and F consists in restricting to graphs which do not contain a 4-cycle. For the proof of the equivalence of these conjectures in Section 4 we first develop in Section 2 a refinement of the technique of contractible subgraphs that was developed in [11] as a common generalization of the closure concept [10] and Catlin's collapsibility technique [3], and in Section 3 a technique that allows to handle the (non)existence of a DC while replacing a subgraph of a graph by another one.

2 Weakly contractible graphs

In this section we introduce a refinement of the contractibility technique from [11] under a special assumption which is automatically satisfied in cubic graphs. We basically follow the terminology and notation of [11].

For a graph H and a subgraph $F \subset H$, $H|_F$ denotes the graph obtained from H by identifying the vertices of F as a (new) vertex v_F, and by replacing the created loops by pendant edges (i.e. edges with one vertex of degree 1). Note that $H|_F$ may contain multiple edges and $|E(H|_F)| = |E(H)|$. For a subset $X \subset V(H)$ and a partition $\mathcal{A}$ of X into subsets, $E(\mathcal{A})$ denotes the set of all edges a_1a_2 (not necessarily in H) such that a_1 and a_2 are in the same element of $\mathcal{A}$, and $H^{\mathcal{A}}$ denotes the graph with vertex set $V(H^{\mathcal{A}}) = V(H)$ and edge set $E(H^{\mathcal{A}}) = E(H) \cup E(\mathcal{A})$ (here the sets $E(H)$ and $E(\mathcal{A})$ are considered to be disjoint, i.e. if $e_1 = a_1a_2 \in E(H)$ and $e_2 = a_1a_2 \in E(\mathcal{A})$, then e_1, e_2 are parallel edges in $H^{\mathcal{A}}$).

Let F be a graph and $A \subset V(F)$. Then F is said to be *A-contractible*, if for every even subset $X \subset A$ (i.e. with $|X|$ even) and for every partition $\mathcal{A}$ of X into two-element subsets, the graph $F^{\mathcal{A}}$ has a DCT containing all vertices of A and all edges of $E(\mathcal{A})$. In particular, the case $X = \emptyset$ implies that an A-contractible graph has a DCT containing all vertices of A.

If H is a graph and $F \subset H$, then a vertex $x \in V(F)$ is said to be a *vertex of attachment of F in H* if x has a neighbor in $V(H) \setminus V(F)$. The set of all vertices of attachment of F in H is denoted by $A_H(F)$. Finally, $\mathrm{dom}_{tr}(H)$ denotes the maximum number of edges of a graph H that are dominated by (i.e. have at least one vertex on) a closed trail in H. Specifically, H has a DCT if and only if $\mathrm{dom}_{tr}(H) = |E(H)|$.

The following theorem shows that a contraction of an $A_H(F)$-contractible subgraph of a graph H does not affect the value of $\mathrm{dom}_{tr}(H)$.

Theorem 4 [11]. *Let F be a connected graph and let $A \subset V(F)$. Then F is A-contractible if and only if*

$$dom_{tr}(H) = dom_{tr}(H|_F)$$

for every graph H such that $F \subset H$ and $A_H(F) = A$.

Specifically, F is A-contractible if and only if, for any H such that $F \subset H$ and $A_H(F) = A$, H has a DCT if and only if $H|_F$ has a DCT (the "only if" part follows by Theorem 4, the "if" part can be easily seen by the definition of A-contractibility).

Let F be a graph and let $A \subset V(F)$. The graph F is said to be *weakly A-contractible*, if for every *nonempty* even subset $X \subset A$ and for every partition $\mathcal{A}$ of X into two-element subsets, the graph $F^{\mathcal{A}}$ has a DCT containing all vertices of A and all edges of $E(\mathcal{A})$.

Thus, in comparison with the contractibility concept as introduced in [11], we do not include the case $X = \emptyset$. This means that we do not require that a weakly A-contractible graph has a DCT containing all vertices of A.

Clearly, every A-contractible graph is also weakly A-contractible. It is easy to see that if F is weakly A-contractible and $|A| \geq 3$, then $d_F(x) \geq 2$ for every $x \in A$.

Examples. **1.** The graphs in Figure 1 are examples of graphs that are weakly A-contractible but not A-contractible (vertices of the set A are double-circled).

2. The triangle C_3 is A-contractible for any subset A of its vertex set.

3. Let C be a cycle of length $\ell \geq 4$, let $x, y \in V(C)$ be nonadjacent and set $A = V(C)$, $X = \{x, y\}$ and $\mathcal{A} = \{\{x, y\}\}$. Then there is no DCT in C containing the edge $xy \in C^{\mathcal{A}}$ and all vertices of A. Hence no cycle C of length at least 4 is weakly $V(C)$-contractible.

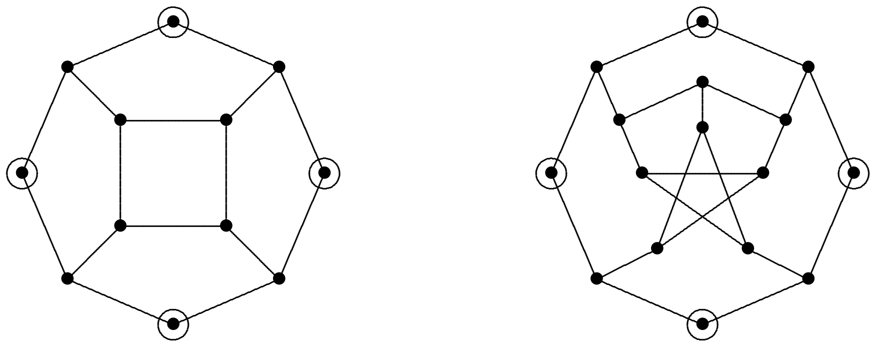

Figure 1

If H is a graph and $F \subset H$, then H_{-F} denotes the graph with vertex set $V(H_{-F}) = V(H) \setminus (V(F) \setminus A_H(F))$ and with edge set $E(H_{-F}) = E(H) \setminus E(F)$ (equivalently, H_{-F} is the graph determined by the edge set $E(H) \setminus E(F)$).

Our next theorem shows that, in a special situation, weak contractibility is sufficient to obtain the equivalence of Theorem 4.

Theorem 5. *Let F be a graph and let $A \subset V(F)$, $|A| \geq 2$. Then F is weakly A-contractible if and only if*

$$dom_{tr}(H) = dom_{tr}(H|_F)$$

for every graph H such that $F \subset H$, $A_H(F) = A$, $d_{H_{-F}}(a) = 1$ for every $a \in A$, and $|V(K) \cap A| \geq 2$ for at least one component K of H_{-F}.

Proof. The proof of Theorem 5 basically follows the proof of Theorem 2.1 of [11].

Let F be a graph and let H be a graph satisfying the assumptions of the theorem. Then every closed trail T in H corresponds to a closed trail in $H|_F$, dominating at least as many edges as T. Hence immediately $\mathrm{dom}_{tr}(H) \leq \mathrm{dom}_{tr}(H|_F)$.

Suppose that F is weakly A-contractible and let T' be a closed trail in $H|_F$ such that T' dominates $\mathrm{dom}_{tr}(H|_F)$ edges and, subject to this condition, T' has maximum length. If $v_F \notin V(T')$, then T' is also a closed trail in H, implying $\mathrm{dom}_{tr}(H|_F) \leq \mathrm{dom}_{tr}(H)$, as requested. Hence we can suppose $v_F \in V(T')$.

If T' is nontrivial, i.e. contains an edge, then the edges of T' determine in H a system of trails $\mathcal{P} = \{P_1, \ldots, P_k\}$, $k \geq 1$, such that every $P_i \in \mathcal{P}$ has endvertices in A (note that all trails in $\mathcal{P}$ are open since $d_{H_{-F}}(a) = 1$ for all $a \in A$). Since $d_{H_{-F}}(a) = 1$ for all $a \in A$, every $x \in A$ is an endvertex of at most one trail from $\mathcal{P}$, and we set $X = \{x \in A_H(F) | \, x$ is an endvertex of some $P_i \in \mathcal{P}\}$ and $\mathcal{A} = \{A_1, \ldots, A_k\}$, where A_i is the (two-element) set of endvertices of P_i, $i = 1, \ldots, k$.

