Forbidden Subgraphs that Imply 2-Factors

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Abstract

The connected forbidden subgraphs and pairs of connected forbidden subgraphs that imply a 2-connected graph is hamiltonian have been characterized by Bedrossian [Be91], and extensions of these excluding graphs for general graphs of order at least 10 were proved by Faudree and Gould [FG97]. In this paper a complete characterization of connected forbidden subgraphs and pairs of connected forbidden subgraphs that imply a 2-connected graph of order at least 10 has a 2-factor will be proved. In particular it will be shown that the characterization for 2-factors is very similar to that for hamiltonian cycles, except there are seven additional pairs. In the case of graphs of all possible orders, there are four additional forbidden pairs not in the hamiltonian characterization, but a claw is part of each pair.

1 Introduction

We will deal only with finite graphs without loops or multiple edges. Notation will be standard, and we will generally follow the notation of Chartrand and Lesniak in [CL96]. The degree of a vertex v in a graph G will be denoted by $d(v)$, and the minimum and maximum degree of vertices in G will be denoted by $\delta(G)$ and $\Delta(G)$ respectively. The independence number of G will be denoted by $\alpha(G)$, the connectivity by $\kappa(G)$, and the clique number by $\omega(G)$.

Given a graph F, a graph G is said to be *F-free* if there is no induced subgraph of G that is isomorphic to F. The graph F is generally called a *forbidden subgraph* of G. In the case of forbidden pairs of graphs, say F and H, we will simply say the graph is FH-free, as opposed to $\{F, H\}$-free. Forbidden singletons and forbidden pairs of connected graphs that imply that a 2-connected graph is hamiltonian have been characterized. Also, similar characterizations have been given for other hamiltonian properties such as traceable, pancyclic, cycle extendable, etc. A collection of forbidden graphs used in results of this type are pictured in Figure 1. The graph obtained from a triangle by attaching disjoint paths of length i, j, and k respectively to the 3 vertices of the triangle will be denoted by $N(i, j, k)$. These graphs are generalized nets, and in particular, $Z_i = N(i, 0, 0)$, $B = N(1, 1, 0)$, and $N = N(1, 1, 1)$. If $i, j \geq 0$, then the graphs $N(i, j, 0)$ are the generalized bulls and will be denoted by just $B(i, j)$.

The following result, which extends the results of Bedrossian in [Be91], gives all forbidden singletons and forbidden pairs that imply hamiltonicity in 2-connected graphs of order at least 10. A survey of results of this kind for other hamiltonian type properties can be found in [F96], and a more general survey on claw-free graphs can be found in [FFR97].

Theorem 1 (Faudree, Gould [FG97]) *The only connected forbidden subgraph F that implies a 2-connected graph G is hamiltonian is P_3. Let X and Y be connected graphs with $X, Y \not\leq P_3$, and let G be a 2-connected graph of order $n \geq 10$. Then, G being XY-free implies that G is hamiltonian if, and only if, up to the order of the pairs, $X = C$ and Y is a subgraph of either $P_6, N, W,$ or Z_3.*

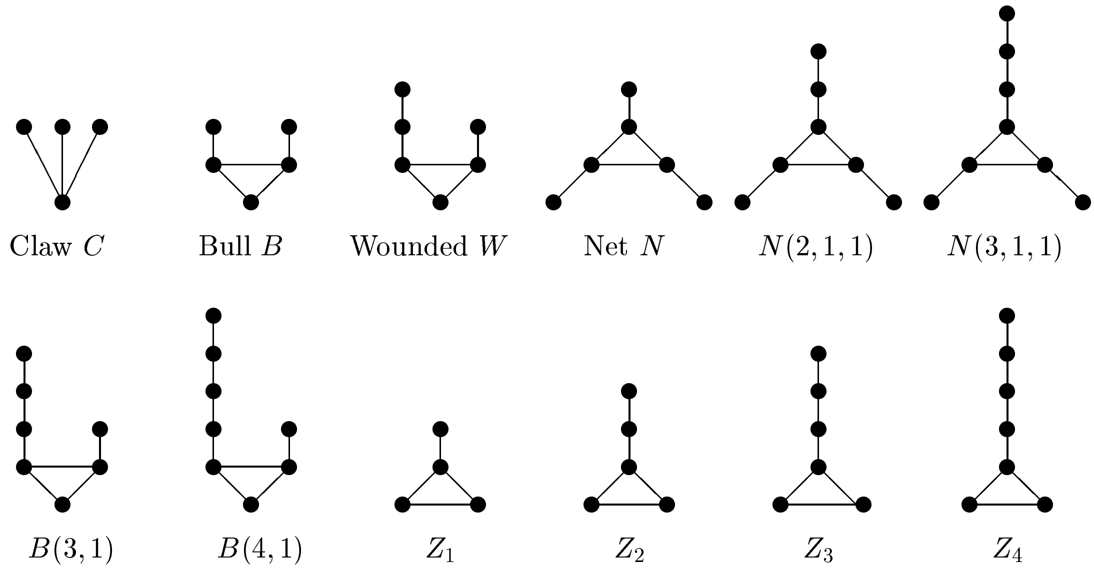


Figure 1

The characterization for 2-factors corresponding to Theorem 1 for hamiltonian cycles is given by the following result, which is the main result of this paper.

Theorem 2 *The only connected forbidden subgraph F that implies a 2-connected graph G has a 2-factor is P_3. Let X and Y be connected graphs with $X, Y \not\leq P_3$, and let G be a 2-connected graph of order $n \geq 10$. Then, G being XY-free implies that G has a 2-factor if and only if, up to the order of the pairs, $X = C$ and Y is a subgraph of either $P_7, Z_4, B(4,1)$, or $N(3,1,1)$, or $X = K_{1,4}$ and $Y = P_4$.*

There are 7 additional pairs of forbidden subgraphs in the characterization for 2-factors not present for hamiltonian cycles; those involving the claw, namely CP_7, CZ_4, $CB(3,1)$, $CB(4,1)$, $CN(2,1,1)$, and $CN(3,1,1)$, as well as the pair $K_{1,4}P_4$. Of course, the graphs P_7 and Z_4 are subgraphs of $B(4,1)$, so there are only two new maximal forbidden subgraphs in Theorem 2, namely $N(3,1,1)$ and $B(4,1)$. Three of these forbidden pairs, namely $CN(3,1,1), CB(4,1)$, and $K_{1,4}P_4$, do not imply the existence of a 2-factor for all graphs, in particular for graphs of order 9 or less. Hence, there are only four possible additional pairs of forbidden graphs implying the existence of a 2-factor when applied to all graphs. Note that the 2-connected condition is necessary. Neither a path P_n nor the graph G_n obtained from a complete graph by attaching an edge has a 2-factor since some vertices of the graph are not on cycles. However, all of the forbidden pair conditions in Theorem 2 are satisfied by either P_n or G_n.

As a consequence of the proof of Theorem 2, there also results a complete characterization of all connected forbidden graphs and connected forbidden pairs of graphs that imply the existence of a 2-factors for all 2-connected graphs, not just 2-connected graphs of order at least 10.

