Closure concept for 2-factors in claw-free graphs

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Abstract

We introduce a closure concept for 2-factors in claw-free graphs that generalizes the closure introduced by the first author. The 2-factor closure of a graph is uniquely determined and the closure operation turns a claw-free graph into the line graph of a graph containing no cycles of length at most 5 and no cycles of length 6 satisfying a certain condition. A graph has a 2-factor if and only if its closure has a 2-factor; however, the closure operation preserves neither the minimum number of components of a 2-factor nor the hamiltonicity or nonhamiltonicity of a graph.

Keywords: closure; 2-factor; claw-free graph; line graph; dominating system.

1 Introduction

By a *graph* we always mean a simple loopless finite undirected graph $G = (V(G), E(G))$. We use standard graph-theoretical notation and terminology and for concepts and notations not defined here we refer the reader to [1].

The degree of a vertex $x \in V(G)$ is denoted $d_G(x)$, and $\delta(G)$ denotes the *minimum degree* of G, i.e. $\delta(G) = \min\{d_G(x)|\ x \in V(G)\}$. An edge of G is a *pendant edge* if some of its vertices is of degree 1. The *distance in* G of two vertices $x, y \in V(G)$ is denoted $\mathrm{dist}_G(x, y)$, and for two subgraphs $F_1, F_2 \subset G$ we denote $\mathrm{dist}_G(F_1, F_2) = \min\{\mathrm{dist}_G(x, y)|\ x \in V(F_1), y \in V(F_2)\}$. If F is a subgraph of G, we simply write $G - F$ for $G - V(F)$.

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For a set of vertices $S \subset V(G)$, $\langle S \rangle_G$ denotes the subgraph *induced* by S, and for a set of edges $D \subset E(G)$, $\langle\langle D \rangle\rangle_G$ denotes the *edge-induced subgraph* determined by the set D. A *clique* is a (not necessarily maximal) complete subgraph of a graph G, and, for an edge $e \in E(G)$, $\omega_G(e)$ denotes the largest order of a clique containing e.

A cycle of length i is denoted C_i, and for a cycle C with a given orientation and a vertex $x \in V(C)$, x^- and x^+ denotes the predecessor and successor of x on C, respectively.

The *girth* of a graph G, denoted $g(G)$, is the length of a shortest cycle in G, and the *circumference* of G, denoted $c(G)$, is the length of a longest cycle in G. A cycle (path) in G having $|V(G)|$ vertices is called a *hamiltonian cycle* (*hamiltonian path*), and a graph containing a hamiltonian cycle (hamiltonian path) is said to be *hamiltonian* (*traceable*), respectively. A *2-factor* in a graph G is a spanning subgraph of G in which all vertices have degree 2. Thus, a hamiltonian cycle is a connected 2-factor.

If H is a graph, then the *line graph* of H, denoted $L(H)$, is the graph with $E(H)$ as vertex set, in which two vertices are adjacent if and only if the corresponding edges have a vertex in common. It is well-known that if G is a line graph (of some graph), then the graph H such that $G = L(H)$ is uniquely determined (with one exception of the graphs C_3 and $K_{1,3}$, for which both $L(C_3)$ and $L(K_{1,3})$ are isomorphic to C_3). The graph H for which $L(H) = G$ will be called the *preimage* of G and denoted $H = L^{-1}(G)$.

Let H be a graph and $e = xy \in E(H)$ an edge of H. Let $H|_e$ be the graph obtained from H by identifying x and y to a new vertex v_e and adding to v_e a (new) pendant edge e'. Then we say that $H|_e$ is obtained from H by *contraction* of the edge e. Note that $|E(H)| = |E(H|_e)|$.

The *neighborhood* of a vertex $x \in V(G)$ is the set $N_G(x) = \{y \in V(G)|\ xy \in E(G)\}$, and for $S \subset V(G)$ we denote $N_G(S) = \cup_{x \in S} N_G(x)$. For a vertex $x \in V(G)$, the graph G^*_x with $V(G^*_x) = V(G)$ and $E(G^*_x) = E(G) \cup \{uv|\ u, v \in N_G(x)\}$ is called the *local completion of* G *at* x.