If T' is trivial (i.e., a one-vertex trail), then we consider a component K of H_{-F} for which $|V(K) \cap A_H(F)| \geq 2$. Let $x_1, x_2 \in V(K) \cap A_H(F)$. If $V(K) \setminus \{x_1, x_2\} \neq \emptyset$ then, since K is connected, K contains a path of length at least 2 with end vertices x_1, x_2, but then we have a contradiction with the maximality of T'. Hence $V(K) = \{x_1, x_2\}$ and $E(K) = \{x_1x_2\}$, and we set $P_1 = x_1x_2$, $\mathcal{P} = \{P_1\}$, $X = \{x_1, x_2\}$ and $\mathcal{A} = \{\{x_1, x_2\}\}$. Note that in both cases the set X is nonempty.

By the weak A-contractibility of F, $F^{\mathcal{A}}$ has a DCT Q, containing all vertices of A and all edges of $E(\mathcal{A})$. The trail Q determines in F a system of trails $Q_1, \ldots, Q_k$ such that every Q_i has its two endvertices in two different elements of $\mathcal{A}$. Now, the trails Q_i together with the system $\mathcal{P}$ form a closed trail in H, dominating at least as many edges as T'. Hence $\mathrm{dom}_{tr}(H|_F) \leq \mathrm{dom}_{tr}(H)$, implying $\mathrm{dom}_{tr}(H|_F) = \mathrm{dom}_{tr}(H)$.

Next suppose that F is not weakly A-contractible (possibly even disconnected). Then, for some nonempty $X \subset A$ and a partition $\mathcal{A}$ of X into two-element sets, $F^{\mathcal{A}}$ has no DCT containing all vertices of A and all edges of $E(\mathcal{A})$. Let $\mathcal{A} = \{\{x_1', x_1''\}, \ldots, \{x_k', x_k''\}\}$, and construct a graph H with $F \subset H$ by replacing the edges of $E(\mathcal{A})$ by k vertex disjoint x_i', x_i''-paths P_i of length at least 3, $i = 1, \ldots, k$, and by attaching a pendant edge to every vertex in $A \setminus X$. Since $X \neq \emptyset$, at least one component K of H_{-F} is a path with end vertices in A, implying $|V(K) \cap A| \geq 2$. Since $F^{\mathcal{A}}$ has no DCT containing all vertices of A and all edges of $E(\mathcal{A})$, H has no DCT. However, clearly $H|_F$ has a DCT and we have $\mathrm{dom}_{tr}(H) < \mathrm{dom}_{tr}(H|_F)$. ■

In the special case of cubic graphs, we have the following corollary.

Corollary 6. *Let F be a graph with $\delta(F) = 2$, $\Delta(F) \leq 3$ and $|A| \geq 2$, where $A = \{x \in V(F) \mid d_F(x) = 2\}$. Then F is weakly A-contractible if and only if*

$$dom_{tr}(H) = dom_{tr}(H|_F)$$

for every cubic graph H such that $F \subset H$, $A_H(F) = A$, and $|V(K) \cap A| \geq 2$ for at least one component K of H_{-F}.

Proof. Clearly $d_{H_{-F}} = 1$ for every $a \in A$, since H is cubic. If F is weakly A-contractible, then $\mathrm{dom}_{tr}(H) = \mathrm{dom}_{tr}(H|_F)$ immediately by Theorem 5. For the rest of the proof, it is sufficient to modify the last part of the proof of Theorem 5 such that the constructed graph H is cubic. To achieve this, it is sufficient to use a copy of the graph in Figure 2(a) instead of each of the paths P_i, and a copy of the graph in Figure 2(b) instead of each of the pendant edges attached to the vertices $a_j \in A \setminus X$. Then there is a component K of H_{-F} with $|V(K) \cap A| \geq 2$ since X is nonempty. The graph $H|_F$ has a closed trail dominating all edges except for the edges different from e_j in the copies attached to the vertices in $A \setminus X$, while in H there is no such closed trail. ■

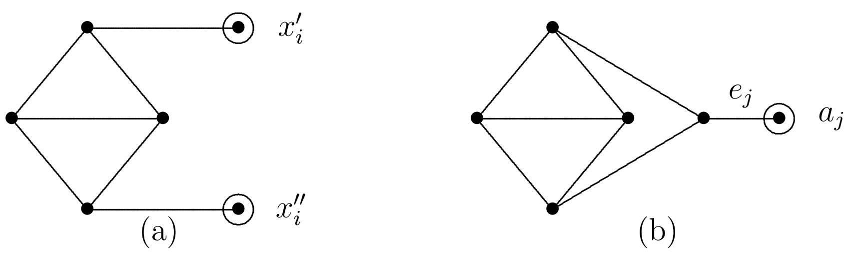

Figure 2

We say that a subgraph $F \subset H$ is a *weakly contractible subgraph of H* if F is weakly $A_H(F)$-contractible. We then have the following corollary.

Corollary 7. *Let H be a cubic graph and let F be a weakly contractible subgraph of H with $\delta(F) = 2$. Then H has a DC if and only if $H|_F$ has a DCT.*

Proof. First note that in a cubic graph every closed trail is a cycle and that a cubic graph with a DC must be essentially 2-edge-connected. Since H is cubic and $\delta(F) = 2$, $A_H(F) = \{x \in V(F) \mid d_F(x) = 2\}$ and the weak contractibility assumption implies F is connected. If every component of H_{-F} contains one vertex from $A_H(F)$, then clearly neither H nor $H|_F$ is essentially 2-edge-connected (since H is cubic) and hence neither H nor $H|_F$ has a DCT. The rest of the proof follows from Corollary 6. ∎

Example. Let H be the graph obtained from three vertex-disjoint copies F_1, F_2, F_3 of the graph F_i from Figure 2(a) by adding edges $x'_1x'_2$, $x'_1x'_3$, $x'_2x'_3$, $x''_1x''_2$, $x''_1x''_3$, $x''_2x''_3$. Then H is cubic, $F_1 \subset H$ is weakly contractible, $H|_{F_1}$ has a DCT, but H has no DC. This example shows that the assumption $\delta(F) = 2$ in Corollaries 6 and 7 cannot be omitted.

3 Replacement of a subgraph

In this section we develop a technique to replace certain subgraphs by others without affecting the (non)existence of a DCT.

Let G be a graph and let $F \subset G$ be a subgraph of G. Let F' be a graph such that $V(F') \cap V(G) = \emptyset$, let $A' \subset V(F')$ be such that $|A'| = |A_G(F)|$ and let $\varphi : A_G(F) \to A'$ be a bijection. Let H be the graph obtained from G_{-F} and F' by identifying each $x \in A_G(F)$ with its image $\varphi(x) \in A'$. We say that the graph H is obtained by *replacement (in G) of F by F' modulo φ* and denote $H = G[F \xrightarrow{\varphi} F']$.

Note that if $H = G[F \xrightarrow{\varphi} F']$ then also clearly $G = H[F' \xrightarrow{\varphi^{-1}} F]$.

Let F be a graph and let $A = \{a_1, \ldots, a_k\} \subset V(F)$. Let $\overline{A}$ be a set with $\overline{A} \cap V(F) = \emptyset$, $|\overline{A}| = |A|$, and set $\overline{A} = \{\overline{a}_1, \ldots, \overline{a}_k\}$. Then $\overline{F}^A$ denotes the graph with vertex set $V(\overline{F}^A) = V(F) \cup \overline{A}$ and with edge set $E(\overline{F}^A) = E(F) \cup \{a_i\overline{a}_i \mid i = 1, \ldots, k\}$ (i.e., $\overline{F}^A$ is obtained from F by attaching a pendant edge to every vertex of A).

The following observation shows that, under certain conditions, the replacement in a graph G of a weakly contractible subgraph by another one affects neither the existence nor the nonexistence of a DCT in G.

