On Diregularity of Digraphs of Defect at Most Two

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Abstract

Since Moore digraphs do not exist for $k \neq 1$ and $d \neq 1$, the problem of finding digraphs of out-degree $d \geq 2$, diameter $k \geq 2$ and order close to the Moore bound, becomes an interesting problem. To prove the non-existence of such digraphs or to assist in their construction (if they exist), we first may wish to establish some properties that such digraphs must possess. In this paper we consider the diregularity of such digraphs. It is easy to show that any digraph with out-degree at most $d \geq 2$, diameter $k \geq 2$ and order one or two less than Moore bound must have all vertices of out-degree d. However, establishing the regularity or otherwise of the in-degree of such a digraph is not easy. In this paper we prove that all digraphs of defect two are either diregular or almost diregular. Additionally, in the case of defect one we present a new, simpler and shorter, proof that a digraph of defect one must be diregular, and in the case of defect two and for $d = 2$ and $k \geq 3$, we present an alternative proof that a digraph of defect two must be diregular.

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1 Introduction

By a *directed graph*, or a *digraph*, we mean a structure $G = (V(G), A(G))$, where $V(G)$ is a finite nonempty set of distinct elements called *vertices*, and $A(G)$ is a set of ordered pairs (u, v) of distinct vertices $u, v \in V(G)$. The elements of $A(G)$ are called *arcs*.

The *order* of the digraph G is the number of vertices in G. An *in-neighbour* (respectively, *out-neighbour*) of a vertex v in G is a vertex u (respectively, w) such that $(u, v) \in A(G)$ (respectively, $(v, w) \in A(G)$). The set of all in-neighbours (respectively, out-neighbours) of a vertex v is called the *in-neighbourhood* (respectively, the *out-neighbourhood*) of v and denoted by $N^-(v)$ (respectively, $N^+(v)$). The *in-degree* (respectively, *out-degree*) of a vertex v is the number of all its in-neighbours (respectively, out-neighbours). If every vertex of a digraph G has the same in-degree (respectively, out-degree) then G is said to be *in-regular* (respectively, *out-regular*). A digraph G is called a *diregular* digraph of degree d if G is in-regular of in-degree d and out-regular of out-degree d.

An alternating sequence $v_0a_1v_1a_2...a_lv_l$ of vertices and arcs in G such that $a_i = (v_{i-1}, v_i)$, for each i, is called a *walk* of length l in G. A walk is *closed* if $v_0 = v_l$. If all the vertices of a $v_0 - v_l$ walk are distinct, then such a walk is called a *path*. A *cycle* is a closed path.

The *distance* from vertex u to vertex v, denoted by $dist(u, v)$, is the length of a shortest path from u to v, if any; otherwise, $dist(u, v) = \infty$. Note that, in general, $dist(u, v)$ is not necessarily equal to $dist(v, u)$. The *in-eccentricity* of v, denoted by $e^-(v)$, is defined as $e^-(v) = max\{dist(u, v) : u \in V\}$ and *out-eccentricity* of v, denoted by $e^+(v)$, is defined as $e^+(v) = max\{dist(v, u) : u \in V\}$. The *in-radius* of G, denoted by $rad^-(G)$, is defined as $rad^-(G) = min\{e^-(v) : v \in V\}$, and the *out-radius* of G, denoted by $rad^+(G)$, is defined as $rad^+(G) = min\{e^+(v) : v \in V\}$. The *diameter* of G, denoted by $diam(G)$, is defined as $diam(G) = max\{e^-(v), e^+(v) : v \in V\}$. The *girth* of a digraph G is the length of a shortest cycle in G.

The well known *degree/diameter* problem for digraphs is to determine the largest possible order $n_{d,k}$ of a digraph, given out-degree at most $d \geq 1$

and diameter $k \geq 1$. There is a natural upper bound on the order of digraphs, given out-degree at most d and diameter k. For any given vertex v of a digraph G, we can count the maximum possible number of vertices at a particular distance from that vertex. Let n_i , for $0 \leq i \leq k$, be the number of vertices at distance i from v. Then $n_i \leq d^i$, for $0 \leq i \leq k$, and consequently,

$$n_{d,k} = \sum_{i=0}^{k} n_i \leq 1 + d + d^2 + \cdots + d^k. \tag{1}$$

The right-hand side of (1), denoted by $M_{d,k}$, is called the *Moore bound*. If the equality sign holds in (1) then the digraph is called a *Moore digraph*. It is well known that Moore digraphs exist only in the cases when $d = 1$ (directed cycles of length $k + 1$, C_{k+1} , for any $k \geq 1$) or $k = 1$ (complete digraphs of order $d + 1, K_{d+1}$, for any $d \geq 1$) [2, 11]. For more details of Moore digraphs and digraphs close to the Moore bound, see the survey [9].

Note that every Moore digraph is diregular (of degree one in the case of C_{k+1} and of degree d in the case of K_{d+1}). Since for $d > 1$ and $k > 1$, there are no Moore digraphs, we are next interested in digraphs of order n 'close' to Moore bound.