Corollary 1 *A connected forbidden subgraph F implies a 2-connected graph G has a 2-factor if and only if $F = P_3$. Let X and Y be connected graphs with $X, Y \not\leq P_3$, and let G be a 2-connected graph of order n. Then, G being XY-free implies that G has a 2-factor if, and only if, up to the order of the pairs, $X = C$ and Y is a subgraph of either $P_7, Z_4, B(3,1)$ or $N(2,1,1)$.*

Theorem 2 will be proved in the next section.

2 PROOFS

The proof of Theorem 2 will be broken into several results. We begin by proving that the conditions of Theorem 2 are necessary for forbidden subgraphs to imply a 2-factor in a 2-connected graph.

Proof: First note that none of the graphs $G_1, G_2, G_3, G_4, G_5, G_6$, and G_7 in Figure 2 have a 2-factor, and that neither G_5 nor G_6 have an induced claw. Any collection of forbidden subgraphs that imply the existence of a 2-factor must have at least one of the subgraphs in the collection as an induced subgraph of each G_i, $(1 \leq i \leq 7)$.

Let H be a connected graph such that G being H-free implies that G has a 2-factor. Thus, H must be a subgraph of each of the graphs G_1, G_3, and G_5 in Figure 2. However, since G_5 has no induced claw C, a path is the only graph common to G_1 and G_5. However, the longest induced path in G_3 is P_3. Hence, H must be a subgraph of P_3, and so $H = P_3$.

Let X and Y be a pair of connected graphs (X and $Y \neq P_3$) such that G being XY-free implies that G has a 2-factor. We will first show that either X or Y must be C or $K_{1,4}$. Assume that this is not true. With no loss of generality we can assume that X is a subgraph of G_1. This implies that either $X = C$, or X contains an induced path P_4. Since both G_2 and G_3 do not have induced P_4's, the graph Y must be an induced subgraph of both G_2 and G_3. Being a subgraph of G_3 implies that Y must be a complete bipartite graph, but the only complete bipartite subgraph of G_2 is a star. Thus, Y is a subgraph of $K_{1,4}$. This verifies the claim, so we can assume that $X = C$ or $K_{1,4}$.

If $X = K_{1,4}$, then Y must be an induced subgraph of G_1, G_5, and G_7, since none of these graphs contains a $K_{1,4}$. However, the only induced graph common to these graphs is a path, and the longest induced path is a P_4. Thus, if $X = K_{1,4}$, then $Y = P_4$. If $X = C$, then Y must be an induced subgraph of G_5, and so Y is either a path or a $N(i,j,k)$ for appropriate i, j, and k. Also, Y must be a subgraph of G_6, since it is claw-free. It is straightforward to check that the longest induced path in G_6 is a P_7, the largest induced Z_i is a Z_4, the maximum generalized bull is a $N(4,1,0)$, and the maximum generalized net is a $N(3,1,1)$. This completes the proof. $\square$

We will next show the forbidden pairs in Theorem 2 imply the existence of a 2-factor in a 2-connected graph. We will start with the pair $K_{1,4}P_4$. Note that for $n \geq 9$ the 2-connected graph $K_2 + (K_2 \cup K_2 \cup K_{n-6})$ has no hamiltonian cycle, but it does have a 2-factor. Also, observe that the graph $K_2 + (3K_2)$ is 2-connected, does not have a 2-factor, and it is a $K_{1,4}P_4$-free graph. Also, this graph has subgraphs of order $5, 6$, and 7 with the same properties. However, the following theorem implies that if G is a 2-connected $K_{1,4}P_4$-free graph of order $n \geq 9$, then G does have a 2-factor.

Theorem 3 *If G is a 2-connected $K_{1,4}P_4$-free graph of order $n \geq 9$, then G has a 2-factor. In fact G has a 2-factor with at most 2 cycles.*

Proof: Select a minimum cutset of G, say A. If $|A| \geq n/2$, then $\delta(G) \geq n/2$ and G is

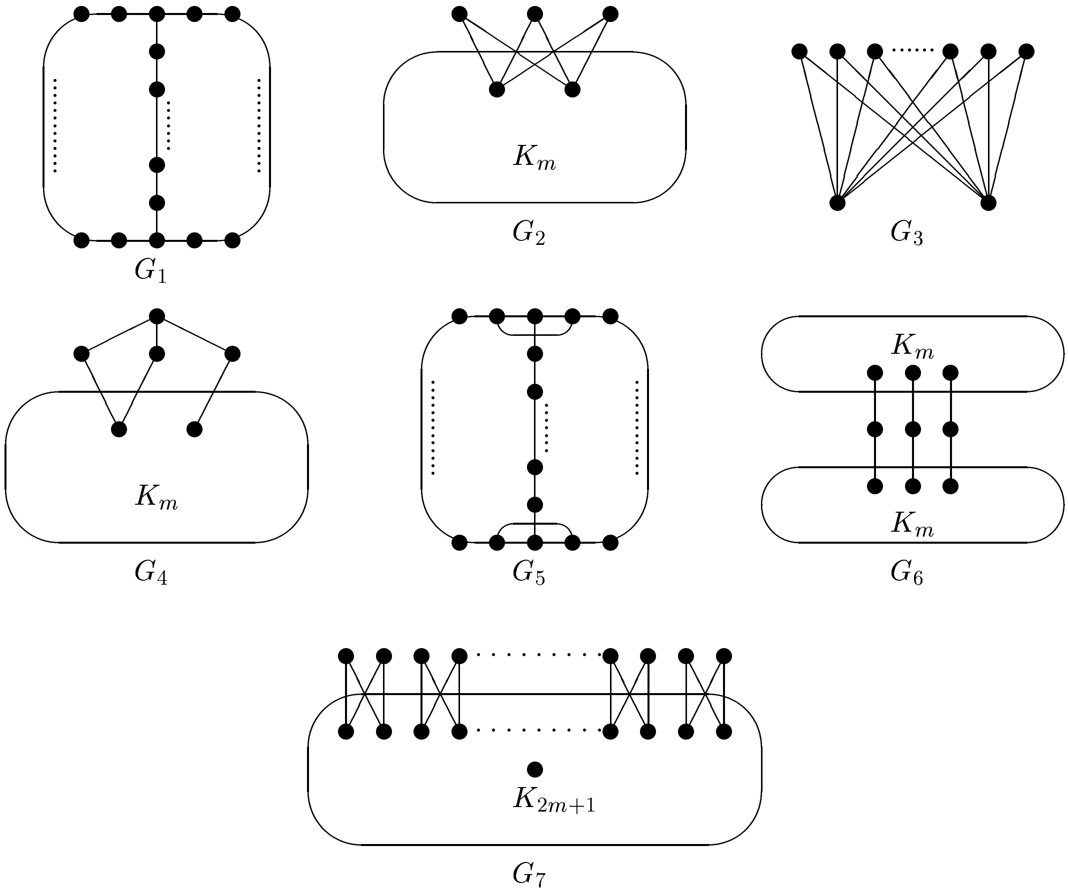


Figure 2

hamiltonian by Dirac's Theorem in [D52]. Thus, we can assume that $\kappa(G) < n/2$. The graph $G - A$ has either 2 or 3 components, since G is $K_{1,4}$-free. Denote these components by $B_1, B_2, \cdots$. The minimality of the cutset implies that each vertex of A has an adjacency in each B_i. In fact, each vertex $a \in A$ is adjacent to all of the vertices in each of the B_i, for otherwise there would be an induced P_4 containing 2 vertices from the B_i, the vertex a, and a vertex from a B_j for $j \neq i$. Since G is $K_{1,4}$-free, the independence number $\alpha(A) \leq 3$. Thus, the vertices of A (or in fact any subset of vertices of A) can always be partitioned into at most 3 paths, since the initial vertices of the paths in any such path system with a minimum number of paths are independent.