Proposition 8. *Let G be a graph with $\delta(G) \geq 1$ and let $F \subset G$ be a weakly contractible subgraph of G such that $|E(F)| \geq 1$, $d_{G_{-F}}(x) = 1$ for every $x \in A_G(F)$ and $G \not\simeq \overline{F}^{A_G(F)}$. Let F', $|E(F')| \geq 1$, be a weakly A'-contractible graph for an $A' \subset V(F')$, and let $\varphi : A_G(F) \to A'$ be a bijection. Then G has a DCT if and only if $G[F \xrightarrow{\varphi} F']$ has a DCT.*

Proof. Set $H = G[F \xrightarrow{\varphi} F']$. For $|A_G(F)| = 0$ the assumptions $G \not\simeq \overline{F}^{A_G(F)}$ and $\delta(G) \geq 1$ imply that G is disconnected and neither G nor H has a DCT. If $|A_G(F)| = 1$ or if $|A_G(F)| \geq 2$ and $|V(K) \cap A_G(F)| = 1$ for every component K of G_{-F}, then neither G nor H can have a DCT since $|E(F)| \geq 1$, $|E(F')| \geq 1$, $d_{G_{-F}}(x) = 1$ for every $x \in A_G(F)$ and $G \not\simeq \overline{F}^{A_G(F)}$. Thus, we can assume that $|A_G(F)| \geq 2$ and there is a component K of G_{-F} such that $|V(K) \cap A_G(F)| \geq 2$. Then, by Theorem 5, G has a DCT if and only if $G|_F$ has a DCT. Similarly, H has a DCT if and only if $H|_{F'}$ has a DCT, but the graphs $G|_F$ and $H|_{F'}$ are, up to the number of pendant edges at v_F ($v_{F'}$), isomorphic. ■

In the special case of cubic graphs, we obtain the following consequence.

Corollary 9. *Let G be a cubic graph and let $F \subset G$ be a weakly contractible subgraph of G with $\delta(F) = 2$. Let F' be a graph with $\delta(F') = 2$ and $\Delta(F') \leq 3$, let $A' = \{x \in V(F')|\ d_{F'}(x) = 2\}$ and suppose that F' is weakly A'-contractible. Let $\varphi : A_G(F) \to A'$ be a bijection. Then the graph $H = G[F \xrightarrow{\varphi} F']$ is cubic and G has a DC if and only if H has a DC.*

Proof. Clearly $A_G(F) = \{x \in V(F)|\ d_F(x) = 2\}$ and since G is cubic, we have $d_{G_{-F}}(x) = 1$ for every $x \in A_G(F)$ and $G \not\simeq \overline{F}^{A_G(F)}$. Since φ is a bijection, H is cubic. By Proposition 8, G has a DCT if and only if H has a DCT, but in cubic graphs every DCT is a DC. ■

Now we consider a similar question if F and/or F' are not contractible. We restrict our observations to cubic graphs.

A connected graph F without multiple edges with $\Delta(F) \leq 3$ will be called a *cubic fragment*. For any cubic fragment F and $i = 1, 2$ we set $A_i(F) = \{x \in V(F)|\ d_F(x) = i\}$ and $A(F) = A_1(F) \cup A_2(F)$ (note that if $F \subset H$, F is connected and H is cubic, then F is a cubic fragment and $A_H(F) = A(F)$). A cubic fragment F is said to be *essential* if $|V(F) \setminus A_1(F)| \geq 2$. It is easy to observe that if F is an essential cubic fragment, the set $V(F) \setminus A_1(F)$ induces (in F) a connected subgraph with at least one edge.

For a cubic fragment F we now introduce the concept of an F-linkage. An F-linkage will be allowed to contain loops. A loop on a vertex v is considered as an edge joining v to itself, and is denoted by an element vv of the edge set. Edges of an F-linkage that are not loops will be referred to as *open edges*.

Let F be a cubic fragment and let B be a graph with $V(B) \subset A(F)$, $E(B) \cap E(F) = \emptyset$, and with components $B_1, \ldots, B_k$. We say that B is an *F-linkage*, if $E(B)$ contains at least one open edge and, for any $i = 1, \ldots, k$,

(*i*) every B_i is a path (of length at least one) or a loop,
(*ii*) if B_i is a path of length at least two, then all interior vertices of B_i are in $A_1(F)$,
(*iii*) if B_i is a loop at a vertex x, then $x \in A_2(F)$.

Let F be a cubic fragment and let B be an F-linkage. Then F^B denotes the graph with vertex set $V(F^B) = V(F)$ and edge set $E(F^B) = E(F) \cup E(B)$. Note that $E(B)$ and $E(F)$ are assumed to be disjoint, i.e. if $h_1 = x_1x_2 \in E(F)$ and $h_2 = x_1x_2 \in E(B)$, then h_1, h_2 are parallel edges of the graph F^B.

Let F_1, F_2 be cubic fragments with $|A(F_1)| = |A(F_2)|$ and let $\varphi : A(F_1) \to A(F_2)$ be a bijection. For any F_1-linkage B, $\varphi(B)$ denotes the graph with vertex set $V(\varphi(B)) = \{\varphi(x)|\ x \in V(B)\}$ and edge set $E(\varphi(B)) = \{\varphi(x)\varphi(y)|\ xy \in E(B)\}$ (note that the sets $E(F_2)$ and $E(\varphi(B))$ are again considered to be disjoint, and we admit $x = y$ in which case $\varphi(x)\varphi(x)$ is a loop at $\varphi(x)$). Note that $\varphi(B)$ is an F_2-linkage.

Let F_1, F_2 be cubic fragments with $|A(F_1)| = |A(F_2)|$ and let $\varphi : A(F_1) \to A(F_2)$ be a bijection. We say that φ is a *compatible mapping* if

(i) $\varphi(A_i(F_1)) = A_i(F_2)$, $i = 1, 2$,

(ii) if B is an F_1-linkage such that F_1^B has a DC containing all open edges of B, then $F_2^{\varphi(B)}$ has a DC containing all open edges of $\varphi(B)$.

For a compatible mapping $\varphi : A(F_1) \to A(F_2)$ we will simply write $\varphi : F_1 \to F_2$.

Let F_1, F_2 be cubic fragments and let $\varphi : A(F_1) \to A(F_2)$ be a bijection such that $\varphi(A_i(F_1)) = A_i(F_2)$, $i = 1, 2$. It is easy to observe that if F_2 is weakly $A(F_2)$-contractible then φ is compatible, and if moreover F_1 is weakly $A(F_1)$-contractible then both φ and φ^{-1} are compatible (note that B cannot contain a path of length at least 2 in this case – this is clear for $|A(F_i)| \leq 2$, and for $|A(F_i)| \geq 3$ this follows from the fact that weak $A(F_i)$-contractibility of F_i then implies $A(F_i) = A_2(F_i)$).

The following example shows that the compatibility of a mapping φ does not imply φ^{-1} is compatible if the F_i's are not weakly contractible.

Example. Let F_1, F_2 be the graphs in Figure 3 and let $\varphi : A(F_1) \to A(F_2)$ be the mapping that maps a_j^1 on a_j^2, $j = 1, 2, 3, 4$. By a straightforward check of all possible

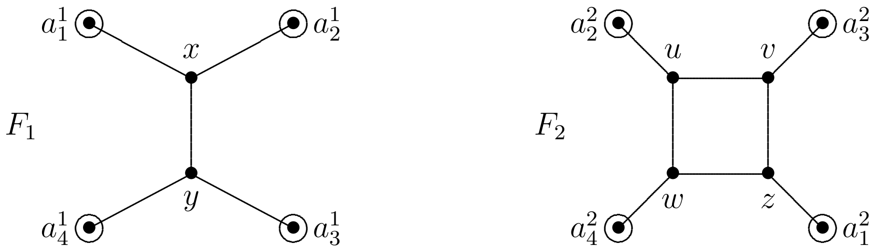

Figure 3

F_1-linkages B and the corresponding DC's in F_1^B and in $F_2^{\varphi(B)}$, we easily see that there are, up to symmetry, the following possibilities.