It is easy to show that a digraph of order n, $M_{d,k} - M_{d,k-1} + 1 \leq n \leq M_{d,k} - 1$, with out-degree at most $d \geq 2$ and diameter $k \geq 2$, must have all vertices of out-degree d. In other words, the out-degree of such a digraph is constant ($= d$). This can be easily seen because if there were a vertex in the digraph with out-degree $d_1 < d$ (i.e., $d_1 \leq d - 1$), then the order of the digraph,

$$\begin{aligned}
n &\leq 1 + d_1 + d_1 d + \cdots + d_1 d^{k-1} \\
&= 1 + d_1(1 + d + \cdots + d^{k-1}) \\
&\leq 1 + (d-1)(1 + d + \cdots + d^{k-1}) \\
&= (1 + d + \cdots + d^k) - (1 + d + \cdots + d^{k-1}) \\
&= M_{d,k} - M_{d,k-1}
\end{aligned}$$

However, establishing the regularity or otherwise of in-degree for digraphs of order close to Moore bound is not easy. It is well known that there exist digraphs of out-degree d and diameter k whose order is just two or three less than the Moore bound and in which *not all* vertices have the same in-degree. In Figure 1 we give two examples of digraphs of diameter 2, out-degree $d = 2, 3$, respectively, and order $M_{d,2} - d$, with vertices not all of the same in-degree.

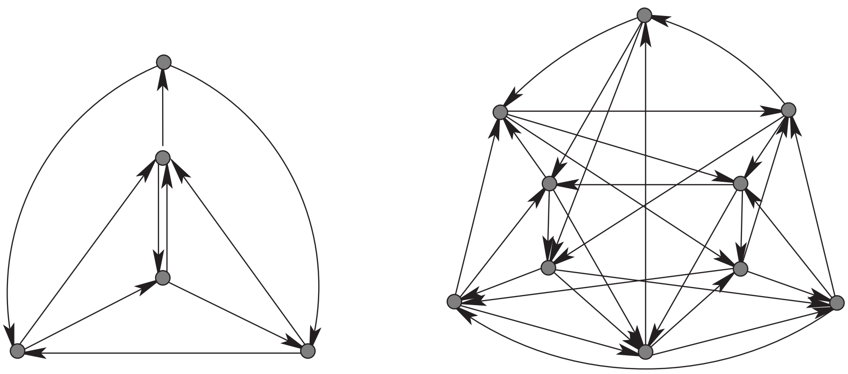

Figure 1: Two examples of non-diregular digraphs.

Miller, Gimbert, Širáň and Slamin [6] considered the diregularity of digraphs of defect one, that is, $n = M_{d,k} - 1$, and proved that such digraphs are diregular. For defect two, diameter $k = 2$ and any out-degree $d \geq 2$, non-diregular digraphs always exist. One such family of digraphs can be generated from Kautz digraphs which contain vertices with identical out-neighbourhoods and so we can apply the vertex deletion scheme, see [7]. This technique can be used on any digraph, of maximum out-degree d, diameter k and order n, that contains two vertices u and v with $N^+(u) = N^+(v)$. In such a case, we delete one of the vertices, say u, and redirect all in-neighbours of u to go to v, thereby obtaining a digraph of maximum out-degree d, diameter $k' \leq k$ and order $n - 1$. Figure 2 shows a digraph obtained by vertex deletion scheme on G after deleting vertex v_{12}.

The notion of 'almost diregularity' was first introduced in [3]. Throughout this paper, let S be the set of all vertices of G whose in-degree is less than d, where d is the average out-degree (or, equivalently, in-degree) of G. Let S' be the set of all vertices of G whose in-degree is greater than d; and let σ^- be the *in-excess*, $\sigma^- = \sigma^-(G) = \sum_{w \in S'}(d^-(w) - d) = \sum_{v \in S}(d - d^-(v))$.

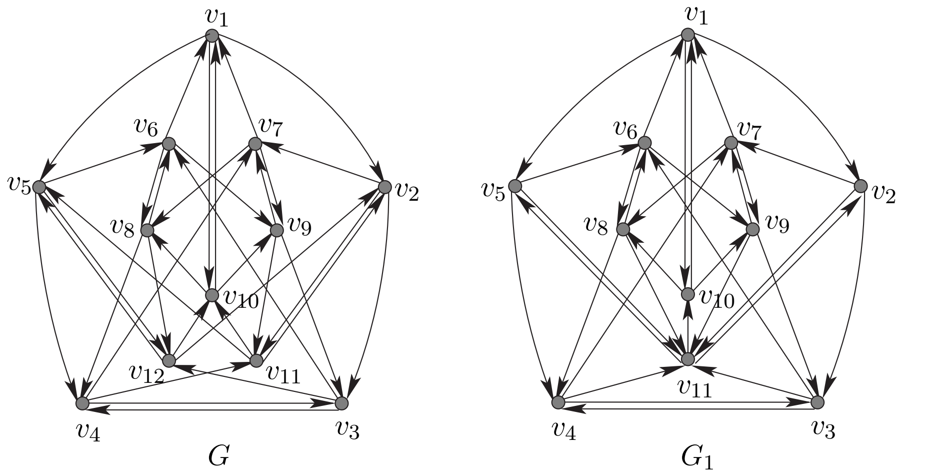

Figure 2: Digraphs G of order 12 and G_1 of order 11.

