On stability of Hamilton-connectedness under the 2-closure in claw-free graphs

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Abstract

We show that, in a claw-free graph, Hamilton-connectedness is preserved under the operation of local completion performed at a vertex with 2-connected neighborhood. This result proves a conjecture by Bollobás et al.

1 Notation and terminology

In this paper, by a *graph* we mean a finite simple undirected graph $G = (V(G), E(G))$. For a vertex $x \in V(G)$, $N_G(x)$ denotes the *neighborhood of x in G*, i.e. $N_G(x) = \{y \in V(G) |\ xy \in E(G)\}$, and $N_G[x]$ denotes the *closed neighborhood of x in G*, i.e. $N_G[x] = N_G(x) \cup \{x\}$. If G, H are graphs, then $H \subset G$ means that H is a *subgraph* of G. The *induced subgraph* of G on a set $M \subset V(G)$ is denoted $\langle M\rangle_G$. By a *clique* we mean a (not necessarily maximal) complete subgraph of G. A vertex $x \in V(G)$ for which $\langle N_G(x)\rangle_G$ is a connected graph (k-connected graph, clique) is said to be *locally connected* (*locally k-connected*, *simplicial*), respectively.

A path with endvertices a, b will be referred to as an *(a, b)-path*. If P is an (a, b)-path and $u \in V(P)$, then $u^{-(P)}$ and $u^{+(P)}$ denotes the *predecessor* and *successor* of u on P (always considered in the orientation from a to b). If no confusion can arise we simply write u^- and u^+. If P is a path and $u, v \in V(P)$, then uPv denotes the (u, v)-subpath of P. If we want to emphasize that a subpath is traversed in the same (opposite) orientation as P, we use the notation $u\overrightarrow{P}v$ or $u\overleftarrow{P}v$, respectively.

Throughout the paper, $\kappa(G)$ denotes the (vertex) *connectivity* of G and $c(G)$ the *circumference* of G (i.e. the length of a longest cycle in G). A graph G is *hamiltonian* if $c(G) = |V(G)|$.

For a graph G and $a, b \in V(G)$, $p(G)$ denotes the length of a longest path in G, $p_a(G)$ the length of a longest path in G with one endvertex at $a \in V(G)$, and $p_{ab}(G)$ the length of a longest (a, b)-path in G. A graph G is *homogeneously traceable* if, for any $a \in V(G)$, G

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has a hamiltonian path with one endvertex at a (i.e., for any $a \in V(G)$, $p_a(G) = |V(G)|$), and G is *Hamilton-connected* if, for any $a, b \in V(G)$, G has a hamiltonian (a, b)-path (i.e., for any $a, b \in V(G)$, $p_{ab}(G) = |V(G)|$).

We say that G is *claw-free* if G does not contain a copy of the *claw* $K_{1,3}$ as an induced subgraph. Whenever we list vertices of an induced claw, its *center* (i.e. the only vertex of degree 3) is always the first vertex of the list.

For further concepts and notations not defined here we refer the reader to [2].

2 Introduction

A locally connected nonsimplicial vertex is called *eligible*. The *local completion* of G at a vertex x is the graph G'_x obtained from G by adding all edges with both vertices in $N_G(x)$ (note that the local completion at x turns x into a simplicial vertex, and preserves the claw-free property of G).

The *closure* $\mathrm{cl}(G)$ of a claw-free graph G is the graph obtained from G by recursively performing the local completion operation at eligible vertices as long as this is possible. We say that G is *closed* if $G = \mathrm{cl}(G)$.

The following was proved in [7].

Theorem A [7]. *For every claw-free graph G:*

(i) $\mathrm{cl}(G)$ *is uniquely determined,*

(ii) $\mathrm{cl}(G)$ *is the line graph of a triangle-free graph,*

(iii) $c(\mathrm{cl}(G)) = c(G)$,

(iv) $\mathrm{cl}(G)$ *is hamiltonian if and only if G is hamiltonian.*

A graph class $\mathcal{C}$ is *stable* if $G \in \mathcal{C}$ implies $\mathrm{cl}(G) \in \mathcal{C}$. If $\mathcal{C}$ is a stable class, then a graph property π is *stable* in $\mathcal{C}$ if G has π if and only if $\mathrm{cl}(G)$ has π, and a graph invariant α is *stable* in $\mathcal{C}$ if $\alpha(G) = \alpha(\mathrm{cl}(G))$, for any $G \in \mathcal{C}$.
Thus, Theorem A says that circumference is a stable invariant and hamiltonicity is a stable property in the class of claw-free graphs.

Let G be the line graph of the multigraph H shown in Figure 1(a). Then G has no hamiltonian (u_1, u_2)-path (where u_1, u_2 are the vertices of $G = L(H)$ that correspond to the edges u_1, u_2 in H), but $\mathrm{cl}(G)$ is Hamilton-connected. This example shows that Hamilton-connectedness is not stable in 3-connected claw-free graphs.

Brandt [3] proved that every 9-connected claw-free graph is Hamilton-connected, and Hu, Tian and Wei [5] improved this result by showing that every 8-connected claw-free graph is Hamilton-connected. Consequently, Hamilton-connectedness is stable in 8-connected claw-free graphs.

The following extension of the closure concept was introduced in [1]. For an integer $k \geq 1$, a locally k-connected nonsimplicial vertex is said to be *k-eligible*, and the *k-closure* of G, denoted $\mathrm{cl}_k(G)$, is the graph obtained from G by recursively performing the local completion operation at k-eligible vertices as long as this is possible. A graph G is *k-closed* if $G = \mathrm{cl}_k(G)$.

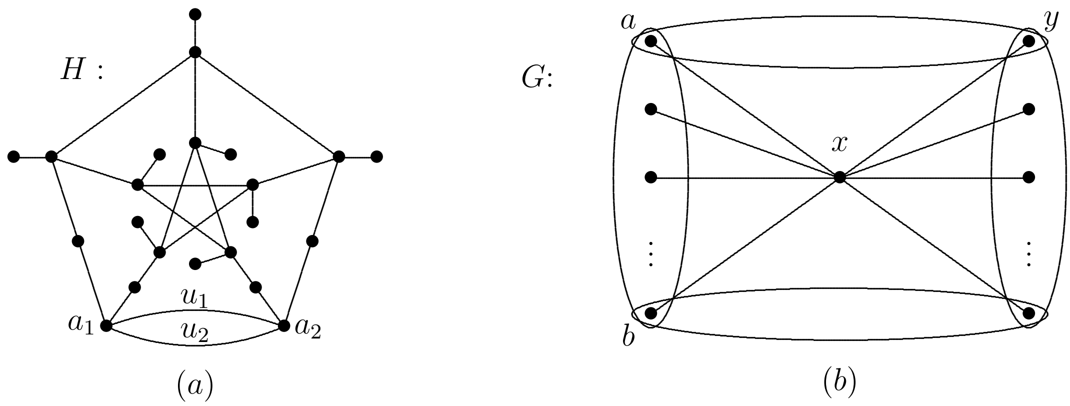

Figure 1

A graph class $\mathcal{C}$ is *k-stable* if $G \in \mathcal{C}$ implies $\mathrm{cl}_k(G) \in \mathcal{C}$. For a k-stable class $\mathcal{C}$, a graph property π is *k-stable in* $\mathcal{C}$ if G has π if and only if $\mathrm{cl}_k(G)$ has π, and a graph invariant α is *k-stable* in $\mathcal{C}$ if $\alpha(G) = \alpha(\mathrm{cl}(G))$, for any $G \in \mathcal{C}$.

The following result is implicit in the proof of the main result of [1].

Proposition B [1]. *Let G be a claw-free graph, let $x \in V(G)$ and let G'_x be the local completion of G at x.*
(*i*) *If $\langle N_G(x)\rangle_G$ is 2-connected, then, for any $a \in V(G)$, $p_a(G) = p_a(G'_x)$;*
(*ii*) *If $\langle N_G(x)\rangle_G$ is 3-connected, then, for any $a, b \in V(G)$, $p_{ab}(G) = p_{ab}(G'_x)$.*

Proposition B then immediately implies (*ii*) and (*iii*) of the following result.