The following proposition, which is easy to observe (see also [9]), shows the relation between the operations of local completion and of contraction of an edge.

Proposition A. *Let H be a graph, $e \in E(H)$, $G = L(H)$, and let $x \in V(G)$ be the vertex corresponding to the edge e. Then $G^*_x = L(H|_e)$.*

Note that if e is in a triangle then $H|_e$ may contain a multiple edge. To avoid necessity of working work with multigraphs in this paper, we will always arrange local completions in such a way that Proposition A is always applied to a triangle-free graph.

We say that a graph is *even* if every its vertex has positive even degree. A connected even graph is called a *circuit*, and the complete bipartite graph $K_{1,m}$ is a *star*. Specifically, the four-vertex star $K_{1,3}$ will be referred to as the *claw*. A subgraph F of a graph H *dominates* H if F dominates every edge of H, i.e. if every edge of H has at least one vertex in $V(F)$. Let $\mathcal{S}$ be a set of edge-disjoint circuits and stars with at least three edges in H. We say that $\mathcal{S}$ is a *dominating system* (abbreviated *d-system*) in H if every edge of H that is not in a star of $\mathcal{S}$ is dominated by a circuit in $\mathcal{S}$. We will use the following result by Gould and Hynds [5].

Theorem B [5]. *Let H be a graph. Then $L(H)$ has a 2-factor with c components if and only if H has a d-system with c elements.*

A graph G is said to be *claw-free* if G does not contain an induced subgraph isomorphic to the claw $K_{1,3}$. It is a well-known fact that every line graph is claw-free, hence the class of claw-free graphs can be considered as a natural generalization of the class of line graphs. For more information on claw-free graphs, see e.g. the survey paper [4].

In the class of claw-free graphs, a closure concept has been introduced in [8] as follows. Let G be a claw-free graph and $x \in V(G)$. We say that x is *locally connected* if $\langle N_G(x)\rangle_G$ is a connected graph, x is *simplicial* if $\langle N_G(x)\rangle_G$ is a clique, and x is *eligible* if x is locally connected and nonsimplicial. The set of eligible or simplicial vertices of a graph G is denoted $\mathrm{EL}(G)$ or $\mathrm{SI}(G)$, respectively. The graph, obtained from G by recursively performing the local completion operation at eligible vertices, as long as this is possible, is called the *closure* of G and denoted $\mathrm{cl}(G)$. (More precisely: there are graphs $G_1, \dots, G_k$ such that $G_1 = G$, $G_{i+1} = (G_i)^*_{x_i}$ for some $x_i \in \mathrm{EL}(G_i)$, $i = 1, \dots, k-1$, $G_k = \mathrm{cl}(G)$ and $\mathrm{EL}(G_k) = \emptyset$.)

The following result summarizes basic properties of the closure.

Theorem C [8]. *For every claw-free graph G:*
- (i) $\mathrm{cl}(G)$ *is uniquely determined,*
- (ii) $\mathrm{cl}(G)$ *is the line graph of a triangle-free graph,*
- (iii) $c(\mathrm{cl}(G)) = c(G)$,
- (iv) $\mathrm{cl}(G)$ *is hamiltonian if and only if G is hamiltonian.*

In [10] it was shown that the closure operation preserves also the existence or nonexistence of a 2-factor. More specifically, the following was proved in [10].

Theorem D [10]. *Let G be a claw-free graph and let $x \in \mathrm{EL}(G)$. If G^*_x has a 2-factor with k components, then G has a 2-factor with at most k components.*

Consequently, the local completion operation performed at eligible vertices preserves the minimum number of components of a 2-factor. Specifically, we obtain the following.

Corollary E [10]. *Let G be a claw-free graph. Then G has a 2-factor if and only if $\mathrm{cl}(G)$ has a 2-factor.*

Further properties of $\mathrm{cl}(G)$ are summarized in the survey paper [3].

In this paper, we significantly strengthen the closure concept such that it still preserves the (non)-existence of a 2-factor.

