Disjoint Hamilton cycles in the star graph

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Abstract

In 1987, Akers, Harel and Krishnamurthy proposed the star graph $\Sigma(n)$ as a new topology for interconnection networks. Hamiltonian properties of these graphs have been investigated by several authors. In this paper, we prove that $\Sigma(n)$ contains $\lfloor n/8 \rfloor$ pairwise edge-disjoint Hamilton cycles when n is prime, and $\Omega(n/\log\log n)$ such cycles for arbitrary n.

1 Introduction

A multicomputer system has nodes that communicate by exchanging messages through an interconnection network. The topology of this network can be conveniently modeled by an undirected graph whose properties determine how efficiently the system can function. In such a system, the order of the graph (the number of its nodes) is pre-determined, while its size (the number of interconnections) is affected by cost and the physical characteristics of the processors.

In systems where all nodes have the same characteristics, the interconnection design calls for a regular graph. Further obvious desired properties of these graphs include the following:

- Fault tolerance: high connectivity.
- Fast communication: small diameter.

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- Symmetry: vertex- and edge-transitivity.
- Recursive structure.

Another desired property is the presence of a Hamilton cycle, or multiple edge-disjoint Hamilton cycles, which increases fault tolerance and plays an important role in many parallel algorithms, such as broadcasting, gossiping or sorting.

The search for disjoint Hamilton cycles in graphs is an active and popular area in graph theory. Recently, this search was carried over to the design of network topologies. The n-cube (a classical interconnection network) is an n-regular graph with 2^n vertices and diameter n. In other words, the degree and diameter of the n-cube are logarithmic in the number of its vertices. The n-cube also has a recursive structure and a Hamiltonian decomposition, i.e., $\lfloor \frac{n}{2} \rfloor$ pairwise edge-disjoint Hamilton cycles. It became a kind of benchmark against which other topologies are compared. In particular, the Hamiltonian properties of various alternatives to the n-cube topology have been investigated by a number of authors. For instance, in [2, 3, 12], multiple disjoint Hamilton cycles are constructed in various tori and in the de Bruijn networks. Micheneau [10] studies disjoint Hamilton cycles in recursive circulant graphs, which were proposed as a new topology for multicomputer networks in [11]. Another recent topology is the locally twisted cube introduced in [19], in which the search for disjoint Hamilton cycles is still going on.

The star-graph topology (defined in the next section) was introduced in 1986 by Akers et al. [1]. Comparing the star-graph topology to the n-cube, the authors concluded that similar desired properties on networks of comparable size are achieved by star graphs with smaller degrees (fewer interconnections) and smaller diameter. As noted above, the n-cube is known to have many edge-disjoint Hamilton cycles. The corresponding question for the star graph was not addressed in [1].

In this paper, we prove that star graphs too have multiple edge-disjoint Hamilton cycles. More precisely, our main result is the following theorem:

Theorem 1. *(i) If n is a prime, then $\Sigma(n)$ contains $\lfloor n/8 \rfloor$ pairwise edge-disjoint Hamilton cycles.*

(ii) For arbitrary n, there are $\Omega(n/\log\log n)$ pairwise disjoint Hamilton cycles in $\Sigma(n)$.

It is likely that star graphs are Hamiltonian decomposable (this was shown in [7] to be true for $\Sigma(5)$, a 4-regular graph of order 120).

2 Definitions, notation and background

In this paper, we use common graph-theoretical definitions and notation. The symmetric group $\mathcal{S}_n$ is the group of permutations of order n. Since we use modular arithmetic extensively, we prefer to use $\mathbb{Z}_n = \{0, \ldots, n-1\}$ as the underlying set on which the permutations are defined. To each permutation π, we associate its *representing word* $[\pi] = \pi(0) \ldots \pi(n-1)$.

Throughout this paper, we write $\circ$ for the group operation in $\mathcal{S}_n$, and define

$$(\sigma \circ \tau)(i) = \sigma(\tau(i)).$$

Observe that if π is a permutation and (i, j) a transposition, then the representing word of $(i, j) \circ \pi$ is obtained from that of π by interchanging the symbols i and j. Similarly, the representing word $[\pi \circ (i, j)]$ is obtained from $[\pi]$ by interchanging the symbols *at positions* i and j (counting from 0).

