# Closure and forbidden pairs for 2-factors

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**Abstract**

Pairs of connected graphs $X, Y$ such that a graph $G$ being 2-connected and $XY$-free implies $G$ is hamiltonian were characterized by Bedrossian. Using the closure concept for claw-free graphs, the first author simplified the characterization by showing that if considering the closure of $G$, the list in the Bedrossian's characterization can be reduced to one pair, namely, $K_{1,3}, N_{1,1,1}$ (where $K_{i,j}$ is the complete bipartite graph, and $N_{i,j,k}$ is the graph obtained by identifying endvertices of three disjoint paths of lengths $i, j, k$ to the vertices of a triangle). Faudree et al. characterized pairs $X, Y$ such that $G$ being 2-connected and $XY$-free implies $G$ has a 2-factor. Recently, the first author et al. strengthened the closure concept for claw-free graphs such that the closure of a graph has stronger properties while still preserving the (non)-existence of a 2-factor. In this paper we show that, using the 2-factor closure, the list of forbidden pairs for 2-factors can be reduced to two pairs, namely, $K_{1,4}, P_4$ and $K_{1,3}, N_{1,1,3}$.

## 1 Notation and terminology

In this paper, by a graph we mean a simple finite undirected graph $G = (V(G), E(G))$, and for notations and terminology not defined here we refer to [3].

Specifically, $C_k$ denotes the cycle on $k$ vertices and $P_k$ the path on $k$ vertices (i.e. of length $k-1$). A *trivial path* is a path having only one vertex, and a path with endvertices $a, b$ is also referred to as an *$(a, b)$-path.* For $x \in V(G)$, $d_G(x)$ denotes the *degree* of $x$, and $\Delta(G)$ stands for the *maximum degree of* $G$, i.e. $\Delta(G) = \max\{d_G(x)|\ x \in V(G)\}$. An edge $e = uv \in E(G)$ is a *pendant edge of* $G$ if $\min\{d_G(u), d_G(v)\} = 1$ and $\max\{d_G(u), d_G(v)\} \geq 3$. The *girth* of a graph $G$, denoted $g(G)$, is the length of a shortest cycle in $G$, and the *circumference* of $G$, denoted $c(G)$, is the length of a longest cycle in $G$. A *clique* in a graph

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$G$ is a (not necessarily maximal) complete subgraph of $G$, and, for an edge $e \in E(G)$, $\omega_G(e)$ denotes the largest order of a clique containing $e$.

We use the notation $H \subset G$ for a *subgraph* $H$ of a graph $G$, and, for a set $M \subset V(G)$, $\langle M \rangle_G$ denotes the *induced subgraph* of $G$ on $M$. If $H$ is an induced subgraph of $G$, then we also use the notation $H \overset{\text{IND}}{\subset} G$.

The *distance* in $G$ of two vertices $x, y \in V(G)$ is denoted $\text{dist}_G(x, y)$, and for two subgraphs $H_1, H_2 \subset G$ we set $\text{dist}_G(H_1, H_2) = \min\{\text{dist}_G(x, y) |\ x \in V(H_1), y \in V(H_2)\}$. If $x \in V(G)$, then the *neighborhood of* $x$ *in* $G$ is the set of all neighbors of $x$ in $G$, i.e. $N_G(x) = \{y \in V(G) |\ xy \in E(G)\}$, and for a subgraph $F \subset G$ we set $N_G(F) = \cup_{x \in V(F)} N_G(x)$. A vertex $x \in V(G)$ is said to be *locally connected* or *simplicial* if $\langle N_G(x) \rangle_G$ is a connected graph or a clique, respectively.

A cycle in $G$ of length $|V(G)|$ is called a *hamiltonian cycle*, and a graph containing a hamiltonian cycle is said to be *hamiltonian*. A *2-factor* in a graph $G$ is a spanning subgraph of $G$ in which all vertices have degree 2. Thus, a hamiltonian cycle is a connected 2-factor.

The *line graph* of a graph $H$ is the graph $G = L(H)$ with vertex set $E(H)$, in which two vertices are adjacent if and only if the corresponding edges of $H$ have a vertex in common. It is a well-known fact that if $G$ is a line graph (of some graph), then the graph $H$ such that $G = L(H)$ is uniquely determined (with one exception of the graphs $C_3$ and $K_{1,3}$, for which both $L(C_3)$ and $L(K_{1,3})$ are isomorphic to $C_3$). The graph $H$ for which $L(H) = G$ will be called the *preimage* of $G$ and denoted $H = L^{-1}(G)$. It is easy to observe that a graph $F$ is a subgraph (not necessarily induced) of a graph $H$ if and only if its line graph $L(F)$ is an induced subgraph of the graph $G = L(H)$. It is also well-known that a line graph $G$ is $k$-connected if and only if its preimage $H = L^{-1}(G)$ is *essentially* $k$*-edge-connected*, i.e., $H$ contains no edge cut $R$ such that $|R| < k$ and at least two components of $G - R$ are nontrivial (i.e. containing at least one edge).

If $\mathcal{C}$ is a class of graphs, we say that a graph $G$ is $\mathcal{C}$*-free* if $G$ does not contain any graph from $\mathcal{C}$ as an induced subgraph. If $\mathcal{C} = \{X_1, \ldots, X_k\}$, we also say that $G$ is $X_1 \ldots X_k$*-free* and the graphs $X_i$ are referred to in this context as *forbidden induced subgraphs*. Specifically, the four-vertex star $K_{1,3}$ will be called the *claw*, and a $K_{1,3}$-free graph will be also said to be *claw-free*. It is a well-known fact (which follows e.g. also from the forbidden subgraph characterization of line graphs by Beineke [2]) that every line graph is claw-free. Other graphs that will be often used as forbidden induced subgraphs are shown in Figure 1 (where in all cases $i, j, k \geq 1$).

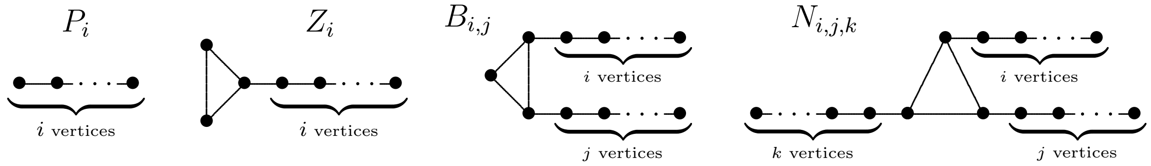


Figure 1: Forbidden induced subgraphs

## 2 Introduction

The first result on forbidden induced subgraphs for cycle and path properties of graphs is by Goodman and Hedetniemi [10] who observed that every 2-connected $K_{1,3}Z_1$-free graph is hamiltonian. Analogous results were then proved for 2-connected $K_{1,3}N_{1,1,1}$-free graphs [7], $K_{1,3}Z_2$-free graphs [11] and $K_{1,3}P_6$-free graphs [5]. Bedrossian [1] characterized all pairs of forbidden subgraphs for hamiltonicity, and later on, Faudree and Gould [9] reconsidered the Bedrossian's characterization (where the 'only if' part is now based on infinite families of graphs).