First consider the case when there are 3 components of $G - A$. Each of the components B_1, B_2 and B_3 is complete, since G is $K_{1,4}$-free. Assume $|B_1| \leq |B_2| \leq |B_3|$. By assumption we have that $|G - A| \geq \lceil n/2 \rceil \geq 5$. If $|B_3| \geq 3$, then arbitrarily select 2 vertices, say a_1 and a_2, of A, and let $A' = A - \{a_1, a_2\}$. A cycle C_1 can easily be constructed containing $B_1 \cup B_2 \cup \{a_1, a_2\}$. Since A' can be partitioned into at most 3 paths (possibly none when $A' = \emptyset$), any hamiltonian cycle C' in B_3 can be extended to a cycle C_2 containing A' by replacing (at most 3) edges in C' by paths in A' along with edges between A' and B_3. Hence, we are left with the case $|B_3| = 2$, and so $|G - A| \leq 6$ and $|A| \geq 3$. The vertices of A can be partitioned into three paths A_1, A_2 and A_3. In this case a hamiltonian cycle of G can be constructed using A_1, A_2 and A_3, hamiltonian paths in B_1, B_2 and B_3 and edges between A and $G - A$.

We are left with the case when $G - A$ has two components B_1 and B_2. Since G is $K_{1,4}$-free, $\alpha(B_1), \alpha(B_2) \leq 2$, and so each of B_1 and B_2 are C-free. Since both B_1 and B_2 are CP_4-free, they are traceable (see [FG97]). Thus, we can choose to partition $G - A$ into either 2 or 3 paths to match the number of paths into which A can partitioned. In either case, a hamiltonian cycle can be constructed using either the 4 paths or the 6 paths and edges in the complete bipartite graph between A and $G - A$. This completes the proof of Theorem 3. $\square$

Before considering pairs of graphs, one of which is a claw C, that implies a 2-connected graph has a 2-factor, we need to recall the closure concept for claw-free graphs introduced by Ryjáček in [R97]. Given a claw-free graph G, the *closure* $cl(G)$ is the graph obtained from G by sequentially replacing each connected neighborhood of a vertex of G by a complete graph on the same vertex set. We say a graph G is closed if $G = cl(G)$. The following was proved in [R97], where $c(H)$ denotes the circumference of the graph H.

Theorem 4 (Ryjáček [R97]) *Let G be a claw-free graph. Then*

(i) the closure $cl(G)$ is well-defined,

(ii) there is a triangle-free graph H such that $cl(G) = L(H)$, and

(iii) $c(G) = c(cl(G))$

Then later Ryjáček, Saito, and Schelp in [RSS99] proved the following relationship between 2-factors in a claw-free graph G and its closure $cl(G)$.

Theorem 5 (Ryjáček, Saito, and Schelp [RSS99]) *A claw-free graph G has a 2-factor with at most k components if and only if the closure $cl(G)$ has a 2-factor with at most k components.*

For a given forbidden graph F the property of being CF-free is said to be *stable* if when G is a graph that is CF-free then the closure $cl(G)$ is also CF-free. The following result of Brousek, Scheirmeyer, and Ryjáček in [BSR99] gives some critical pairs of interest that are stable.

Theorem 6 (Brousek, Schiermeyer, and Ryjáček [BSR99]) *For $i, j, k \geq 1$, the properties CP_i-free, CZ_i-free, and $CN(i, j, k)$-free are all stable. However, the property $CB(i, j)$-free is not stable.*

This means that when considering a condition that implies a CP_i-free, CZ_i-free, or $CN(i, j, k)$-free claw-free graph G has a 2-factor, the graph G can be assumed to be closed. Note also that if a C-free graph is closed, then the neighborhood of each vertex is either a complete graph or the disjoint union of 2 complete graphs.

Nearly all 2-connected claw-free graphs of very small order have 2-factors and in most cases are also hamiltonian. This follows from a result of Brousek [B98] on minimal 2-connected claw-free non-hamiltonian graphs. Let $\mathcal{P}$ denote the class of graphs obtained by taking two vertex disjoint triangles, pairing the vertices of the triangles, and joining each pair with vertex disjoint paths of length at least two or a triangle. For example the graph obtained by joining the three pairs by a path with $i \geq 3$ vertices, a path with $j \geq 3$ vertices, and a triangle T will be denoted by $P_{i,j,T} \in \mathcal{P}$. In Figure 3 are examples of $P_{4,3,3} \in \mathcal{P}$ and $P_{4,T,T} \in \mathcal{P}$. Brousek [B98] proved the following result.

Theorem 7 (Brousek [B98]) *A graph G is a minimal 2-connected non-hamiltonian claw-free graph if and only if $G \in \mathcal{P}$.*

Since all of the graphs in $\mathcal{P}$ have at least 9 vertices and those of order 9 are the 4 graphs between L and L^* in Figure 4, this gives the following lemma.

$P_{4,3,3}$ $P_{4,T,T}$

Figure 3

Lemma 1 *Let G be a 2-connected claw-free graph of order n. If $n \leq 8$, then G is hamiltonian. If G has order 9, then G has a 2-factor unless $G = L$ of Figure 4, and G is hamiltonian unless G is one of the 4 graphs between L and the graph L^* in Figure 4.*

L L^*

Figure 4

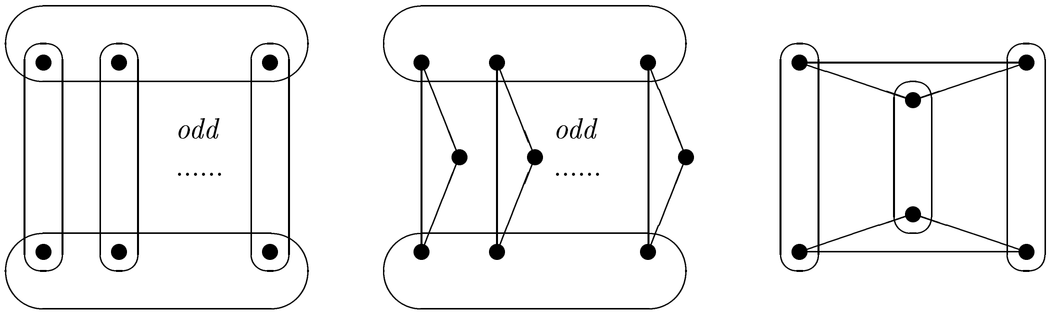
In [BFR99] Brousek, Favaron, and Ryjáček proved a series of theorems using forbidden subgraphs that implied either a graph was hamiltonian or is a member of some special families of graphs. In order to state these results, we picture in Figure 5 three special families of graphs. In each case an oval in Figure 5 represents a complete graph with at least 3 vertices and the remark "odd" indicates that the total number of maximal cliques in that graph is odd. Note that none of the graphs in Figure 5 is hamiltonian, but each has a 2-factor.