2 Closure concept

Let C_k be a cycle of even length $k \geq 4$. Two edges $e_1, e_2 \in E(G)$ are said to be *antipodal in C_k*, if they are at maximum distance in C_k (i.e., $\mathrm{dist}_{C_k}(e_1, e_2) = k/2-1$). An even cycle C_k in a graph G is said to be *edge-antipodal*, abbreviated EA, if $\min\{\omega_G(e_1), \omega_G(e_2)\} = 2$ for any two antipodal edges $e_1, e_2 \in E(C_k)$. Analogously, two vertices $x_1, x_2 \in V(C_k)$ are *antipodal in C_k* if they are at maximum distance in C_k (i.e. $\mathrm{dist}_{C_k}(x_1, x_2) = k/2$), and

C_k is said to be *vertex-antipodal*, abbreviated VA, if $\min\{d_G(x_1), d_G(x_2)\} = 2$ for any two antipodal vertices $x_1, x_2 \in V(C_k)$.

Let G be a claw-free graph. A vertex $x \in V(G)$ is said to be *2f-eligible*, if x satisfies one of the following:

(i) $x \in \mathrm{EL}(G)$,

(ii) $x \notin \mathrm{EL}(G)$ and x is in an induced cycle of length 4 or 5 or in an induced EA-cycle of length 6.

The set of all 2f-eligible vertices of G will be denoted $\mathrm{EL}^{2f}(G)$.

We say that a graph $\mathrm{cl}^{2f}(G)$ is a *2-factor-closure* (abbreviated 2f-closure) of a claw-free graph G, if there is a sequence of graphs $G_1, \ldots, G_k$ such that

(i) $G_1 = G$,

(ii) $G_{i+1} = (G_i)^*_{x_i}$ for some $x_i \in \mathrm{EL}^{2f}(G_i)$, $i = 1, \ldots, k-1$,

(iii) $G_k = \mathrm{cl}^{2f}(G)$ and $\mathrm{EL}^{2f}(G_k) = \emptyset$.

Thus, the 2f-closure of a claw-free graph G is obtained by recursively repeating the local completion operation at 2f-eligible vertices, as long as this is possible. In the next section we will show that, for a given claw-free graph G, its 2f-closure is uniquely determined, which will justify the notation $\mathrm{cl}^{2f}(G)$.

The graph G in Figure 1 is an example of a claw-free graph with a complete 2f-closure, in which $\mathrm{EL}(G) = \emptyset$. Note that G is nonhamiltonian and $G - x$ is nontraceable, while $\mathrm{cl}^{2f}(G)$ is complete and $\mathrm{cl}^{2f}(G-x)$ is traceable. Hence $\mathrm{cl}^{2f}(G)$ preserves neither the (non)-hamiltonicity nor the (non)-traceability of a graph. Moreover, since G is nonhamiltonian and $\mathrm{cl}^{2f}(G)$ is complete, this example also shows that $\mathrm{cl}^{2f}(G)$ does not preserve the minimum number of components of a 2-factor, i.e., an analogue of Theorem D is not true for $\mathrm{cl}^{2f}(G)$. However, in Section 4 we will prove the analogue of Corollary E for $\mathrm{cl}^{2f}(G)$.

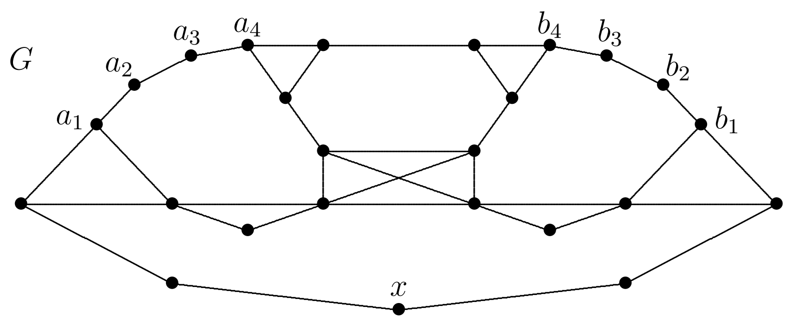

Figure 1

3 Uniqueness of the closure

We recall some definitions and facts from [6] that will be helpful to prove the uniqueness of $\mathrm{cl}^{2f}(G)$ as a special case of a more general setting.