**Theorem A [1], [9].** *Let $X, Y$ be connected graphs with $X, Y \not\simeq P_3$ and let $G$ be a 2-connected graph of order $n \geq 10$ that is not a cycle. Then, $G$ being $XY$-free implies $G$ is hamiltonian if and only if (up to a symmetry) $X = K_{1,3}$ and $Y$ is an induced subgraph of at least one of the graphs $P_6$, $Z_3$, $B_{1,2}$ or $N_{1,1,1}$.*

The first author [12] introduced a closure concept for claw-free graphs as follows. The *local completion* of a graph $G$ at a vertex $x \in V(G)$ is the graph $G^*_x = (V(G), E(G) \cup \{uv|\ u, v \in N_G(x)\}))$, and a vertex $x \in V(G)$ is *eligible* if $x$ is locally connected and nonsimplicial. The set of all eligible vertices of $G$ is denoted $\mathrm{EL}(G)$. The *closure* $\mathrm{cl}(G)$ of a claw-free graph $G$ is the graph obtained by recursively performing the local completion operation at eligible vertices as long as this is possible (i.e., more precisely, there is a sequence of graphs $G_1, \ldots, G_k$ such that $G_1 = G$, $G_{i+1} = (G_i)^*_{x_i}$ for some $x_i \in \mathrm{EL}(G_i)$, $i = 1, \ldots, k-1$, $G_k = \mathrm{cl}(G)$ and $\mathrm{EL}(G_k) = \emptyset$). A graph $G$ is *closed* if $G = \mathrm{cl}(G)$.

The following result summarizes basic properties of the closure.

**Theorem B [12].** *For every claw-free graph $G$:*

($i$) $\mathrm{cl}(G)$ *is uniquely determined,*

($ii$) $\mathrm{cl}(G)$ *is the line graph of a triangle-free graph,*

($iii$) $c(\mathrm{cl}(G)) = c(G)$,

($iv$) $\mathrm{cl}(G)$ *is hamiltonian if and only if $G$ is hamiltonian.*

Using the closure concept, it was shown in [13] that the list of forbidden subgraphs in the characterization, given in Theorem A, can be reduced (with one simple class of exceptions) to just one graph, namely the graph $N_{1,1,1}$, and the structure of closures of such graphs was fully described. Let $\mathcal{F}^{\mathrm{cl}}$, $\mathcal{C}^N_1$ and $\mathcal{C}^N_2$ be the families of graphs shown in Figure 2 (where circular and elliptical parts represent cliques). The following theorem summarizes Theorems 6 and 8 of [13].

**Theorem C [13].** *Let $G$ be a 2-connected $XY$-free graph, where $X, Y$ is a pair of connected graphs such that $G$ being $XY$-free implies $G$ is hamiltonian. Then $G$ satisfies each of the following:*

($i$) *$G$ is claw-free and $\mathrm{cl}(G)$ is $N_{1,1,1}$-free or $\mathrm{cl}(G) \in \mathcal{F}^{\mathrm{cl}}$,*

($ii$) *$G$ is claw-free and $\mathrm{cl}(G) \in \mathcal{F}^{\mathrm{cl}} \cup \mathcal{C}^N_1 \cup \mathcal{C}^N_2$.*

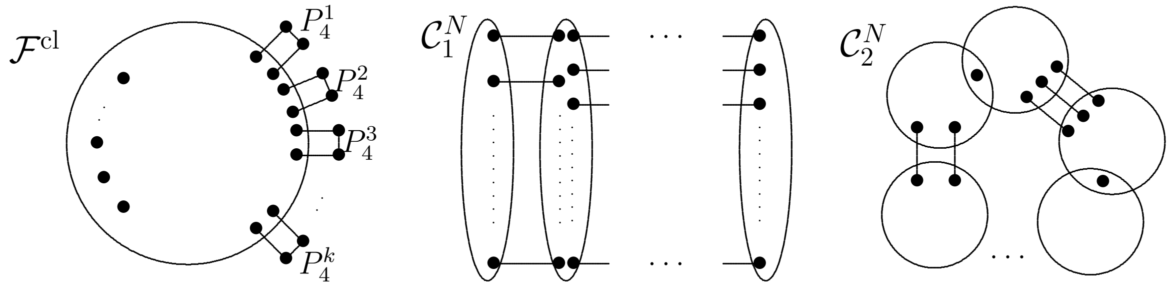


Figure 2: Exception classes for Theorem C

In [4], the closure concept for hamiltonicity was strengthened in the following way. Let $G$ be a closed claw-free graph. A $k$-cycle $C$ in $G$ is said to be *eligible* if $4 \leq k \leq 6$ and at least $k-3$ nonconsecutive edges of $C$ are contained in no clique of order at least 3. For an eligible cycle $C$ in $G$, the graph $G^*_C = (V(G), E(G) \cup \{uv|\ u, v \in N_G(C)\})$ is called the *cycle-completion of* $G$ *at* $C$.

Let now $G$ be a claw-free graph. A graph $\mathrm{cl}^C(G)$ is said to be a *cycle closure* of $G$, if there is a sequence of graphs $G_1, \ldots, G_t$ such that $G_1 = \mathrm{cl}(G)$, $G_{i+1} = \mathrm{cl}((G_i)^*_C)$ for some eligible cycle $C$ in $G_i$, $i = 1, \ldots, t-1$, and $G_t = \mathrm{cl}^C(G)$ contains no eligible cycle. It was shown in [4] that, for any claw-free graph $G$, $\mathrm{cl}^C(G)$ is uniquely determined and $c(G) = c(\mathrm{cl}^C(G))$.

Let $k \geq 4$ be an integer and let $K_1, \ldots, K_k$ be vertex-disjoint cliques of order $|V(K_i)| = r_i \geq 2$ with $x_i, y_i \in V(K_i)$, $x_i \neq y_i$, $i = 1, \ldots, k$. The graph, obtained by identifying $y_i$ with $x_{i+1}$, $i = 1, \ldots, k$ (indices modulo $k$) will be denoted $C^{r_1,\ldots,r_k}$. We set $\mathcal{C}^k = \{C^{r_1,\ldots,r_k}|\ r_i \geq 2,\ i = 1, \ldots, k\}$, and for an integer $t \geq 4$ we further denote $\mathcal{C}^{\geq t} = \cup_{k \geq t} \mathcal{C}^k$ (see Figure 3). It is easy to see that all graphs in $\mathcal{C}^{\geq t}$ are 2-connected, $K_{1,3}N_{1,1,1,}$-free and closed. The following theorem shows that, using the cycle closure, the characterization given in Theorem A can be further simplified.

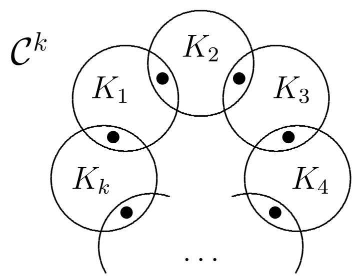


Figure 3: 2-connected closed $K_{1,3}N_{1,1,1,}$-free graphs

**Theorem D [13].** *Let $G$ be a 2-connected $XY$-free graph of order $n \geq 11$, where $X, Y$ is a pair of connected graphs such that $G$ being $XY$-free implies $G$ is hamiltonian. Then $G$ satisfies each of the following:*

($i$) *$G$ is claw-free and $\mathrm{cl}^C(G)$ is $N_{1,1,1}$-free,*

($ii$) *$G$ is claw-free and $\mathrm{cl}^C(G)$ is either complete or belongs to $\mathcal{C}^{\geq 4}$.*

The characterization of pairs of forbidden subgraphs for 2-factors corresponding to Theorem A for hamiltonian cycles was given by Faudree et al. [8] in the following result. Note that the list in the characterization does not contain any $Z_i$ since, as shown in [8], the largest $Z_i$ implying a 2-factor is the $Z_4$ which is an induced subgraph of the $B_{1,4}$.