$E(B)$	DC in F_1^B	DC in $F_2^{\varphi(B)}$
$a_1^1a_4^1$	$a_1^1a_4^1yxa_1^1$	$a_1^2a_4^2wuvza_1^2$
$a_1^1a_2^1$	not existing	not existing
$a_1^1a_2^1, a_2^1a_4^1$	$a_1^1a_2^1a_4^1yxa_1^1$	$a_1^2a_2^2a_4^2wuvza_1^2$
$a_1^1a_3^1, a_3^1a_2^1$	not existing	$a_1^2a_3^2a_2^2uwza_1^2$
$a_1^1a_2^1, a_2^1a_3^1, a_3^1a_4^1$	$a_1^1a_2^1a_3^1a_4^1yxa_1^1$	$a_1^2a_2^2a_3^2a_4^2wuvza_1^2$
$a_1^1a_4^1, a_4^1a_3^1, a_3^1a_2^1$	$a_1^1a_4^1a_3^1a_2^1xa_1^1$	$a_1^2a_4^2a_3^2a_2^2uwza_1^2$
$a_1^1a_4^1, a_2^1a_3^1$	$a_1^1a_4^1ya_3^1a_2^1xa_1^1$	$a_1^2a_4^2wua_2^2a_3^2vza_1^2$
$a_1^1a_2^1, a_3^1a_4^1$	not existing	$a_1^2a_2^2uva_3^2a_4^2wza_1^2$

We conclude that $\varphi : A(F_1) \to A(F_2)$ is a compatible mapping, but there is no compatible mapping of $A(F_2)$ onto $A(F_1)$. Note that this mapping φ will play an important role in the proof of our main result in Section 4.

The following result shows that the replacement of a subgraph of a cubic graph modulo a compatible mapping does not affect the existence of a DC.

Theorem 10. *Let G be a cubic graph and let C be a DC in G. Let $F \subset G$ be an essential cubic fragment such that $G - F$ is not edgeless, and let F' be a cubic fragment such that $V(F') \cap V(G) = \emptyset$ and there is a compatible mapping $\varphi : F \to F'$. Then the graph $G' = G[F \xrightarrow{\varphi} F']$ is a cubic graph having a DC C' such that $E(C) \setminus E(F) = E(C') \setminus E(F')$.*

(Note that if both φ and φ^{-1} are compatible and both F and F' are essential, then G has a DC if and only if $G' = G[F \xrightarrow{\varphi} F']$ has a DC.)

Proof. By the compatibility of φ, $A_1(F') = \varphi(A_1(F))$ and $A_2(F') = \varphi(A_2(F))$, hence G' is cubic. Let C be a DC in G. We show that G' has a DC C' with $E(C) \setminus E(F) = E(C') \setminus E(F')$.

We first observe that $E(C) \cap E(F) \neq \emptyset$. Since F is essential, there is an edge $xy \in E(F)$ with $d_F(x) \geq 2$ and $d_F(y) \geq 2$. Then one of x, y (say, x) is on C. Since $d_F(x) \geq 2$, x has a neighbor x_1 in F, $x_1 \neq y$. Then, since $d_G(x) = 3$, the edge xy or xx_1 is in $E(C) \cap E(F)$.

Let C_F and C_{-F} denote the subgraph of C induced by the edge set $E(C) \cap E(F)$ and $E(C) \cap E(G_{-F})$, respectively. Since $E(C) \cap E(F) \neq \emptyset$ and $G - F$ is not edgeless, C_{-F} is a nonempty system of paths. Let $P_1, \dots, P_k$ be the components of C_{-F}. Then:

- the endvertices of every P_i are in $A(F)$,
- the interior vertices of every P_i are in $A_1(F)$ or in $V(G) \setminus V(F)$,

$i = 1, \dots, k$.

We define an F-linkage B as follows:

(i) for every P_i, let P_i^B be the path obtained from P_i by replacing every maximal subpath of P_i with all interior vertices in $V(G) \setminus V(F)$ by a single edge (with both vertices in $A(F)$),

(ii) for every vertex $x \in A(F) \setminus V(C_{-F})$ which is on C_F (note that such a vertex x must be in $A_2(F)$), let e_x be a loop at x,

(iii) B is the graph with components $\{P_i^B | i = 1, \dots, k\} \cup \{e_x | x \in A_2(F) \setminus V(C_{-F}) \cap V(C)\}$.

It is immediate to observe that the graph F^B has a DC C^B containing all open edges of B. By the compatibility of φ, the graph $(F')^{\varphi(B)}$ has a DC C'^B containing all open edges of the graph $\varphi(B)$.

Let $C'_{F'}$ denote the subgraph of C'^B induced by the edge set $E(C'^B) \cap E(F')$. Then $C'_{F'}$ is a system of paths, and the edges in $E(C'_{F'}) \cup E(C_{-F})$ determine a cycle C' in $G' = G[F \xrightarrow{\varphi} F']$ with $E(C) \setminus E(F) = E(C') \setminus E(F')$. Note that, by the construction, $V(C) \cap A(F) \subset V(C') \cap A(F')$ (this is clear for vertices x with $d_{C_{-F}}(x) \geq 1$, and for vertices x with $d_{C_{-F}}(x) = 0$ this follows from the fact that both C^B and C'^B dominate all loops in B and in $\varphi(B)$, respectively).

It remains to show that C' is a DC in G'. Thus, let $xy \in E(G')$.

If $x, y \in V(G') \setminus V(F') = V(G) \setminus V(F)$, then x or y is on C_{-F}, implying x or y is on C' since $C_{-F} \subset C'$.

If $x, y \in V(F') \setminus A(F')$, then x or y is on $C'_{F'}$, implying x or y is on C' since $C'_{F'} \subset C'$.

Up to symmetry, it remains to consider the case $x \in A(F') = \varphi(A(F))$. If $x \in V(C)$, then also $x \in V(C')$ since $V(C) \cap A(F) \subset V(C') \cap A(F')$, as observed above. Hence we can suppose that $x \notin V(C)$, implying $y \in V(C)$. If $y \in A(F')$, then similarly $y \in V(C')$ and we are done, hence $y \notin A(F')$. Then either $y \in V(F') \setminus A(F')$, or $y \in V(G') \setminus V(F')$. But then, in the first case y is on $C'_{F'}$ since C' is dominating in $(F')^{\varphi(B)}$, and in the second case y is on C_{-F} since C is dominating in G. In either case this implies $y \in V(C')$. ■

The following result shows that the existence of a compatible mapping is not affected by a replacement of a subgraph by another one modulo a compatible mapping.

Proposition 11. *Let X, F be essential cubic fragments such that there is a compatible mapping $\psi : X \to F$. Let $F_1 \subset F$ be an essential cubic fragment, and let F_2 be a cubic fragment such that $V(F) \cap V(F_2) = \emptyset$ and there is a compatible mapping $\varphi : F_1 \to F_2$. Let $F' = F[F_1 \stackrel{\varphi}{\to} F_2]$. Then there is a compatible mapping $\psi' : X \to F'$.*

Proof. For any $x \in A(X)$ set

$$\psi'(x) = \begin{cases} \psi(x) & \text{if} \quad x \in \psi^{-1}(A(F) \setminus A(F_1)), \\ \varphi(\psi(x)) & \text{if} \quad x \in \psi^{-1}(A(F) \cap A(F_1)). \end{cases}$$

Then $\psi' : A(X) \to A(F')$ is a bijection, and $\psi' : A_i(X) \to A_i(F')$, $i = 1, 2$, by the compatibility of ψ and φ. Let B be an X-linkage such that X^B has a DC containing all open edges of B. By the compatibility of ψ, the graph $F^{\psi(B)}$ has a DC C containing all open edges of $\psi(B)$. We need to show that $(F')^{\psi'(B)}$ has a DC containing all open edges of $\psi'(B)$. We will construct a cubic graph H such that $F \subset H$, H has a DC that coincides with C on F, and the structure of $H - F$ implies that an application of Theorem 10 to H yields the required DC in $(F')^{\psi'(B)}$.