Definition 1. *Given a group* Γ *and a subset* $X \subset \Gamma$ *such that* $X = X^{-1}$*, the* Cayley graph $Cay(\Gamma, X)$ *is the graph with vertex set* Γ *whose edges join* g *to* gx *for all* $g \in \Gamma$ *and* $x \in X$*.*

Thus, the Cayley graph $Cay(\Gamma, X)$ is an $|X|$-regular graph of order $|\Gamma|$. In the symmetric group $\mathcal{S}_n$ any transposition (i, j) is its own inverse, and thus any set of transpositions $X \subset \mathcal{S}_n$ defines a Cayley graph $Cay(\mathcal{S}_n, X)$. We are now ready to define the star graph:

Definition 2. *The* star graph $\Sigma(n)$ *is the Cayley graph* $Cay(\mathcal{S}_n, X_0)$ *where* $X_0 = \{(0, 1), (0, 2), \ldots, (0, n-1)\} \subset \mathcal{S}_n$*.*

The star graph $\Sigma(n)$ is $(n-1)$-regular and bipartite, with one color class consisting of the even permutations and the other color class of the odd ones. Note that the edges of $\Sigma(n)$ connect permutations whose representing words differ by the transposition of the leading symbol with some other symbol.

The star graph $\Sigma(4)$ is shown in Figure 1. (Vertex labels in this and similar figures are the representing words of the associated permutations.)

An arbitrary generating set Y of transpositions in $\mathcal{S}_n$ may be conveniently represented by a graph on $\mathbb{Z}_n$ with an edge ij for each transposition $(i, j) \in Y$. This justifies the term ‘star graph’ since the graph representing X is a star (see Figure 2a). Analogously, we may define the *path graph*, $\Pi(n)$, as the Cayley graph

$$\Pi(n) = \mathrm{Cay}(\mathcal{S}_n, \{(0, 1), (1, 2), \ldots, (n-2, n-1)\})$$

where the generating set is a path on $\mathbb{Z}_n$ (Figure 2b). The graph $\Pi(4)$ is shown in Figure 3.

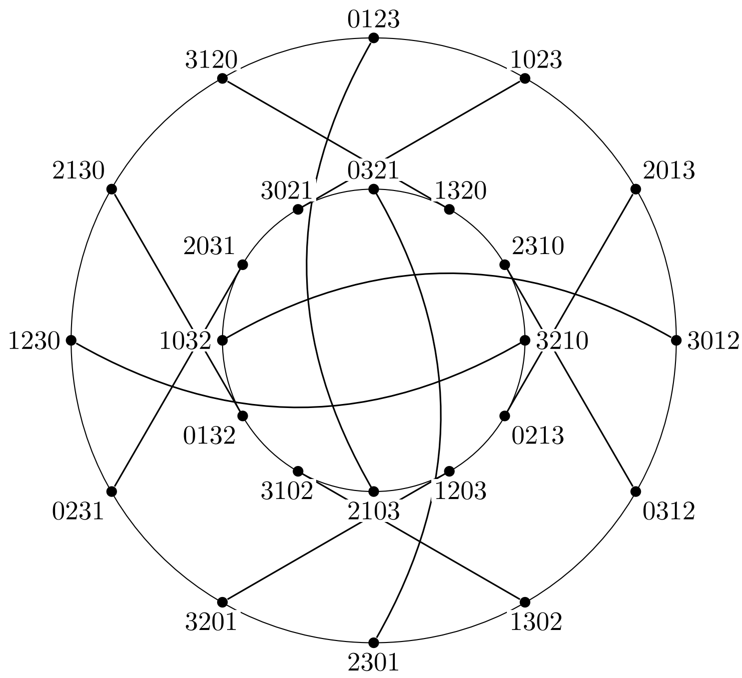

Figure 1: The star graph $\Sigma(4)$.

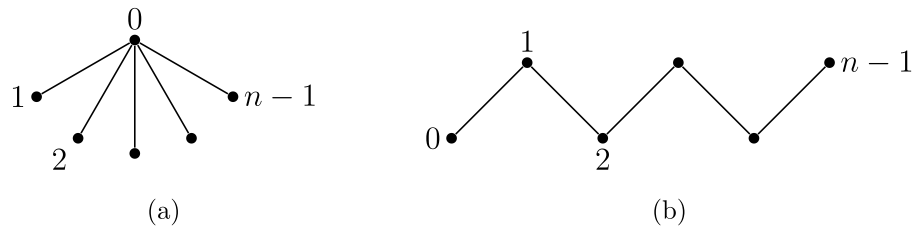

Figure 2: The generating sets of (a) $\Sigma(n)$ and (b) $\Pi(n)$.