**Theorem E [8].** *Let $X$ and $Y$ be connected graphs with $X, Y \not\simeq P_3$, and let $G$ be a 2-connected graph of order $n \geq 10$. Then, $G$ being $XY$-free implies that $G$ has a 2-factor if and only if, up to the order of the pairs, either $\{X, Y\} = \{K_{1,4}, P_4\}$, or $X = K_{1,3}$ and $Y$ is an induced subgraph of at least one of the graphs $P_7$, $B_{1,4}$ or $N_{1,1,3}$.*

The closure concept for claw-free graphs, introduced in [12], was strengthened in [14] such that it still preserves the (non)-existence of a 2-factor in $G$ (while hamiltonian properties are not preserved).

Let $C$ be a cycle of even length $k \geq 4$ in a graph $G$. Two edges $e_1, e_2 \in E(G)$ are said to be *antipodal in $C$*, if they are at maximum distance in $C$, i.e. if $\mathrm{dist}_C(e_1, e_2) = k/2 - 1$. An even cycle $C$ in a graph $G$ is said to be *edge-antipodal in $G$*, abbreviated EA, if $\min\{\omega_G(e_1), \omega_G(e_2)\} = 2$ for any two antipodal edges $e_1, e_2 \in E(C)$. Analogously, two vertices $x_1, x_2 \in V(C)$ are *antipodal in $C$* if they are at maximum distance in $C$, i.e. if $\mathrm{dist}_C(x_1, x_2) = k/2$, and $C$ is said to be *vertex-antipodal in $G$*, abbreviated VA, if $\min\{d_G(x_1), d_G(x_2)\} = 2$ for any two antipodal vertices $x_1, x_2 \in V(C)$. It is easy to observe that an even (not necessarily induced) cycle $C$ in a graph $H$ is VA in $H$ if and only if the cycle $C' = L(C)$ is an induced EA-cycle in $G = L(H)$.

A vertex $x$ in a claw-free graph $G$ is *2f-eligible*, if either $x \in \mathrm{EL}(G)$, or $x \notin \mathrm{EL}(G)$ and $x$ is in an induced cycle of length 4 or 5 or in an induced EA-cycle of length 6. The set of all 2f-eligible vertices of $G$ is denoted $\mathrm{EL}^{2f}(G)$. A graph $\mathrm{cl}^{2f}(G)$ is a *2-factor-closure* (abbreviated 2f-closure) of a claw-free graph $G$, if there is a sequence of graphs $G_1, \ldots, G_k$ such that $G_1 = G$, $G_{i+1} = (G_i)^*_{x_i}$ for some $x_i \in \mathrm{EL}^{2f}(G_i)$, $i = 1, \ldots, k-1$, $G_k = \mathrm{cl}^{2f}(G)$ and $\mathrm{EL}^{2f}(G_k) = \emptyset$ (i.e., the 2f-closure of a claw-free graph $G$ is obtained by recursively repeating the local completion operation at 2f-eligible vertices, as long as this is possible). A graph $G$ is *2f-closed* if $G = \mathrm{cl}^{2f}(G)$. The following result summarizes basic properties of the 2f-closure.

**Theorem F [14].** *Let $G$ be a claw-free graph. Then*

($i$) *the closure $\mathrm{cl}^{2f}(G)$ is uniquely determined,*

($ii$) *there is a graph $H$ such that*

- ($\alpha$) *$L(H) = \mathrm{cl}^{2f}(G)$,*
- ($\beta$) *$g(H) \geq 6$,*
- ($\gamma$) *$H$ does not contain any vertex-antipodal cycle of length 6,*

($iii$) *$G$ has a 2-factor if and only if $\mathrm{cl}^{2f}(G)$ has a 2-factor.*

In the main result of this paper, Theorem 11, we use the 2f-closure to reduce the list given in the characterization in Theorem E in a way similar to that in which Theorems C and D reduce the list given in Theorem A. Namely, we show that if $X, Y$ is a pair of connected graphs such that a 2-connected graph $G$ being $XY$-free implies $G$ has a 2-factor, then every 2-connected $XY$-free graph $G$ of order at least 10 satisfies each of the following:

($i$) $G$ is $K_{1,4}P_4$-free, or $G$ is claw-free and $\mathrm{cl}^{2f}(G)$ is $N_{1,1,3}$-free,

($ii$) $G$ is $K_{1,4}P_4$-free, or $G$ is claw-free and $\mathrm{cl}^{2f}(G)$ is complete or belongs to $\mathcal{C}^{\geq 6}$.

# 3 Results

In this section we show that if a 2-connected graph $G$ is $K_{1,3}P_7$-free, $K_{1,3}B_{1,4}$-free or $K_{1,3}N_{1,1,3}$-free, then its 2f-closure $\mathrm{cl}^{2f}(G)$ is always $K_{1,3}N_{1,1,3}$-free, and we fully describe the structure of 2-connected 2f-closed $K_{1,3}N_{1,1,3}$-free graphs. In our proofs, we will often use the graphs $L^{-1}(N_{1,1,3})$ and $L^{-1}(B_{1,4})$ shown in Figure 4.

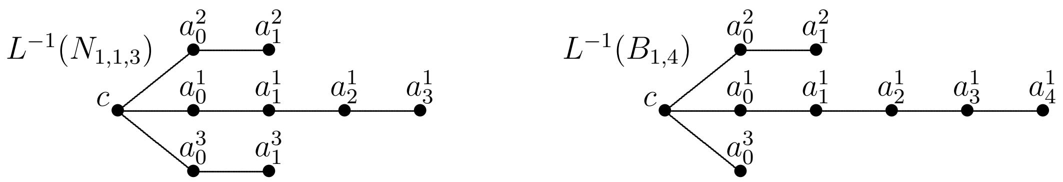


Figure 4: The preimages of $N_{1,1,3}$ and $B_{1,4}$

If $F$ is a graph and $e \in E(F)$, we say that a graph $F'$ is obtained from $F$ by *subdivision of* $e$, if $F'$ is isomorphic to the graph obtained by replacing $e$ by a path of length 2. Let $e = xy \in E(F)$ be a pendant edge of a graph $F$ with $d_F(x) = 1$ and $d_F(y) \geq 3$. We say that a graph $F'$ is obtained from $F$ by *rotation of* $e$ if $F'$ is isomorphic to the graph obtained from the graph $F - x$ by subdivision of one of the edges containing the vertex $y$. Whenever we speak of a rotation of an edge $e$, it is always understood that $e$ is a pendant edge.

The following result shows the way subdivisions and rotations of edges of a graph are related to local completions.

**Proposition 1.** *Let $G$ be a claw-free graph, $G_x^*$ the local completion of $G$ at a vertex $x \in V(G)$, and let $F$ be a connected triangle-free graph with $\Delta(F) \leq 3$ such that $H = L(F) \overset{\text{IND}}{\subset} G_x^*$. Then either $H \overset{\text{IND}}{\subset} G$, or there is a graph $F'$ such that $H' = L(F') \overset{\text{IND}}{\subset} G$ and $F'$ is obtained from $F$ by subdivision or rotation of an edge.*

**Proof.** Let $H \overset{\text{IND}}{\subset} G_x^*$ be such that $H = L(F)$ for some connected triangle-free graph $F$ with $\Delta(F) \leq 3$, and set $B = (E(G_x^*) \setminus E(G)) \cap E(H)$. If $B = \emptyset$, then $H \overset{\text{IND}}{\subset} G$ and we are done, hence suppose $B \neq \emptyset$. Since $H$ is induced, all edges in $B$ are in one maximal clique $K_B$ of $H$, and since $\langle N_G(x) \rangle_{G_x^*}$ is a clique, $x$ has no neighbors outside $K_B$. Note that all maximal cliques in $H$ are of order 2 or 3 (since $\Delta(F) \leq 3$) and edge-disjoint (since $F$ is triangle-free). For a vertex $v \in V(H)$ let $v_F$ denote the edge of $F = L^{-1}(H)$, corresponding to $v$.