Theorem C [1]. *For every claw-free graph G,*
(*i*) $\mathrm{cl}_k(G)$ *is uniquely determined,*
(*ii*) $\mathrm{cl}_2(G)$ *is homogeneously traceable if and only if G is homogeneously traceable,*
(*iii*) $\mathrm{cl}_3(G)$ *is Hamilton-connected if and only if G is Hamilton-connected.*

Thus, homogeneous traceability is 2-stable and hamilton-connectedness is 3-stable in the class of claw-free graphs.

Let now G be the graph in Figure 1(b) (where the ovals represent cliques on at least three vertices). Then G has no hamiltonian (a, b)-path, the vertex x is 2-eligible, and there is a hamiltonian (a, b)-path in the local completion G'_x of G at x. This example shows that the property "having a hamiltonian (a, b)-path for given $a, b \in V(G)$" is not 2-stable. However, neither G nor its 2-closure are Hamilton-connected. This observation motivated in [1] the following conjecture.

Conjecture D [1]. *Hamilton-connectedness is 2-stable in the class of claw-free graphs.*

It should be noted here that in [6] the author claimed to give an infinite family of counterexamples to Conjecture D. However, the behavior of the graphs constructed in [6] is similar to that of the graph in Figure 1(b), i.e., they show that the property of "having a hamiltonian (a, b)-path for given $a, b \in V(G)$" is not 2-stable, but do not disprove Conjecture D.

3 Results

In the graph G of Figure 1(b), the vertex x is locally 2-connected in G, the graph G does not have a hamiltonian (a, b)-path while G'_x does, and there is another pair of vertices u, v (in this example, $u = a$ and $v = y$) for which there is no hamiltonian (u, v)-path in G'_x. The following theorem shows that this essentially has always to be the case.

Theorem 1. *Let $x \in V(G)$ a locally 2-connected vertex of a claw-free graph G and let G'_x be the local completion of G at x. Then G is Hamilton-connected if and only if G'_x is Hamilton-connected.*

Theorem 1 immediately implies the following theorem, which is the main result of this paper and gives an affirmative answer to Conjecture D.

Theorem 2. *Hamilton-connectedness is 2-stable in the class of claw-free graphs.*

Proof of Theorem 2 follows immediately from Theorem 1. ∎

Note that, in [8], Theorem 2 is one of the main tools that allow to develop a closure concept for Hamilton-connectedness, which is then used to show that every 7-connected claw-free graph is Hamilton-connected.

Before proving Theorem 1, we first give several auxiliary structural results.

Fouquet [4] proved that in a connected claw-free graph G with independence number at least 3 the neighborhood of every vertex either can be covered by two cliques or contains an induced C_5. On the other hand, by Proposition B, if $x \in V(G)$ is such that $\kappa(\langle N_G(x)\rangle_G) \geq 3$, then G'_x is Hamilton-connected if and only if G is Hamilton-connected and there is nothing to do. The following statement describes in more detail the structure of the neighborhood of x in the difficult case, i.e. when $\kappa(\langle N_G(x)\rangle_G) = 2$.

Lemma 3. *Let G be a claw-free graph, let $x \in V(G)$ be such that $\kappa(\langle N_G(x)\rangle_G) = 2$, let $R = \{r_1, r_2\}$ be a cutset of $\langle N_G(x)\rangle_G$ and let N_1, N_2 be the components of $\langle N_G(x)\rangle_G - R$. Then x and R satisfy exactly one of the following:*

- (a) *$V(\langle N_G(x)\rangle_G)$ can be covered by two cliques,*
- (b) *$\langle N_G(x)\rangle_G$ contains an induced C_5 and, up to a symmetry,*
  - (i) *N_1, N_2 are cliques,*
  - (ii) *for every $y \in V(N_1)$, both $yr_1 \in E(G)$ and $yr_2 \in E(G)$,*
  - (iii) *$r_1r_2 \notin E(G)$,*
  - (iv) *for every $y \in V(N_2)$, $yr_1 \in E(G)$ or $yr_2 \in E(G)$,*
  - (v) *there are $z_1, z_2 \in V(N_2)$ such that $r_iz_i \in E(G)$ but $r_iz_{3-i} \notin E(G)$, $i = 1, 2$ (i.e., r_i is the only neighbor of z_i in R).*

Proof. Suppose that x and $N_G(x)$ satisfy the assumptions of the lemma and $V(\langle N_G(x)\rangle_G)$ cannot be covered by two cliques; we verify the conditions (i) – (v) of (b). Denote N_1, N_2 the components of $\langle N_G(x)\rangle_G - R$.

(i) If $u_1, u_2 \in V(N_1)$, $u_1u_2 \notin E(G)$, then, for some $v \in V(N_2)$, $\langle\{x, u_1, u_2, v\}\rangle_G$ is a claw. Hence N_1 (and symetrically also N_2) is a clique.

(*ii*) We first observe that each of r_1, r_2 is adjacent to all vertices of at least one of N_1, N_2 since if e.g. there are $u_1 \in V(N_1)$ and $u_2 \in V(N_2)$ such that both $r_1u_1 \notin E(G)$ and $r_1u_2 \notin E(G)$, then $\langle\{x, u_1, r_1, u_2\}\rangle_G$ is a claw (and symmetrically for r_2). Choose the notation such that r_1 is adjacent to all vertices of N_1. If r_2 is adjacent to all vertices of N_2, then $\langle V(N_1) \cup \{r_1\}\rangle_G$ and $\langle V(N_2) \cup \{r_2\}\rangle_G$ are two cliques covering $V(\langle N_G(x)\rangle_G)$, a contradiction. Hence r_2 is adjacent to all vertices of N_1.

(*iii*) If $r_1r_2 \in E(G)$, then $\langle V(N_1) \cup \{r_1, r_2\}\rangle_G$ and N_2 are two cliques covering $V(\langle N_G(x)\rangle_G)$, a contradiction.

(*iv*) If both $yr_1 \notin E(G)$ and $yr_2 \notin E(G)$ for some $y \in V(N_2)$, then $\langle\{x, r_1, y, r_2\}\rangle_G$ is a claw.

(*v*) If there is no such z_1, then r_2 is adjacent to all vertices of N_2 and $V(\langle N_G(x)\rangle_G)$ can be covered by two cliques, a contradiction. Symmetrically for z_2.

Finally, $C = r_1z_1z_2r_2ur_1$, where u is an arbitrary vertex of N_1, is an induced C_5 in $\langle N_G(x)\rangle_G$. ■

Corollary 4. *Let G be a claw-free graph, let $x \in V(G)$ be such that $\kappa(\langle N_G(x)\rangle_G) = 2$ and let $R = \{r_1, r_2\}$ be a cutset of $\langle N_G(x)\rangle_G$. Then there are sets $K_1, K_2 \subset N_G(x)$ such that:*

(1) $K_1 \cap K_2 = \emptyset$ *and* $K_1 \cup K_2 = N_G(x)$,

(2) $|K_i| \geq 2$, $i = 1, 2$,

(3) *there is exactly one of the following two possibilities:*

  (*a*) $\langle K_i\rangle_G$ *is a clique,* $i = 1, 2$,

  (*b*) $V(\langle N_G(x)\rangle_G)$ *contains an induced* C_5 *and*

    (*i*) $R \subset K_1$,

    (*ii*) $r_1r_2 \notin E(G)$,

    (*iii*) $\langle K_1\rangle_G + r_1r_2$ *and* $\langle K_2\rangle_G$ *are cliques,*

    (*iv*) *for every* $y \in K_2$, $yr_1 \in E(G)$ *or* $yr_2 \in E(G)$,

    (*v*) *there are* $z_1, z_2 \in K_2$ *such that* $r_iz_i \in E(G)$ *but* $r_iz_{3-i} \notin E(G)$, $i = 1, 2$ *(i.e.,* r_i *is the only neighbor of* z_i *in* R*).*

Proof. Suppose first that $V(\langle N_G(x)\rangle_G)$ can be covered by two cliques A_1, A_2. If both $|V(A_1)| \geq 2$ and $|V(A_2)| \geq 2$, we set $K_i = V(A_i)$, $i = 1, 2$, and we are done. Hence suppose that e.g. $|V(A_1)| = 1$ with $V(A_1) = \{a\}$. By the 2-connectedness of $\langle N_G(x)\rangle_G$, there are $b_1, b_2 \in V(A_2)$ such that $ab_1, ab_2 \in E(G)$. If $|V(A_2)| = 2$, then $\langle N_G(x)\rangle_G$ is a triangle and there is no cutset R. Hence $|V(A_2)| \geq 3$ and we set $K_1 = \{a, b_1\}$ and $K_2 = V(A_2) \setminus \{b_1\}$.

