NOTE

On cycle lengths in claw-free graphs with complete closure

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Abstract

We show a construction that gives an infinite family of claw-free graphs of connectivity $\kappa = 2, 3, 4, 5$ with complete closure and without a cycle of a given fixed length. This construction disproves a conjecture by the first author, A. Saito and R.H. Schelp.

1 Introduction

We consider finite simple undirected graphs $G = (V(G), E(G)$ and for concepts and notations not defined here we refer to [4]. Specifically, for $x, y \in V(G)$, $\mathrm{dist}_G(x, y)$ denotes the distance of x and y in G. The *girth* of G is the smallest length of a cycle in G. A graph G is *hamiltonian* if G contains a *hamiltonian cycle*, i.e. a cycle of length $|V(G)|$, and G is *pancyclic* if G contains cycles C_λ of all lengths λ, $3 \leq \lambda \leq |V(G)|$. A subpath of a cycle C with endvertices $x, y \in V(C)$ will be called a *segment* and denoted xCy. A graph G is *claw-free* if G does not contain an induced subgraph isomorphic to the *claw* $K_{1,3}$.

The following concepts were introduced in [14]. A vertex $x \in V(G)$ is *eligible* if its *neighborhood* $N_G(x) = \{y \in V(G)|\ xy \in E(G)\}$ induces a connected noncomplete graph, and x is *simplicial* if the subgraph induced by $N_G(x)$ is complete. The *local completion* of G at a vertex x is the graph obtained from G by adding all edges with both vertices in

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$N_G(x)$ (note that the local completion at x turns x into a simplicial vertex). The *closure* $\mathrm{cl}(G)$ of a claw-free graph G is the graph obtained from G by recursively performing the local completion operation at eligible vertices as long as this is possible. It was proved in [14] that for every claw-free graph G

(*i*) $\mathrm{cl}(G)$ is uniquely determined,

(*ii*) $\mathrm{cl}(G)$ is the line graph of a triangle-free graph,

(*iii*) $\mathrm{cl}(G)$ is hamiltonian if and only if G is hamiltonian.

However, as shown in [5], for any integer $k \geq 2$ there is a k-connected nonpancyclic claw-free graph with pancyclic closure.

Specifically, if $\mathrm{cl}(G)$ is complete (hence pancyclic), then G is hamiltonian by (*iii*). As shown in [15], such a graph G contains also a cycle of length $|V(G)| - 1$, but there are nonpancyclic graphs with complete closure. The following conjecture was posed in [15].

Conjecture A [15]. *Let c_1, c_2 be fixed constants. Then for large n, any claw-free graph G of order n whose closure is complete contains cycles C_i for all i, where $3 \leq i \leq c_1$ and $n - c_2 \leq i \leq n$.*

In our main result, Theorem 1, we disprove Conjecture A by giving infinite families of counterexamples.

2 Main result

Theorem 1. *Let κ, λ be integers, $2 \leq \kappa \leq 5$, $\lambda \geq 33$ if $\kappa \in \{2, 3, 4\}$ and $\lambda \geq 52$ if $\kappa = 5$. Then there is an infinite family of claw-free graphs of connectivity κ with complete closure and not containing a cycle of length λ.*

As mentioned in the introduction, Theorem 1 disproves one part of Conjecture A in the sense that c_1 does not have to be large. However, we believe that the second part of Conjecture A is true, and that such a construction as shown in Theorem 1 is possible only for connectivities $\kappa \leq 5$. Thus, we conjecture the following.

Conjecture 2. *Let c be a fixed constant. Then for large n, any claw-free graph G of order n whose closure is complete contains cycles C_i for all i, $n - c \leq i \leq n$.*

Conjecture 3. *Every 6-connected claw-free graph with complete closure is pancyclic.*

3 Proof of Theorem 1

Case 1: $\kappa = 2$. Let $\ell \geq 3$, $p \geq 4$ be integers, let J_1^i, J_2^i, $i = 0, 1, \ldots, 2p-1$ be vertex-disjoint copies of the graphs shown in Figure 1, and let $G_{p,\ell}^2$ be the graph obtained by identifying $z_{2j} := (b_2^{2j} \equiv b_2^{2j+1})$, $z_{2j+1} := (b_4^{2j+1} \equiv b_4^{2j+2})$, $x_{2j} := (b_3^{2j} \equiv a_1^{2j})$, $x_{2j+1} := (b_3^{2j+1} \equiv a_2^{2j})$, $y_{2j} := (b_1^{2j+1} \equiv a_1^{2j+1})$, $y_{2j+1} := (b_1^{2j+2} \equiv a_2^{2j+1})$, and by relabeling $v_{2j} := c_1^{2j}$, $v_{2j+1} := c_2^{2j}$,