Let $B_1, \ldots, B_k$ be the components of $\psi(B)$, and choose the notation such that

- $B_1, \ldots, B_p$ ($p \geq 1$) are paths, $V(B_j) = \{x_j^0, \ldots, x_j^{\ell_j}\}$ (i.e. B_j is of length ℓ_j), $j = 1, \ldots, p$;
- if none of $B_1, \ldots, B_k$ is a loop, then $\ell = 0$, otherwise $B_{p+1}, \ldots, B_{p+\ell}$ are loops, $V(B_{p+j}) = \{x_{p+j}\}$, $j = 1, \ldots, \ell$;
- if $A(F) \setminus V(\psi(B)) = \emptyset$, then $f = 0$, otherwise $A(F) \setminus V(\psi(B)) = \{x_{p+\ell+1}, \ldots, x_{p+\ell+f}\}$.

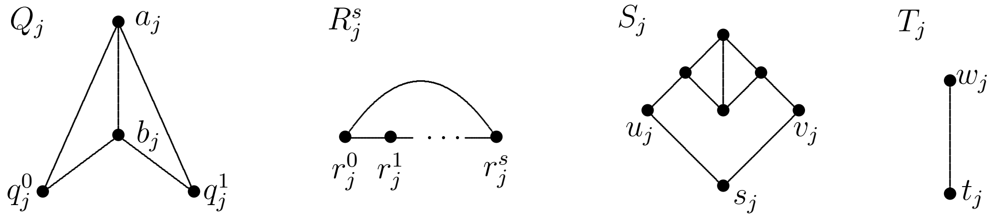

Figure 4

Thus, we have $k = p + \ell$ and $V(\psi(B)) = \cup_{j=1}^{p+\ell}(V(B_j))$.

Let Q_j, R_j^s ($s \geq 2$), S_j and T_j be the graphs shown in Figure 4. We construct a cubic graph H containing F by the following construction:

- take the graph F with the labeling of vertices of $A(F)$ defined above;
- for each B_j with $1 \leq j \leq p$, $\ell_j = 1$, take one copy of Q_j and for $i = 0, 1$ identify $x_j^i = q_j^i$ if $x_j^i \in A_1(F)$ or add the edge $x_j^i q_j^i$ if $x_j^i \in A_2(F)$, respectively,
- for each B_j with $1 \leq j \leq p$, $\ell_j > 1$, take one copy of R_j^s for $s = \ell_j$ and
  - for $i = 0$ and $i = \ell_j$ identify $x_j^i = r_j^i$ if $x_j^i \in A_1(F)$ or add the edge $x_j^i r_j^i$ if $x_j^i \in A_2(F)$, respectively,
  - for $1 \leq i \leq \ell_j - 1$ identify $x_j^i = r_j^i$;
- for each B_j with $p+1 \leq j \leq p+\ell$ (if $\ell > 0$) take one copy of S_j, add the edge $x_j s_j$, and if $\ell \geq 2$, then for $j \geq p+2$ add the edge $v_{j-1}u_j$;
- for each x_j with $p + \ell + 1 \leq j \leq p + \ell + f$ (if $f > 0$) do the following:
  - if $x_j \in A_1(F)$, take one copy of S_j, identify $x_j = s_j$ and if $f \geq 2$, then for $j \geq p + \ell + 2$ add the edge $v_{j-1}u_j$ (if $x_{j-1} \in A_1(F)$), or the edge $w_{j-1}u_j$ (if $x_{j-1} \in A_2(F)$), respectively;
  - if $x_j \in A_2(F)$, take one copy of T_j, identify $x_j = t_j$ and if $f \geq 2$, then for $j \geq p + \ell + 2$ add the edge $v_{j-1}w_j$ (if $x_{j-1} \in A_1(F)$), or the edge $w_{j-1}w_j$ (if $x_{j-1} \in A_2(F)$), respectively;
  - if $x_{p+\ell+1} \in A_2(F)$, then relabel $w_{p+\ell+1}$ as $u_{p+\ell+1}$ and if $x_{p+\ell+f} \in A_2(F)$, then relabel $w_{p+\ell+f}$ as $v_{p+\ell+f}$;
- if $\ell \neq 0$, then
  - for $\ell_1 = 1$ remove the edge $q_1^0 a_1$ and add the edges $q_1^0 u_{p+1}$ and $a_1 v_{p+\ell}$,
  - for $\ell_1 > 1$ remove the edge $r_1^0 r_1^1$ and add the edges $r_1^0 u_{p+1}$ and $r_1^1 v_{p+\ell}$;
- if $f \neq 0$, then
  - for $\ell_1 = 1$ remove the edge $b_1 q_1^1$ and add the edges $b_1 u_{p+\ell+1}$ and $q_1^1 v_{p+\ell+f}$,
  - for $\ell_1 > 1$ remove the edge $r_1^{\ell_1 - 1} r_1^{\ell_1}$ and add the edges $r_1^{\ell_1 - 1} u_{p+\ell+1}$ and $r_1^{\ell_1} v_{p+\ell+f}$.

Then H is a cubic graph, $F \subset H$, $A_H(F) = A(F)$, and it is straightforward to check that H has a DC C^H such that $E(C^H) \cap E(F) = E(C) \cap E(F)$.

Let C^H_{-F} denote the subgraph of C^H induced by the edge set $E(C^H) \cap E(H_{-F})$. Then the structure of the graphs Q_j, R^s_j, S_j and T_j implies the following properties of C^H_{-F}:

- if $1 \le j \le p$ and $i = 0$ or $i = \ell_j$, then $d_{C^H_{-F}}(x^i_j) = 1$,
- if $1 \le j \le p$ and $1 \le i \le \ell_j - 1$, then $d_{C^H_{-F}}(x^i_j) = 2$,
- if $\ell > 0$ and $p + 1 \le j \le p + \ell$, then $d_{C^H_{-F}}(x_j) = 0$ and x_j has no neighbor on C^H_{-F},
- if $f > 0$ and $p + \ell + 1 \le j \le p + \ell + f$, then $d_{C^H_{-F}}(x_j) = 0$ and all neighbors of x_j in H_{-F} are on C^H_{-F}.

Set $H' = H[F_1 \overset{\varphi}{\to} F_2]$. By the compatibility of φ and by Theorem 10, H' has a DC $C^{H'}$ such that $E(C^{H'}) \setminus E(F_2) = E(C^H) \setminus E(F_1)$. Specifically, $F' \subset H'$ and $E(C^{H'}) \setminus E(F') = E(C^H) \setminus E(F)$. Let $C^{H'}_{F'}$ and $C^{H'}_{-F'}$ denote the subgraph of $C^{H'}$ induced by $E(C^{H'}) \cap E(F')$ and $E(C^{H'}) \cap E(H'_{-F'})$, respectively. Then $C^{H'}_{-F'} = C^H_{-F}$ and from the above properties of C^H_{-F} we obtain the following properties of $C^{H'}_{F'}$:

- if $1 \le j \le p$ and $i = 0$ or $i = \ell_j$, then $d_{C^{H'}_{F'}}(x^i_j) = 1$,
- if $1 \le j \le p$ and $1 \le i \le \ell_j - 1$, then $d_{C^{H'}_{F'}}(x^i_j) = 0$ and all edges of F' with at least one vertex in $N_{F'}(x^i_j)$ have at least one vertex on $C^{H'}$,
- if $\ell > 0$ and $p + 1 \le j \le p + \ell$, then $d_{C^{H'}_{F'}}(x_j) = 2$,
- if $f > 0$ and $p + \ell + 1 \le j \le p + \ell + f$, then either $d_{C^{H'}_{F'}}(x_j) = 2$, or $d_{C^{H'}_{F'}}(x_j) = 0$ and all neighbors of x_j in F' are on $C^{H'}_{F'}$.

This implies that $C^{H'}_{F'}$ together with the open edges of $\psi'(B)$ determines the required DC in $(F')^{\psi'(B)}$ containing all open edges of $\psi'(B)$. ■

For a cubic fragment F with $A(F) = A_2(F)$ we will simply write $\overline{F}^{A(F)} = \overline{F}$. If F_1, F_2 are cubic fragments with $A(F_i) = A_2(F_i)$, $i = 1, 2$ and $\varphi : A(F_1) \to A(F_2)$ is a bijection, then $\overline{\varphi}$ denotes the bijection $\overline{\varphi} : A(\overline{F_1}) \to A(\overline{F_2})$ defined by $\overline{\varphi}(\overline{a}) = \overline{\varphi(a)}$, $a \in A(F_1)$.