If $\langle N_G(x)\rangle_G$ contains an induced C_5, then, by Lemma 3, we set $K_1 = V(N_1) \cup \{r_1, r_2\}$ and $K_2 = V(N_2)$. The rest is clear. ■

Note that, for a given x, neither the cutset R nor the decomposition of $N_G(x)$ into K_1 and K_2 are, in general, uniquely determined; however, K_1 and K_2 are uniquely determined for a given R if $\langle N_G(x)\rangle_G$ contains an induced C_5.

For the proof of Theorem 1 we will further need some special notation and one more structural result characterizing the situations when $p_{ab}(G) < p_{ab}(G'_x)$.

For a given (a,b)-path P in a graph G, a vertex $x \in V(G)$ and $i = 0,1,2$ we denote $V_i^x(P) = \{y \in V(P) \cap N_G(x);\ |\{y^-, y^+\} \cap N_G[x]| = i\}$. If $V_1^x(P) \neq \emptyset$, then a_P^x (b_P^x) denotes the first (last) vertex of P which is in $V_1^x(P)$, respectively (if the vertex x is clear from the context, we will simply denote $V_1(P)$, a_P and b_P). Thus, equivalently, a_P (b_P) is the first (last) vertex of an (a,b)-path P for which the edge $a_P a_P^+$ ($b_P^- b_P$) is in $\langle N_G(x)\rangle_G$. Analogously we define $V_i^x(C) = \{y \in V(C) \cap N_G(x);\ |\{y^-, y^+\} \cap N_G[x]| = i\}$ for a cycle $C \subset G$.

Proposition 5. *Let G be a claw-free graph, let $x \in V(G)$ be such that $\kappa(\langle N_G(x)\rangle_G) = 2$, let G' be the local completion of G at x and let $a, b \in V(G)$, $a \neq b$. Then $p_{ab}(G) < p_{ab}(G')$ if and only if $\{a,b\}$ is a cutset of $\langle N_G(x)\rangle_G$ and, for every longest (a,b)-path P' in G',*

(1) *$x \in V(P')$,*
(2) *$|\{a, a_{P'}, b, b_{P'}\}| = 4$,*
(3) *$a_{P'} b_{P'} \in E(G)$,*
(4) *if C is the component of $\langle N_G(x)\rangle_G - R$ not containing $a_{P'}$, then $V(C) \setminus V_0^x(P') \neq \emptyset$,*
(5) *there are no two vertices $u, v \in V_1^x(P')$ such that u, v are in different components of $\langle N_G(x)\rangle_G - R$ and all interior vertices of the subpath $uP'v$ of the path P' are in $(V(G) \setminus N_G[x]) \cup V_0^x(P')$.*

Moreover, if $p_{ab}(G) < p_{ab}(G')$, then there are vertices $\bar{a}, \bar{b} \in N_G(x)$ such that

(6) *$\bar{a} = a_{P'}$ and $\bar{b} = b_{P'}$ for any longest (a,b)-path P' in G',*
(7) *$a\bar{a} \in E(G)$ and $b\bar{b} \in E(G)$.*

Proof of Proposition 5 is lengthy and technical and it is thus postponed to Section 4. ■

Note that statement (6) of Proposition 5 equivalently says that if $p_{ab}(G) < p_{ab}(G')$, then, for given vertices a, b are the vertices $a_{P'}$, $b_{P'}$ uniquely determined (i.e., do not depend on the choice of the (a,b)-path P'). In the rest of this section we will keep the notation $\bar{a}$, $\bar{b}$ for these vertices given by (6) of Proposition 5.

Proposition 6. *Let G be a claw-free graph, let $x \in V(G)$ be such that $\kappa(\langle N_G(x)\rangle_G) = 2$, let G' be the local completion of G at x and let $a, b \in V(G)$, $a \neq b$. If $p_{ab}(G) < p_{ab}(G')$, then, for every hamiltonian cycle C in G', $E(C) \cap \{a\bar{a}, b\bar{b}\} = \emptyset$.*

Proof. Let $a, b \in V(G)$ be such that $p_{ab}(G) < p_{ab}(G')$. By Proposition 5, $a, b \in N_G(x)$. Let C be a hamiltonian cycle in G' and suppose, to the contrary, that $a\bar{a} \in E(C)$ (the proof for $b\bar{b} \in E(C)$ is symmetric). Let $Q_1, \ldots, Q_k$ denote nontrivial components of the graph obtained from C by removing all edges with both vertices in $N_G[x]$, q_i^1, q_i^2 the endvertices of Q_i, $i = 1, \ldots, k$, and set $A = \{q_i^j;\ i = 1, \ldots, k,\ j = 1, 2\}$. Then $A \subset N_G(x)$ and every Q_i is a path with endvertices in A and with interior vertices in $V_0^x(C) \cup (V(G) \setminus N_G[x])$.

1. Suppose first that $a \notin A$. If $a \notin \cup_{i=1}^k V(Q_i)$, then, using edges in $\langle N_G[x]\rangle_{G'}$, we can connect the paths $Q_1, \ldots, Q_k$ to obtain a hamiltonian (a,b)-path P' in G' with $a = a_{P'}$, contradicting Proposition 5 (2) (recall that $\langle N_G[x]\rangle_{G'}$ is a clique). Hence $a \in V_0^x(C)$, but then, considering the claw at $\langle\{a, a^{-(C)}, a^{+(C)}, x\}\rangle_G$ we have $a^{-(C)} a^{+(C)} \in E(G)$, and replacing in C the path $a^{-(C)} a a^{+(C)}$ by the edge $a^{-(C)} a^{+(C)}$ we are in the previous situation.

2. Hence $a \in A$. Symmetrically, $b \in A$ (note that in the proof of $a \in A$ we have not used the assumption that $a\bar{a} \in E(C)$). Choose the notation such that $a = q_1^1$. By the assumption, $a\bar{a} \in E(C)$, implying $q_1^2 \neq \bar{a}$.

If $q_1^2 \neq b$, then the paths $Q_1, \dots, Q_k$ can be interconnected in $\langle N_G[x]\rangle_{G'}$ to obtain a hamiltonian (a, b)-path P' in G' with $a_{P'} = q_1^2 \neq \bar{a}$, contradicting (6) of Proposition 5. Hence $q_1^2 = b$.

By Proposition 5, $R = \{a, b\}$ is a cutset of $\langle N_G(x)\rangle_G$, hence there are $y_1, y_2 \in N_G(x) \cap N_G(a)$ such that $y_1 \neq y_2$, $y_1y_2 \notin E(G)$ and $y_1 \neq b \neq y_2$. Set $a^+ = a^{+(Q_1)}$ (note that $a^+ \notin N_G(x)$). From the claw at $\langle\{a, a^+, y_1, y_2\}\rangle_G$ we then have $a^+y_1 \in E(G)$ or $a^+y_2 \in E(G)$; choose the notation such that $a^+y_1 \in E(G)$. We have 3 possibilities.

a) $y_1 \notin \cup_{i=1}^k V(Q_i)$. Then we set $Q'_1 = y_1a^+Q_1b$ and for the system of paths $Q'_1, Q_2, \dots, Q_k$ we are in case 1.

b) $y_1 \in V_0^x(C)$. Then $y_1 \in V_0^x(Q_j)$ for some j, $2 \leq j \leq k$; choose the notation such that $j = 2$. From the claw at $\langle\{y_1, y_1^{-(Q_2)}, y_1^{+(Q_2)}, x\}\rangle_G$ we have $y_1^{-(Q_2)}y_1^{+(Q_2)} \in E(G)$. We set $Q'_2 = q_2^1Q_2y_1^{-(Q_2)}y_1^{+(Q_2)}Q_2q_2^2$ and for the system of paths $Q_1, Q'_2, Q_3, \dots, Q_k$ we are in subcase 2a).

c) $y_1 \in A$. We choose the notation such that $y_1 = q_2^2$, set $Q'_2 = q_2^1Q_2q_2^2a^+Q_1b$, and for the system of paths $Q'_2, Q_3, \dots, Q_k$ we are in case 1. ∎

Now we can prove stability of Hamilton-connectedness under cl_2.