Similarly, let R be the set of all vertices of G whose out-degree is less than d. Let R' be the set of all vertices of G whose out-degree is greater than d. We define the *out-excess*, $\sigma^+ = \sigma^+(G) = \sum_{w \in R'}(d^+(w) - d) = \sum_{v \in R}(d - d^+(v))$. A digraph of average in-degree d is called *almost in-regular* if the in-excess is at most equal to d. Similarly, a digraph of average out-degree d is called *almost out-regular* if the out-excess is at most equal to d. A digraph is *almost diregular* if it is almost in-regular and almost out-regular. Note that if $\sigma^- = 0$ (respectively, $\sigma^+ = 0$) then G is in-regular (respectively, out-regular).

To start with, as a "warm up", we present a new proof that a digraph of defect one must be diregular. This proof is simpler and shorter than the original one presented in [6]. This is followed by the main result of this paper, namely, that all digraphs of defect two, out-degree $d \geq 2$ and diameter $k \geq 2$ are out-regular and almost in-regular. Finally, we present an alternative proof that a digraph of defect two, out-degree $d = 2$ and diameter $k \geq 3$ is diregular. This proof is again simpler than the original one presented in [12].

2 Results

Let G be a digraph of out-degree d, diameter k and order $M_{d,k} - 1$. Since the order of G is $M_{d,k}-1$, using a counting argument, it is easy to show that for each vertex u of G there exists exactly one vertex $r(u)$ in G with the property that there are two $u \to r(u)$ walks in G of lengths not exceeding k. The vertex $r(u)$ is called the *repeat* of u; this concept was introduced in [8]. If G is a diregular digraph then it follows from [1] that the mapping $v \to r(v)$ is an *automorphism* of the digraph G.

Let G be a digraph of out-degree $d \geq 2$, diameter $k \geq 3$ and order $M_{d,k}-2$. Using a counting argument, it is easy to show that, for each vertex u of G, there exist exactly two vertices $r_1(u)$ and $r_2(u)$ (not necessarily distinct) in G with the property that there are two $u \to r_i(u)$ walks, for $i = 1, 2$, in G of length not exceeding k. The vertices $r_i(u)$, $i = 1, 2$, are the repeats of u. If $r_1(x) = r_2(x) = r(x)$ then $r(x)$ is called a double repeat.

We will use the following notation throughout.

Notation 2.1 *Let $\mathcal{G}(d, k, \delta)$ be the set of all digraphs of maximum out-degree d, diameter k and defect δ, that is, order $n = M_{d,k} - \delta$. Then we refer to any digraph $G \in \mathcal{G}(d, k, \delta)$ as a (d, k, δ)-digraph.*

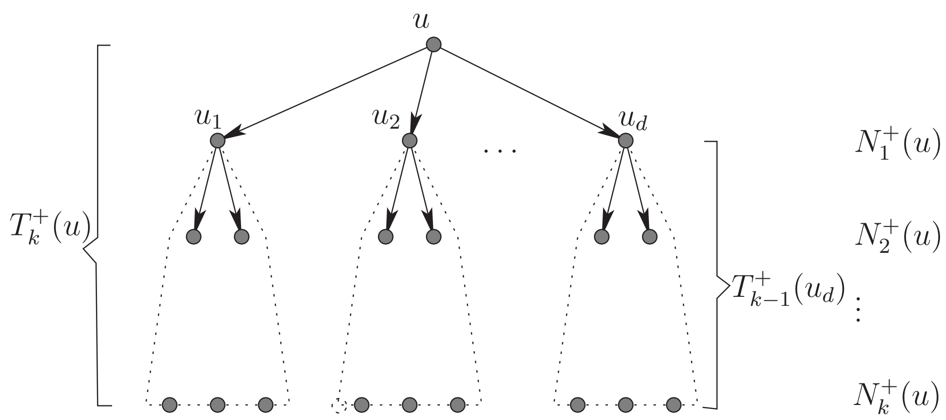

Figure 3: Multiset $T_k^+(u)$

We will also use the following notation throughout. For each vertex u of a digraph G described above, and for $1 \leq s \leq k$, let $T_s^+(u)$ be the multiset of all endvertices of directed paths in G of lengths at most s which start at u. Similarly, by $T_s^-(u)$ we denote the multiset of all starting vertices of directed paths of lengths at most s in G which terminate at u. Observe that the vertex u is in both $T_s^+(u)$ and $T_s^-(u)$, as it corresponds to a path of zero length. Let $N_s^+(u)$ be the set of all endvertices of directed paths in G of length exactly s which start at u. Similarly, by $N_s^-(u)$ we denote the set of all starting vertices of directed paths of length exactly s in G which terminate at u. If $s = 1$, the sets $T_1^+(u) \setminus \{u\}$ and $T_1^-(u) \setminus \{u\}$ represent the out- and in-neighbourhoods of the vertex u in the digraph G; we denote these neighbourhoods simply by $N^+(u)$ and $N^-(u)$, respectively. We illustrate the notations $T_s^+(u)$ and $N_s^+(u)$ in Figure 3.