Case 1: $|B| = 1$. Let $e = uv \in B$. If $\omega_H(e) = 2$, then $uxv$ is an induced path in $G$ and $x$ has no other neighbors in $V(H)$. Set $H' = (V(H) \cup \{x\}, (E(H) \setminus \{uv\}) \cup \{ux, vx\})$. Then $H' \overset{\text{IND}}{\subset} G$ and $H' = L(F')$, where $F'$ is isomorphic to the graph obtained from $F$ by subdivision of $u_F$ or of $v_F$.

Thus, let $\omega_H(e) = 3$. Let $\{u, v, w\} = K_B$ and set $H' = (V(H), E(H) \setminus \{uv\})$. Similarly as before, $H' \overset{\text{IND}}{\subset} G$. If $d_H(w) = 3$, then also $d_{H'}(w) = 3$, and $\langle\{w, u, v, w'\}\rangle_{H'}$, where $w'$ is the third neighbor of $w$ in $H$, is an induced claw in $G$, a contradiction. Hence $d_{H'}(w) = d_H(w) = 2$. Then $w$ is a simplicial vertex in $H$, implying $w_F$ is a pendant edge of $F$. Then $uwv$ is an induced path in $G$, hence $H' = L(F')$, where in $F'$ the edges $u_{F'}, w_{F'}, v_{F'}$ determine a path of length 3. Hence $F'$ is obtained by rotation of $w_F$.

Case 2: $|B| = 2$. Then necessarily $|V(K_B)| = 3$. Let $V(K_B) = \{u, v, w\}$ and choose the notation such that $B = \{uw, vw\}$ (i.e., $uv \in E(G)$). Set $H' = (V(H) \cup \{x\}, (E(H) \setminus \{uw, vw\} \cup \{xu, xv, xw\})$. Then $H' \overset{\text{IND}}{\subset} G$ and $H' = L(F')$, where $F'$ is obtained from $F$ by replacing the edge $w_F$ by the path of length 2 determined by the edges $w_{F'}, x_{F'}$. Thus, $F'$ is a subdivision of $F$.

Case 3: $|B| = 3$. Then $V(K_B) = \{u, v, w\}$ is independent in $G$. Since $V(K_B) \subset N_G(x)$, $\langle\{x, u, v, w\}\rangle_G$ is a claw, a contradiction. ■

The following result that was originally proved in [6] for $\text{cl}(G)$. We include its (short) proof here since we use the result here with a different type of closure and we prove it in a slightly more general setting.

**Proposition 2.** *Let $G$ be a claw-free graph, $x \in V(G)$, $i, j, k \geq 1$ integers, and let $G^*_x$ be the local completion of $G$ at $x$.*

*(i) If $G$ is $P_i$-free, then $G^*_x$ is $P_i$-free,*

*(ii) if $G$ is $N_{i,j,k}$-free, then $G^*_x$ is $N_{i,j,k}$-free.*

Note that Proposition 2 immediately implies that if $G$ is $K_{1,3}P_i$-free or $K_{1,3}N_{i,j,k}$-free, then so is $\text{cl}^{2f}(G)$.

**Proof.** If $H \overset{\text{IND}}{\subset} G^*_x$ with $H \simeq P_i$, then, since $P_i = L(P_{i+1})$, by Proposition 1 either $H \overset{\text{IND}}{\subset} G$, or $H' \overset{\text{IND}}{\subset} G$, where $H'$ is the line graph of a subdivision of $P_{i+1} = L^{-1}(H)$. In both cases, $G$ is not $P_i$-free.

Similarly, if $H \overset{\text{IND}}{\subset} G^*_x$ with $H \simeq N_{i,j,k}$, then, since $L^{-1}(N_{i,j,k})$ is a connected triangle-free graph with maximum degree at most 3 and with no pendant edges (recall that we suppose $i, j, k \geq 1$; for $L^{-1}(N_{1,1,3})$ see Figure 4), by Proposition 1 we have either $H \overset{\text{IND}}{\subset} G$, or $H' \overset{\text{IND}}{\subset} G$, where $H'$ is the line graph of a subdivision of $L^{-1}(N_{i,j,k})$. Since every subdivision of $L^{-1}(N_{i,j,k})$ contains a subgraph isomorphic to $L^{-1}(N_{i,j,k})$, $G$ is not $N_{i,j,k}$-free. ■

It should be noted here that a result similar to Proposition 2 is not true in the case of the class of $K_{1,3}B_{i,j}$-free graphs. The graph in Figure 5 is an example of a 2-connected $K_{1,3}B_{i,j}$-free graph $G$ for which $\text{cl}^{2f}(G)$ contains an induced $B_{i,j}$.

For our next results we will need some definitions and notation.

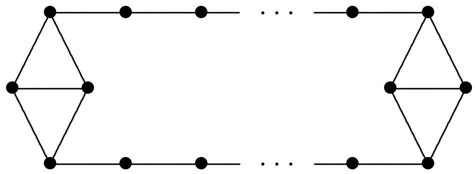

Figure 5: A 2-connected $K_{1,3}B_{i,j}$-free graph

Let $k \geq 4$ and $i_1, \ldots, i_k$ be nonnegative integers. The graph, obtained from a cycle $C$ of length $k$ with $V(C) = \{z_1, \ldots, z_k\}$ and $k$ (not necessarily nontrivial) vertex-disjoint paths $P_1, \ldots, P_k$ with $|V(P_j)| = i_j + 1$ and $P_j = y_0^j y_1^j \ldots y_k^j$ by identifying $z_j = y_0^j$, $j = 1, \ldots, k$, is called an $(i_1, \ldots, i_k)$-*sun*, or, briefly, a *sun* (see Figure 6). The cycle $C$ is called the *disc* and the paths $P_1, \ldots, P_k$ the *beams* of the sun, and for a beam $P_j$, the vertex $z_j$ is called its *root*. The family of all suns will be denoted $\mathcal{S}$.

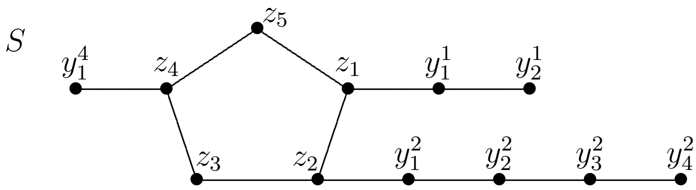


Figure 6: A $(2, 4, 0, 1, 0)$-sun

Now we can handle the case when $G$ is $K_{1,3}P_7$-free.