Proof of Theorem 1. Suppose, to the contrary, that G' is Hamilton-connected but G is not. Then $p_{ab}(G) < p_{ab}(G')$ for some $a, b \in V(G)$, $a \neq b$. By Proposition 5, there are uniquely determined vertices $\bar{a}$, $\bar{b}$ such that $|\{a, \bar{a}, b, \bar{b}\}| = 4$ and $a\bar{a}, b\bar{b} \in E(G)$. If P is a hamiltonian $(a, \bar{a})$-path in G', then $C = P + a\bar{a}$ is a hamiltonian cycle in G' with $a\bar{a} \in E(C)$, contradicting Proposition 6. ∎

4 Proof of Proposition 5

We first prove one simple lemma that will be useful throughout the proof.

Lemma 7. *Let G be a claw-free graph, $x \in V(G)$, let G' be the local completion of G at x and let P' be a longest (a, b)-path in G' (for some $a, b \in V(G)$, $a \neq b$) such that $x \in V(P')$. Then there is a longest (a, b)-path P'' in G' such that*

(i) $V(P'') = V(P')$,

(ii) $V_0^x(P'') = \emptyset$,

(iii) *for every subpath $Q' = uP'v$ of P' with $u, v \in N_G(x) \setminus V_0^x(P')$ and interior vertices in $(V(G') \setminus N_G[x]) \cup V_0^x(P')$ the corresponding subpath $Q'' = uP''v$ of P'' satisfies $V(Q'') = V(Q') \setminus V_0^x(P')$,*

(iv) *the vertices in $V_1^x(P')$ and $V_1^x(P'')$ occur on P' and P'' in the same order.*

Proof. Let $y \in V_0^x(P')$. Then $xy^-, xy^+ \notin E(G)$, and from the claw at $\langle\{y, y^-, y^+, x\}\rangle_G$ we have $y^-y^+ \in E(G)$. The lemma then immediately follows from the fact that $\langle N_G(x)\rangle_{G'}$ is a clique. ∎

Note that (iii) yields a system of vertex-disjoint paths Q_i, $i = 1, \ldots, k$, with endvertices in $N_G(x)$, $V_0^x(Q_i) = \emptyset$, and with $V(P') = (\cup_{i=1}^k V(Q_i)) \cup N_G[x]$.

I. We first show that if $R = \{a, b\}$ is a cutset of $\langle N_G(x)\rangle_G$ and every longest (a, b)-path in G' satisfies the conditions (1) – (5) of Proposition 5, then $p_{ab}(G) < p_{ab}(G')$. Let, to the contrary, $p_{ab}(G) = p_{ab}(G')$, and let P be a longest (a, b)-path in G. Then P is a longest (a, b)-path also in G', hence P satisfies (1) – (5).

We define a graph G^+ by $G^+ = G$ if $\langle N_G(x)\rangle_G$ can be covered by two cliques, and $G^+ = G + ab$ if $\langle N_G(x)\rangle_G$ contains an induced C_5. By Corollary 4, there are $K_1^+, K_2^+ \subset V(G)$ such that

- $|K_i^+| \geq 2$ and $\langle K_i^+\rangle_{G^+}$ is a clique, $i = 1, 2$,
- $K_1^+ \cap K_2^+ = \emptyset$ and $K_1^+ \cup K_2^+ = N_G(x)$,
- if $\langle N_G(x)\rangle_G$ contains an induced C_5, then both a and b are in the same K_i^+.

Choose the notation such that $a \in K_1^+$.

We have several structural observations.

(i) $N_G(x) \subset V(P)$. This follows from (1) and from the fact that $\langle N_G(x)\rangle_{G'}$ is a clique.

(ii) $a^+, b^- \notin N_G(x)$. If e.g. $a^+ \in N_G(x)$, then $a = a_P$, contradicting (2).

(iii) *a^+ has a neighbor in K_2^+, and b^- has a neighbor in that of K_1^+, K_2^+ which does not contain b.* Since $R = \{a, b\}$ is a cutset of $\langle N_G(x)\rangle_G$, a has a neighbor $\tilde{a}$ in K_2^+. If a^+ has a neighbor $\tilde{a}_1$ in $K_1^+ \setminus \{a, b\}$, then, using (5) and Lemma 7 we get a contradiction with (2). Hence $a^+\tilde{a} \in E(G)$ since otherwise $\langle\{a, a^+, \tilde{a}, \tilde{a}_1\}\rangle_G$ is a claw. The proof for b^- is symmetric.

(iv) $a_P, b_P \in K_2^+$. If $a_P \in K_1^+$, then, using (iii) and Lemma 7, we have a contradiction with (2). Hence $a_P \in K_2^+$, and by (3) (and since $\{a, b\}$ is a cutset) also $b_P \in K_2^+$.

(v) $b \in K_1^+$. If $b \in K_2^+$, then $b, b_P \in K_2^+$ and we have a contradiction with (2) by a similar argument.

Let now $s \in K_1^+ \setminus (\{a, b\} \cup V_0^x(P))$ (such a vertex s exists by (4)). By (i), $s \in V(P)$, hence $s \in V_1^x(P) \cup V_2^x(P)$. By (iv), $a_P, b_P \in K_2^+$, and, by the definition of a_P, all interior vertices of the paths aPa_P and b_PPb are in $V_0^x(P)$ or outside $N_G[x]$. Hence there are vertices $c_1, c_2 \in V_1^x(P) \cap (K_1^+ \setminus \{a, b\})$ and $d_1, d_2 \in V_1^x(P) \cap K_2^+$ such that the vertices $a, a_P, d_1, c_1, s, c_2, d_2, b_P, b$ occur on P in this order (not excluding the possibility that some of them can coincide). One of the subpaths d_1Pc_1, c_2Pd_2 (say, d_1Pc_1) can be of length 2 with x as the only interior vertex, but the existence of c_2Pd_2 contradicts (5).

II. Now we show that, conversely, $p_{ab}(G) < p_{ab}(G')$ implies that $\{a, b\}$ is a cutset of $\langle N_G(x)\rangle_G$, every longest (a, b)-path P' in G' satisfies the conditions (1) – (5) of Proposition 5 and, moreover, (6) and (7) also holds. Thus, suppose that $p_{ab}(G) < p_{ab}(G')$, let P' be a longest (a, b)-path in G' and let $R = \{r_1, r_2\}$ be a cutset of $\langle N_G(x)\rangle_G$.

Observe that if $V_1^x(P') = \emptyset$, then $P' \subset G$, contradicting the assumption $p_{ab}(G) < p_{ab}(G')$. Hence $V_1^x(P') \neq \emptyset$ and then, by the maximality of P' and since $\langle N_G(x)\rangle_{G'}$ is a clique, we have $N_G[x] \subset V(P')$. Now we introduce some special terminology and notations and prove several auxiliary statements.

Similarly as in the first part of the proof, we set $G^+ = G$ if $\langle N_G(x)\rangle_G$ can be covered by two cliques, and $G^+ = G + r_1r_2$ if $\langle N_G(x)\rangle_G$ contains an induced C_5. Then, by Corollary 4, there are $K_1^+, K_2^+ \subset V(G)$ such that $|K_i^+| \geq 2$ and $\langle K_i^+\rangle_{G^+}$ is a clique, $i = 1, 2$, $K_1^+ \cap K_2^+ = \emptyset$ and $K_1^+ \cup K_2^+ = N_G(x)$, and if $\langle N_G(x)\rangle_G$ contains an induced C_5, then both r_1 and r_2 are in the same K_i^+. Unlike in the first part, we choose the notation such that $a_{P'} \in K_2^+$.