In the proof of Proposition 14 we will also need the following statement showing that the existence (or nonexistence) of a compatible mapping is not affected by adding pendant edges to vertices of attachment.

Proposition 12. *Let F_1, F_2 be cubic fragments with $|A(F_1)| = |A(F_2)|$ and $A(F_i) = A_2(F_i)$, $i = 1, 2$, and let $\varphi : A(F_1) \to A(F_2)$ be a bijection. Then φ is compatible if and only if $\overline{\varphi} : A(\overline{F_1}) \to A(\overline{F_2})$ is compatible.*

Proof. Set $A(F_1) = \{a_1, \ldots, a_k\}$. Suppose first that φ is compatible and let $\overline{B}$ be an $\overline{F_1}$-linkage such that there is a DC $\overline{C}$ in $(\overline{F_1})^{\overline{B}}$ containing all open edges of $\overline{B}$. Since $A(\overline{F_1}) = A_1(\overline{F_1})$, all components of $\overline{B}$ are paths. We define an F_1-linkage B as follows:

(i) $a_i a_j \in E(B)$, $i \neq j$, if and only if $\overline{B}$ has a component which is an $\overline{a_i}, \overline{a_j}$-path,

(ii) $a_i a_i \in E(B)$ if and only if $\overline{a_i} \in A(\overline{F_1}) \setminus V(\overline{B})$.

(This means that vertices in $A(F)$ corresponding to internal vertices of paths in $\overline{B}$ will not be in $V(B)$, and vertices corresponding to vertices not in $V(\overline{B})$ will have loops in B).

Since $\overline{C}$ dominates all edges of $\overline{F_1}$ (including the edges $a_i\overline{a_i}$ with $\overline{a_i} \notin V(\overline{B})$), it is straightforward to see that removing from $\overline{C}$ the edges of $\overline{B}$ and the pendant edges of $\{a_i\overline{a_i},\ i = 1, \ldots, k\} \cap E(\overline{C})$, and adding the open edges of B results in a DC C in F_1^B, containing all open edges of B. Using the compatibility of φ we obtain a DC in $F_2^{\varphi(B)}$ containing all open edges of $\varphi(B)$, and adding the pendant edges and all edges of $\overline{\varphi}(\overline{B})$ yields a required DC in $(\overline{F_2})^{\overline{\varphi}(\overline{B})}$.

Conversely, let $\overline{\varphi} : A(\overline{F_1}) \to A(\overline{F_2})$ be compatible and let B be an F_1-linkage. Since $A(F_1) = A_2(F_1)$, B contains no paths of length more than one. Suppose the notation is chosen such that $E(B) = \{a_1a_2, \ldots, a_{2p-1}a_{2p}, a_{2p+1}a_{2p+1}, \ldots, a_{2p+\ell}a_{2p+\ell}\}$, where $2p + \ell \leq k$. Then we define $\overline{B}$ as the graph which has as components the path $a_1a_{2p+\ell+1} \ldots a_ka_2$ and (if $p > 1$) the edges $a_{2i-1}a_{2i}$, $i = 2, \ldots, p$. Rest of the proof is similar as above. ■

4 Equivalence of Conjectures A, B, C, D, E, F

Before proving our main result, Theorem 3, we first prove several auxiliary statements that describe the structure of potential counterexamples to Conjecture D.

Proposition 13. *If Conjecture D is not true, then there is an essential cubic fragment F such that*

(i) $|A_2(F)| = |A(F)| = 4$,

(ii) *there is a cyclically 4-edge-connected cubic graph G such that $F \subset G$,*

(iii) *there is no compatible mapping $\varphi : C_4 \to F$.*

Proof. Let G be a counterexample to Conjecture D, i.e. a cyclically 4-edge-connected cubic graph having no DC, let $e = uv \in E(G)$ and set $F = G - \{u, v\}$. Then F is an essential cubic fragment with $|A_2(F)| = |A(F)| = 4$. Let, to the contrary, $\varphi : C_4 \to F$ be a compatible mapping and set $G' = G[F \xrightarrow{\varphi^{-1}} C_4]$. Then G' is isomorphic to one of the graphs in Figure 5, hence G' has a DC. But then, by Theorem 10, the graph $G = G'[C_4 \xrightarrow{\varphi} F]$ has a DC, a contradiction. ■

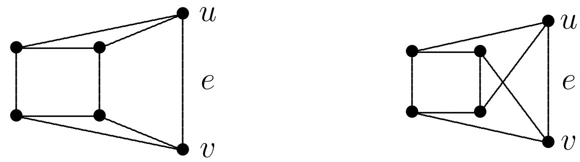

Figure 5

Proposition 14. *Let F be an essential cubic fragment such that*

(i) $|A_2(F)| = |A(F)| = 4$,

(ii) *there is a cyclically 4-edge-connected cubic graph G such that $F \subset G$,*

(iii) *there is no compatible mapping $\varphi : C_4 \to F$,*

(iv) *subject to (i), (ii) and (iii), $|V(F)|$ is minimal.*

Then F is essentially 3-edge-connected and contains no cycle of length 4.

Proof. Recall that a cubic graph is cyclically 4-edge-connected if and only if it is essentially 4-edge-connected (see [5]).

We first show that F is essentially 3-edge-connected. Suppose the contrary. By definition, F is connected. Denote $A(F) = \{a_1, a_2, a_3, a_4\}$, and let f_i denote the edge in $E(G) \setminus E(F)$ incident with a_i, $i = 1, 2, 3, 4$. If F has a cut edge e, then some nontrivial (i.e. containing at least one edge) component of $F - e$ contains at most two vertices a_i, but then e together with the corresponding edges f_i is an essential edge cut in G of size at most 3, a contradiction. Hence F has no cut edge. (Note that F has also no cut vertex since G is cubic.)

Thus, let $R = \{e_1, e_2\} \subset E(F)$ be an essential edge cut of F, and let F_1, F_2 be nontrivial components of $F - R$. Denote $e_i = b_i^1 b_i^2$ with $b_i^j \in V(F_j)$, $i, j = 1, 2$. If $|V(F_1) \cap A(F)| = 1$, then we set $V(F_1) \cap A(F) = \{x\}$ and observe that the edges e_1, e_2 and the only edge of G_{-F} incident to x form an essential edge cut of G of size 3, a contradiction. We obtain a similar contradiction for $|V(F_1) \cap A(F)| = 0$, hence $|V(F_1) \cap A(F)| \geq 2$. Symmetrically, $|V(F_2) \cap A(F)| \geq 2$, implying $|V(F_1) \cap A(F)| = |V(F_2) \cap A(F)| = 2$. Thus, we can suppose the notation is chosen such that $a_1, a_2 \in V(F_1)$ and $a_3, a_4 \in V(F_2)$.

If $|V(F_1)| > 4$, then there is a compatible mapping $\varphi : C_4 \to F_1$ by the minimality of F. Let $\widetilde{C}$ be a copy of C_4 and set $H = F[F_1 \xrightarrow{\varphi^{-1}} \widetilde{C}]$. Then $|V(H)| < |V(F)|$ and, by the minimality of F, there is a compatible mapping $\psi : C_4 \to H$. By Proposition 11 (with $X := C_4$, $F := H$, $F_1 := \widetilde{C}$ and $F_2 := F_1$), there is a compatible mapping $\psi' : C_4 \to H[\widetilde{C} \xrightarrow{\varphi} F_1] = F$, a contradiction. Hence $|V(F_1)| \leq 4$ and, symmetrically, $|V(F_2)| \leq 4$.

Now, since G is cyclically 4-edge-connected, either $\{a_1, a_2\} \cap \{b_1^1, b_2^1\} = \emptyset$, or (up to symmetry), $a_1 = b_1^1$ and $a_2 = b_2^1$. Hence F_1 is a single edge or a cycle of length 4. Similarly, F_2 is a single edge or a cycle of length 4. Thus, F is isomorphic to one of the graphs shown in Figure 6. However, it is straightforward to check that for each of these graphs there is a compatible mapping $\varphi : C_4 \to F$, a contradiction. Thus, F is essentially 3-edge-connected.