**Theorem 3.** *Let $G$ be a 2-connected $K_{1,3}P_7$-free graph. Then $\mathrm{cl}^{2f}(G)$ is $K_{1,3}N_{1,1,3}$-free.*

**Proof.** Let, to the contrary, $G$ be a 2-connected $K_{1,3}P_7$-free graph such that $\mathrm{cl}^{2f}(G)$ is not $K_{1,3}N_{1,1,3}$-free. By Proposition 2, we can suppose that $G$ is 2f-closed. By Theorem F, there is a graph $H = L^{-1}(G)$ with $g(H) \geq 6$. Since $G$ is not $K_{1,3}N_{1,1,3}$-free, $H$ contains a subgraph $F$ (not necessarily induced) such that $F \simeq L^{-1}(N_{1,1,3})$. Let the vertices of $F$ be labeled as indicated in Figure 4. The graph $G$ is 2-connected, hence $H$ is essentially 2-edge-connected. Thus, the edge $ca_0^1$ cannot be a cut-edge of $H$, hence there is a path $P = d_0 d_1 \ldots d_k$, $k \geq 1$, such that, up to a symmetry, $d_0 \in \{c, a_0^2, a_1^2\}$, $d_k \in \{a_0^1, a_1^1, a_2^1, a_3^1\}$ and $d_i \in V(H) \setminus V(F)$ for $1 \leq i \leq k-1$. The path $P$ then together with the unique $(d_0, d_k)$-path $P_F$ in $F$ determines a cycle of length at least 6, containing $c$. Then $\langle V(C) \cup \{a_0^3, a_1^3\}\rangle_H$ is a sun containing a $P_8$, hence $G = L(H)$ contains an induced $P_7$, a contradiction. ■

We next consider the case when $G$ is $K_{1,3}B_{1,4}$-free. The proof is more complicated in this case since, as already noted, $G$ being $K_{1,3}B_{1,4}$-free does not imply $\mathrm{cl}^{2f}(G)$ is $K_{1,3}B_{1,4}$-free and the proof will require some more definitions.

We say that a sun $S \in \mathcal{S}$ is *good* if $S$ contains a (not necessarily induced) subgraph $T \subset S$ such that $T = L^{-1}(B_{1,4})$ (see Figure 4). The family of all good suns will be denoted $\mathcal{S}_G$. For $S \in \mathcal{S}_G$, the beam of $S$ containing the center (i.e. the only vertex of degree 3) of $T$ is called the *main beam* of $S$, and if an $S \in \mathcal{S}_G$ contains more subgraphs that are isomorphic to $T$, we will always suppose that $T$ is chosen such that the length of the main beam is maximum. Beams that are not main are called *side beams*. For example, the $(2,4,0,1,0)$-sun in Figure 6 contains a $T$ centered at $z_1$ or at $z_4$. Since the choice of the center of $T$ at $z_1$ maximizes the length of the beam containing the center, we consider the beam $z_1y_1^1y_2^1$ as the main beam and the others as the side ones.

Let $S \in \mathcal{S}_G$. We say that $S$ is a *good sun of type A*, if the main beam of $S$ has length at least two. The family of all good suns of type A will be denoted $\mathcal{S}_G^A$.

Whenever we speak of a rotation of an edge $e$ of a sun, it is always understood that $e$ is a pendant edge, i.e. the only edge of a beam of length 1, and we will also sometimes simply speak of rotation of a beam of length 1.

The following observation is straightforward.

**Proposition 4.** *Let $S \in \mathcal{S}_G$ and let $S' \in \mathcal{S}$ be obtained from $S$ by subdivision of an edge or by rotation of a side beam. Then $S' \in \mathcal{S}_G$. Moreover, if $S \in \mathcal{S}_G^A$, then also $S' \in \mathcal{S}_G^A$.* ■

Let $S \in \mathcal{S}_G$. We say that $S$ is a *good sun of type B* if, for every sequence $S_1, \dots, S_k$ such that $S_1 = S$ and $S_{i+1}$ is obtained from $S_i$ by subdivision or rotation of an edge, either

($i$) $S_i \in \mathcal{S}_G$, $i = 1, \dots, k$, or

($ii$) there is a $k_0$, $1 \le k_0 \le k$, such that $S_i \in \mathcal{S}_G$ for $i = 1, \dots, k_0 - 1$ and $S_{k_0} \in \mathcal{S}_G^A$.

The family of all good suns of type B will be denoted $\mathcal{S}_G^B$.

Note that, by Proposition 4, for the proof that $S \in \mathcal{S}_G^B$ it is enough to verify ($i$) and ($ii$) for all sequences $S_1, \dots, S_k$ such that $S_{i+1}$ is obtained from $S_i$ by rotation of an edge. Also by Proposition 4, $\mathcal{S}_G^A \subset \mathcal{S}_G^B$, and, for example, for the sun $S$ of Figure 6 we have $S \in \mathcal{S}_G^A$ and $S - y_2^1 \in \mathcal{S}_G^B \setminus \mathcal{S}_G^A$.

From the definitions of the classes $\mathcal{S}$, $\mathcal{S}_G$, $\mathcal{S}_G^A$, $\mathcal{S}_G^B$ and from Propositions 1 and 4 we now easily conclude the following result that will be crucial for handling the class of $K_{1,3}B_{1,4}$-free graphs. Here, for a class $\mathcal{C}$ we denote $L(\mathcal{C}) = \{L(G) |\ G \in \mathcal{C}\}$.

**Theorem 5.** *Let $G$ be a claw-free graph and let $G_x^*$ be the local completion of $G$ at a vertex $x \in V(G)$.*

($i$) *If $G$ is $L(\mathcal{S})$-free, then $G_x^*$ is $L(\mathcal{S})$-free.*

($ii$) *If $G$ is $L(\mathcal{S}_G)$-free, then $G_x^*$ is $L(\mathcal{S}_G)$-free.*

($iii$) *If $G$ is $L(\mathcal{S}_G^A)$-free, then $G_x^*$ is $L(\mathcal{S}_G^A)$-free.*

($iv$) *If $G$ is $L(\mathcal{S}_G^A \cup \mathcal{S}_G^B)$-free, then $G_x^*$ is $L(\mathcal{S}_G^A \cup \mathcal{S}_G^B)$-free.*

**Proof.** ($i$) If $L(S) \overset{\text{IND}}{\subset} G_x^*$ for some $S \in \mathcal{S}$, then, since every sun is a connected triangle-free graph with maximum degree at most 3, Proposition 1 implies $L(S') \overset{\text{IND}}{\subset} G$, where $S'$ is obtained by subdivision or rotation of an edge.

The proof of parts ($ii$), ($iii$) and ($iv$) is similar using Proposition 4 and the definition of the class $\mathcal{S}_G^B$. ■

Now we are ready to turn our attention to $K_{1,3}B_{1,4}$-free graphs.

**Proposition 6.** *Let $G$ be a $K_{1,3}B_{1,4}$-free graph. Then $\mathrm{cl}^{2f}(G)$ contains no induced subgraph $F$ such that $L^{-1}(F) \in \mathcal{S}_G^A \cup \mathcal{S}_G^B$.*

**Proof.** If $F \overset{\text{IND}}{\subset} \mathrm{cl}^{2f}(G)$ is such that $L^{-1}(F) \in \mathcal{S}_G^A \cup \mathcal{S}_G^B$, then, by Theorem 5 and by induction, there is an $F' \overset{\text{IND}}{\subset} G$ such that $L^{-1}(F') \in \mathcal{S}_G^A \cup \mathcal{S}_G^B$. Thus, $L^{-1}(F)$ contains a subgraph $T \simeq L^{-1}(B_{1,4})$, contradicting the fact that $G$ is $B_{1,4}$-free. ■

**Proposition 7.** *Let $G$ be a 2-connected 2f-closed claw-free graph. If $G$ is not $K_{1,3}N_{1,1,3}$-free, then $G$ contains an induced subgraph $F$ such that $L^{-1}(F) \in \mathcal{S}_G^A \cup \mathcal{S}_G^B$.*

**Proof** of Proposition 7 is postponed to Section 4. ■

Now we can prove that the 2f-closure of a 2-connected $K_{1,3}B_{1,4}$-free graph must be $N_{1,1,3}$-free.