Let $\mathcal{C}$ be a class of graphs and let $\mathcal{P}$ be a function on $\mathcal{C}$ such that, for any $G \in \mathcal{C}$, $\mathcal{P}(G) \subset 2^{V(G)}$ (i.e., $\mathcal{P}(G)$ is a set of subsets of $V(G)$). For any $X \subset V(G)$ let G^*_X denote the local completion of G at X, i.e. the graph with $V(G^*_X) = V(G)$ and $E(G^*_X) = E(G) \cup \{uv|\ u, v \in X\}$ (thus, the previous notation G^*_x means that, for a vertex $x \in V(G)$, we simply write G^*_x for $G^*_{N_G(x)}$).

We say that a graph F is a $\mathcal{P}$-*extension* of G, denoted $G \preceq F$, if there is a sequence of graphs $G_0 = G, G_1, \ldots, G_k = F$ such that $G_{i+1} = (G_i)^*_{X_i}$ for some $X_i \in \mathcal{P}(G_i)$. Clearly, for any graph G there is a $\preceq$-maximal $\mathcal{P}$-extension H, and in this case we say that H is a $\mathcal{P}$-closure of G. If a $\mathcal{P}$-closure is uniquely determined then it is denoted by $\mathrm{cl}_{\mathcal{P}}(G)$. Finally, a function $\mathcal{P}$ is *non-decreasing* (on a class $\mathcal{C}$), if, for any $H, H' \in \mathcal{C}$, $H \preceq H'$ implies that for any $X \in \mathcal{P}(H)$ there is an $X' \in \mathcal{P}(H')$ such that $X \subset X'$.

The following result was proved in [6]. For the sake of completeness, we include its (short) proof here.

Theorem F [6]. *If $\mathcal{P}$ is a non-decreasing function on a class $\mathcal{C}$, then, for any $G \in \mathcal{C}$, a $\mathcal{P}$-closure of G is uniquely determined.*

Proof. Let $H \neq H'$ be $\mathcal{P}$-closures of G, let $G = G_0, G_1, \ldots, G_k = H'$ be such that $G_{i+1} = (G_i)^*_{X_i}$ for some $X_i \in \mathcal{P}(G_i)$, and let s be a smallest integer such that $G_s \not\subset H$. Since $G_{s-1} \subset H$ and $\mathcal{P}$ is non-decreasing, there is $X \in \mathcal{P}(H)$ such that $X_{s-1} \subset X$. Since H is $\preceq$-maximal, we have $H^*_X = H$, a contradiction. ■

It is easy to see that $\mathcal{P}(G) = \{N_G(x)|\ x \in \mathrm{EL}^{2f}(G) \cup \mathrm{SI}(G)\}$ is a non-decreasing function on the class $\mathcal{C}$ of claw-free graphs, and $\mathrm{cl}_{\mathcal{P}}(G)$ equals the 2f-closure of G. This immediately implies the following fact.

Proposition 1. *For any claw-free graph G, the 2f-closure of G is uniquely determined.* ■

4 Properties of the closure

The following result summarizes basic properties of the 2f-closure.

Theorem 2. *Let G be a claw-free graph. Then*

(*i*) *the closure $\mathrm{cl}^{2f}(G)$ is uniquely determined,*

(*ii*) *there is a graph H such that*

 (α) $L(H) = \mathrm{cl}^{2f}(G)$,

 (β) $g(H) \geq 6$,

 (γ) *H does not contain any vertex-antipodal cycle of length 6,*

(*iii*) *G has a 2-factor if and only if $\mathrm{cl}^{2f}(G)$ has a 2-factor.*

Proof. (i) Part (i) follows immediately from Proposition 1.