Next we show that

($*$) F contains no subgraph $\widetilde{F}$, $\widetilde{F} \neq F$, with $|V(\widetilde{F})| > 4$ and $|A_2(\widetilde{F})| = |A(\widetilde{F})| = 4$.

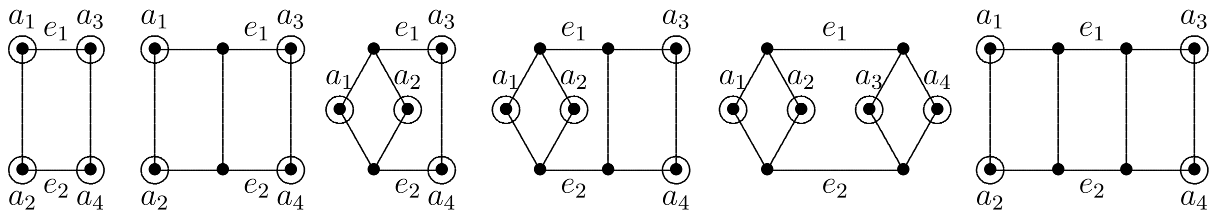

Figure 6

Thus, let $\widetilde{F}$ be such a subgraph. By the minimality of F, there is a compatible mapping $\varphi : C_4 \to \widetilde{F}$. Let $\widetilde{C}$ be a copy of C_4 and set $H = F[\widetilde{F} \xrightarrow{\varphi^{-1}} \widetilde{C}]$. By the minimality of F, there is a compatible mapping $\psi : C_4 \to H$. By Proposition 11 (with $X := C_4$, $F := H$, $F_1 := \widetilde{C}$ and $F_2 := \widetilde{F}$), there is a compatible mapping $\psi' : C_4 \to H[\widetilde{C} \xrightarrow{\varphi} \widetilde{F}] = F$, a contradiction. Hence there is no such $\widetilde{F}$.

Finally, we show that F contains no cycle of length 4. Let, to the contrary, $Y \subset F$ be a copy of C_4 (note that possibly $V(Y) \cap A(F) \neq \emptyset$). Let $\overline{F}$ be the graph obtained from F by attaching a pendant edge to each vertex in $A(F)$, and let F_1 and F_2 be the graphs shown in Figure 3 (recall that we already know there is a compatible mapping $\varphi : F_1 \to F_2$). Let $\overline{Y}$ be the (only) subgraph of $\overline{F}$ such that $Y \subset \overline{Y}$ and $\overline{Y}$ is isomorphic to F_2, let T be a copy of F_1 and let $\varphi : T \to \overline{Y}$ be a compatible mapping. Set $\overline{F}' = \overline{F}[\overline{Y} \xrightarrow{\varphi^{-1}} T]$ (i.e., $\overline{F} = \overline{F}'[T \xrightarrow{\varphi} \overline{Y}]$), and let F' be the graph obtained from $\overline{F}'$ by removing the 4 pendant edges. Then F' is a cubic fragment with $|A(F')| = |A_2(F')| = 4$.

We show that there is no compatible mapping $\psi : C_4 \to F'$. Let, to the contrary, $\psi : C_4 \to F'$ be compatible. By adding pendant edges to $A(C_4)$ and $A(F')$ and by Proposition 12, there is a compatible mapping $\overline{\psi} : \overline{C_4} \to \overline{F}'$. Thus, we have $\overline{\psi} : \overline{C_4} \to \overline{F}'$, $T \subset \overline{F}'$ and $\varphi : T \to \overline{Y}$. By Proposition 11, there is a compatible mapping $\overline{\psi}' : \overline{C_4} \to \overline{F}$. By removing the pendant edges and by Proposition 12 we obtain a compatible mapping $\psi' : C_4 \to F$, a contradiction. Thus, there is no compatible mapping $\psi : C_4 \to F'$.

By the minimality of F, the graph F' (and hence also $\overline{F}'$) cannot be a subgraph of a cyclically 4-edge-connected cubic graph. Thus, there is an edge cut R' of $\overline{F}'$ such that $|R'| \leq 3$ and at least one component X' of $\overline{F}' - R'$ contains a cycle and has minimum degree 2 (if such an R' does not exist then, identifying the vertices of degree 1 of $\overline{F}'$ with vertices of a C_4, we get a cyclically 4-edge-connected cubic graph containing $\overline{F}'$, a contradiction). However, there is no such edge cut in $\overline{F}$. Since $\overline{F}' = \overline{F}[\overline{Y} \xrightarrow{\varphi^{-1}} T]$, R' contains the edge $e = xy \in E(T)$ with $d_T(x) = d_T(y) = 3$ and some two edges f_1, $f_2 \in E(\overline{F}') \backslash E(T)$. Suppose the vertices of T are labeled such that $A_1(T) = \{a_1, a_2, a_3, a_4\}$, $E(T) = \{a_1x, a_2x, a_3y, a_4y, xy\}$ and $a_1, a_2, x \in V(X')$. Then $R'' = \{f_1, f_2, a_3y, a_4y\}$ is an edge cut in $\overline{F}'$ such that $|R''| = 4$ and $X' + e$ is a component of $\overline{F}' - R''$. Let e_1 (e_2, e_3, e_4) denote the pendant edge of $\overline{Y}$ which corresponds to the edge a_1x (a_2x, a_3y, a_4y) $\in E(T)$, respectively, in the mapping φ. Then $R = \{f_1, f_2, e_3, e_4\}$ is an edge cut of $\overline{F}$ such that the component X of $\overline{F} - R$ containing X' and Y has $|V(X)| > 4$ and $|A_2(X)| = |A(X)| = 4$.

By ($*$) (and since $F \not\simeq C_4$, implying $e_1, e_2 \in E(F)$), F contains no such graph as a proper subgraph, hence $X = F$. But then $\{e_1, e_2\}$ is an edge cut of F, contradicting the fact that F is essentially 3-edge-connected. Hence F contains no cycle of length 4. ■

Proposition 15. *If Conjecture D is not true, then there is an essential cubic fragment F such that*

(*i*) *F contains no cycle of length 4,*

(*ii*) *there is a cyclically 4-edge-connected cubic graph G such that $F \subset G$,*

(*iii*) *$|A_2(F)| = |A(F)| = 4$ and $A(F)$ is independent,*

(*iv*) *there is a compatible mapping $\varphi : F \to C_4$.*

Proof. By Propositions 13 and 14, there is an essential cubic fragment H such that H contains no cycle of length 4, $|A_2(H)| = |A(H)| = 4$, there is a cyclically 4-edge-connected cubic graph G such that $H \subset G$, and there is no compatible mapping $\psi : C_4 \to H$. Let H be minimal with these properties. Since $A(H) = A_2(H)$, by the nonexistence of a compatible mapping $\psi : C_4 \to H$, H is not weakly $A(H)$-contractible. Hence there is a nonempty even set $X \subset A(H)$ and a partition $\mathcal{A}$ of X into two-element subsets such that $H^{\mathcal{A}}$ has no DCT containing all vertices of $A(H)$ and all edges of $E(\mathcal{A})$. Set $A(H) = \{a_1, a_2, a_3, a_4\}$ and suppose the notation is chosen such that $\mathcal{A} = \{\{a_1, a_2\}\}$ if $|X| = 2$ or $\mathcal{A} = \{\{a_1, a_2\}, \{a_3, a_4\}\}$ if $|X| = 4$. Then the graph H^B has no DC containing all open edges of B for either $E(B) = \{a_1a_2, a_3a_3, a_4a_4\}$ or $E(B) = \{a_1a_2, a_3a_4\}$.

Let H, H' be two copies of H (with a corresponding labeling $A(H') = \{a'_1, a'_2, a'_3, a'_4\}$), and let F be the cubic fragment obtained from H and H' by adding the edges $a_1a'_1$ and $a_2a'_2$. Recall that H contains no cycle of length 4. Since H is essentially 3-edge-connected by Proposition 14, the set $\{a_1, a_2, a_3, a_4\}$ (and hence also $\{a'_1, a'_2, a'_3, a'_4\}$) is independent. Hence F also contains no cycle of length 4, and the set $A(F) = \{a_3, a_4, a'_3, a'_4\}$ is independent. It remains to prove that there is a compatible mapping $\varphi : F \to C_4$.