Hamiltonian properties of path graphs are well-studied, albeit under a different name: permutation Gray codes. In fact, a *permutation Gray code* is nothing but a Hamilton cycle in $\Pi(n)$. Put in a more traditional way, it is a list of all permutations of length n, such that no permutation is repeated and neighboring permutations differ by a transposition of adjacent entries. The first results on the existence of such a Gray code were due to Johnson [8] and Trotter [18] (see also [17]) who produced an explicit algorithm to find one, the celebrated Johnson–Trotter algorithm. (For background on various types of combinatorial Gray codes, we refer the reader to the excellent survey by Savage [15]. More information on permutation Gray codes may be found in [16].)

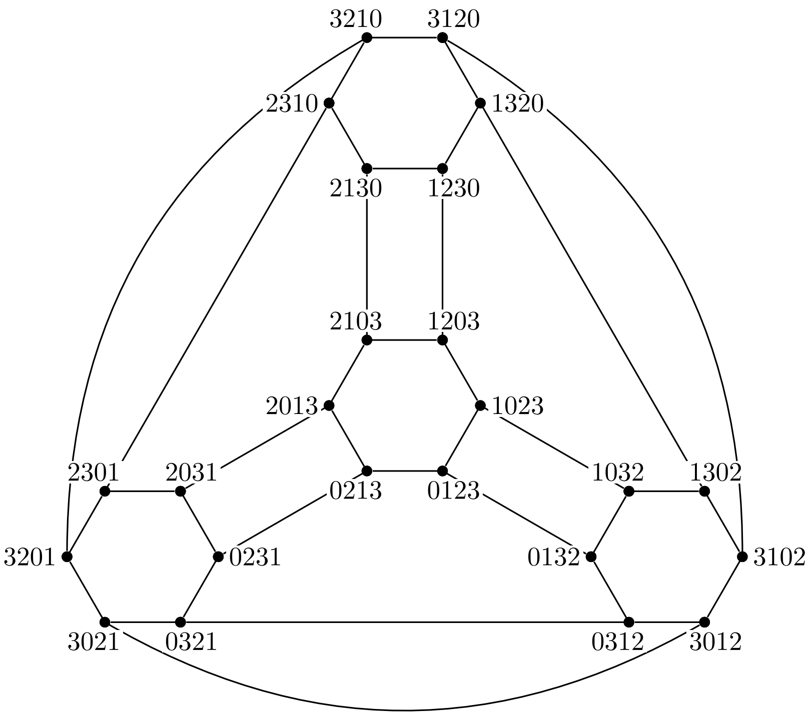

Figure 3: The path graph Π(4).

Results on the existence of Hamilton cycles in path graphs are generalized by the following theorem of Kompel'makher and Liskovets [9]:

Theorem 2. *Let X be any generating set of transpositions in $\mathcal{S}_n$, where $n \geq 3$. Then* $\mathrm{Cay}(\mathcal{S}_n, X)$ *is Hamiltonian.*

In particular, we may replace the path representing the generating set of $\Pi(n)$ by any tree on $\mathbb{Z}_n$ and still obtain a Hamiltonian Cayley graph. We remark that it is an open problem whether $\mathrm{Cay}(\mathcal{S}_n, X)$ is Hamiltonian for any generating set X of *involutions* (see, e.g., [13]).

Theorem 2 implies that the star graphs $\Sigma(n)$ ($n \geq 3$) are Hamiltonian, but more can be said about their Hamiltonian properties. As in [6], a bipartite graph G is said to be *strongly Hamiltonian-laceable* if every two vertices x, y are joined by a Hamilton path if they are from distinct color classes, and by a path of length $|V(G)| - 2$ if they are from the same color class. The following result was proved in [6]:

Theorem 3. *For every $n \geq 4$, the star graph $\Sigma(n)$ is strongly Hamiltonian-laceable.*

3 Disjoint Hamilton cycles in star graphs

Throughout this section, we assume that $n \geq 5$.