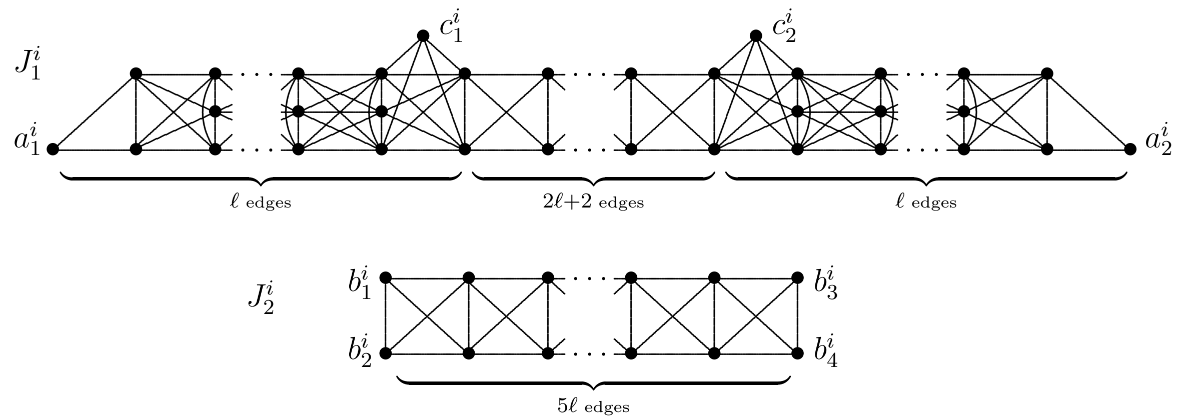

Figure 1

$w_{2j} := c_1^{2j+1}$, $w_{2j+1} := c_2^{2j+1}$, $j = 0, 1, \ldots, p-1$, where the notation $x := (b \equiv a)$ means that the vertex x is obtained by identifying the vertices a and b, and all indices are taken modulo $2p$ (see Figure 2).