An (a, b)-path P' in G' is said to be a *private path* if P' satisfies the following conditions:

(i) P' is a longest (a, b)-path in G',
(ii) $x \in V(P')$,
(iii) $V_0^x(P') = \emptyset$,
(iv) subject to (i), (ii) and (iii), $|\{a, a_{P'}, b, b_{P'}\}|$ is minimum.

Note that, by (i) and (ii), a private path contains all vertices of $N_G[x]$. By Lemma 7, for any longest (a, b)-path in G' containing x there is a private (a, b)-path in G' with the same vertex set. Moreover, it is clear that if (1), (2), (3), (5) and (6) of Proposition 5 are satisfied for any private (a, b)-path in G', then these conditions also hold for any longest (a, b)-path in G'. The conditions (2) of Proposition 5 and (iv) of the definition of private path then imply that every longest (a, b)-path P' in G' with $V_0^x(P') = \emptyset$ is private in this case. These observations together with the fact that (7) does not depend on P' imply that it is sufficient to verify (1) – (7) for all private (a, b)-paths in G'.

Thus, suppose that P' is a private (a, b)-path in G'. We denote $Q' = a_{P'}P'b_{P'}$ the $(a_{P'}, b_{P'})$-subpath of P', $Q'_1, \ldots, Q'_k$ the nontrivial components of the graph obtained from Q' by removing edges with both ends in $N_G[x]$ and q_i^1, q_i^2 the endvertices of Q'_i, $i = 1, \ldots, k$ (where the numbering of Q'_i and q_i^j is chosen in the orientation from $a_{P'}$ to $b_{P'}$).

Let further S denote the system of subsets of $N_G(x)$ defined by $S = S^{(1)} \cup S^{(2)}$, where $S^{(2)} = \{\{q_i^1, q_i^2\}|\ i = 1, \ldots, k\}$ and $S^{(1)} = \{\{u\}|\ u \in N_G(x) \setminus (\{a, a_{P'}, b, b_{P'}\} \cup (\cup_{s \in S^{(2)}} s)\}$. For $S' \subset S$ we set $V(S') = \cup_{s \in S'} s$. This means that $N_G(x)$ consists of $V(S)$, $a_{P'}$, $b_{P'}$, and possibly a or b (or both), and any longest (a, b)-path in G' (and, to obtain a contradiction, also in G), has to contain all elements of $S^{(1)}$ and all paths represented by pairs of their endvertices in $S^{(2)}$. We further denote $S_i = \{s \in S|\ V(s) \subset K_i^+\}$, $i = 1, 2$, and $S_{12} = \{s \in S|\ V(s) \cap K_i^+ \neq \emptyset,\ i = 1, 2\}$ (thus, $S = S_1 \cup S_2 \cup S_{12}$ and $S_{12} \subset S^{(2)}$).

The fact that $\langle N_G(x) \rangle_{G'}$ is a clique will allow us to use this notation to simplify description of paths in G': whenever, in the description of a path, a subset S' of S occurs, this means that all elements of $S^{(1)} \cap S'$ and all paths represented by elements of $S^{(2)} \cap S'$ (if any) have to be included using appropriate edges of the clique $\langle N_G(x) \rangle_{G'}$. For two consecutive elements u, v of such a description of a path, we will use the notation $\widehat{uv}$ to indicate that we do not exclude the possibility $u = v$.

Claim 8. *Let P' be a private (a, b)-path in G'. If $a \in N_G(x)$ and $a \neq a_{P'}$, then a^+ has no neighbor in $V(S)$, and, symmetrically, if $b \in N_G(x)$ and $b \neq b_{P'}$, then b^- has no neighbor in $V(S)$.*

Proof. Suppose a^+ is adjacent to $u \in V(s)$, $s \in S$. Then for the path $\tilde{P} = asa^+P'a_{P'}(S \setminus s)xb_{P'}b$ (recall that $\langle N_G[x] \rangle_{G'}$ is a clique) we have $a = a_{\tilde{P}}$, contradicting the assumption that P' is private. The proof for b^- is symmetric.

□

Claim 9. *Let P' be a longest (a,b)-path in G', $x \in V(P')$. If $a \in N_G(x)$, $a \neq a_{P'}$, and there are $y_1, y_2 \in N_G(x) \cap N_G(a)$ such that*

(i) $y_1, y_2 \notin E(G)$,

(ii) $y_i \neq a_{P'}$, and if $b = b_{P'}$, then also $y_i \neq b_{P'}$, $i = 1, 2$,

then P' is not a private (a,b)-path in G'.

Proof. Suppose that P' satisfies the assumptions of Claim 9. Considering the claw at $\langle\{a, a^+, y_1, y_2\}\rangle_G$, we obtain (possibly after renumbering y_1 and y_2) that $a^+y_1 \in E(G)$. By Claim 8, $y_1 \notin V(S)$ (otherwise we are done), hence $y_1 \in \{b, b_{P'}\}$. Moreover, $b \neq b_{P'}$, for otherwise by (ii) we have $y_1 \notin \{a, a_{P'}, b, b_{P'}\}$, implying $y_1 \in V(S)$, a contradiction.

Case 1: $y_1 = b_{P'}$. Then the path $P'' = a(S \cup \{x\})a_{P'}\overleftarrow{P'}a^+y_1\overrightarrow{P'}b$ is a longest (a,b)-path in G' with $a = a_{P''}$, hence P' is not private.

Case 2: $y_1 = b$. Then $b \neq b_{P'}$ implies $b^- \notin N_G[x]$ and $a \neq a_{P'}$ implies $a^+ \notin N_G[x]$. From the claw at $\langle\{y_1, x, b^-, a^+\}\rangle_G$ we have $b^-a^+ \in E(G)$. The path $P'' = a(S \cup \{x\})a_{P'}\overleftarrow{P'}a^+b^-\overleftarrow{P'}b_{P'}b$ then satisfies $a = a_{P''}$, hence P' is not private. $\square$

Claim 10. *Let $\{a,b\} = R = \{r_1, r_2\}$ and let P_1', P_2' be private (a,b)-paths in G' such that $a_{P_1'} \neq a \neq a_{P_2'}$, $b_{P_1'} \neq b \neq b_{P_2'}$ and $\{a_{P_1'}, a_{P_2'}\} \subset K_j^+$ for some $j \in \{1,2\}$. Then $a_{P_1'} = a_{P_2'}$.*

Proof. Since $\{a,b\}$ is a cutset of $\langle N_G(x)\rangle_G$, there are $w \in K_j^+$ and $z \in K_{3-j}^+$ such that $aw, az \in E(G)$. If $w \neq a_{P_1'}$, then, applying Claim 9 to P_1', we get that P_1' is not private, a contradiction. Hence $w = a_{P_1'}$. Analogously $w = a_{P_2'}$, implying $a_{P_1'} = a_{P_2'}$. $\square$

In general, $\langle N_G(x)\rangle_G$ can have more 2-element cutsets. If this is the case, we suppose that the cutset $R = \{r_1, r_2\}$ is chosen such that, for the given private (a,b)-path P',

(i) $|R \cap (\{a,b\} \setminus \{a_{P'}, b_{P'}\})|$ is maximum,

(ii) if $|R \cap (\{a,b\} \setminus \{a_{P'}, b_{P'}\})| = 0$, then $|R \cap \{a_{P'}, b_{P'}\}|$ is maximum.

Let now H be the graph with $V(H) = N_G[x]$ and $E(H) = E(\langle N_G(x)\rangle_G) \cup S^{(2)}$, and let H' be the local completion of H at x (i.e., H is a clique with some vertices belonging to $V(S)$ and some edges belonging to $S^{(2)}$). It is now clear that every longest (a,b)-path P' in G' defines an $(a_{P'}, b_{P'})$-path Q' in H' such that Q' contains x and all elements of S (i.e., all edges in $S^{(2)}$ and all vertices in $V(S^{(1)})$. To reach a contradiction, i.e. to find an (a,b)-path P in G with $V(P) = V(P')$, it is sufficient to find an $(a_{P'}, b_{P'})$-path Q in H containing x and all elements of S.