(ii) By (i), the 2f-closure does not depend on the order of 2f-eligible vertices used during the construction of $\mathrm{cl}^{2f}(G)$. Thus, we can first apply local completion to eligible vertices, obtaining $\mathrm{cl}(G)$, and then apply local completion to 2f-eligible vertices of $\mathrm{cl}(G)$. In some steps, it is possible that again $\mathrm{EL}(G_i) \neq \emptyset$ and, if this occurs, we choose x_i such that $x_i \in \mathrm{EL}(G_i)$, as long as this is possible. Let $G_1, \ldots, G_k$ be the resulting sequence of graphs and $x_1, \ldots, x_{k-1}$ the corresponding sequence of 2f-eligible vertices, i.e. $G_1 = G$, $G_k = \mathrm{cl}^{2f}(G)$, $G_{i+1} = (G_i)^*_{x_i}$ and $x_i \in \mathrm{EL}^{2f}(G_i)$, $i = 1, \ldots, k-1$. Then, any time when $x \in \mathrm{EL}(G_i)$, the subsequence of eligible vertices yields a triangle-free graph by Theorem C and thus, any time when $x_i \in \mathrm{EL}^{2f}(G_i) \setminus \mathrm{EL}(G_i)$, the choice of x_i guarantees that $G_i = L(H_i)$ for some triangle-free graph H_i. Then, by Proposition A, $G_{i+1} = (G_i)^*_{x_i} = L(H_i|_{e_i})$, where e_i is the edge of H_i corresponding to the vertex $x_i \in V(G_i)$, and the fact that H_i is triangle-free guarantees that $H_i|_{e_i}$ is a graph (i.e. the contraction of e_i does not create a multiple edge). By induction, each G_i is a line graph. Since $L^{-1}(C_i) = C_i$, and the preimage of an EA-C_6 is a VA-C_6, the graph $H = L^{-1}(\mathrm{cl}^{2f}(G))$ has the required properties.

(iii) Clearly, every 2-factor in G is a 2-factor in $\mathrm{cl}^{2f}(G)$, hence we need to prove that if $\mathrm{cl}^{2f}(G)$ has a 2-factor then G has a 2-factor.

Similarly as in part (ii) of the proof, we can construct $\mathrm{cl}^{2f}(G)$ such that we first apply local completion to eligible vertices as long as this is possible, and we obtain $\overline{G} = \mathrm{cl}(G)$ and the triangle-free graph $\overline{H} = L^{-1}(\overline{G})$. The 2f-closure of G is then obtained by applying local completion to 2f-eligible vertices. In the i-th step of the construction we then have $G_{i+1} = (G_i)^*_{v_i}$, where $v_i \in \mathrm{EL}^{2f}(G_i)$. If $v_i \in \mathrm{EL}(G_i)$, we are done by Theorem D, hence suppose that $\mathrm{EL}(G_i) = \emptyset$ and v_i is in an induced cycle C_G. By the definition of the 2f-closure, C_G is a C_4, a C_5 or an EA-C_6.

Let $H = L^{-1}(G_i)$, $C = L^{-1}(C_G)$, and let $e = xy \in E(H)$ be the edge corresponding to v_i. Then $e \in E(C)$ and C is a C_4, a C_5 or a VA-C_6. We will suppose that C is oriented such that $x = y^+$. By Proposition A, we have $L^{-1}((G_i)^*_{v_i}) = H|_e$, thus, by Theorem B, it remains to prove the following claim.

Claim 3. *If $H|_e$ has a d-system, then H has a d-system.*

We set $H' = H|_e$ and denote by v_e the vertex obtained by contracting $e = xy$, and by e' the pendant edge (corresponding to e) attached to v_e.

Let $\mathcal{S}'$ be a d-system in H', and let $B(\mathcal{S}')$ and $St(\mathcal{S}')$ be the set of circuits and the set of stars in $\mathcal{S}'$, respectively. Note that in the spanning subgraph (of H')

$$D' = (V(H'), \bigcup_{B \in B(\mathcal{S}')} E(B)),$$

every vertex has even degree (possibly zero). We can suppose that there is no star in $\mathcal{S}'$ whose center has positive even degree in D' because all the edges of such a star are dominated by the circuit passing through the center. Since e' is a pendant edge in H', $e' \notin E(D')$, hence there exists either a star in $St(\mathcal{S}')$ whose center is v_e, or a circuit in $B(\mathcal{S}')$ passing through v_e . If there is a star in $St(\mathcal{S}')$ whose center is v_e , we denote this star by T'; otherwise let T' be an empty graph, i.e., $V(T') = \emptyset$. Let S be the set of the subgraphs in H corresponding to the stars in $St(\mathcal{S}') \setminus \{T'\}$ and D the spanning subgraph in H corresponding to D'. Notice that all elements in S are stars in H and $d_D(x) \equiv d_D(y) \pmod 2$.