For $i \in \{1, \dots, \lfloor n/2 \rfloor\}$, let $C^i(n)$ be the set of all edges $\sigma\tau$ of $\Sigma(n)$ such that $\sigma(0) - \tau(0)$ is congruent to $\pm i$ modulo n. It is easy to observe the following (the inner and outer cycles in Figure 1 may be a helpful illustration for $n = 4$):

Observation 4. *The graph $C^1(n)$ is a 2-factor of $\Sigma(n)$.*

We remark that a similar property holds for $C^j(n)$ if $n > 2j$, but we will not need this.

Lemma 5. *If F is a cycle of $C^1(n)$ and $i, j \in \{0, \dots, n-1\}$ distinct integers, then F contains exactly one vertex π with $\pi(0) = i$ and $\pi(n-1) = j$.*

Proof. Start at any vertex $\rho = \rho_0$ of F and let $k = \rho_0(0)$. Follow F to $\rho_1 := (k, k+1) \circ \rho$ (recall that $(k, k+1)$ denotes a transposition) and continue along F through (say) $\rho_2, \rho_3, \dots$. The change at each step is that $\rho_{i+1}(0) = \rho_i(0) + 1$ and there is exactly one ℓ such that $\rho_{i+1}(\ell) = \rho_i(\ell) - 1$; all other values of ρ_{i+1} are the same as in ρ_i. Moreover, the value at ℓ will not change for the $n-2$ permutations following $\rho_{i+1}(\ell)$. This implies that

$$\rho_i(j) = \rho_{i+n-1}(j) + 1$$

for all $j \in \mathbb{Z}_n$. A consequence of this is that for any i, $\rho_i = \rho_{i+n(n-1)}$, and in fact it is easy to see that $|F| = n(n-1)$.

Observe that as i increases, $\rho_i(n-1)$ is always constant for a block of $n-1$ steps and then drops by one. Within the block, $\rho_i(0)$ takes all the $n-1$ possible values different from $\rho_i(n-1)$. It follows that we eventually encounter a permutation $\pi = \rho_t$ such that $\pi(0) = i$ and $\pi(n-1) = j$. Since $|F| = n(n-1)$, this cannot happen more than once. □

Note that by this lemma, the length of each cycle of $C^1(n)$ is $n(n-1)$ and hence there are $(n-2)!$ such cycles.

Secondly, Lemma 5 implies that every cycle F of $C^1(n)$ contains a unique vertex π with $\pi(n-1) = n-1$ and $\pi(0)$ equal to a given $k \in \{0, \dots, n-2\}$. We let this vertex be denoted by $\sigma^k(F)$. Furthermore, we call $\sigma^0(F)$ the *signature* of F. If π is the signature of a cycle F of $C^1(n)$, we write just π^k for $\sigma^k(F)$.

Observation 6. *Let F be a cycle of $C^1(n)$ with signature σ and $0 \le k \le n-2$. The vertex σ^k is given by*

$$\sigma^k(i) = \begin{cases} k & \text{if } i = 0, \\ \sigma(i) - 1 & \text{if } i \ge 1 \text{ and } \sigma(i) \le k, \\ \sigma(i) & \text{otherwise.} \end{cases}$$

Lemma 7. *Let $\sigma \in \mathcal{S}_n$ be a signature and let $\tau = (k, k+1) \circ \sigma$, where $1 \le k \le n-3$. Then τ is also a signature and the following holds:*

(i) $\tau^{k+1} = (k-1, k+1) \circ \sigma^{k-1}$,

(ii) σ^{k-1} *and* τ^{k+1} *are joined by an edge in* $C^2(n)$*, and the same holds for* σ^{k+1} *and* τ^{k-1}.

Proof. The permutation τ is a signature since $\tau(0) = 0$ and $\tau(n-1) = n-1$. We prove (i). By Observation 6 applied to τ^{k+1}, we see that

$$\tau^{k+1}(i) = \begin{cases} k+1 & \text{if } i = 0, \\ \sigma(i) - 1 & \text{if } i \ge 1 \text{ and } \sigma(i) \le k-1, \\ k & \text{if } \sigma(i) = k, \\ k-1 & \text{if } \sigma(i) = k+1, \\ \sigma(i) & \text{otherwise.} \end{cases}$$

Comparing this with the statement of Observation 6 for σ^{k-1}, we see that $\tau^{k+1} = (k-1, k+1) \circ \sigma^{k-1}$ as claimed.