Theorem 8 [BFR99] *Let G be a 2-connected graph.*

(i) If G is CP_7-free, then G is either hamiltonian or $cl(G) \in \mathcal{F}_1$.

(ii) If G is CZ_4-free, then G is either hamiltonian or $G \in \{P_{3,T,T}, P_{3,3,T}, P_{3,3,3}, P_{4,T,T}\}$, or $cl(G) \in \mathcal{F}_2$.

(iii) If G is $CN_{1,1,2}$-free, then G is either hamiltonian or $cl(G) \in \mathcal{F}_3$.



$\mathcal{F}_1$ $\mathcal{F}_2$ $\mathcal{F}_3$

Figure 5

With this result we can give easy and straightforward proofs to three results on forbidden subgraph conditions that imply the existence of 2-factors.

Theorem 9 *If G is a 2-connected CP_7-free graph of order $n \geq 3$, then G has a 2-factor.*

Proof: By Theorem 8 (i), either G is hamiltonian or $cl(G) \in \mathcal{F}_1$. In the first case G has a 2-factor and in the second case $cl(G)$ has a 2-factor. However, by Theorem 5, G also has a 2-factor. This completes the proof of Theorem 9. $\square$

Theorem 10 *If G is a 2-connected CZ_4-free graph of order $n \geq 3$, then G has a 2-factor unless G is of order 9 and $G = L$ as in Figure 4.*

Proof: By Theorem 8 (ii), either G is hamiltonian or $G \in \{P_{3,T,T}, P_{3,3,T}, P_{3,3,3}, P_{4,T,T}\}$ or $cl(G) \in \mathcal{F}_2$. In each of the cases, both G and $cl(G)$ have a 2-factor, except for the one graph $L = P_{3,3,3}$. This completes the proof of Theorem 10. $\square$

Theorem 11 *If G is a 2-connected $CN(2,1,1)$-free graph of order $n \geq 3$, then G has a 2-factor.*

Proof: By Theorem 8 (iii), either G is hamiltonian or $cl(G) \in \mathcal{F}_3$. In the first case case G has a 2-factor and in the second case $cl(G)$ has a 2-factor. However, by Theorem 5 G also has a 2-factor. This completes the proof of Theorem 11. □

Next we prove a similar result for $CN(3,1,1)$-free graphs. Note that in the special case of $CN(2,2,1)N(3,1,1)$-free graphs, we can also get a simple proof from the results in [BFR99], but the $CN(3,1,1)$-free case is not handled there.

Theorem 12 *If G is a 2-connected $CN(3,1,1)$-free graph of order $n \geq 3$, then G has a 2-factor unless $n = 9$ and $G = L$.*

Proof: By Lemma 1, the only 2-connected claw-free graph of order $n \leq 9$ that does not have a 2-factor is the $N(3,1,1)$-free graph L. Hence we can suppose that $n \geq 10$.

Let G be a 2-connected $CN(3,1,1)$-free graph of order $n \geq 10$ having no 2-factor. By Theorems 5 and 6, we can suppose G is closed. Thus, by Theorem 4, there is a triangle-free graph H such that $G = L(H)$ (we will also write $H = L^{-1}(G)$). Since G is $CN(3,1,1)$-free, H contains no subgraph (not necessarily induced) that is isomorphic to the graph $L^{-1}(N(3,1,1))$ (see Figure 6).

$L^{-1}(N(3,1,1))$

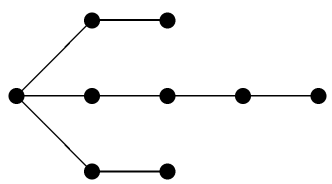


Figure 6

Note that G being 2-connected implies H is essentially 2-edge-connected, i.e., H has no cutedge the removal of which results in a graph with at least two nontrivial components. Also, by a result of Harary and Nash-Williams [HNW65], G is hamiltonian if and only if H contains a dominating closed trail, i.e., a closed trail T such that every edge of H has at least one vertex on T. In the proofs, we will use similar constructions to obtain 2-factors in $G = L(H)$.

The graph G cannot be hamiltonian and thus, by Theorem 7, G contains an induced subgraph $F = P_{i,j,k} \in \mathcal{P}$. If one of the i, j, k is at least 4, then F (and hence also G) contains an induced $N(3,1,1,)$, hence each of the i, j, k is either 3 or T. Then the graph H contains as a subgraph (not necessarily induced) a graph D isomorphic to $L^{-1}(P_{3,3,3})$, $L^{-1}(P_{3,3,T})$, $L^{-1}(P_{3,T,T})$ or $L^{-1}(P_{T,T,T})$. We will always refer to the vertices of these subgraphs as labeled in Figure 7.

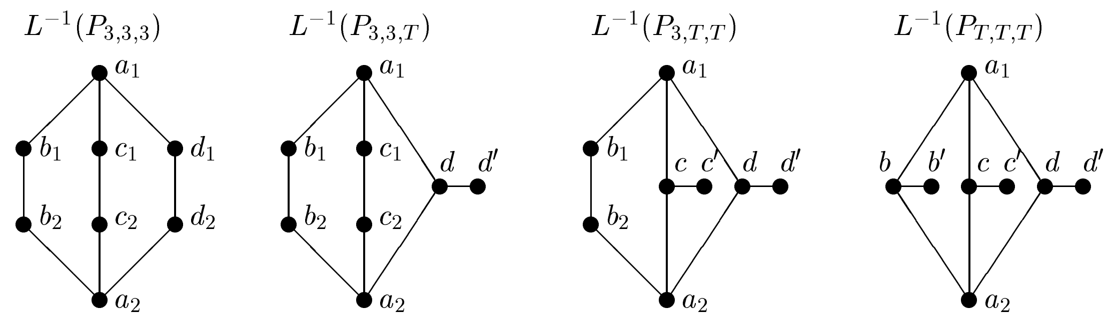


Figure 7

Case 1: $D = L^{-1}(P_{3,3,3})$. Since $|E(D)| = 9$ and $|V(G)| \geq 10$, there is an edge $xy \in E(H) \setminus E(D)$ such that $x \in V(D)$. First suppose that $y \notin V(D)$. Then, up to a symmetry, $x = a_1$ or $x = b_1$, but then in the first case the edges $\{a_2b_2, b_2b_1, a_2d_2, d_2d_1, a_2c_2, c_2c_1, c_1a_1, a_1y\}$ and in the second case the edges $\{a_1b_1, b_1y, a_1c_1, c_1c_2, a_1d_1, d_1d_2, d_2a_2, a_2b_2\}$ determine a copy of $L^{-1}(N(3,1,1))$ in H. Hence all edges in $E(H) \setminus E(D)$ have both ends in $V(D)$. By symmetry and since H is triangle-free, we can suppose $x = b_1$ and $y = c_2$, but then $(a_1, c_1, c_2, b_1, b_2, a_2, d_2, d_1, a_1)$ is a hamiltonian cycle in H, implying $G = L(H)$ has a 2-factor.