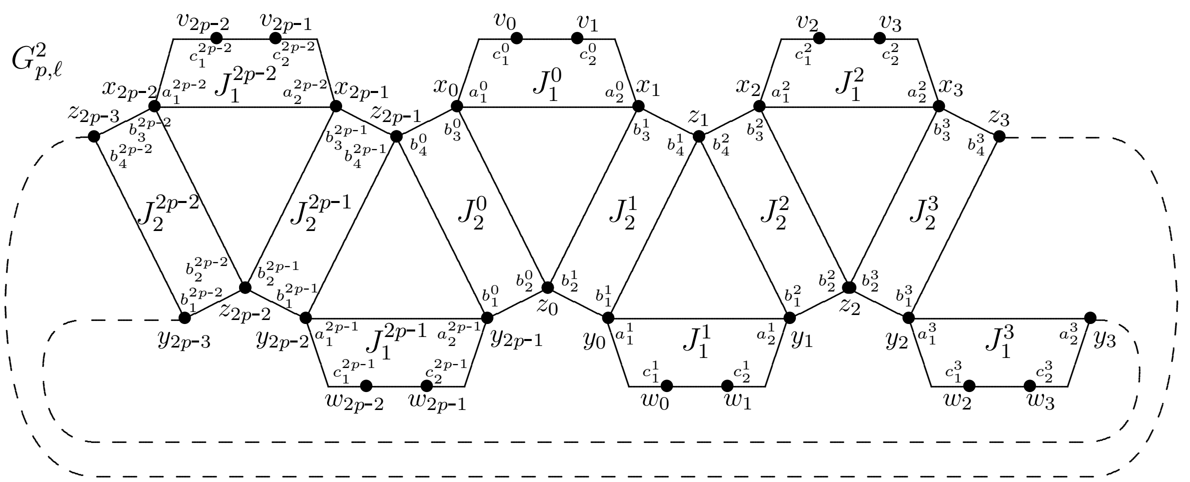

Figure 2

It is straightforward to check that $G_{p,\ell}^2$ is a claw-free graph of connectivity 2 with complete closure. Since $|V(J_1^i)| = |V(J_2^i)| = 10\ell + 2$, each copy of J_1^i and also each copy of J_2^i contains all cycles C_λ for $3 \le \lambda \le 10\ell + 2$. Any cycle of the next possible length $\lambda > 10\ell + 2$ has to contain a path from two copies of J_2^i and one copy of J_1^i, implying $\lambda \ge 14\ell + 2$. Thus, $G_{p,\ell}^2$ contains no C_λ for $10\ell + 3 \le \lambda \le 14\ell + 1$. For fixed $\ell \ge 3$ and arbitrary $p \ge 4$ this gives the required infinite family.

Case 2: $\kappa = 3$. We omit the proof in this case since the required graphs can be easily obtained from the construction for $\kappa = 4$. Details are left to the reader.

Case 3: $\kappa = 4$. Let n, s be positive integers such that s divides n, let C be a cycle of length n, and let $c_0, \ldots, c_{s-1}$ be integers. Then $n_s(c_0 \ldots, c_{s-1})$ denotes (cf. [8]) the graph obtained from C by adding all edges joining v_i to v_{i+c_j}, where $j \equiv i \pmod{s}$. It is shown in [7] that, for example, $14_2(5, -5)$ is the Heawood graph and $24_3(12, 7, -7)$ is the McGee $(3,7)$-cage (i.e. minimal cubic graph of girth 7), but, more importantly, the graph $272_{16}(45,59,-119,-89,101,-109,-72,72,109,-101,89,119,-59,-45,72,-72)$ is the cubic hamiltonian Cayley graph of girth 13, originally discovered by Hoare [9] (see also [1], p. 21). It is easy to see that, for any integer $r \geq 1$, the graph $H_r = (272r)_{16}(45,59,-119,-89,101,-109,-72,72,109,-101,89,119,-59,-45,72,-72)$ is also a cubic hamiltonian graph of girth 13.

Now, let $r \geq 1$, $\ell \geq 3$, set $p = 136r$, let H_r^i $(i = 1,2)$ be two vertex-disjoint copies of H_r with $V(H_r^i) = \{v_0^i, v_1^i, \ldots, v_{2p-1}^i\}$, let $C_{H_r^i}$ be hamiltonian cycle in H_r^i and set $\overline{H_r^i} = (V(H_r^i), E(H_r^i) \setminus E(C_{H_r^i}))$. We construct a graph $G_{r,\ell}^4$ from $\overline{H_r^1}$, $\overline{H_r^2}$ and the graph $G_{p,\ell}^2$ by identifying $v_i := (v_i \equiv v_i^1)$ and $w_i := (w_i \equiv v_i^2)$, $i = 0,1,\ldots,2p-1$. Then $G_{r,\ell}^4$ is clearly claw-free (since it is obtained by attaching edges to simplicial vertices of $G_{r,\ell}^2$), and has complete closure (since so does $G_{r,\ell}^2$). Moreover, $G_{r,\ell}^4$ is 4-connected: by the construction of $G_{r,\ell}^2$, a vertex cut R of size at most 3 would have to contain a pair of vertices x_{2i}, x_{2i+1} (or, symmetrically, y_{2i}, y_{2i+1}) for some i, but then the third vertex in R is adjacent to both v_{2i} and v_{2i+1} (or w_{2i} and w_{2i+1}), contradicting the fact that H_r is cubic.

It remains to show that $G_{r,\ell}^4$ contains no cycle C_λ for $10\ell + 3 \leq \lambda \leq 14\ell + 1$. Let, to the contrary, C_λ be such a cycle. By part 2, C_λ has to contain at least one edge from $\overline{H_r^1}$ or $\overline{H_r^2}$. Let k be the number of such edges in C, and set $V_1 = \{v_0, \ldots, v_{2p-1}\}$, $V_2 = \{w_0, \ldots, w_{2p-1}\}$. Since $\mathrm{dist}_{G_{r,\ell}^2}(a,b) \geq 2\ell + 2$ for any $a, b \in V_1 \cup V_2$, we have $14\ell + 2 > |E(C)| \geq k + k(2\ell + 2) = k(2\ell + 3)$, from which $k < \frac{14\ell+2}{2\ell+3} < 7$. Hence $k \leq 6$.