We present our new results concerning the diregularity of digraphs of order close to Moore bound in the following sections.

2.1 Diregularity of $(d, k, 1)$-digraphs

In this section we present a new proof of the diregularity of a digraph of defect one with out-degree $d \geq 2$ and diameter $k \geq 2$. Let S be the set of all vertices of G whose in-degree is less than d. Let S' be the set of all vertices of G whose in-degree is greater than d; and let σ be the *in-excess*, $\sigma^- = \sum_{w \in S'}(d^-(w) - d) = \sum_{v \in S}(d - d^-(v))$.

Lemma 2.1 *Let $G \in \mathcal{G}(d, k, 1)$. Let S be the set of all vertices of G whose in-degree is less than d. Let $u \in G$ be an arbitrary vertex. Then $S \subseteq N^+(r(u))$.*

Proof. Let $v \in S$. Consider an arbitrary vertex $u \in V(G)$, $u \neq v$, and let $N^+(u) = \{u_1, u_2, ..., u_d\}$. Since the diameter of G is equal to k, the vertex v must occur in each of the sets $T_k^+(u_i)$, $i = 1, 2, ..., d$. It follows that for each i there exists a vertex $x_i \in \{u\} \cup T_{k-1}^+(u_i)$ such that $x_i v$ is an arc of G. Since the in-degree of v is less than d, the in-neighbours x_i of v are not all distinct. This implies that there exists some vertex which occurs at least twice in $T_k^+(u)$. Such a vertex must be a repeat of u. As G has defect 1, there is exactly one vertex of G which is the repeat of u, namely, $r(u)$. Therefore, $S \subseteq N^+(r(u))$. $\square$

Since every vertex in G has out-degree d, we have the following corollary of Lemma 2.1.

Corollary 2.1 *$|S| \leq d$.*

Additionally, no other vertices of in-neighbours of v can occur twice in $T_k^+(u)$, that is, the vertices x_i, for $i = 1, \ldots, d-1$, are mutually distinct. Therefore, we have the following corollary.

Corollary 2.2 *Let $v \in S$. Then the in-degree of v is $d-1$.*

Lemma 2.2 *Let $x \in S'$. Then x is the repeat of every vertex in G.*

Proof. Let $v \in S$ and $x \in S'$. First we consider the number of distinct vertices in the multiset $T_k^-(u)$ where $u \in V(G)\backslash\{x, v\}$. By diameter assumption, vertex x must occur at distance at most k to u. Since x goes to vertex of in-degree $d-1$, we have

$$\begin{aligned} |T_k^-(u)| &\leq 1 + d + d^2 + ... + d^k - 1 \\ &= M_{d,k} - 1. \end{aligned}$$

This implies that every $u \in V(G)\backslash\{x, v\}$ is not a repeat of any vertex in G.

We now consider the number of distinct vertices in the multiset $T_k^-(v_1)$, where $v_1 \in S$. To reach v_1 from all the other vertices in G, the number of distinct vertices in $T_k^-(v_1)$ must be

$$|T_k^-(v_1)| \leq \sum_{t=0}^{k} |N_t^-(v_1)|. \tag{2}$$

To estimate the above sum we can observe the following inequality

$$|N_t^-(v_1)| \leq \sum_{u \in N_{t-1}^-(v_1)} d^-(u) = d|N_{t-1}^-(v_1)| + \varepsilon_t, \tag{3}$$

where $2 \leq t \leq k$ and $\varepsilon_2 + \varepsilon_3 + \ldots + \varepsilon_k \leq \sigma$. Since $d^-(v_1) = d - 1$, we have $|N^-(v_1)| = |N_1^-(v_1)| = d - 1$. It is not difficult to see that a safe upper bound on the sum of $|T_k^-(v_1)|$ is obtained from Inequality (3) by setting $\varepsilon_2 = d$, and $\varepsilon_t = 0$ for $3 \leq t \leq k$. This gives

$$\begin{aligned}
|T_k^-(v_1)| &\leq 1 + |N_1^-(v_1)| + |N_2^-(v_1)| + |N_3^-(v_1)| + \ldots + |N_k^-(v_1)| \\
&= 1 + (d-1) + (d(d-1) + \varepsilon_2) + (d(d(d-1) + \varepsilon_2) + \varepsilon_3) \\
&\quad (1 + d + \cdots + d^{k-3}) \\
&= 1 + (d-1) + (d(d-1) + d) + (d(d(d-1) + d) + 0) \\
&\quad (1 + d + \cdots + d^{k-3}) \\
&= 1 + d - 1 + d^2 + d^3(1 + d + \cdots + d^{k-3}) \\
&= M_{d,k} - 1.
\end{aligned}$$

This implies that every vertex $v_1 \in S$ is not a repeat of any vertex in G. Hence, the only repeat of every vertex in G is $x \in S'$. □

Corollary 2.3 $|S| = d$.