To prove (ii), observe that σ^{k-1} and $\tau^{k+1} = (k-1, k+1) \circ \sigma^{k-1}$ are joined by an edge in $\Sigma(n)$ since the latter permutation may be written as $\sigma^{k-1} \circ (0, i)$, where $\sigma^{k-1}(i) = k+1$. This edge belongs to $C^2(n)$ since $k-1$ and $k+1$ differ by 2. The second part of (ii) follows similarly from (i) with the roles of σ and τ reversed. □

Let $C^{12}(n)$ be the spanning subgraph of $\Sigma(n)$ with edge set $C^1(n) \cup C^2(n)$. Furthermore, let $C^{12/1}(n)$ be the graph obtained from $C^{12}(n)$ by contracting each cycle of $C^1(n)$ to a single vertex, discarding all loops and replacing multiple edges by single edges. Somewhat surprisingly, $C^{12/1}(n)$ contains a spanning copy of $\Pi(n-2)$, as shown by the following lemma.

Lemma 8. *For any permutation π of length $n-2$, define a permutation $\overline{\pi}$ of length n by setting*

$$\overline{\pi}(i) = \begin{cases} 0 & \text{if } i = 0, \\ \pi^{-1}(i-1) + 1 & \text{if } i = 1, \dots, n-2, \\ n-1 & \text{if } i = n-1. \end{cases}$$

Let φ be the mapping from $\Pi(n-2)$ to $C^{12/1}(n)$ sending each vertex π to the vertex corresponding to the cycle of $C^1(n)$ with signature $\overline{\pi}$. Then the following holds:

(i) if $\rho = \pi \circ (k-1, k)$, where $1 \le k \le n-3$, then $\overline{\rho}^{k+1}$ is adjacent to $\overline{\pi}^{k-1}$ in $C^2(n)$,

(ii) φ is an isomorphism of $\Pi(n-2)$ with a spanning subgraph of $C^{12/1}(n)$.

Proof. (i) Let $\rho = \pi \circ (k-1, k)$. It is straightforward to verify that $\overline{\rho} = (k, k+1) \circ \overline{\pi}$. Thus, the claim follows from Lemma 7(ii).

(ii) If π and ρ are adjacent vertices of $\Pi(n-2)$, then by (i), $\varphi(\pi)$ and $\varphi(\rho)$ are adjacent in $C^{12/1}(n)$. Hence φ is an injective homomorphism. Since the graphs in question have the same number of vertices, φ has the stated property. □

Can we use a Hamilton cycle H in $\Pi(n-2)$ (which is known to exist) and Lemma 8 to build up a Hamilton cycle in $C^{12}(n)$? It turns out that we can, provided that H satisfies an additional constraint. Let us call H *a doubly adjacent Gray code* if for every vertex π of $\Pi(n-2)$, the two neighbours of π on H can be written as $\pi \circ (k-1, k)$ and $\pi \circ (k, k+1)$ for some $k \in \{1, \dots, n-4\}$. By a deep result of Compton and Williamson [5] (see also [4]), doubly adjacent Gray codes exist in all path graphs except the trivial cases $\Pi(1)$ and $\Pi(2)$ (see Figure 4 for an illustration):