Similarly as with G^+, we set $H^+ = H$ if $\langle N_G(x)\rangle_G$ can be covered by two cliques, and $H^+ = H + r_1r_2$ if $\langle N_G(x)\rangle_G$ contains an induced C_5. We proceed in two steps: in Step A, we for a given $(a_{P'}, b_{P'})$-path Q' in H' either find an $(a_{P'}, b_{P'})$-path Q^+ in H^+ containing x and all elements of S, or verify the conditions (1) – (7) of Proposition 5; in Step B, we complete the proof by showing that in each case when there is an $(a_{P'}, b_{P'})$-path Q^+ in H^+ containing x and all elements of S, there also is such a path Q in H.

Step A: H' **to** H^+**.**

Let Q' be an $(a_{P'}, b_{P'})$-path in H' containing x and all elements of S.

Case 1: $b_{P'} \in K_1^+$ (recall that the notation is chosen such that $a_{P'} \in K_2^+$.) Then $Q^+ = a_{P'}S_2S_{12}xS_1b_{P'}$ is an $(a_{P'}, b_{P'})$-path in H^+ containing x and all elements of S.

Case 2: $b_{P'} = x$ (and hence also necessarily $b = x$).

a) If $S_{12} \neq \emptyset$, then for an $s \in S_{12}$ the path $Q^+ = a_{P'}S_2sS_1(S_{12} \setminus \{s\})xb_{P'}$ has the required properties.

b) If $S_{12} = \emptyset$ and there is an edge $uv \in E(G)$ with $u \in V(S_1)$ and $v \in V(S_2)$, then we set $Q^+ = a_{P'}S_2vuS_1xb_{P'}$.

c) Hence $S_{12} = \emptyset$ and there is no edge $uv \in E(G)$ with $u \in V(S_1)$ and $v \in V(S_2)$. Recall that $|K_1^+| \geq 2$, $|K_2^+| \geq 2$, and there are only 2 vertices, namely a and $a_{P'}$, that are not in $V(S_1) \cup V(S_2)$. If $\{a, a_{P'}\}$ is a cutset of $\langle N_G(x)\rangle_G$, then both K_1^+ and K_2^+ contains a vertex in $V(S_1) \cup V(S_2)$ and, by Claim 9, P' is not private, a contradiction. Hence $\{a, a_{P'}\}$ is not a cutset, but then there is an edge uv with $u \in V(S_1)$ and $v \in V(S_2)$ and we are in subcase 2b.

Case 3: $\{a_{P'}, b_{P'}\} \subset K_2^+$.

a) If $S_{12} \neq \emptyset$, then for some $s \in S_{12}$ we set $Q^+ = a_{P'}S_2sS_1(S_{12} \setminus \{s\})xb_{P'}$.

b) If $S_{12} = \emptyset$ and there is an $uv \in E(G)$ with $u \in V(S_1)$ and $v \in \{a_{P'}, b_{P'}\} \cup V(S_2)$, we set $Q^+ = a_{P'}S_2xS_1ub_{P'}$ if $v = b_{P'}$ and $Q^+ = \widehat{a_{P'}v}uS_1xS_2b_{P'}$ otherwise.

c) Hence $S_{12} = \emptyset$ and there is no edge $uv \in E(G)$ with $u \in V(S_1)$ and $v \in \{a_{P'}, b_{P'}\} \cup V(S_2)$. If $S_1 = \emptyset$, then $K_1^+ = \{a, b\}$, implying $Q^+ = Q'$, hence $S_1 \neq \emptyset$. By the 2-connectedness of $\langle N_G(x)\rangle_G$ and since $|K_i^+| \geq 2$, $i = 1, 2$, there are two vertex-disjoint edges e_1, e_2 between K_1^+ and K_2^+.

If $a = a_{P'}$, then a, $a_{P'}$ and $b_{P'}$ are in K_2^+, hence one of e_1, e_2 has a vertex in $V(S_1)$ and we are in subcase 3b. Hence $a \neq a_{P'}$ and, symmetrically, $b \neq b_{P'}$. The nonexistence of an edge $uv \in E(G)$ with $u \in V(S_1)$ and $v \in \{a_{P'}, b_{P'}\} \cup V(S_2)$ then implies that $\{a, b\}$ is a cutset of $\langle N_G(x)\rangle_G$. By the choice of R, we have $R = \{a, b\}$.

Let now $y_1 \in K_1^+$ be such that $y_1 \neq b$ and $y_1a \in E(G)$ (such an y_1 exists since $\{a, b\} = R$). If $a \in K_2^+$, then for $y_2 = b_{P'}$ we have a contradiction with Claim 9, hence $a \in K_1^+$. Analogously we observe that $a_{P'}$ is the only neighbor of a in K_2^+. Symmetrically, $b \in K_1^+$ and $b_{P'}$ is the only neighbor of b in K_2^+.

Summarizing, we have the following facts:

- $x \in V(P')$, verifying condition (1) of Proposition 5,
- $a \neq a_{P'}$ and $b \neq b_{P'}$, implying $|\{a, a_{P'}, b, b_{P'}\}| = 4$, thus verifying (2),
- $a_{P'}, b_{P'} \in K_2^+$, hence $a_{P'}b_{P'} \in E(G)$, implying (3),
- $S_{12} = \emptyset$, implying (5),
- $S_1 \neq \emptyset$, hence also $V(C) \setminus V_0^x(P') = (K_1^+ \setminus \{a, b\}) \setminus V_0^x(P') \neq \emptyset$ (since the case when $(K_1^+ \setminus \{a, b\}) \subset V_0^x(P')$ can be transformed in an obvious way to the case $S_1 = \emptyset$); this also establishes (4).

Moreover, by Claim 10, $a_{P'}$ and $b_{P'}$ are uniquely determined, verifying (6), and the fact that $aa_{P'} \in E(G)$ and $bb_{P'} \in E(G)$ implies (7).

Step B: H^+ to H.

In this part we complete the proof by showing that in each case when there is an $(a_{P'}, b_{P'})$-path Q^+ in H^+ containing x and all elements of S, there also is such a path Q in H.

If $\langle N_G(x)\rangle_G$ can be covered by two cliques, then $H^+ = H$ and there is nothing to do, hence in the rest of the proof suppose that $\langle N_G(x)\rangle_G$ contains an induced C_5. Let $K_1, K_2 \subset N_G(x)$ be the sets given in Corollary 4 (note that, specifically, $R \subset K_1$, and $\{K_1, K_2\} = \{K_1^+, K_2^+\}$), and (if necessary) relabel the sets S_1, S_2 in accordance with the labeling of K_1, K_2,

Claim 11. *Let P' be a private (a, b)-path in G'. If $\langle N_G(x)\rangle_G$ contains an induced C_5 and $a \in N_G(x)$, then at least one of the following holds:*

1. $a = a_{P'}$,
2. $aa_{P'} \in E(G)$,
3. $b = b_{P'}$, $b \neq x$, $ab \in E(G)$.

Proof. Choose $y_1, y_2 \in N_G(x) \cap N_G(a)$ such that $y_1y_2 \notin E(G)$. This is always possible: for $a \in K_1 \setminus R$ we choose $\{y_1, y_2\} = R$, for $a \in R$ we choose $y_1 \in K_1 \setminus R$, $y_2 \in K_2$, and for $a \in K_2$ we choose $y_1 \in R$ and $y_2 \in K_2$ such that $y_1y_2 \notin E(G)$ (such vertices exist since if r_1 or r_2 is adjacent to all vertices in K_2 then $\langle N_G(x)\rangle_G$ can be covered by two cliques).

We suppose that $a \neq a_{P'}$ and $aa_{P'} \notin E(G)$, and we show that this implies condition 3. If $b = x$ (and hence also $b_{P'} = b = x$), then $\{y_1, y_2\} \subset S$, and the fact that $y_1y_2 \notin E(G)$ and Claim 8 imply that $\langle\{a, a^+, y_1, y_2\}\rangle_G$ is a claw, a contradiction. Hence $b \neq x$. If $b \neq b_{P'}$, then, by Claim 9, P' is not private, a contradiction. Hence $b = b_{P'}$ and $b \neq x$. Now, if $ab \notin E(G)$, then also $ab_{P'} \notin E(G)$ (and hence also $y_i \neq b_{P'}$, $i = 1, 2$), and by Claim 9, P' is not private, a contradiction. Thus, we have $ab \in E(G)$, $b = b_{P'}$ and $b \neq x$, verifying condition 3. □

Let now Q^+ be an $(a_{P'}, b_{P'})$-path in H^+ containing x and all elements of S.