Theorem 2.1 [6] *Every digraph of defect one is diregular.*

Proof. Let $v \in S$ and $x \in S'$. Consider the multiset $T_k^+(x)$. By Lemma 2.1, it follows that $N^+(x) = \{v_1, v_2, \ldots, v_d\}$, where $v_{i \in \{1,2,\ldots,d\}} \in S$. By Lemma 2.2, x is a selfrepeat. Therefore, one of the vertices $v_1, v_2, \ldots, v_d$ has distance $k - 1$ to x. Without loss of generality, we suppose that $dist(v_1, x) = k - 1$. But then v_1 also is a selfrepeat, and this contradicts the fact that x is the only repeat in G. □

2.2 Diregularity of $(d, k, 2)$-digraphs

In this section we present a new result concerning the in-regularity of digraphs of defect two for any out-degree $d \geq 2$ and diameter $k \geq 2$. Let S be the set of all vertices of G whose in-degree is less than d. Let S' be the set of all vertices of G whose in-degree is greater than d; and let σ be the *in-excess*, $\sigma^- = \sum_{w \in S'}(d^-(w) - d) = \sum_{v \in S}(d - d^-(v))$.

Lemma 2.3 *Let $G \in \mathcal{G}(d, k, 2)$. Let S be the set of all vertices of G whose in-degree is less than d. Then $S \subseteq N^+(r_1(u)) \cup N^+(r_2(u))$, for any vertex $u \in G$.*

Proof. Let $v \in S$. Consider an arbitrary vertex $u \in V(G)$, $u \neq v$, and let $N^+(u) = \{u_1, u_2, ..., u_d\}$. Since the diameter of G is equal to k, the vertex v must occur in each of the sets $T_k^+(u_i)$, $i = 1, 2, ..., d$. It follows that, for each i, there exists a vertex $x_i \in \{u\} \cup T_{k-1}^+(u_i)$ such that $x_i v$ is an arc of G. Since the in-degree of v is less than d then the in-neighbours x_i of v are not all distinct. This implies that there exists some vertex which occurs at least twice in $T_k^+(u)$. Such a vertex must be a repeat of u. As G has defect 2, there are at most two vertices of G which are repeats of u, namely, $r_1(u)$ and $r_2(u)$. Therefore, $S \subseteq N^+(r_1(u)) \cup N^+(r_2(u))$. □

Combining Lemma 2.3 with the fact that every vertex in G has out-degree d gives

Corollary 2.4 *$|S| \leq 2d$.*

In principle, we might expect that the in-degree of $v \in S$ could attain any value between 1 and $d - 1$. However, the next lemma asserts that the in-degree cannot be less than $d - 1$.

Lemma 2.4 *Let $G \in \mathcal{G}(d, k, 2)$. If $v_1 \in S$ then $d^-(v_1) = d - 1$.*

Proof. Let $v_1 \in S$. Consider an arbitrary vertex $u \in V(G)$, $u \neq v_1$, and let $N^+(u) = \{u_1, u_2, ..., u_d\}$. Since the diameter of G is equal to k, the vertex v_1 must occur in each of the sets $T_k^+(u_i)$, $i = 1, 2, ..., d$. It follows that for each i there exists a vertex $x_i \in \{u\} \cup T_{k-1}^+(u_i)$ such that $x_i v_1$ is an arc of G. If $d^-(v_1) \leq d - 3$ then there are at least three repeats of u, which is impossible. Suppose that $d^-(v_1) \leq d - 2$. By Lemma 2.3, the in-excess must satisfy

$$\sigma^- = \sum_{x \in S'} (d^-(x) - d) = \sum_{v_1 \in S} (d - d^-(v_1)) = |S| \leq 2d.$$

We now consider the number of vertices in the multiset $T_k^-(v_1)$. To reach v_1 from all the other vertices in G, the number of distinct vertices in $T_k^-(v_1)$ must be

$$|T_k^-(v_1)| \leq \sum_{t=0}^{k} |N_t^-(v_1)|. \tag{4}$$

To estimate the above sum we can observe the inequality

$$|N_t^-(v_1)| \leq \sum_{u \in N_{t-1}^-(v_1)} d^-(u) = d|N_{t-1}^-(v_1)| + \varepsilon_t, \tag{5}$$

where $2 \leq t \leq k$ and $\varepsilon_2 + \varepsilon_3 + \ldots + \varepsilon_k \leq \sigma^-$. Suppose $d^-(v_1) = d_1 \leq d-2$, that is, $|N^-(v_1)| = |N_1^-(v_1)| = d_1 \leq d-2$. It is not difficult to see that a safe upper bound on the sum of $|T_k^-(v_1)|$ is obtained from inequality (5) by setting $\varepsilon_2 = 2d$, and $\varepsilon_t = 0$ for $3 \leq t \leq k$. This gives

$$\begin{aligned}
|T_k^-(v_1)| &\leq 1 + |N_1^-(v_1)| + |N_2^-(v_1)| + |N_3^-(v_1)| + \ldots + |N_k^-(v_1)| \\
&= 1 + d_1 + (dd_1 + \varepsilon_2) + (d(dd_1 + \varepsilon_2) + \varepsilon_3) \\
&\quad (1 + d + \cdots + d^{k-3}) \\
&\leq 1 + (d-2) + (d(d-2) + 2d) + (d(d(d-2) + 2d) + 0) \\
&\quad (1 + d + \cdots + d^{k-3}) \\
&= 1 + d - 2 + d^2 + d^3(1 + d + \cdots + d^{k-3}) \\
&= M_{d,k} - 2.
\end{aligned}$$