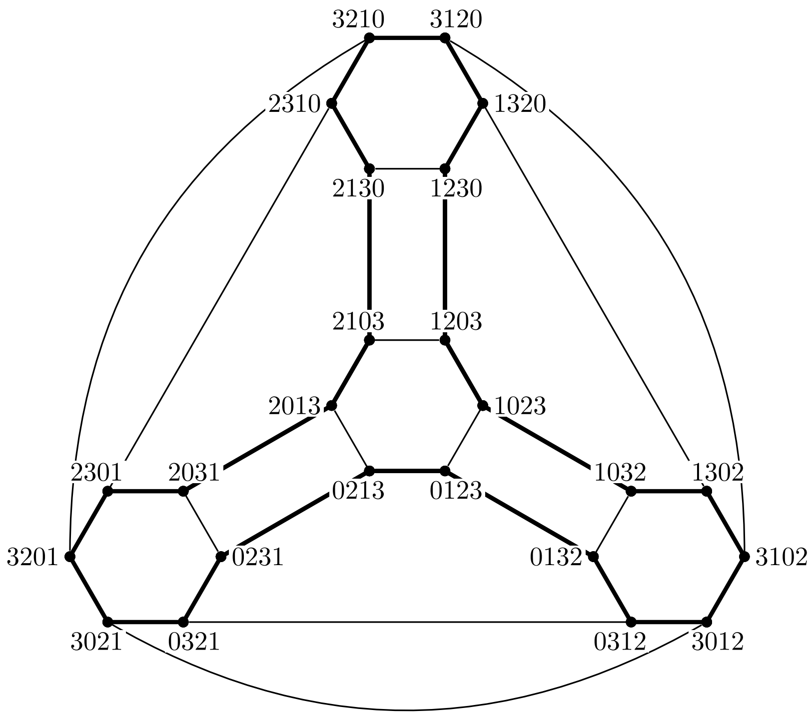

Figure 4: A doubly adjacent Gray code in $\Pi(4)$ (bold).

Theorem 9 ([5]). *For every $m \geq 3$, the path graph $\Pi(m)$ admits a doubly adjacent Gray code.*

With Theorem 9, it is not difficult to derive the following:

Lemma 10. *For $n \geq 5$, the graph $C^{12}(n)$ is Hamiltonian.*

Proof. Let $F = \pi_0, \ldots, \pi_{N-1}$ be a doubly adjacent Gray code in $\Pi(n-2)$, where $N = (n-2)!$. Fix $i \in \{0, \ldots, N-1\}$. We have that $\pi_i = \pi_{i-1} \circ (k-1, k)$ for some $k \in \{1, \ldots, n-3\}$ (indices modulo N). Moreover, one of the following cases hold:

(a) $\pi_{i+1} = \pi_i \circ (k, k+1)$ and $k \leq n-4$,

(b) $\pi_{i+1} = \pi_i \circ (k-2, k-1)$ and $k \geq 2$.

In each case, we define a subpath P_i of the desired Hamilton cycle in $\Sigma(n)$; informally, P_i will be used as the replacement for the edge $\pi_i\pi_{i+1}$ of F. Let D_i be the cycle of $C^1(n)$ with signature $\overline{\pi_i}$ (using the notation of Lemma 8). Furthermore, let us call $\overline{\pi_i}^{k+1}$ the *reference vertex of D_i*.

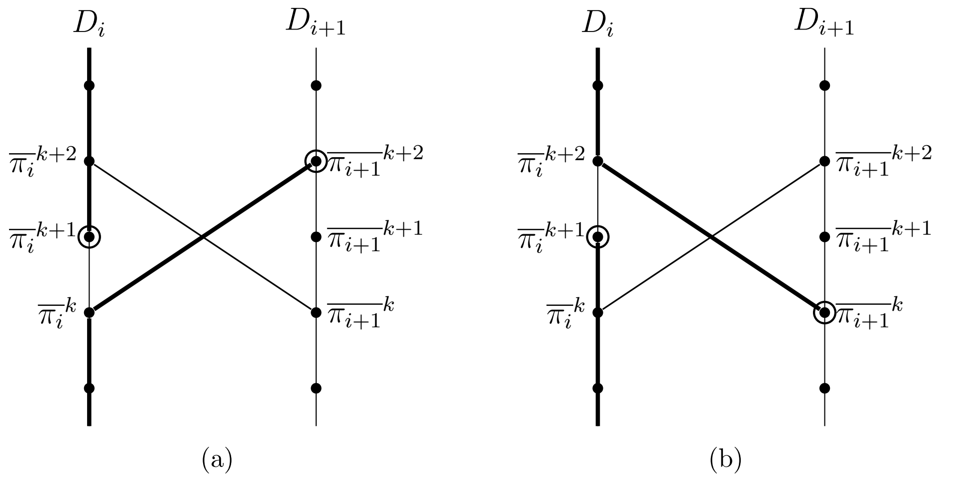

Figure 5: The path P_i (bold) in cases (a) and (b) in the proof of Lemma 10. The reference vertices are circled.