Case 2: $D = L^{-1}(P_{3,3,T})$. Let $xy \in E(H) \setminus E(D)$. If $y \notin V(D)$ and $x \in \{a_1, a_2\}$, say, $x = a_1$, or if $y \notin V(D)$ and $x \in \{b_1, b_2, c_1, c_2\}$, say, $x = b_1$, then $\{a_2b_2, b_2b_1, a_2d, dd', a_2c_2, c_2c_1, c_1a_1, a_1y\}$ or $\{a_1b_1, b_1y, a_1d, dd', a_1c_1, c_1c_2, c_2a_2, a_2b_2\}$ gives an $L^{-1}(N(3,1,1))$. Thus, d is a cutvertex of H.

If there is a path P of length 3 outside $D - d'$ with endvertex d, say, $P = (d, u_1, u_2, u_3)$ (not excluding the possibility that $d' \in \{u_1, u_2, u_3\}$), then we have an $L^{-1}(N(3,1,1))$ at

$\{a_1b_1, b_1b_2, a_1c_1, c_1c_2, a_1d, du_1, u_1u_2, u_2u_3\}$, a contradiction. This immediately implies that if $x, y \notin V(D - d')$, then one of x, y (say, x) is adjacent to d. Since the removal of the edge xd cannot separate xy from D, there is a path $P = (u_1, \ldots, u_k)$ such that $u_1 \in \{x, y\}$, $u_k \in \{d, d'\}$ and $u_2, \ldots, u_{k-1} \in V(H) \setminus V(D)$. But then in each of the possible cases we get a path of length 3 outside $D - d'$ since H is triangle-free.

This implies that all edges in $E(H) \setminus E(D)$ are incident to d or have both ends in $V(D)$, but then the cycle $(a_1, c_1, c_2, a_2, b_2, b_1, a_1)$ together with the star centered at d determine two cycles in $G = L(H)$ that can be extended to a 2-factor of G.

Case 3: $D = L^{-1}(P_{3,T,T})$. Let again $xy \in E(H) \setminus E(D)$ and suppose that $y \notin V(D)$. Then immediately $x \notin \{a_1, a_2\}$, for if e.g. $x = a_1$, then $\{a_2d, dd', a_2c, cc', a_2b_2, b_2b_1, b_1a_1, a_1y\}$ gives an $L^{-1}(N(3,1,1))$. Thus, a_1, a_2 have no neighbor outside D.

Suppose that xy is at distance 2 from D, and let $xz \in E(H)$ for some $z \in V(D)$ (note that we already know that $z \notin \{a_1, a_2\}$). If $z = b_1$, then we have an $L^{-1}(N(3,1,1))$ at $\{a_2d, dd', a_2c, ca_1, a_2b_2, b_2b_1, b_1x, xy\}$, hence $z \neq b_1$. Symmetrically, $z \neq b_2$. If $z = d'$, then we have an $L^{-1}(N(3,1,1))$ at $\{a_1b_1, b_1b_2, a_1c, ca_2, a_1d, dd', d'x, xy\}$, hence $z \neq d'$ and, symmetrically, $z \neq c'$. Note that a symmetric argument implies that there is no edge between any of x, y and c', d'.

By symmetry, it remains to consider the possibility $z = d$. For the sake of brevity, we merge this possibility with the case that $x = d'$ (i.e., we suppose that x is adjacent to d, not excluding the possibility $x = d'$). By the connectivity assumption, there is a path $P = (u_1, \ldots, u_k)$, $k \geq 2$, outside D with $u_1 \in \{x, y\}$ and $u_k \in \{d, c\}$. If $u_k = d$, then $k \geq 3$ for $u_1 = y$ or $k \geq 4$ for $u_1 = x$ (since H is triangle-free) and we get an $L^{-1}(N(3,1,1))$ in a way similar to that in the case $z = d'$. Thus, $u_k = c$. We distinguish the following possibilities.

Case	$L^{-1}(N(3,1,1))$
$u_1 = y$, $k \geq 3$	$\{da_1, a_1b_1, da_2, a_2b_2, dx, xy, yu_2, u_2u_3\}$
$u_1 = y$, $k = 2$	$\{da_1, a_1b_1, da_2, a_2b_2, dx, xy, yc, cc'\}$
$u_1 = x$, $k \geq 4$	$\{da_1, a_1b_1, da_2, a_2b_2, dx, xu_2, u_2u_3, u_3u_4\}$
$u_1 = x$, $k = 3$	$\{da_1, a_1b_1, da_2, a_2b_2, dx, xu_2, u_2c, cc'\}$

Hence we have $u_1 = x$ and $k = 2$ (i.e., $xc \in E(H)$) as the only remaining possibility.

We can summarize that if there is an edge $xy \in E(H) \setminus E(D)$ with $y \notin V(D)$, then there are the following possibilities:

(i) xy is at distance 2 from D, x is adjacent to both c and d, and y has no neighbor in D, or

(ii) $x \in \{b_1, b_2, c, d\}$.

Let B denote the set of all edges at distance 2 from D, and let $x_1y_1, x_2y_2 \in B$. Then, by (i), we have $x_1c, x_2c, x_1d, x_2d \in E(H)$, but then $\{ca_2, a_2b_2, cx_1, x_1y_1, cx_2, x_2d, da_1, a_1b_1\}$ is an $L^{-1}(N(3,1,1))$, unless $x_1 = x_2$. Thus, if $B \neq \emptyset$, then there is a vertex x such that $xc, xd \in V(H)$ and every edge in B contains x. This implies that if $B \neq \emptyset$, then the cycle $(a_1, b_1, b_2, a_2, c, x, d, a_1)$ contains at least one vertex of every edge of H and hence $G = L(H)$ is hamiltonian, and if $B = \emptyset$, then the cycle $(a_1, b_1, b_2, a_2, c, a_1)$ together with the star centered at d correspond to cycles in G that can be extended to a 2-factor of G.

Case 4: $D = L^{-1}(P_{T,T,T})$. Suppose there is an edge $xy \in E(H) \setminus E(D)$ at distance 2 from $D' = D - \{b', c', d'\}$, and x has a neighbor in D'.

First observe that neither x nor y can be adjacent to any of a_1, a_2, for if e.g. $xa_1 \in E(H)$, then $\{a_2b, bb', a_2c, cc', a_2d, da_1, a_1x, xy\}$ gives an $L^{-1}(N(3,1,1))$ (where we suppose that $y \neq b'$ and $y \neq c'$, otherwise we interchange the roles of the b's, c's and d's accordingly). We show that there are two vertices $u, v \in \{b, c, d\}$ such that either $xu, xv \in E(H)$ or $xu, yv \in E(H)$. By symmetry, suppose that $xb \in E(H)$, but neither x nor y is adjacent to any of c, d (note that we do not exclude the possibility that $x = b'$). By the connectivity assumption, there is a path $P = (u_1, \ldots, u_k)$ such that $u_1 \in \{x, y\}$ and $u_k \in \{b, c, d\}$. If $u_k = b$, then for $u_1 = x$ we have $k \geq 4$, and for $u_1 = y$ we have $k \geq 3$ since H is triangle-free. Then, in the first case $\{a_1d, da_2, a_1c, cc', a_1b, bx, xu_2, u_2u_3\}$ and in the second case $\{a_1d, da_2, a_1c, cc', a_1b, bx, xy, yu_2\}$ gives an $L^{-1}(N(3,1,1))$. Hence $u_k \in \{c, d\}$, say, $u_k = c$. By the assumption, neither x nor y is adjacent to c, implying $k \geq 3$. Then, in the first case the subgraph given by $\{a_1d, dd', a_1b, ba_2, a_1c, cu_{k-1}, u_{k-1}u_{k-2}, \ldots, xy\}$ contains an $L^{-1}(N(3,1,1))$, and in the second case $\{a_1d, dd', a_1c, ca_2, a_1b, bx, xy, yu_2\}$ gives an $L^{-1}(N(3,1,1))$.