Since $\varepsilon_2 = 2d$, $\varepsilon_t = 0$ for $3 \leq t \leq k$, and G contains a vertex of in-degree $d-2$, it follows that $|S| = d$. Let $S = \{v_1, v_2, \ldots, v_d\}$. Every v_i, for $i = 2, 3, \ldots, d$, has to reach v_1 at distance at most k. Since v_1 and every v_i have exactly the same in-neighbourhood, v_1 is forced to be a selfrepeat. This implies that v_1 occurs twice in the multiset $T_k^-(v_1)$. Hence $|T_k^-(v_1)| < M_{d,k} - 2$, which is a contradiction. Therefore, $d^-(v_1) = d-1$, for any $v_1 \in S$. $\square$

Lemma 2.5 *If S is the set of all vertices of G whose in-degree is $d-1$ then $|S| \leq d$.*

Proof. Suppose $|S| \geq d+1$. Then there exists $v_i \in S$ such that $d^-(v_i) = d-1$, for $i = 1, 2, \ldots, d+1$. The in-excess $\sigma^- = \sum_{v \in S}(d - d^-(v)) \geq d+1$. This implies that $|S'| \geq 1$. However, we cannot have $|S'| = 1$. For a contradiction, suppose $S' = \{x\}$. To reach v_1 (and v_i, $i = 2, 3, \ldots, d+1$) from all the other vertices in G, we must have $x \in \bigcap_{i=1}^{d+1} N^-(v_i)$, which is impossible as the out-degree of x is d. Hence $|S'| \geq 2$.

Let $u \in V(G)$ and $u \neq v_i$. To reach v_i from u, we must have $\bigcup_{i=1}^{d+1} N^-(v_i) \subseteq \{r_1(u), r_2(u)\}$. Since G has out-degree d, it follows that $|\bigcup_{i=1}^{d+1} N^-(v_i)| = d$. Let $r_1(u) = x_1$ and $r_2(u) = x_2$. Without loss of generality, we suppose $x_1 \in \bigcup_{i=1}^{d} N^-(v_i)$ and $x_2 \in N^-(v_{d+1})$. Now consider the multiset $T_k^+(x_1)$. Since every v_i, for $i = 1, 2, \ldots, d$, respectively, must reach $\{v_{j \neq i}\}$, for $j = 1, 2, \ldots, d+1$, within distance at most k, then x_1 occurs three times in $T_k^+(x_1)$, otherwise x_1 will have at least three repeats, which is impossible. This implies that x_1 is a double selfrepeat. Since two of v_i, say v_k and v_l, for $k, l \in \{1, 2, \ldots, d+1\}$, occur in the walk joining two selfrepeats then v_k and v_l are selfrepeats. Then it is not possible for the d out-neighbours of x_1 to reach v_{d+1}. $\square$

Theorem 2.2 *For $d \geq 2$ and $k \geq 3$, every $(d, k, 2)$-digraph is out-regular and almost in-regular. Moreover, if $k = 2$ then $d - 1 \leq |S| \leq d$ and if $k \geq 3$ then $|S| = d$.*

Proof. The out-regularity of $(d, k, 2)$-digraphs was established in the Introduction. Hence we only need to prove that every $(d, k, 2)$-digraph is almost in-regular. If $S = \emptyset$ then $(d, k, 2)$-digraph is diregular. By Lemma 2.4, if $S \neq \emptyset$ then all vertices in S have in-degree $d - 1$. This gives

$$\sigma = \sum_{x \in S'}(d^-(x) - d) = \sum_{v \in S}(d - d^-(v)) = |S| \leq 2d.$$

Take an arbitrary vertex $v \in S$; then $|N^-(v)| = |N_1^-(v)| = d - 1$. By the diameter assumption, the union of all the sets $N_t^-(v)$ for $0 \leq t \leq k$ is the entire vertex set $V(G)$ of G, which implies that

$$|V(G)| \leq \sum_{t=0}^{k} |N_t^-(v)|. \tag{6}$$

To estimate the above sum we can observe that

$$|N_t^-(v)| \leq \sum_{u \in N_{t-1}^-(v)} d^-(u) = d|N_{t-1}^-(v)| + \varepsilon_t, \tag{7}$$

where $2 \leq t \leq k$ and $\varepsilon_2 + \varepsilon_3 + \ldots + \varepsilon_k \leq \sigma$.