Claim 12. *If $V(S_1) \setminus R \neq \emptyset$, then there is an $(a_{P'}, b_{P'})$-path Q in H containing x and all elements of S.*

Proof. Choose $s \in V(S_1)$ and set $s^- = s^{-(Q^+)}$ and $s^+ = s^{+(Q^+)}$. If $\{s^-, s^+\} = R$, then $r_1r_2 \notin E(Q^+)$ and we are done, hence $\{s^-, s^+\} \neq R$.

1. If $s \in V(S_1 \cap S^{(1)})$, then we obtain the path Q by replacing in Q^+ the path s^-ss^+ by the edge s^-s^+ and the edge r_1r_2 by the path r_1sr_2 (not excluding the possibility that some of s^-, s^+ can coincide with some of r_1, r_2).

2. Let $s \in V(S_1 \cap S^{(2)})$. Then $s \in s_1$ for some $s_1 \in V(S_1 \cap S^{(2)})$, and we choose the notation such that $s_1 = \{s, s^+\}$ (if this is not the case, we interchange a, b). If $s^+ \in R$ (say, $s^+ = r_1$), then $s^- \notin R$ (otherwise $r_1r_2 \notin E(G^+)$), and we obtain Q by replacing in Q^+ the path $s^-s(s^+ = r_1)r_2$ by the path $s^-(s^+ = r_1)sr_2$. If $s^+ \notin R$, then $\{s^-, s^{++}\} \neq R$ (otherwise $r_1r_2 \notin E(Q^+)$), and we replace the path $s^-ss^+s^{++}$ by the edge s^-s^{++} and the edge r_1r_2 by the path $r_1ss^+r_2$ or $r_1s^+sr_2$ (not excluding the case that some of s^-, s^{++} can coincide with some of r_1, r_2). □

Claim 13. *If $\{a,b\} \subset N_G(x)$, $\{a,b\} \not\subset K_2$ and $|\{a, a_{P'}, b, b_{P'}\}| = 4$, then there is an $(a_{P'}, b_{P'})$-path Q in H containing x and all elements of S.*

Proof. Clearly $\{a,b\} \cap R = \emptyset$, since otherwise $r_1r_2 \notin E(Q^+)$, and by Claim 12 we can suppose $V(S_1) \setminus R = \emptyset$.

1. If $\{a,b\} \subset K_1 \setminus R$, then the application of Claim 9 to a and b (with $y_1 = r_1$ and $y_2 = r_2$) gives $R = \{a_{P'}, b_{P'}\}$, implying $r_1r_2 \notin E(Q^+)$.

2. Hence $a \in K_1 \setminus R$ and $b \in K_2$, implying $ab \notin E(G)$. Application of Claim 9 to a gives $a_{P'} \in R$; choose the notation such that $a_{P'} = r_1$. By Claim 10 (applied to b) then $bb_{P'} \in E(G)$. Since $R \neq \{a_{P'}, b_{P'}\}$ (otherwise $r_1r_2 \notin E(Q^+)$), we have $b_{P'} \in K_2$.

If $br_1 \in E(G)$, then Claim 9 applied to b (with $y_1 = r_1$ and y_2 being a vertex in K_2 with $y_2r_1 \notin E(G)$), implies that $b_{P'}$ is the only neighbor of r_2 in K_2 that is not adjacent to r_1, and we set $Q = a_{P'}S_2S_{12}xb_{P'}$ if $r_2 \in V(S_{12})$ and $Q = a_{P'}S_2S_{12}xr_2b_{P'}$ otherwise (not excluding the case that $S_{12} = \emptyset$).

Hence $br_1 \notin E(G)$, implying $br_2 \in E(G)$. By Claim 9 applied to b we then analogously get that $b_{P'}$ is the only neighbor of r_1 in K_2 that is not adjacent to r_2, and then $Q = a_{P'}xS_{12}S_2b_{P'}$ if $r_2 \in V(S_{12})$ and $Q = a_{P'}xr_2S_{12}S_2b_{P'}$ otherwise, where we do not exclude the possibility $S_{12} = \emptyset$ and we choose the first vertex $u \in V(S_{12})$ (i.e., $u = x^{+(Q)}$ or $u = r_2^{+(Q)}$, respectively) such that $u \in K_1$ if $|S_{12}|$ is odd and $u \in K_2$ if $|S_{12}|$ is even. $\square$

Now observe that if $R \cap (\{a,b\} \setminus \{a_{P'}, b_{P'}\}) \neq \emptyset$, or if $R = \{a_{P'}, b_{P'}\}$, then again $r_1r_2 \notin E(Q^+)$ and we are done. Hence in the remaining part of the proof we suppose that the following conditions are satisfied:

($*$) $R \cap (\{a,b\} \setminus \{a_{P'}, b_{P'}\}) = \emptyset$,

($**$) $R \neq \{a_{P'}, b_{P'}\}$.

For $s \in S_{12}$ we will denote $s = \{s_1, s_2\}$, where $s_1 \in K_1$ and $s_2 \in K_2$.

<u>Case 1: $\{a_{P'}, b_{P'}\} \subset K_2$.</u> We choose the notation such that $r_1a_{P'} \in E(G)$ (this is possible by Corollary 4).

1. First suppose that $|S_{12}| \geq 2$. Let $s, s' \in S_{12}$, and choose the notation such that if r_1 is some of s_1, s_1', then $r_1 = s_1$. Then we set $Q = a_{P'}\widehat{r_1s_1}s_2S_2s_2'\widehat{s_1'r_2}(S_{12} \setminus \{s,s'\})xb_{P'}$ (where we do not exclude the possibility that $r_2 \in V(S_{12} \setminus \{s,s'\})$).

2. Hence $|S_{12}| \leq 1$. By Claim 11, we have $a \notin K_1 \setminus R$, since $a \in K_1 \setminus R$ would imply $a \neq a_{P'}$, $aa_{P'} \notin E(G)$, and if $b = b_{P'}$ then also $ab \notin E(G)$, contradicting Claim 10. Symmetrically, $b \notin K_1 \setminus R$. By Claim 12 we have $(K_1 \setminus R) \cap V(S_1) = \emptyset$, implying $K_1 \setminus R \subset V(S_{12})$. Hence $|S_{12}| = 1$. Let $S_{12} = \{s\}$, and then $Q = a_{P'}r_1xr_2s_1s_2S_2b_{P'}$.

<u>Case 2: $\{a_{P'}, b_{P'}\} \subset K_1$.</u> We choose the notation such that $b_{P'} \notin R$, and if $a_{P'} \in R$, then $a_{P'} = r_1$ (see the assumption ($**$)).

1. If $|S_{12}| \geq 2$, let $s, s' \in S_{12}$, and choose the notation such that $r_2 \notin s$. Then $Q = \widehat{a_{P'}r_1s_1}s_2S_2s_2'\widehat{s_1'r_2}(S_{12} \setminus \{s,s'\})xb_{P'}$ (where the notation $\widehat{a_{P'}r_1s_1}$ means that r_1 can coincide with $a_{P'}$ or s_1).

2. Next suppose $|S_{12}| = 1$, let $S_{12} = \{s\}$ and choose the notation such that $r_1 \notin s$. Then $Q = \widehat{a_{P'}r_1}xS_2s_2\widehat{s_1r_2}b_{P'}$.

3. Hence $|S_{12}| = 0$. By the choice of notation and by $(*)$ and $(**)$ we have $r_2 \in V(S_1)$. If $\{a, b\} \subset K_2$, then we have $a \neq a_{P'}$, $b \neq b_{P'}$ and $bb_{P'} \notin E(G)$, contradicting Claim 11 (applied to b). Hence at most one of a, b is in K_2.