**Theorem 8.** *Let $G$ be a 2-connected $K_{1,3}B_{1,4}$-free graph. Then $\mathrm{cl}^{2f}(G)$ is $N_{1,1,3}$-free.*

**Proof.** If $\mathrm{cl}^{2f}(G)$ is not $N_{1,1,3}$-free, then, by Proposition 7, there is an $F \overset{\text{IND}}{\subset} \mathrm{cl}^{2f}(G)$ with $L^{-1}(F) \in \mathcal{S}_G^A \cup \mathcal{S}_G^B$, contradicting Proposition 6. ■

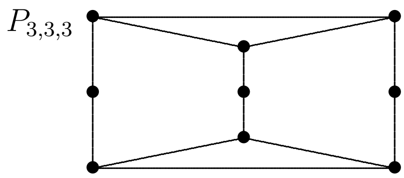


Figure 7: The only noncomplete 2-connected 2f-closed $K_{1,3}N_{1,1,3}$-free graph not in $\mathcal{C}^{\geq 6}$

In our next result we describe the structure of 2-connected 2f-closed $N_{1,1,3}$-free graphs. Here $\mathcal{C}^{\geq 6}$ is the family of graphs defined in Section 2 (see Figure 3) and $P_{3,3,3}$ is the graph shown in Figure 7.

**Theorem 9.** *Let $G$ be a 2-connected 2f-closed claw-free graph. Then $G$ is $N_{1,1,3}$-free if and only if $G$ is either complete or $G \in \mathcal{C}^{\geq 6} \cup \{P_{3,3,3}\}$.*

**Proof** of Theorem 9 is postponed to Section 4. ■

**Corollary 10.** *Let $Y \in \{P_7, B_{1,4}, N_{1,1,3}\}$ and let $G$ be a 2-connected $K_{1,3}Y$-free graph. Then either $\mathrm{cl}^{2f}(G)$ is complete or $\mathrm{cl}^{2f}(G) \in \mathcal{C}^{\geq 6} \cup \{P_{3,3,3}\}$.*

**Proof** follows immediately from Proposition 2 and from Theorems 3, 8 and 9. ■

Our final theorem summarizes results given in Theorem E, Proposition 2, Theorem 3, Theorem 8 and in Corollary 10.

**Theorem 11.** *Let $G$ be a 2-connected $XY$-free graph of order $n \geq 10$, where $X, Y$ is a pair of connected graphs such that $G$ being $XY$-free implies $G$ has a 2-factor. Then $G$ satisfies each of the following:*

*(i) $G$ is $K_{1,4}P_4$-free, or $G$ is claw-free and $\mathrm{cl}^{2f}(G)$ is $N_{1,1,3}$-free,*

*(ii) $G$ is $K_{1,4}P_4$-free, or $G$ is claw-free and $\mathrm{cl}^{2f}(G)$ is either complete or belongs to $\mathcal{C}^{\geq 6}$.* ■

# 4 Proofs

Throughout this section, for the description of a subgraph $T$ of $H$ isomorphic to $L^{-1}(N_{1,1,3})$ (see Figure 4) we will use the notation $T(P_1; P_2; P_3)$, where the three paths $P_1, P_2, P_3$ determining $T$ are ordered such that $P_1$ is the longest one, and the center of $T$ (i.e. the common vertex of $P_1$, $P_2$ and $P_3$) is always the first one.

For a sun with vertex set $\{x_1, \ldots, x_s\}$ we will use notation $S(x_1 \ldots x_s)$, where in the list of vertices we always list first the vertices of the disc, followed by lists of vertices of the beams, where the vertices of beams are ordered starting with the root and the beams are ordered in the order of the vertices of the disc; lists are separated with semicolons. If a sun is good, we always start the list of vertices of the disc with the root of the main beam (and, consequently, the main beam is also the first one in the list of beams). For example, the (good of type A) sun in Figure 6 will be denoted $S(z_1z_2z_3z_4z_5; z_1y_1^1y_2^1; z_2y_1^2y_2^2y_3^2y_4^2; z_4y_1^4)$.

**Proof of Proposition 7**. Let $G$ be a 2-connected 2f-closed claw-free graph and suppose that $G$ is not $N_{1,1,3}$-free. By Theorem F, there is a graph $H = L^{-1}(G)$ such that $g(H) \geq 6$ and $H$ contains no VA-cycle of length 6. Since $G$ is not $N_{1,1,3}$-free, $H$ contains a subgraph $F$ (not necessarily induced) such that $F \simeq L^{-1}(N_{1,1,3})$. We will use the labeling of vertices of $F$ as indicated in Figure 4. The graph $G$ is 2-connected and hence $H$ is essentially 2-edge-connected. Thus, the edge $ca_0^1$ cannot be a cut-edge of $H$, implying there is a path $P = d_0d_1 \ldots d_k$, $k \geq 1$, such that, up to a symmetry, $d_0 \in \{c, a_0^2, a_1^2\}$, $d_k \in \{a_0^1, a_1^1, a_2^1, a_3^1\}$ and $d_i \in V(H) \setminus V(F)$ for $1 \leq i \leq k-1$. We will show that in each of the possible cases $H$ contains a sun $S \in \mathcal{S}_G^A \cup \mathcal{S}_G^B$. We list all possible cases in the following table in which the first two columns describe the case, in the third column we give minimum length of the path $P$ which follows from the fact that $g(H) \geq 6$, and in the last column we indicate the sun obtained in this case.

| Case | | Minimum | The sun $S$ containing | |
|---|---|---|---|---|
| $d_0$ | $d_k$ | length $k$ of $P$ | a copy of $L^{-1}(B_{1,4})$ | |
| $a_0^2$ | $a_0^1$ | 4 | $S(d_k \dots d_0 c; d_k a_1^1 a_2^1; d_0 a_1^2) \in \mathcal{S}_G^A$ | |
| $a_0^2$ | $a_1^1$ | 3 | $S(c a_0^1 d_k \dots d_0; c a_0^3 a_1^3; d_k a_2^1) \in \mathcal{S}_G^A$ | |
| $a_0^2$ | $a_2^1$ | 2 | $S(d_k \dots d_0 c a_0^1 a_1^1; d_k a_3^1; c a_0^3 a_1^3) \in \mathcal{S}_G^B$ | |
| $a_0^2$ | $a_3^1$ | 1 | $S(c a_0^1 a_1^1 a_2^1 d_k \dots d_0; c a_0^3 a_1^3) \in \mathcal{S}_G^A$ | if $k \geq 2$ |
| $a_1^2$ | $a_0^1$ | 3 | $S(c d_k \dots d_0 a_0^2; c a_0^3 a_1^3) \in \mathcal{S}_G^A$ | if $k \geq 4$ |
| $a_1^2$ | $a_1^1$ | 2 | $S(d_k \dots d_0 a_0^2 c a_0^1; d_k a_2^1 a_3^1; c a_0^3) \in \mathcal{S}_G^A$ | |
| $a_1^2$ | $a_2^1$ | 1 | $S(d_k \dots d_0 a_0^2 c a_0^1 a_1^1; d_k a_3^1; c a_0^3 a_1^3) \in \mathcal{S}_G^B$ | |
| $a_1^2$ | $a_3^1$ | 1 | $S(c a_0^1 a_1^1 a_2^1 d_k \dots d_0 a_0^2; c a_0^3 a_1^3) \in \mathcal{S}_G^A$ | |
| $c$ | $a_0^1$ | 5 | $S(d_0 \dots d_k; d_0 a_0^3 a_1^3) \in \mathcal{S}_G^A$ | if $k \geq 6$ |
| $c$ | $a_1^1$ | 4 | $S(d_0 \dots d_k a_0^1; d_0 a_0^3 a_1^3; d_k a_2^1) \in \mathcal{S}_G^A$ | |
| $c$ | $a_2^1$ | 3 | $S(d_k \dots d_0 a_0^1 a_1^1; d_k a_3^1; d_0 a_0^3 a_1^3) \in \mathcal{S}_G^B$ | |
| $c$ | $a_3^1$ | 2 | $S(d_0 \dots d_k a_2^1 a_1^1 a_0^1; d_0 a_0^3 a_1^3) \in \mathcal{S}_G^A$ | if $k \geq 3$ |