For any $\{u, v\} \subset \{b, c, d\}$ denote

$B^1_{u,v} = \{x, y \in E(H)|\ x, y \notin V(D'), xu \in E(H), xv \in E(H)\}$, and

$B^2_{u,v} = \{x, y \in E(H)|\ x, y \notin V(D'), xu \in E(H), yv \in E(H)\}$.

We have shown that every edge xy with $x, y \notin V(D')$ belongs to some $B^j_{u,v}$ (if some edge belongs to more $B^j_{u,v}$'s, we choose one of them).

Let $x_1y_1 \in B^1_{u,v}$ and $x_2y_2 \in B^2_{u,v}$ for some $u, v \in \{b, c, d\}$, say, $u = b$ and $v = c$. Then x_1y_1 and x_2y_2 have no vertex in common since H is triangle-free, but then $\{ca_2, a_2d, cx_1, x_1y_1, cy_2, y_2x_2, x_2b, ba_1\}$ is an $L^{-1}(N(3,1,1))$. This proves that for any $\{u, v\} \subset \{b, c, d\}$, at most one of $B^1_{u,v}$, $B^2_{u,v}$ is nonempty.

Next suppose that $|B^2_{u,v}| \geq 2$ for some $\{u, v\} \subset \{b, c, d\}$, say, $u = b$, $v = c$, and let $x_1y_1, x_2y_2 \in B^2_{b,c}$. Then $x_1 \neq y_2$ and $x_2 \neq y_1$ since H is triangle-free, and then $\{cy_1, y_1x_1, ca_2, a_2d, cy_2, y_2x_2, x_2b, ba_1\}$ if $x_1 \neq x_2$, $y_1 \neq y_2$, or $\{a_2b, ba_1, a_2d, dd', a_2c, cy_1, y_1x_1, x_1y_2\}$ if $x_1 = x_2$, $y_1 \neq y_2$, gives an $L^{-1}(N(3,1,1))$ (the case $x_1 \neq x_2$, $y_1 = y_2$ is symmetric). Hence $|B^2_{u,v}| \leq 1$ for any $\{u, v\} \subset \{b, c, d\}$.

Similarly, suppose $|B^1_{u,v}| \geq 2$ for some $\{u, v\} \subset \{b, c, d\}$, say, $u = b$, $v = c$, and let $x_1y_1, x_2y_2 \in B^1_{b,c}$. Again clearly $x_1 \neq y_2$ and $x_2 \neq y_1$ since H is triangle-free. If $x_1 \neq x_2$, then $\{bx_1, x_1y_1, bx_2, x_2y_2, ba_1, a_1d, da_2, a_2c\}$ for $y_1 \neq y_2$, or $\{a_2c, ca_1, a_2d, dd', a_2b, bx_1, x_1y_1, y_1x_2\}$ for $y_1 = y_2$ gives an $L^{-1}(N(3,1,1))$. This proves that for every nonempty $B^1_{u,v}$ there is a vertex x_{uv} such that $x_{uv}u, x_{uv}v \in E(H)$ and every edge in $B^1_{u,v}$ contains x_{uv}.

Specifically, for any $\{u, v\} \subset \{b, c, d\}$, either $B^1_{u,v} = B^2_{u,v} = \emptyset$, or exactly one of $B^1_{u,v}$, $B^2_{u,v}$ is nonempty and there is a u, v-path P_{uv} of length 2 or 3 such that P_{uv} is internally vertex-disjoint from D' and every edge in the $B^1_{u,v}$ ($B^2_{u,v}$) has at least one vertex on P_{uv}. We now have, up to symmetry, the following possibilities.

a) $B^j_{u,v} = \emptyset$ for any $\{u, v\} \subset \{b, c, d\}$ and $j \in \{1, 2\}$. Then the cycle (a_1, b, a_2, c, a_1) together with the star centered at d give two cycles in G that can be extended to a 2-factor of G.

b) $B^{j_0}_{b,c} \neq \emptyset$ for some $j_0 \in \{1, 2\}$, $B^j_{c,d} = B^j_{b,d} = \emptyset$ for all $j \in \{1, 2\}$. Then the cycle $(a_1, b, P_{bc}, c, a_2, d, a_1)$ gives a hamiltonian cycle in G.

c) $B^{j_1}_{b,c} \neq \emptyset$ and $B^{j_2}_{c,d} \neq \emptyset$ for some $j_1, j_2 \in \{1, 2\}$, $B^j_{b,d} = \emptyset$ for all $j \in \{1, 2\}$. Then the closed trail $(a_1, b, P_{bc}, c, P_{cd}, d, a_2, c, a_1)$ gives a hamiltonian cycle in G.

d) $B_{b,c}^{j_1} \neq \emptyset$, $B_{c,d}^{j_2} \neq \emptyset$ and $B_{b,d}^{j_3} \neq \emptyset$ for some $j_1, j_2, j_3 \in \{1, 2\}$. Then the closed trail $(b, P_{bc}, c, P_{cd}, d, a_1, b, P_{bd}, d, a_2, b)$ gives a hamiltonian cycle in G.

This completes the proof of Theorem 12. $\square$

We will next prove the following theorem which is another forbidden subgraph sufficient condition for a graph to have a 2-factor.

Theorem 13 *If G is a 2-connected graph of order $n \geq 3$ that is C-free and C_i-free for all $i \geq 6$, then G has a 2-factor.*

The following lemma will be useful in the proof of Theorem 13.

Lemma 2 *Let $k \geq 4$ be an integer and let G be a graph that is C-free and C_i-free for all $i \geq k$. Then, $cl(G)$ is also C-free and C_i-free for all $i \geq k$.*

Proof: In [R97] is was shown that $cl(G)$ is C-free. Suppose that G is not C_ℓ-free for some $\ell \geq k$. Let $G = G_1 < G_2 < \cdots < G_s = cl(G)$ be a sequence of graphs that yields the closure $cl(G)$. Assume that G_r is the first graph in the sequence that yields the first induced cycle $C = (v_1, v_2, \cdots, v_\ell, v_1)$ for some $\ell \geq k$. Then, G_{r-1} has no induced cycles of length $i \geq k$, and G_r is obtained from G_{r-1} by replacing the connected neighborhood N of a vertex $x_i \in G_{r-1}$ by a complete graph on N. The cycle C and the complete graph induced by N have precisely two vertices and the edge between them in common, which without loss of generality is v_1v_2, since C would be induced in G_{r-1} if there were no edges and C would not be induced in G_r if there were at least two edges. Thus, x_i is adjacent to precisely v_1 and v_2 in G_{r-1} and $v_1v_2 \notin E(G_{r-1})$. This implies that $C^* = (v_1, x_i, v_2, \cdots, v_\ell, v_1)$ is an induced cycle in G_{r-1}, a contradiction. This completes the proof of Lemma 2. $\square$

Proof: (of Theorem 13) By Lemma 2 and Theorem 5 we can assume that G is closed. Therefore by Theorem 4 there is a triangle-free graph H such that $G = L(H)$. Since any cycle of length p in H determines an induced cycle of length p in G, the only cycles in H

are of length 4 and 5. There is no cutedge e of H such that $H - e$ has two components each containing an edge, since this would imply that G is not 2-connected. Thus, the only cutedges of H are pendant edges, and the deletion of the pendant edges of H results in a 2-edge connected graph H^* with only cycles of length 4 and 5.