We observe that there is a $v \in V(S_2)$ such that $r_2v \in E(G)$: for $\{a, b\} \cap K_2 = \emptyset$ this follows from Corollary 4, and for $\{a, b\} \cap K_2 = \{u\}$ the nonexistence of such a v implies that u is the only neighbor of r_2 in K_2, but then the cutset $\{u, r_1\}$ of $\langle N_G(x)\rangle_G$ contradicts the choice of R. Thus, let $v \in V(S_2)$ be such that $r_2v \in E(G)$. If $v \in V(S_2 \cap S^{(1)})$, we set $s = \{v\}$ and then $Q = \widehat{a_{P'}r_1}x(S_2 \setminus \{s\})vr_2b_{P'}$; if $v \in V(S_2 \cap S^{(2)})$, then $s = \{v, v'\} \in S_2$ for some $v' \in V(S_2)$ and then $Q = \widehat{a_{P'}r_1}x(S_2 \setminus \{s\})v'vr_2b_{P'}$.

Case 3: $a_{P'} \in K_1$, $b_{P'} \in K_2$. We choose the notation such that if $a_{P'} \in R$, then $a_{P'} = r_1$.

1. If $|S_{12}| \geq 2$, let $s, s' \in S_{12}$, and choose the notation such that $r_2 \notin s$. Then $Q = \widehat{a_{P'}r_1s_1}s_2s_2'\widehat{s_1'r_2}(S_{12} \setminus \{s, s'\})xS_2b_{P'}$.

2. If $|S_{12}| = 1$, let $S_{12} = \{s\}$ and choose the notation such that $r_1 \notin s$. Then $Q = \widehat{a_{P'}r_1}xr_2s_1s_2S_2b_{P'}$.

3. Hence $|S_{12}| = 0$. We distinguish two subcases.

a) $a_{P'} \in R$ (i.e. $a_{P'} = r_1$). Since $K_1 \setminus R \neq \emptyset$, by Claim 12 we have $K_1 \setminus R \subset \{a, b\}$. By Claim 13, $K_1 \setminus R \neq \{a, b\}$, hence $K_1 \setminus R = \{a\}$ or $K_1 \setminus R = \{b\}$. Since $|K_2| \geq 2$, we have $S_2 \neq \emptyset$ (one of a, b is in $K_1 \setminus R$ and $b \in K_2$, $b \neq b_{P'}$ is not possible by Claim 13).

If $r_2s \notin E(G)$ for all $s \in V(S_2)$, then $b_{P'}$ is the only neighbor of r_2 in K_2 (since by Claim 13 necessarily $a = a_{P'}$ or $b = b_{P'}$), but then $\{a_{P'}, b_{P'}\}$ is a cutset of $\langle N_G(x)\rangle_G$ contradicting the choice of R. Hence there is a $u \in V(S_2)$ such that $r_2u \in E(G)$. If $u \in V(S_2 \cap S^{(1)})$, we set $s = \{u\}$ and then $Q = a_{P'}xr_2u(S_2 \setminus \{s\})b_{P'}$; if $u \in V(S_2 \cap S^{(2)})$, then $s = \{u, u'\} \in S_2$ for some $u' \in V(S_2)$ and then $Q = a_{P'}xr_2uu'(S_2 \setminus \{s\})b_{P'}$.

b) $a_{P'} \notin R$. If $|K_2| = 2$, then $\{b_{P'}, r_1\}$ or $\{b_{P'}, r_2\}$ is a cutset of $\langle N_G(x)\rangle_G$, contradicting the choice of R; hence $|K_2| \geq 3$. If $\{a, b, b_{P'}\} \subset K_2$ with $b \neq b_{P'}$, then we have $a \neq a_{P'}$, $b \neq b_{P'}$ and $aa_{P'} \notin E(G)$, contradicting Claim 11. Hence there is a $u \in V(S_2)$. We choose the notation such that $r_2u \in E(G)$, set $u' = u$ and $s = \{u\}$ if $u \in V(S_2 \cap S^{(1)})$ or $s = \{u, u'\} \in S_2$ if $u \in V(S_2 \cap S^{(2)})$, and then $Q = a_{P'}r_1xr_2uu'(S_2 \setminus \{s\})b_{P'}$.

Case 4: $a_{P'} \in K_1$, $b_{P'} = x$. We choose the notation such that if $a_{P'} \in R$, then $a_{P'} = r_1$; recall then $x = b_{P'}$ implies $x = b_{P'} = b$.

1. If $|S_{12}| \geq 2$, let $s, s' \in S_{12}$, and choose the notation such that $r_2 \notin s$. Then $Q = \widehat{a_{P'}r_1}s_1s_2S_2s_2'\widehat{s_1'r_2}(S_{12} \setminus \{s, s'\})b_{P'}$.

2. If $|S_{12}| = 1$, let $S_{12} = \{s\}$ and choose the notation such that $r_1 \notin s$. If $a \in K_2$ and a is the only neighbor of r_1, then $\{a, r_2\}$ is a cutset of $\langle N_G(x)\rangle_G$, contradicting the choice of R. Hence there is a $u \in K_2$ such that $u \in V(S_2 \cup S_{12})$ and $r_1u \in E(G)$.

a) If there is such a $u \in V(S_2)$, then $Q = \widehat{a_{P'}r_1}uu'(S_2 \setminus \{s'\})s_2s_1r_2b_{P'}$, where $u' = u$ and $s' = \{u\}$ if $u \in V(S_2 \cap S^{(1)})$ or $s' = \{u, u'\} \in S_2$ if $u \in V(S_2 \cap S^{(2)})$.

b) If such a $u \in V(S_2)$ does not exist, then $u = s_2$ (where $s = \{s_1, s_2\}$ is the only element of S_{12}), and by Corollary 4 we have $r_2v \in E(G)$ for every $v \in V(S_2)$ (since $r_1v \notin E(G)$). Then $Q = \widehat{a_{P'}r_1}s_2\widehat{s_1r_2}S_2b_{P'}$ (not excluding the possibility $S_2 = \emptyset$).

3. It remains to consider the case $|S_{12}| = \emptyset$. If $a_{P'} \in R$, then by Claim 12 we have $K_1 \setminus R = \{a\}$, and if $a_{P'} \notin R$, then, by Claim 12 and Claim 11, $K_1 \setminus R = \{a, a_{P'}\}$ (not

excluding the possibility $a = a_{P'}$). Then $Q = \widehat{a_{P'}r_1}S_2r_2b_{P'}$ (it is straightforward to check that this is always possible if we keep an element of $V(S_2)$ that is nonadjacent to r_1 as the last one).

<u>Case 5: $a_{P'} \in K_2$, $b_{P'} = x$.</u> We choose the notation such that $a_{P'}r_1 \in E(G)$ (this is always possible by Corollary 4).

1. If $|S_{12}| \geq 2$, let $s, s' \in S_{12}$, and choose the notation such that $r_2 \notin s$. Then $Q = a_{P'}\widehat{r_1s_1}s_2S_2s_2'\widehat{s_1'r_2}(S_{12} \setminus \{s, s'\})b_{P'}$.

2. Let $|S_{12}| \leq 1$. Then, by Claim 11 and by $(*)$, $a \in K_2$, and since $K_1 \setminus R \neq \emptyset$, we have $K_1 \setminus R \subset V(S_{12})$. Hence $|S_{12}| = 1$, set $S_{12} = \{s\}$.

a) If $S_2 = \emptyset$, then, by Corollary 4, either $s_2r_1 \in E(G)$ and then $Q = a_{P'}r_1s_2s_1r_2b_{P'}$, or $s_2r_2 \in E(G)$ and then $Q = a_{P'}r_1s_1s_2r_2b_{P'}$.

b) If $S_2 \neq \emptyset$, we choose $u \in V(S_2)$ and denote $u' = u$ and $s' = \{u\}$ if $u \in V(S_2 \cap S^{(1)})$ or $s' = \{u, u'\} \in S_2$ if $u \in V(S_2 \cap S^{(2)})$. By Corollary 4, either $r_1u \in E(G)$ and then $Q = a_{P'}r_1uu'(S_2 \setminus \{s'\})s_2s_1r_2b_{P'}$, or $r_2u \in E(G)$ and then $Q = a_{P'}r_1s_1s_2(S_2 \setminus \{s'\})u'ur_2b_{P'}$. ■

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