It is not difficult to see that a safe upper bound on the sum of $|V(G)|$ is obtained from inequality (7) by setting $\varepsilon_2 = \sigma = |S|$, and $\varepsilon_t = 0$, for $3 \leq t \leq k$; note that the latter is equivalent to assuming that *all* vertices from $S \setminus \{v\}$ are contained in $N_k^-(v)$ and that all vertices of S' belong to $N_1^-(v)$. This way we successively obtain

$$\begin{aligned}
|V(G)| &\leq 1 + |N_1^-(v)| + |N_2^-(v)| + |N_3^-(v)| + \ldots + |N_k^-(v)| \\
&\leq 1 + (d-1) + (d(d-1) + |S|)(1 + d + \cdots + d^{k-2}) \\
&= d + d^2 + \cdots + d^k + (|S| - d)(1 + d + \cdots + d^{k-2}) \\
&= M_{d,k} - 2 + (|S| - d)(1 + d + \cdots + d^{k-2}) + 1.
\end{aligned}$$

But G is a digraph of order $M_{d,k} - 2$ which implies that

$$\begin{aligned}
(|S| - d)(1 + d + \cdots + d^{k-2}) + 1 &\geq 0 \\
(|S| - d)\frac{d^{k-1} - 1}{d - 1} + 1 &\geq 0 \\
|S| &\geq d - \frac{d-1}{d^{k-1} - 1}
\end{aligned}$$

If $k = 2$ and $d \geq 3$ then $|S| \geq d - 1$. Since we also have $1 \leq |S| \leq d$, we get $d - 1 \leq |S| \leq d$. If $k \geq 3$ and $d \geq 3$ then $|S| \geq d$ as $0 < \frac{d-1}{d^{k-1}-1} < 1$. This implies $|S| = d$. That is, in both cases G is almost in-regular. □

2.3 Diregularity of $(2, k, 2)$-digraph

The proof of the non-existence of digraphs of defect two for $d = 2$ and $k \geq 3$ in [5] relies on the diregularity of such digraphs which was proved by Slamin and Miller in [12]. Here we apply the results from the previous section to present a somewhat simpler proof.

In the case of diameter $k = 2$, there are four non-isomorphic digraphs of defect two of out-degree 2 with vertices not all of the same in-degree, as shown in Figure 4.

Let S be the set of all vertices of G whose in-degree is 1; let S' be the set of all vertices of G whose in-degree is greater than 2. Applying Theorem 2.2 asserts that, for $d = 2$ and $k \geq 3$, every $(2, k, 2)$-digraph is out-regular and almost in-regular.

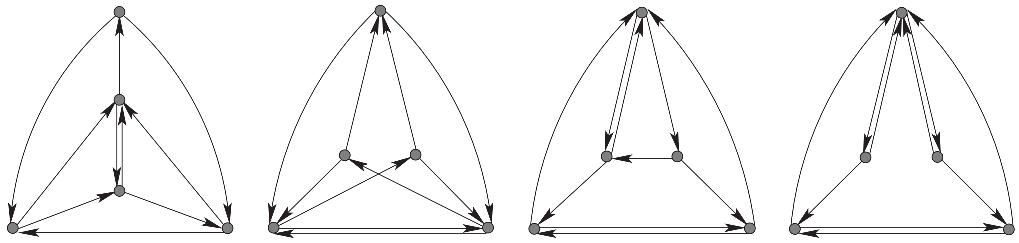

Figure 4: Four non-isomorphic almost in-regular digraphs of order $M_{2,2} - 2$.

We will next prove that $(2, k, 2)$-digraphs must be diregular if $k \geq 3$.

Theorem 2.3 [12] *Every $(2, k, 2)$-digraph is diregular, for $k \geq 3$.*

Proof. Let $G \in \mathcal{G}(2, k, 2)$, $k \geq 3$. By Theorem 2.2, if G is an almost diregular digraph which is not diregular then $|S| = 2$. Let $S = \{v_1, v_2\}$. Suppose $N^-(v_1) = \{x_1\}$ and $N^-(v_2) = \{x_2\}$. Then the in-excess $\sigma^- = \sum_{v \in S}(d - d^-(v)) = 2$. This implies that $1 \leq |S'| \leq 2$. Suppose $|S'| = 2$. Then $S' = \{x_1, x_2\}$. If $d^-(x_1) = 3$ then it is not possible to reach v_1 from all the other vertices in G.

Therefore, $|S'| = 1$, $x_1 = x_2 (= x)$ and $d^-(x) = 4$. We first consider the multisets $T_k^+(v_1)$ and $T_k^+(v_2)$. Since v_1 must reach v_2 within distance at most k and at the same time v_2 also must reach v_1 within distance at most k, vertex x must occur at distance exactly $k - 1$ from both v_1 and v_2. It follows that x occurs three times in the multiset $T_k^+(x)$, which means that x is a double selfrepeat. Vertex x is also a repeat for every other vertex in G. Let $y_i \in N^-(x)$, for all $i = 1, 2, 3, 4$. Then two of y_i occur at distance $k - 2$ from v_1 (respectively, v_2). Without loss of generality, we suppose that $y_1 \in N_{k-2}^+(v_2)$ and $y_2 \in N_{k-2}^+(v_1)$. It follows that y_1 and y_2 are each a selfrepeat exactly once.