It is sufficient to show that H^* contains a (not necessarily connected) spanning subgraph F such that

(*i*) $d_F(x)$ is even for every $x \in V(H^*)$,

(*ii*) the set $\{x \in V(H^*) |\ d_F(x) = 0\}$ is independent in H^* (i.e., every edge of H^* has at least one vertex in a nontrivial component of F).

Indeed, if F is such a subgraph, then every nontrivial component of F yields a cycle in G. For every vertex x with $d_F(x) = 0$ which is contained in an edge in $E(H) \setminus E(H^*)$, we have $d_H(x) \geq 3$ and hence the star in H centered at x gives a cycle of length at least 3 in G. By the condition (*ii*), this system of cycles can be extended to a 2-factor of G.

Obviously, it is sufficient to show the existence of such a subgraph in every block of H^*, hence we can suppose that H^* is 2-connected.

If H^* contains no C_5, then H^* is bipartite. In fact, H^* is complete bipartite, since the existence of nonadjacent vertices in opposite parts of a 2-connected bipartite graph implies the existence of a cycle of length at least 6. Thus, in addition, H^* is isomorphic to a $K_{2,s}$ for some $s \geq 2$, and the existence of F is straightforward.

Thus, H^* contains a cycle $D = C_5$. Set $D = (v_1, v_2, v_3, v_4, v_5, v_1)$, and let $u \in V(H^*) \setminus V(D)$. Since H^* is 2-connected, there are two internally vertex-disjoint paths from u to two distinct vertices of D. Then, the only possibility that does not imply the existence of a cycle of length at least 6 is that both paths are of length 1 and their endvertices are nonconsecutive on D. By symmetry, we can suppose that $uv_1 \in E(H^*)$ and $uv_3 \in E(H^*)$.

If there is an edge $u_1u_2 \in E(H^*)$ such that $u_1, u_2 \notin V(D)$, then, by the previous observation, both u_1 and u_2 must have two nonconsecutive neighbors on D, but this immediately implies the existence of a cycle of length at least 6 (note that the neighbors of u_1, u_2 on D must be distinct since H^* is triangle-free). Hence the set $V(H^*) \setminus V(D)$ is independent in H^* and the existence of F follows. □

Note that the graph $L = P_{3,3,3}$ shows that Theorem 13 cannot be extended to $i \geq 7$.

With Theorem 13 and the following technical lemma we will be prepared to prove an additional sufficient forbidden subgraph condition for the existence of a 2-factor using a generalized bull.

Lemma 3 *Let G be a 2-connected claw-free graph of order $n \geq 3$, $D = (v_1, v_2, \cdots, v_p, v_1)$ an induced cycle of length $p \geq 6$, S the vertices of G a distance 1 from D, and T the vertices of G a distance at least 2 from D. The following is true for G.*

(i) S can be partitioned into p sets S_i such that each vertex in S_i is adjacent to v_i and v_{i+1} but not v_{i-1}.

(ii) Each set S_i induces a complete subgraph of G.

(iii) A vertex in S_i with an adjacency in T is adjacent to precisely v_i and v_{i+1} in D.

(iv) For any $S' \subseteq S$, the graph spanned by $D \cup S'$ is hamiltonian.

(v) If $T = \emptyset$, then G is hamiltonian.

Proof: No vertex u of S can be adjacent to vertex v_i of D and non-adjacent to both v_{i-1} and v_{i+1}, since that would give a claw centered at v_i in D. Also, no vertex u of S can be adjacent to all of the vertices of D, since this would give a claw centered at u. Thus, there must be some i such that $uv_{i-1} \notin E(G)$, but $uv_i, uv_{i+1} \in E(G)$. Thus, $u \in S_i$. Of course, it is possible that u could be in some other S_j, but if this occurs, arbitrarily choose one of them. This gives a partition of S.

If $u_1, u_2 \in S_i$, then to avoid a claw centered at v_i implies $u_1u_2 \in E(G)$. Thus, S_i must span a complete subgraph. If a vertex $u \in S_i$ has an adjacency in T, then to avoid a claw u cannot be adjacent to 2 independent vertices of D. Hence, u has precisely 2 adjacencies in D, and they are v_i and v_{i+1}.

Since each S_i is complete, there is a path Q_i from v_i to v_{i+1} that contains all of the vertices of $S_i \cap S'$. A hamiltonian cycle can be formed from the paths $Q_1, Q_2, \cdots, Q_p$, which implies that the graph spanned by $D \cup S'$ is hamiltonian. If $T = \emptyset$, then $G = D \cup S$ is hamiltonian by the argument of the previous paragraph. This completes the proof of Lemma 3. $\square$

Theorem 14 *If G is a 2-connected $CB(4,1)$-free graph of order $n \geq 3$, then G has a 2-factor unless $n = 9$ and $G = L$.*

Proof: By Lemma 1, the only 2-connected claw-free graph of order $n \leq 9$ that does not have a 2-factor is the $B(4,1)$-free graph L. Hence we can assume that $n \geq 10$. Also, by Theorem 13 there must be a cycle of length at least 6. We will assume that G does not have a 2-factor and show that this leads to a contradiction.

Assume the notation of Lemma 3 and let D be an induced cycle of maximum length, say p. In the case when $T = \emptyset$, a contradiction is reached since G is hamiltonian by Lemma 3. Hence we can assume that $T \neq \emptyset$.

We will first show that the cycle D has at most 6 vertices, so assume that $p \geq 7$. There is a vertex u of distance 2 from D, and with no loss of generality we can assume we have the path $P = (u, u', v_1)$ with $u' \in S_1$. This gives an induced $B(4,1)$ with the vertices $\{u, u', v_1, v_2, v_3, v_4, v_5, v_6\}$. This contradiction implies that $p = 6$.