Suppose first that both x and y have positive degree in D. Then there exists a circuit in $B(\mathcal{S}')$ passing through v_e, and there is no star in $St(\mathcal{S}')$ with center at v_e. If both x and y have positive even degree in D, then D and S determine a d-system in H since the edge e is dominated in H by any of the circuits passing through x and y. Similarly, if both x and y have positive odd degree, then $D + e$ and S determine a d-system in H.

Hence we suppose that $d_D(x) = 0$ or $d_D(y) = 0$. By symmetry, let $d_D(y) = 0$. If $C - \langle\langle E(D) \cap E(C)\rangle\rangle_H$ is edgeless (i.e., all edges of C have at least one vertex with positive degree in D), then $d_D(x) \geq 2$ and $d_D(y^-) \geq 2$. If T' has no edge whose corresponding edge in H is incident to y, then D and S determine a d-system of H since the edges $e = xy$ and yy^- are dominated by the circuits in D passing through x and y^-, respectively. If T' has an edge whose corresponding edge in H is incident to y, then D and the set of stars which obtained by adding to S the star consisting of xy, yy^- and all the corresponding edges incident to y, determine a d-system in H. Note that in the last case (i.e. if we added a star), the number of elements of the d-system under consideration is increased (and in this case also the minimum number of components of a 2-factor can be increased).

Therefore we suppose $C - \langle\langle E(D) \cap E(C)\rangle\rangle_H$ contains an edge. This implies

$$|E(D) \cap E(C)| \leq |E(C)| - 3. \tag{1}$$

Let $\widetilde{D} = \langle\langle (E(D) \cup E(C)) \setminus (E(D) \cap E(C))\rangle\rangle_H$. As in the above, we can construct a d-system in H if $C - \langle\langle E(\widetilde{D}) \cap E(C)\rangle\rangle_H$ is edgeless. Indeed, in this case $d_{\widetilde{D}}(x) \geq 2$ and $d_{\widetilde{D}}(y) \geq 2$ since $e \in E(\widetilde{D})$. Therefore neither x nor y are singletons in $\widetilde{D}$. If there is a vertex $x_i \in C - \langle\langle E(\widetilde{D}) \cap E(C)\rangle\rangle_H$ such that some edges incident to x_i have no vertex in $\widetilde{D}$, then we construct a star from all such edges and the edges $x_i^- x_i$, $x_i x_i^+$. Let S_1 be the set of all such stars for vertices in $C - \langle\langle E(\widetilde{D}) \cap E(C)\rangle\rangle_H$ and S_2 the set of all stars in S whose centers are on C. Then $\widetilde{D}$ and $(S \setminus S_2) \cup S_1$ determine a d-system in H.

Therefore we suppose $C - \langle\langle E(\widetilde{D}) \cap E(C)\rangle\rangle_H$ contains an edge. This implies

$$|E(C)| - |E(D) \cap E(C)| \leq |E(C)| - 3$$

and hence by (1),

$$3 \leq |E(D) \cap E(C)| \leq |E(C)| - 3 \leq 3.$$

As all the equalities hold, $|C| = 6$ and $|E(D) \cap E(C)| = 3$. Furthermore, the three edges in $E(D) \cap E(C)$ should be adjacent, i.e., these edges determine a path in C (otherwise $C - \langle\langle E(D) \cap E(C)\rangle\rangle_H$ is edgeless). The endvertices of this path are antipodal on C and, since each of them has positive even degree in D, their degrees in H are greater than two. This implies C is not vertex-antipodal, a contradiction. ■

Corollary 4. *Let G be a claw-free graph in which every locally disconnected vertex is in an induced cycle of length 4 or 5, or in an induced EA-C_6. Then G has a 2-factor.*

Proof. If G satisfies the assumptions of the theorem, then every nonsimplicial vertex of G is 2f-eligible, hence $\mathrm{cl}^{2f}(G)$ is complete and G has a 2-factor by Theorem 2. ■

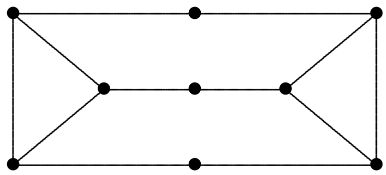

Figure 2

Consider the graph G in Figure 2. The graph G has no 2-factor, and applying local completion at any of its vertices would start a process that results in a complete graph. Each vertex of G is in some cycle of length 6, but neither of these cycles is antipodal. Hence this example shows that the antipodality condition cannot be omitted.