Note that in the third, seventh and eleventh case we obtain a sun $S$ such that the rotation of its beam of length 1 results in a good sun $S'$ with disc of length 7 and main beam of length 2. Clearly $S' \in \mathcal{S}_G^A$, hence $S \in \mathcal{S}_G^B$ by Proposition 4 and by the remark before Proposition 1.

The remaining possibilities are:

$$\begin{aligned} d_0 = a_0^2, &\quad d_k = a_3^1, \quad k = 1; \\ d_0 = a_1^2, &\quad d_k = a_0^1, \quad k = 3; \\ d_0 = c, &\quad d_k = a_0^1, \quad k = 5; \\ d_0 = c, &\quad d_k = a_3^1, \quad k = 2. \end{aligned}$$

We consider these cases separately, starting with the second one.

Case 1: $d_0 = a_1^2, d_k = a_0^1$ and $k = 3$. We first observe that if $d_1 a_1^3 \in E(H)$, then we have $S(d_1 d_2 d_3 c a_0^2 d_0; d_1 a_1^3 a_0^3; d_3 a_1^1 a_2^1 a_3^1) \in \mathcal{S}_G^A$; hence we can suppose that $d_1 a_1^3 \notin E(H)$. Let $S^* = S(c d_3 d_2 d_1 d_0 a_0^2; c a_0^3 a_1^3)$ (note that $S^* \notin \mathcal{S}_G$). Clearly, $S^* \overset{\text{IND}}{\subset} H$, since any edge $xy \in E(H) \setminus E(S^*)$ with $x, y \in V(S^*)$ creates a cycle of length at most 5. The cycle $C = c a_0^2 d_0 d_1 d_2 d_3 c$ (of length 6) cannot be vertex-antipodal and hence at least one of the vertices $d_0, d_1, d_2$ must have a neighbor $u$ outside $S^*$. If $u$ is a neighbor of $d_0$, then $S(c d_3 d_2 d_1 d_0 a_0^2; c a_0^3 a_1^3; d_0 u) \in \mathcal{S}_G^A$; if $u$ is a neighbor of $d_1$, then $S(d_1 d_0 a_0^2 c d_3 d_2; d_1 u; c a_0^3 a_1^3) \in \mathcal{S}_G^B$, and if $u$ is a neighbor of $d_2$, then $S(c d_3 d_2 d_1 d_0 a_0^2; c a_0^3 a_1^3; d_2 u) \in \mathcal{S}_G^A$.

Case 2: $d_0 = c, d_k = a_0^1$ and $k = 5$. If we set $F' = T(c a_0^1 a_1^1 a_2^1 a_3^1; c d_1 d_2; c a_0^3 a_1^3)$ and $P' = d_2 d_3 d_4 d_5$, then for $F'$ and $P'$ we are back in Case 1.

Case 3: $d_0 = c, d_k = a_3^1$ and $k = 2$. If $a_2^1 a_1^2 \in E(H)$, then $S(a_2^1 a_1^1 a_0^1 d_0 d_1 d_2; a_2^1 a_1^2 a_0^2; d_0 a_0^3 a_1^3) \in \mathcal{S}_G^A$; hence we can suppose that $a_2^1 a_1^2 \notin E(H)$ and, symmetrically, $a_2^1 a_1^3 \notin E(H)$. Then there is no edge $xy \in E(H) \setminus (E(F) \cup E(P))$ with $x, y \in V(F) \cup V(P)$ (otherwise we get a cycle of length at most 5). The cycle $C = d_0 d_1 d_2 a_2^1 a_1^1 a_0^1 d_0$ (of length 6) cannot be vertex-antipodal; hence at least one of the vertices $a_1^1$, $a_2^1$, $d_2$ has a neighbor $u \in V(H) \setminus (V(F) \cup$

$V(P)$). If $ua_1^1 \in E(H)$, then $S(d_0d_1d_2a_2^1a_1^1a_0^1; d_0a_0^3a_1^3; a_1^1u) \in \mathcal{S}_G^A$; if $ua_2^1 \in E(H)$, then $S(a_2^1a_1^1a_0^1d_0d_1d_2; a_1^2u; d_0a_0^3a_1^3) \in \mathcal{S}_G^B$ and if $ud_2 \in E(H)$, then $S(d_0d_1d_2a_2^1a_1^1a_0^1; d_0a_0^3a_1^3; d_2u) \in \mathcal{S}_G^A$.

Case 4: $d_0 = a_0^2, d_k = a_3^1$ and $k = 1$. Since $H$ is essentially 2-edge-connected, the edge $ca_0^3$ cannot be a cut-edge, implying there is a path $P' = d_0'd_1' \dots d_t'$, $t \geq 1$, such that $d_0' \in \{a_0^3, a_1^3\}$, $d_t' \in V(F) \setminus \{a_0^3, a_1^3\}$ and $d_i' \in V(H) \setminus V(F)$ for $1 \leq i \leq t-1$.

If $d_t' \in \{a_0^1, a_1^1, a_2^1, a_3^1\}$, then each of these possibilities is symmetric to some of the previous cases, except for the case when $d_0' = a_0^3$, $d_t' = a_3^1$ and $t = 1$; but then $C = a_0^3a_3^1a_0^2ca_0^3$ is a cycle of length 4, a contradiction. Hence it remains to consider the cases when $d_t' \in \{c, a_0^2, a_1^2\}$. We first make the following easy observations.