In case (a), we let P_i start in the reference vertex of D_i, continue through $\overline{\pi_i}^{k+2}$ along D_i as far as $\overline{\pi_i}^{k}$, and then follow the edge $\overline{\pi_i}^{k}\overline{\pi_{i+1}}^{k+2}$ of $C^2(n)$ (which exists by Lemma 7). See Figure 5 for an illustration. In case (b), P_i starts in the reference vertex of D_i, follows D_i through $\overline{\pi_i}^{k}$ to $\overline{\pi_i}^{k+2}$, and takes the edge $\overline{\pi_i}^{k+2}\overline{\pi_{i+1}}^{k}$.

Observe that the ending vertex of P_i is, in both cases, the reference vertex of D_{i+1}. Since this is also the starting vertex of P_{i+1}, all the paths P_j may be concatenated. Moreover, the resulting cycle in $\Sigma(n)$ is Hamiltonian, because each P_j spans all of the cycle D_j. □

We can now proceed to the proof of our main result.

Proof of Theorem 1. We may clearly continue to assume (as we do throughout Section 3) that $n \geq 5$. Let $U \subset \mathbb{Z}_n^* = \{1, \ldots, n-1\}$ be the set of elements relatively prime to n. For technical reasons, we extend the definition of the sets $C^i(n)$ to let i range over all of $\mathbb{Z}_n^*$ by setting, for $i = 1, \ldots, \lfloor (n-1)/2 \rfloor$,

$$C^{-i}(n) := C^i(n).$$

(Throughout this proof, all arithmetic on the indices is performed modulo n.)

Claim 1. *If $j \in U$, then $C^j(n) \cup C^{2j}(n)$ is Hamiltonian.*

Define a permutation ψ on V by

$$\psi(i) = ij \bmod n.$$

The mapping $\pi \mapsto \psi \circ \pi$ on $\mathcal{S}_n$ is an automorphism of $\Sigma(n)$ that carries each edge set $C^i(n)$ to $C^{ij}(n)$. Since we know that $C^1(n) \cup C^2(n)$ is Hamiltonian by Lemma 10, the same follows for $C^j(n) \cup C^{2j}(n)$. This proves the claim.

Thus, to prove the theorem, it suffices to find an appropriately large set $X \subset U$ such that for all $x \in X$, the sets $Q(x) := \{x, -x, 2x, -2x\}$ are pairwise disjoint. Define an undirected graph H on U whose edge set is

$$E(H) = \{xy : \ x, y \in U, x \neq y \text{ and } Q(x) \cap Q(y) \neq \emptyset\}\,.$$

Claim 2. *The maximum degree $\Delta(H)$ of H is at most 9. If n is a prime then $\Delta(H) \leq 7$.*

Note first that by elementary arithmetic, for any $x \in \mathbb{Z}_n^*$ there are at most two $z \in \mathbb{Z}_n^*$ such that $x = 2z$. Moreover, if n is a prime, then there is exactly one such z.

Let x be any vertex of H. Its neighbors are the three elements of $Q(x) \setminus \{x\}$, the at most four elements z with $2z = \pm x$, and the at most two elements z' with $2z' = \pm 2x$ besides x and $-x$. Thus, $\Delta(H) \leq 9$. The improvement for prime n comes from the fact that in this case, there are only two z such that $2z = \pm x$. The proof of the claim is complete.

We now prove part (i) of the theorem. Let n be a prime. Since $\Delta(H) \leq 7$, one can employ the straightforward greedy method to find an independent set X of size at least $\lfloor (n-1)/8 \rfloor = \lfloor n/8 \rfloor$ in H. By the independence of X, the sets $Q(x)$ for $x \in X$ are pairwise disjoint and so there are $\lfloor n/8 \rfloor$ disjoint Hamilton cycles corresponding to the elements of X.

Part (ii) is proved similarly: in this case, $\Delta(H) \leq 9$ and we can find an independent set X in H such that $|X| \geq |U|/10 = \varphi(n)/10$, where φ is the Euler function. By the results in [14, paragraph I.8],

$$\varphi(n) = \Omega\Big(\frac{n}{\log\log n}\Big),$$

which implies that $|X|$ is $\Omega(n/\log\log n)$ as well and the proof is complete. □

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