If $E(C) \cap E(\overline{H_r^i}) \neq \emptyset$ for both $i = 1,2$, then, since $a \in V_1$ and $b \in V_2$ implies $\mathrm{dist}_{G_{r,\ell}^2}(a,b) \geq 7\ell$, we have $|E(C)| \geq 14\ell + 2$, a contradiction. Hence we can suppose that C contains no edges from $\overline{H_r^2}$. Set $V(C) \cap V_1 = \{v_{i_0}, v_{i_1}, \ldots, v_{i_{2k-1}}\}$, where the labels are chosen along C and $v_{i_{2j+1}}v_{i_{2j+2}} \in E(\overline{H_r^1})$, $j = 0,\ldots,k-1$. Then $|i_{2j} - i_{2j+1}| \leq \frac{1}{2\ell+2}\mathrm{dist}_{G_{r,\ell}^2}(v_{i_{2j}}, v_{i_{2j+1}})$ for any j, $0 \leq j \leq k-1$ (indices modulo $2p$). Every segment $v_{i_{2j}}Cv_{i_{2j+1}}$ of C corresponds in H to a subpath of the hamiltonian cycle in H. These paths together with the edges $v_{i_{2j+1}}v_{i_{2j+2}} \in E(\overline{H_r^1})$ determine in H a closed walk W of length $|E(W)| = k + \sum_{j=0}^{k-1}|i_{2j} - i_{2j+1}| \leq k + \frac{1}{2\ell+2}\sum_{j=0}^{k-1}\mathrm{dist}_{G_{r,\ell}^2}(v_{i_{2j}}, v_{i_{2j+1}}) \leq k + \frac{1}{2\ell+2}\sum_{j=0}^{k-1}|E(v_{i_{2j}}Cv_{i_{2j+1}})| = \frac{2\ell+1}{2\ell+2}k + \frac{1}{2\ell+2}(k + \sum_{j=0}^{k-1}|E(v_{i_{2j}}Cv_{i_{2j+1}})|) < \frac{2\ell+1}{2\ell+2}k + \frac{14\ell+2}{2\ell+2} < k + 7 \leq 13$. Thus, H_r contains a closed walk, and hence also a cycle, of length at most 12, a contradiction.

<u>Case 4: $\kappa = 5$.</u> The construction for $\kappa = 5$ is similar as above with two main differences: to achieve 5-connectedness,

(*i*) the graph H_r has to be 3-connected,

(*ii*) the structure of the subgraphs J_1^i, J_2^i is different.

We begin with (*i*). The existence of 3-connected hamiltonian cubic graphs of large girth is guaranteed by the following probabilistic results, where $\mathcal{G}_{n,d}$ denotes the uniform probability space of d-regular graphs on n vertices, dn even (see also [16]).

Fact 1 ([2], [17]). For fixed $d \geq 3$, any $G \in \mathcal{G}_{n,d}$ is asymptotically almost surely d-connected.

Fact 2 ([12], [13]). For fixed $d \geq 3$, any $G \in \mathcal{G}_{n,d}$ is asymptotically almost surely hamiltonian.

Fact 3 ([3], [18]). For fixed d, let $X_i = X_{i,n}$ $(i \geq 3)$ be the number of cycles of length i in a graph in $\mathcal{G}_{n,d}$. For fixed $k \geq 3$, $X_3, \ldots, X_k$ are asymptotically independent Poisson random variables with means $\lambda_i = \frac{(d-1)^i}{2i}$.

From Facts 1-3 we easily conclude the following consequence.

Fact 4. There is an infinite family of cubic hamiltonian 3-connected graphs with arbitrarily large fixed girth.

Thus, let $\ell \geq 6$ be an integer and let H be an arbitrary 3-connected hamiltonian cubic graph with girth $g(H) \geq (3\ell+11)^2$. Set $t = |V(H)|$.