First we show that the graph F^B has no DC containing all open edges of B for $E(B) = \{a_3a_3, a_4a_4, a'_3a'_4\}$. To the contrary, let C be such a DC. Then $(E(C) \cap E(H)) \cup \{a_1a_2\}$ is a DC in H^B containing all open edges of B for $E(B) = \{a_1a_2, a_3a_3, a_4a_4\}$, and $(E(C) \cap E(H')) \cup \{a'_1a'_2, a'_3a'_4\}$ is a DC in $H'^{B'}$ containing all open edges of B' for $E(B') = \{a'_1a'_2, a'_3a'_4\}$, which is not possible. Thus, there is no such DC in F^B. Symmetrically, $F^{B'}$ has no DC containing all open edges of B' for $E(B') = \{a'_3a'_3, a'_4a'_4, a_3a_4\}$. Let Y be a copy of C_4 with vertices labeled b_3, b_4, b'_3, b'_4 such that $b_3b_4 \notin E(Y)$ and $b'_3b'_4 \notin E(Y)$. Then it is straightforward to check that $Y^{B''}$ has a DC containing all open edges of B'' for all Y-linkages B'' except for the cases $E(B'') = \{b_3b_3, b_4b_4, b'_3b'_4\}$ and $E(B'') = \{b'_3b'_3, b'_4b'_4, b_3b_4\}$. Hence the mapping $\varphi : A(F) \to A(Y)$ that maps a_i on b_i and a'_i on b'_i, $i = 3, 4$, is a compatible mapping. ■

Note that we do not know any example of a cubic fragment with the properties given in Proposition 15. Moreover, we believe that such a graph in fact does not exist.

Now we are ready to prove the main result of this paper, Theorem 3.

Proof of Theorem 3. Clearly, Conjecture E implies Conjecture F. By Theorem 2, it is sufficient to show that Conjecture F implies Conjecture D. Thus, suppose Conjecture D is not true, and let F be an essential cubic fragment as given by Proposition 15. Let G be a counterexample to Conjecture D, i.e. a cyclically 4-edge-connected cubic graph without a DC. For any cycle C of length 4 in G, choose a compatible mapping of F on C, and let G' be the graph obtained by recursively replacing every cycle of length 4 by a copy of F. Then G' is a cubic graph of girth $g(G') \geq 5$ and, by Theorem 10, G' has no DC. Moreover, G' is cyclically 4-edge-connected since any cycle-separating edge cut in G' of size at most three would imply the existence of such an edge cut in G. If G' is not 3-edge-colorable, G' is a snark and we are done. Otherwise, we use the following fact and construction by Kochol [7].

Claim [7]. *If a cubic graph G contains the graph H of Figure 7 as an induced subgraph, then G is not 3-edge-colorable.*

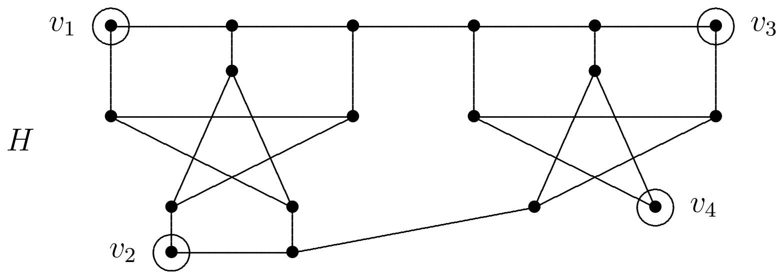

Figure 7

We use the claim as follows. Let $xy \in E(G')$, let x', x'' (y', y'') be the neighbors of x (of y) different from y (x), respectively, and let G'_i, $i = 1, 2, 3$, be three copies of the graph $G' - x - y$ (where x'_i, x''_i, y'_i, y''_i are the copies of x', x'', y', y'' in G'_i), $i = 1, 2, 3$. Then the graph $\bar{G}$ obtained from G'_1, G'_2, G'_3 and H by adding the edges x'_1v_3, x''_1v_4, $y'_1x'_2$, $y''_1x''_2$, $y'_2x'_3$, $y''_2x''_3$, y'_3v_1 and y''_3v_2 is a cyclically 4-edge-connected graph of girth $g(\bar{G}) \geq 5$. By the claim, $\bar{G}$ is not 3-edge-colorable. It remains to show that $\bar{G}$ has no DC.

Let, to the contrary, C be a DC in $\bar{G}$. Then it is easy to check that for some $i \in \{1, 2, 3\}$, the intersection of C with G'_i is either a path with one end in $\{x'_i, x''_i\}$ and the second in $\{y'_i, y''_i\}$, or two such paths. But, in both cases, the path(s) can be easily extended to a DC in G', a contradiction. ■

5 Concluding remarks

1. Note that our proof of the equivalence of Conjecture F with Conjectures A – E is based on properties (compatible mappings) that are specific for the C_4. This means that our proof cannot be directly extended to obtain higher girth restrictions.

2. We pose the following conjecture and show it is equivalent with Conjectures A – F.

Conjecture G. *Every cyclically 4-edge-connected cubic graph contains a weakly contractible subgraph F with $\delta(F) = 2$.*

Theorem 16. *Conjecture G is equivalent with Conjectures A, B, C, D, E and F.*

Proof. We first show that Conjecture G implies Conjecture D. Suppose Conjecture G is true and let G be a minimum counterexample to Conjecture D. Hence G has no DC. Let $F \subset G$ be a weakly contractible subgraph of G with $\delta(F) = 2$ and set $A = A_G(F)$. Note that $A \neq \emptyset$ since $\delta(F) = 2$. By Corollary 7, the graph $G|_F$ has no DCT. If $|A| \leq 3$, then every edge in G_{-F} has at least one vertex in A since G is essentially 4-edge-connected. But then $G|_F$ has a (trivial) DCT, a contradiction. Hence $|A| \geq 4$.

We use the following operation (see [5]). Let H be a graph, let $v \in V(H)$ be of degree $d = d_H(v) \geq 4$, and let $x_1, \ldots, x_d$ be an ordering of the neighbors of v (allowing repetition in case of multiple edges). Let H' be the graph obtained by adding edges $x_i y_i$, $i = 1, \ldots, d$, to the disjoint union of the graph $H - v$ and the cycle $y_1 y_2 \ldots y_d y_1$. Then H' is said to be an *inflation of H at v*. The following fact was proved in [5].

Claim [5]. *Let H be an essentially 4-edge-connected graph of minimum degree $\delta(G) \geq 3$ and let $v \in V(H)$ be of degree $d(v) \geq 4$. Then some inflation of H at v is essentially 4-edge-connected.*

Now let G' be an essentially 4-edge-connected inflation at v_F of the graph obtained from $G|_F$ by deleting its pendant edges. Then G' is a cubic graph having no DC (since otherwise $G|_F$ would have a DCT). Since no cycle of length $\ell \geq 4$ is weakly contractible, F is not a cycle, and since $\delta(F) = 2$, we have $|A_G(F)| < |E(F)|$. But then $|E(G')| < |E(G)|$, contradicting the minimality of G.

For the rest of the proof, it is sufficient to show that Conjecture D implies Conjecture G. Indeed, if C is a dominating cycle in G, $e = uv \in E(C)$ and $A = \{u, v\}$, then the graph F with $V(F) = V(G)$ and $E(F) = E(G) \setminus \{e\}$ is a weakly A-contractible subgraph of G. ■

It should be noted here that the last part of the proof of Theorem 16 is based on a construction with $|A| = 2$, which forces $G - F$ be empty (G_{-F} is a one edge graph) since G is cubic and cyclically 4-edge-connected. It is straightforward to observe that the following stronger statement implies Conjectures A – G. However, we do not know whether these statements are equivalent.

Conjecture H. *Every cyclically 4-edge-connected cubic graph G contains a weakly contractible subgraph F with $|A_G(F)| \geq 4$.*

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