5 Concluding remarks

1. If $x \in \mathrm{EL}^{2f}(G) \setminus \mathrm{EL}(G)$, then x is in an induced cycle C, where C is a C_4, a C_5 or an EA-C_6, and applying local completion at x turns C into an induced cycle the length of which is one less. Eventually, all vertices in $N_G(V(C))$ induce a clique in $\mathrm{cl}^{2f}(G)$. This simple observation shows that the construction of $\mathrm{cl}^{2f}(G)$ can be speeded up such that, in each step when an induced C_4, C_5 or an EA-C_6 is identified, all vertices in $N_G(V(C))$ are covered with a clique.

2. The 2f-closure can be slightly extended as follows. A *branch* in a graph G is a path in G with all interior vertices of degree 2 and with (distinct) endvertices of degree different from 2. The *length* of a branch is the number of its edges. If $x \in V(G)$ is of $d_G(x) = 2$ and $N_G(x) = \{y_1, y_2\}$, we say that the graph with vertex set $V(G) \setminus \{x\}$ and edge set $(E(G) \setminus \{xy_1, xy_2\}) \cup \{y_1y_2\}$ is obtained by *suppressing* x. The graph obtained from G by suppressing $k-2$ interior vertices in each branch of length $k \geq 3$ is called the *suppresion* of G and denoted $\mathrm{supp}(G)$. It is easy to see that $\mathrm{supp}(G)$ is unique (up to isomorphism), and in $\mathrm{supp}(G)$ both neighbors of every vertex of degree 2 have degree different from 2. The following observation is also straightforward.

Proposition 5. *Let G be a graph. Then G has a 2-factor if and only if* $\mathrm{supp}(G)$ *has a 2-factor.* ■

Thus, it is possible to slightly extend the 2f-closure by setting $\mathrm{cl}_S^{2f}(G) = \mathrm{cl}^{2f}(\mathrm{supp}(G))$. This straightforward extension allows to handle some cycles of arbitrarily large length (for example, the paths $a_1a_2a_3a_4$ and $b_1b_2b_3b_4$ in Figure 1 can be arbitrarily long), however, the drawback of this approach is that possibly $|V(\mathrm{cl}_S^{2f}(G))| \neq |V(G)|$. We leave the technical details to the reader.

3. Combining the observations made in Remarks 1 and 2 with the approach used in [2] we can alternatively define the closure as follows. Let C be an induced cycle in G of length k, and let C_S be the corresponding cycle in $\mathrm{supp}(G)$. We say that C is *2f-eligible in G* if $k \in \{4, 5\}$, or if $k = 6$ and C is edge-antipodal in G, and C is *2fc-eligible in G* if C_S is 2f-eligible in $\mathrm{supp}(G)$. The *local completion of G at C* is the graph G_C^* with

$V(G^*_C) = V(G)$ and $E(G^*_C) = E(G) \cup \{uv|\ u, v \in V(C) \cup N(V(C))\}$, and a graph $\mathrm{cl}^{2f}_C(G)$ is said to be a *2fc-closure of* G if there is a sequence of graphs $G_1, \dots, G_t$ such that

(*i*) $G_1 = \mathrm{cl}(G)$,

(*ii*) $G_{i+1} = \mathrm{cl}((G_i)^*_{C_i})$ for some 2fc-eligible cycle C_i in G_i, $i = 1, \dots, t-1$,

(*iii*) $G_t = \mathrm{cl}^{2f}_C(G)$ contains no 2fc-eligible cycle.

The following facts are easy to see.