No vertex u_1 of T can be a distance 5 from D. For example, assume that $P = (u_1, u_2, u_3, u_4, u_5, v_1)$ is such a distance path and that $u_5 \in S_1$. Then, there is an induced $B(4,1)$ using the vertices $\{u_1, u_2, u_3, u_4, u_5, v_1, v_2, v_3\}$. Hence, we can assume that all vertices in T are a distance at most 4 from D. For $1 \leq i \leq 6$ and $2 \leq j \leq 4$ let T_{ij} be the vertices of T that are a distance j from D such that one of the distance paths to D contains a vertex in S_i. Each vertex in T is in some T_{ij}. For each i let S_i' be the vertices of S_i that are adjacent to a vertex of T. Consider the graph induced by $S_1' \cup T_{12} \cup T_{13} \cup T_{14}$. If S_1' consists of a single vertex, then T_{12} is complete since G is claw-free. If there is a vertex $w \in T_{12}$ and vertices $u_1, u_2 \in S_1'$ such that $wu_1 \in E(G)$ but $wu_2 \notin E(G)$, then there is an induced $B(4,1)$ using the vertices $\{w, u_1, u_2, v_2, v_3, v_4, v_5, v_6\}$. This gives a contradiction that implies that each vertex in T_{12} is adjacent to all of the vertices of S_1', and so T_{12} is also complete. The exact same argument applies for T_{13} relative to T_{12} and for T_{14} relative to T_{13} as well. Thus we can conclude for all i and j that each T_{ij} is complete and there are complete bipartite graphs between T_{i3} and T_{i4}, between T_{i2} and T_{i3} and also between S_i' and T_{i2}.

Consider the case when there is a $w_1 \in T_{12}$ and a $w_3 \in T_{32}$. Let u_1 and u_3 be vertices in S'_1 and S'_3 respectively. Then, there is an induced $B(4,1)$ using the vertices $\{w_3, u_3, v_3, v_4, v_5, v_6, v_1, u_1\}$, unless either $u_1u_3 \in E(G)$ or $u_1w_3 \in E(G)$. To avoid an induced claw, $u_1u_3 \in E(G)$ implies that $u_1w_3 \in E(G)$, and hence we can assume that $w_3 \in T_{12}$. Consequently, each vertex in T_{33} is also in T_{13}, and each vertex in T_{34} is also in T_{14}. By symmetry, we further have $T_{52} \subset T_{12}$, $T_{53} \subset T_{13}$ and $T_{54} \subset T_{14}$. The same argument applied for $i = 2$ gives $T_{ij} \subset T_{2j}$ for $i = 4, 6$ and $j = 2, 3, 4$. We have shown that the vertices of T are partitioned into 6 sets (some could be empty), namely $T = T_{12} \cup T_{13} \cup T_{14} \cup T_{22} \cup T_{23} \cup T_{24}$.

Also, observe that if $u_1 \in S'_1$ and $u_2 \in S'_2$, then $u_1u_2 \notin E(G)$, since this implies the existence of an induced cycle of length 7, namely, the cycle $(u_1, u_2, v_3, v_4, v_5, v_6, v_1, u_1)$. Likewise, no vertex of $S'_1 \cup T_{12} \cup T_{13} \cup T_{14}$ is adjacent to any vertex of $S'_2 \cup T_{22} \cup T_{23} \cup T_{24}$.

Our next objective is to show that the vertices of T can be covered by a system of vertex disjoint cycles that are disjoint from the cycle D. This will verify that G has a 2-factor, since the cycle D can be expanded to a cycle that contains the remaining vertices of G by Lemma 3.

By the 2-connectedness of G, $T_{ij} \neq \emptyset$ implies $|T_{i(j-1)}| \geq 2$ for $i = 1, 2$ and $j = 3, 4$. This immediately implies there is a cycle that covers all of $T_{i2} \cup T_{i3} \cup T_{i4}$ if $T_{i3} \neq \emptyset$, $i = 1, 2$. If $T_{i3} = \emptyset$ and $|T_{i2} \cup S'_i| \geq 3$, then there is a cycle spanning the set $T_{i2} \cup S'_i$. Hence we are left with the cases that either $T_{i3} = \emptyset$ or $|T_{i2}| = |S'_i| = 1$, $i = 1, 2$, and at least one of T_{12}, T_{22} is nonempty (otherwise we are done by Lemma 3).

Suppose that $T_{12} \neq \emptyset$ and set $T_{12} = \{w_1\}$ and $S'_1 = \{u_1\}$. By the connectivity assumption, w_1 must be adjacent to a vertex $u_{i'} \in S'_{i'}$ for some $i' \neq 1$. We already know that $i' \notin \{2, 6\}$. If $i' = 3$, then we must have $u_1u_3 \in E(G)$ to avoid the induced cycle $(w_1, u_3, v_4, v_5, v_6, v_1, u_1, w_1)$ of length 7, but then there is the triangle (u_1, u_3, w_1, u_1). The case $i' = 5$ is symmetric and hence it remains to consider the case $i' = 4$. Then, by a symmetric argument (in which S'_4 plays the role of S'_1) we conclude that $|S'_4| = 1$. Set $S'_4 = \{u_4\}$.

Now, if $T_{22} \neq \emptyset$, say, $T_{22} = \{w_2\}$ and $S'_2 = \{u_2\}$, then, by the same argument we get that w_2 is adjacent to a $u_5 \in S'_5$ and $S'_5 = \{u_5\}$, but then there is a cycle of length greater than 6, namely $(u_1, w_1, u_4, v_5, u_5, w_2, u_2, v_2, u_1)$ if $u_4u_5 \notin E(G)$, or $(u_1, w_1, u_4, u_5, w_2, u_2, v_2, u_1)$ if $u_4u_5 \in E(G)$, respectively. This contradiction proves that $T_{22} = \emptyset$ and we conclude that

$T = T_{12} = \{w_1\}$.

Suppose $S_2 \neq \emptyset$ and let $u_2 \in S_2$. To avoid the $B(4,1)$ induced by $\{w_1, u_4, v_4, v_5, v_6, v_1, v_2, u_2\}$, we must have $u_2 v_4 \in E(G)$. Hence all vertices in S_2 are adjacent to v_4, implying there is a cycle that covers all of $S_2 \cup S_3 \cup \{v_2, v_3, v_4\}$, and this cycle together with a cycle obtained by applying Lemma 3 to the cycle $(u_1, w_1, u_4, v_5, v_6, v_1, u_1)$ gives a 2-factor in G. Hence we get $S_2 = \emptyset$, and, similarly, $S_3 = S_5 = S_6 = \emptyset$. Finally, if there is a vertex $u \in S_1 \setminus S_1'$, then the set $\{v_3, v_2, u, v_1, v_6, v_5, u_4, w_1\}$ induces a $B(4,1)$. Hence $S_1 \setminus S_1' = \emptyset$ and, symmetrically, $S_4 \setminus S_4' = \emptyset$. This implies that G is isomorphic to the graph $L = P_{3,3,3}$, contradicting the assumption that $n \geq 10$. $\square$

Theorem 2 is an immediate consequence of Theorems 3, 9, 10, 12, and 14.

Remark. Note that Theorem 2 can be stated in a slightly stronger form, since all 2-connected graphs of order $n \geq 3$ that satisfy the forbidden subgraph conditions have a 2-factor except for a very limited number of graphs. In the case of claw-free graphs the only exception is the graph $L = P_{3,3,3}$ in Figure 3. In the case of $K_{1,4}P_4$-free graphs it is straightforward to verify that there are only 8 exceptions, namely the graphs

$$H + (K_i \cup K_j \cup K_k),$$

where $H = K_2$ or $\overline{K}_2$ and $1 \leq i \leq j \leq k \leq 2$.

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