Claim 1. *If $H$ contains two edge-disjoint cycles with a common vertex, then $H$ contains a sun $S \in \mathcal{S}_G^A$.*

Proof. If $C_1 = x_1x_2 \dots x_rx_1$ and $C_2 = x_1y_2 \dots y_sx_1$ are two cycles with $r, s \geq 6$ and $V(C_1) \cap V(C_2) = x_1$, then $S(x_1x_2 \dots x_r; x_1y_2y_3y_4y_5y_6) \in \mathcal{S}_G^A$. □

Claim 2. *If $H$ contains two cycles with exactly one common edge, then $H$ contains a sun $S \in \mathcal{S}_G^A \cup \mathcal{S}_G^B$.*

Proof. Let $C_1 = x_1x_2x_3 \dots x_rx_1$ and $C_2 = x_1x_2y_3 \dots y_sx_1$ be two cycles with $r, s \geq 6$, $V(C_1) \cap V(C_2) = \{x_1, x_2\}$ and $E(C_1) \cap E(C_2) = x_1x_2$. If $r \geq 7$ or $s \geq 7$, then we immediately have a $S \in \mathcal{S}_G^A$; hence suppose $r = s = 6$. The only possible edges $uv \in E(H) \setminus (E(C_1) \cup E(C_2))$ with $u, v \in V(C_1) \cup V(C_2)$ that do not create a cycle of length at most 5 are the edges $x_4y_5$ and $x_5y_4$, but in the first case $S(x_4x_5x_6x_1x_2x_3; x_4y_5y_4; x_2y_3) \in \mathcal{S}_G^A$ and the second case is symmetric. Thus, suppose there is no edge outside $E(C_1) \cup E(C_2)$ with both ends in $V(C_1) \cup V(C_2)$. Since $C_1$ cannot be vertex-antipodal, at least one of the vertices $x_3$, $x_4$, $x_5$ has a neighbor $u \in V(H) \setminus (V(C_1) \cup V(C_2))$. If $ux_3 \in E(H)$, then $S(x_1x_2x_3x_4x_5x_6; x_1y_6y_5; x_3u) \in \mathcal{S}_G^A$; if $ux_4 \in E(H)$, then $S(x_4x_5x_6x_1x_2x_3; x_4u; x_1y_6y_5) \in \mathcal{S}_G^B$, and the case $ux_5 \in E(H)$ is symmetric. □

Now we can easily see that in each of our cases $H$ contains a subgraph consisting of two cycles with a common vertex or edge and hence one of the claims applies.

| Case $d_0'$ | Case $d_t'$ | Minimum length $t$ of $P'$ | Claim applicable to this case |
|---|---|---|---|
| $a_0^3$ | $c$ | 5 | Claim 1 |
| $a_0^3$ | $a_0^2$ | 4 | Claim 2 |
| $a_0^3$ | $a_1^2$ | 3 | Claim 2 |
| $a_1^3$ | $c$ | 4 | Claim 1 |
| $a_1^3$ | $a_0^2$ | 3 | Claim 2 |
| $a_1^3$ | $a_1^2$ | 2 | Claim 2 |

In each of the possible cases, we have found a sun $S \in \mathcal{S}_G^A \cup \mathcal{S}_G^B$. ■

**Proof of Theorem 9**. It is immediate to see that every graph $G \in \mathcal{C}^{\geq 6} \cup \{P_{3,3,3}\}$ is 2-connected, 2f-closed and $K_{1,3}N_{1,1,3}$-free. Let thus, conversely, $G$ be a 2-connected 2f-closed

$K_{1,3}N_{1,1,3}$-free graph, suppose that $G$ is not complete and let $H = L^{-1}(G)$. Since $G$ is 2f-closed, $g(H) \geq 6$ and $H$ contains no VA-cycle of length 6. Since $G$ is $N_{1,1,3}$-free, $H$ contains no subgraph $T$ (not necessarily induced) such that $T \simeq L^{-1}(N_{1,1,3})$.

The graph $G$ is 2-connected noncomplete, implying $H$ contains a cycle. If $C = x_1 \dots x_k x_1$ is a cycle in $H$ and $e = x_1x_s \in E(H)$ is a chord of $C$, then $s \geq 6$ and $k \geq s+4$ since $g(H) \geq 6$, but then $T(x_1x_2x_3x_4x_5; x_1x_sx_{s+1}; x_1x_kx_{k-1}) \simeq L^{-1}(N_{1,1,3})$, a contradiction. Hence all cycles in $H$ are chordless.

We first consider the case when $H$ contains a cycle $C = x_1 \dots x_k x_1$ of length $k \geq 7$. If there is an edge $e = uv \in E(H)$ with $u, v \in V(H) \setminus V(C)$, then the edge $e$ and the notation of the vertices of $C$ can be chosen such that $ux_1 \in E(H)$, but then $T(x_1x_2x_3x_4x_5; x_1uv; x_1x_kx_{k-1}) \simeq L^{-1}(N_{1,1,3})$, a contradiction. Hence every edge of $H$ has at least one vertex on $C$.

Let $u \in V(H) \setminus V(C)$ be a vertex of degree $d_H(u) \geq 2$, and let $x_1, x_s \in V(C)$ be neighbors of $u$. Clearly $s \geq 5$ and $k \geq s+3$ since $g(H) \geq 6$. If $s \geq 6$, then $T(x_1x_2x_3x_4x_5; x_1ux_s; x_1x_kx_{k-1}) \simeq L^{-1}(N_{1,1,3})$; hence $s = 5$. Symmetrically, $k = s+3 = 8$. If $d_H(u) \geq 3$, then $u$ has a neighbor $v \in V(H) \setminus V(C)$ (since all cycles in $H$ are chordless), but then $T(x_1x_2x_3x_4x_5; x_1uv; x_1x_8x_7) \simeq L^{-1}(N_{1,1,3})$; hence $d_H(u) = 2$. Similarly $d_H(x_4) = 2$ (otherwise, for a vertex $v \in N_H(x_4) \setminus V(C)$ we have $T(x_1x_2x_3x_4v; x_1ux_5; x_1x_8x_7) \simeq L^{-1}(N_{1,1,3})$), and, symmetrically, $d_H(x_2) = 2$. But then the cycle $x_1x_2x_3x_4x_5ux_1$ is a VA-cycle of length 6 in $H$, a contradiction. Thus, all vertices in $V(H) \setminus V(C)$ are of degree 1, implying $G = L(H) \in \mathcal{C}^{\geq 7} \subset \mathcal{C}^{\geq 6}$.

It remains to consider the case when all cycles in $H$ are of length 6 (and chordless). Let thus $C = x_1x_2x_3x_4x_5x_6x_1 \subset H$.

If there is an edge $e = uv \in E(H)$ with $u, v \in V(H) \setminus V(C)$, then, by the 2-connectedness of $G = L(H)$, there is a path $P$ (not necessarily containing the edge $e$) with endvertices on $C$ and interior vertices outside $C$. Choose the notation such that $P = x_1z_1 \dots z_py$, where $z_1, \dots, z_p \in V(H) \setminus V(C)$ and $y \in V(C)$. If $y = x_2$ or $y = x_3$ (or, symmetrically, $y = x_5$ or $y = x_6$), then, for any value of $p$, the graph $H$ contains a cycle of length different from 6. Thus, there are only two possibilities that do not create a cycle of length different from 6, namely, $y = x_1$ and $p = 5$ or $y = x_4$ and $p = 2$. But in the first case immediately $T(x_1x_2x_3x_4x_5; x_1z_1z_2; x_1z_5z_4) \simeq L^{-1}(N_{1,1,3})$; hence we have $y = x_4$ and $p = 2$. If there is a vertex $w \in V(H) \setminus (V(C) \cup V(P))$, then, by symmetry, we can suppose that $wx_6 \in E(H)$ or $wx_4 \in E(H)$, but then we have $T(x_1x_2x_3x_4x_5; x_1z_1z_2; x_1x_6w) \simeq L^{-1}(N_{1,1,3})$ or $T(x_1x_2x_3x_4w; x_1z_1z_2; x_1x_6x_5) \simeq L^{-1}(N_{1,1,3})$, respectively. Hence $V(H) = V(C) \cup V(P)$, implying $G = L(H) \simeq P_{3,3,3}$.

Thus, the remaining possibility is that all edges of $H$ have at least one vertex in $V(C)$. But then all vertices in $V(H) \setminus V(C)$ are of degree 1 (otherwise we get a cycle of length at most 5), implying $G = L(H) \in \mathcal{C}^{6} \in \mathcal{C}^{\geq 6}$. ■

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