Theorem 6. *Let G be a claw-free graph. Then*

(*i*) *the closure $\mathrm{cl}^{2f}_C(G)$ is uniquely determined,*

(*ii*) $\mathrm{cl}^{2f}(G) \subset \mathrm{cl}^{2f}_C(G)$ *and* $\mathrm{cl}^{2f}(G) = \mathrm{cl}^{2f}_C(G)$ *if and only if G has no branches of length $k \geq 3$,*

(*iii*) *G has a 2-factor if and only if $\mathrm{cl}^{2f}_C(G)$ has a 2-factor.* ■

4. We show another alternative way of introducing the closure that gives a concept slightly weaker, but in some situations easier to use.

For $x \in V(G)$ and a positive integer k, let $N^k_G(x) = \{y \in V(G)|\ 1 \leq \mathrm{dist}_G(x, y) \leq k\}$, and set $\mathrm{EL}^k(G) = \{x \in V(G)|\ \langle N^k_G(x)\rangle_G$ is connected noncomplete$\}$. The vertices in $\mathrm{EL}^k(G)$ will be called *k-distance-eligible* (note that $\mathrm{EL}^1(G) = \mathrm{EL}(G)$).

For a claw-free graph G, let $\mathrm{cl}^{d2}(G)$ be the graph obtained from G by local completions at 2-distance-eligible vertices, as long as such a vertex exists. It is straightforward to observe that $x \in \mathrm{EL}^2(G)$ if and only if $x \in V(G)$ is either eligible (i.e. $x \in \mathrm{EL}(G)$), or x is in an induced cycle of length 4 or 5. Thus, the following facts are straightforward.

Theorem 7. *Let G be a claw-free graph. Then*

(*i*) *the closure $\mathrm{cl}^{d2}(G)$ is uniquely determined,*

(*ii*) *there is a graph H with $g(H) \geq 6$ such that $L(H) = \mathrm{cl}^{d2}(G)$,*

(*iii*) *G has a 2-factor if and only if $\mathrm{cl}^{d2}(G)$ has a 2-factor.* ■

A graph G is N^2-locally connected if, for every $x \in V(G)$, $\langle N^2_G(x)\rangle_G$ is a connected graph. Clearly, if G is N^2-locally connected, then $\mathrm{cl}^{d2}(G)$ is a complete graph. Hence the following result by Li and Liu [7] is an immediate corollary of Theorem 7.

Theorem G [7]. *Every N^2-locally connected claw-free graph with $\delta(G) \geq 2$ has a 2-factor.*

The graph G in Figure 3 is an example of a graph that does not satisfy the assumptions of Theorem G, but $\mathrm{cl}^{d2}(G)$ is a complete graph (and hence G has a 2-factor by Theorem 7).

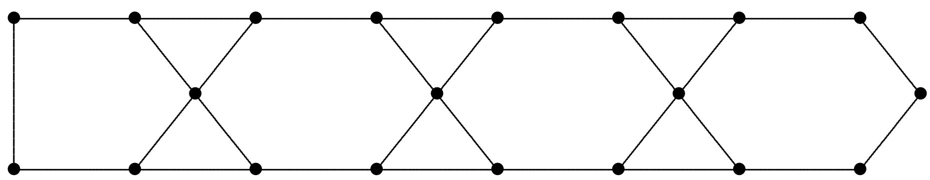

Figure 3

Consider the graph G in Figure 4. Clearly, G is claw-free and has no 2-factor. The vertex x is eligible in G (i.e., $x \in \mathrm{EL}(G)$), hence also $x \in \mathrm{EL}^2(G)$. However, applying the local completion operation to the whole distance 2-neighborhood $N^2(x)$ would result in a graph that has a 2-factor. This example shows that modifying the 2-distance closure such that, in each step, $N^2(x)$ of a vertex $x \in \mathrm{EL}^2(G)$ is covered with a clique, would result in closure that does not preserve the (non)-existence of a 2-factor.

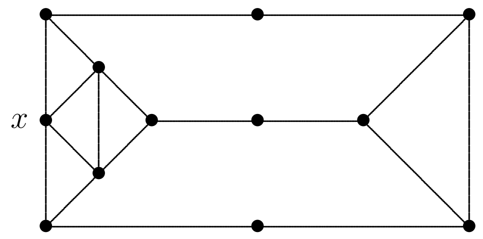

Figure 4

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