Let S_1 and S_2 be two multisets. We denote $S = S_1 \uplus S_2$ the multiset defined as follows. If x occurs n_1 times in S_1 and n_2 times in S_2 then x occurs exactly n_1+n_2 times in S. Consider the multiset $T^+_k(y_1) = V(G) \uplus \{x\} \uplus \{y_1\}$. Alternatively, we can express $T^+_k(y_1) = T^+_{k-1}(c_1) \uplus T^+_{k-1}(x) \uplus \{y_1\}$. Combining these two equations gives

$$V(G) \uplus \{x\} \quad = \quad T^+_{k-1}(c_1) \uplus T^+_{k-1}(x) \tag{8}$$

Consider the multiset $T^+_k(y_2) = V(G) \uplus \{x\} \uplus \{y_2\}$. Similarly, we can express $T^+_k(y_2) = T^+_{k-1}(c_2) \uplus T^+_{k-1}(x) \uplus \{y_2\}$. Combining these two equations gives

$$V(G) \uplus \{x\} \quad = \quad T^+_{k-1}(c_2) \uplus T^+_{k-1}(x) \tag{9}$$

From Equations (8) and (9), it follows that $T^+_{k-1}(c_1) = T^+_{k-1}(c_2)$. Since $N^+_{k-l-1}(c_2) \in T^+_{k-1}(x)$, we get $c_1 = c_2$, otherwise y_1 has at least three repeats, namely, $\{y_1\} \uplus \{x\} \uplus \{u | u \in N^+_{k-l-1}(c_2) \cap T^+_{k-1}(c_2)\}$, which is impossible.

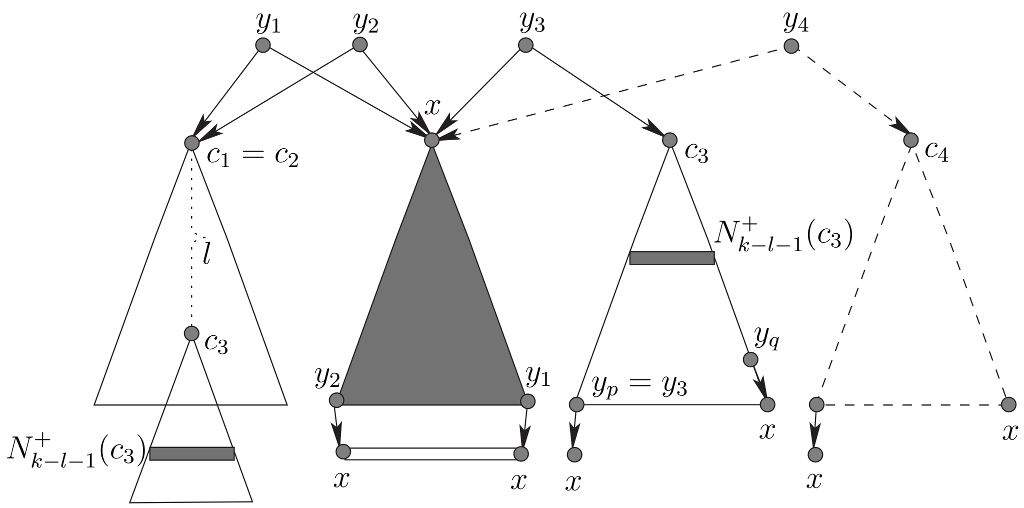

Figure 5: Illustration for the case $|S| = 2$.

We now consider the multiset $T^+_k(y_3) = V(G) \uplus \{x\} \uplus \{r(y_3)\}$. We have also $T^+_k(y_3) = T^+_{k-1}(c_3) \uplus T^+_{k-1}(x) \uplus \{y_3\}$. Combining these two equations gives

$$V(G) \uplus \{x\} \quad = \quad T^+_{k-1}(c_3) \uplus T^+_{k-1}(x) \uplus \{y_3\} - \{r(y_3)\} \tag{10}$$

We need to show that $r(y_3) = y_3$. We consider the multiset $T^+_{k-1}(c_3)$. Since y_1 and y_2 are each repeat exactly once, that is, $r(y_1) = y_1$ and $r(y_2) = y_2$, it follows $y_1, y_2 \notin T^+_{k-1}(c_3)$. Vertex y_q must not be y_3, otherwise there exist a cycle of length $k-1$ in G, which is impossible. This implies that $y_p = y_3$. It follows that y_3 occurs twice in the multiset $T^+_k(y_3)$, which means that y_3 is a selfrepeat. Then Equation (10) gives

$$V(G) \uplus \{x\} \quad = \quad T^+_{k-1}(c_3) \uplus T^+_{k-1}(x) \tag{11}$$

By combining Equations (8) and (11), we get $T^+_{k-1}(c_1) = T^+_{k-1}(c_3)$. Since $N^+_{k-l-1}(c_3) \in T^+_{k-1}(x)$, see Figure 5, we have $c_1 = c_3$, otherwise y_1 has at least three repeats, namely, $\{y_1\} \uplus \{x\} \uplus \{u | u \in N^+_{k-l-1}(c_3) \cap T^+_{k-1}(c_3)\}$, which is impossible. Therefore, $c_1 = c_2 = c_3 (= c)$. Since $c_1 \in N^+(y_1)$, $c_2 \in N^+(y_2)$ and $c_3 \in N^+(y_3)$, it follows that $c \in N^+(y_1) \cap N^+(y_2) \cap N^+(y_3)$. This implies that $S' = \{x, c\}$, which is a contradiction. $\square$

We conclude this paper with a conjecture.

Conjecture 2.1 *All digraphs of out-degree $d \geq 2$ and defect 2 are diregular, for diameter $k \geq 3$.*

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