We construct a graph $\widetilde{G^5_{p,\ell}}$ by the same construction as used for $G^2_{p,\ell}$, but instead of the graphs J^i_1, J^i_2 of Figure 1 we use the graphs $\overline{J^i_1}$, $\overline{J^i_2}$ and $\overline{J^i_3}$ of Figure 3, where we choose

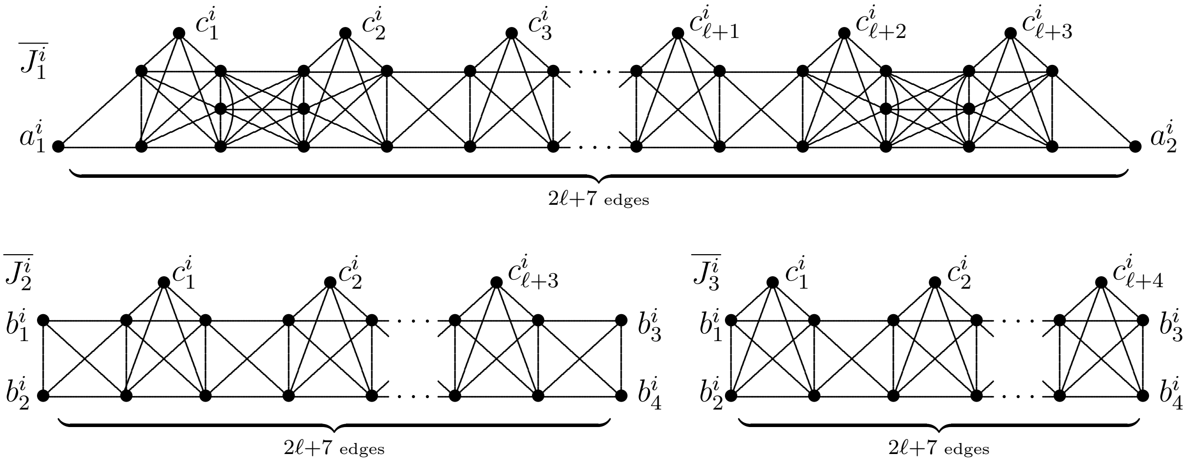

Figure 3

integers p, r such that $t = 4(\ell+3)p + r$, $0 \leq r < 4(\ell+3)$, and instead of J^i_2 we use $p - r$ copies of $\overline{J^i_2}$ and r copies of $\overline{J^i_3}$ (this guarantees that the total number of vertices c^i_j in $\widetilde{G^5_{p,\ell}}$ equals t). We relabel the copies of the graphs $\overline{J^i_1}$, $\overline{J^i_2}$ and $\overline{J^i_3}$ as shown in Figure 4 and relabel the vertices c^i_j in $J_0, \ldots, J_{4p-1}$ by $v_0, \ldots, v_{t-1}$ such that if $v_{j_1} \in V(J_{i_1})$ and $v_{j_2} \in V(J_{i_2})$, then $i_1 < i_2$ implies $j_1 < j_2$ (vertices inside the subgraphs $J_0, \ldots, J_{4p-1}$ are labeled arbitrarily). This labeling guarantees that any two consecutive vertices v_i are at distance (in $\widetilde{G^5_{p,\ell}}$) at least 2, and in any three consecutive subgraphs J_i any two vertices v_i are at distance at least 2. Since any three consecutive subgraphs J_i contain at most $3\ell+11$ vertices v_i, this implies that for any v_{i_1}, v_{i_2} we have $|i_1 - i_2| \leq \frac{3\ell+11}{2} \mathrm{dist}_{\widetilde{G^5_{p,\ell}}}(v_{i_1}, v_{i_2})$ (indices modulo t).

Now, let $C_H = x_0, \ldots, x_{t-1}x_0$ be a hamiltonian cycle in H and set $\overline{H} = (V(H), E(H) \setminus E(C_H))$. We construct the graph $G^5_{p,\ell}$ from $\widetilde{G^5_{p,\ell}}$ and $\overline{H}$ by identifying $v_i := (v_i \equiv x_i)$, $i = 0, \ldots, t-1$. As before, $G^5_{p,\ell}$ is a claw-free graph with complete closure. It is easy to observe that the 3-connectedness of H guarantees $G^5_{p,\ell}$ is 5-connected: a vertex cut R of size at most 4 in $\widetilde{G^5_{p,\ell}}$ would imply that the set $\{v_0, v_1, \ldots, v_{t-1}\}$ can be partitioned

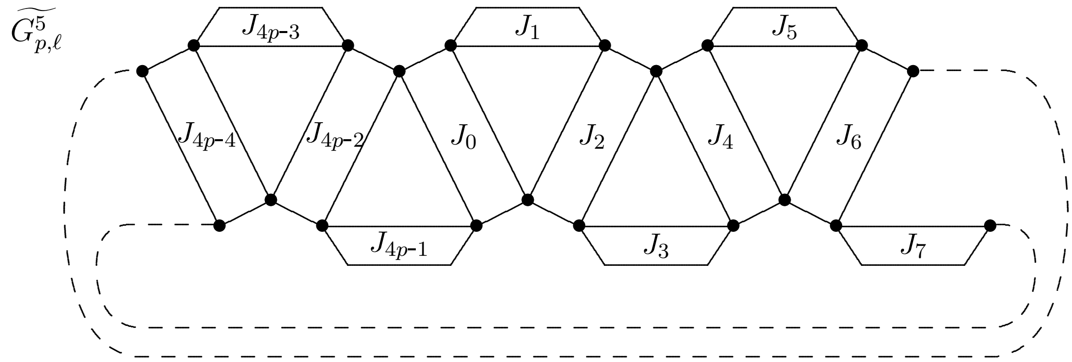

Figure 4

into two subsets V_1, V_2 in such a way that there are no edges between V_1, V_2 and the corresponding subsets in $V(H)$ partition C_H into two segments, but thus contradicts the 3-connectedness of H.

Since $|V(\overline{J_1^i})| = 5\ell + 21$ and $|V(\overline{J_2^i})| + 1 = |V(\overline{J_3^i})| = 5\ell + 20$, $G^5_{p,\ell}$ contains cycles C_λ for all λ, $3 \leq \lambda \leq 5\ell + 21$, and the next possible cycle length in $\widetilde{G^5_{p,\ell}}$ is $3(2\ell + 7) = 6\ell + 21$. We show that also $G^5_{p,\ell}$ contains no C_λ for $5\ell + 22 \leq \lambda \leq 6\ell + 20$. For $\ell \geq 6$ this will give the required family.

Thus, let C be such a cycle, and let k be the number of edges from $\overline{H}$ in C. As in the previous case, the cycle C corresponds to a closed walk W in H of length $|E(W)| = k + \sum_{j=0}^{k-1} |i_{2j} - i_{2j+1}| \leq k + \frac{3\ell+11}{2} \sum_{j=0}^{k-1} \mathrm{dist}_{\widetilde{G^5_{r,\ell}}}(v_{i_{2j}}, v_{i_{2j+1}}) \leq k + \frac{3\ell+11}{2} \sum_{j=0}^{k-1} |E(v_{i_{2j}} C v_{i_{2j+1}})| < \frac{3\ell+11}{2}(k + \sum_{j=0}^{k-1} |E(v_{i_{2j}} C v_{i_{2j+1}})|) \leq \frac{3\ell+11}{2}(6\ell + 20) < (3\ell + 11)^2$, a contradiction. ■

Remarks. 1. It is easy to observe that in specific "small" cases our construction can be slightly improved. There are some more scattered missing cycle lengths, namely $23 \leq \lambda \leq 29$ for $\kappa = 2, 3, 4$, and $\lambda \in \{32, 37, 38, 42, 43, 44, 47, 48, 49, 50\}$ for $\kappa = 5$, and there are easy modifications that give some further small missing cycles for small connectivities (for example, for $\kappa = 2$ it is possible to obtain an infinite family without a C_9), but the drawback is that this makes the construction split into more cases. We leave these straightforward details to the reader.

2. Cubic connected graphs of order n for infinitely many n, with girth g in $\Omega(\log_2 n)$, are constructed independently by Chiu [6] and Morgenstern [10]. Morgenstern's examples are all non-bipartite and with girth $g \geq (2/3)\log_2 n$, in Chiu two sequences are constructed one with non-bipartite members, the other with balanced bipartite. All these graphs are Cayley graphs and hence, by a result of Nedela and Škoviera [11], they are cyclically 3-edge-connected and hence also 3-connected. It is, however, an open problem if infinitely many examples are hamiltonian. Yet the examples are likely to be so since they are also Ramanujan graphs. The smallest of graphs in Chiu, which is bipartite and of order 24, can be seen 3-connected, hamiltonian, and of girth 4. The smallest of cubic graphs in Morgenstern can be seen to be of order 60.

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