Thomassen's conjecture implies polynomiality of 1-Hamilton-connectedness in line graphs

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Abstract

A graph G is 1-Hamilton-connected if $G - x$ is Hamilton-connected for every $x \in V(G)$, and G is 2-edge-Hamilton-connected if the graph $G + X$ has a hamiltonian cycle containing all edges of X for any $X \subset E^+(G) = \{xy|\ x, y \in V(G)\}$ with $1 \leq |X| \leq 2$. We prove that Thomassen's conjecture (every 4-connected line graph is hamiltonian, or, equivalently, every snark has a dominating cycle) is equivalent to the statements that every 4-connected line graph is 1-Hamilton-connected and/or 2-edge-Hamilton-connected. As a corollary, we obtain that Thomassen's conjecture implies polynomiality of both 1-Hamilton-connectedness and 2-edge-Hamilton-connectedness in line graphs. Consequently, proving that 1-Hamilton-connectedness is NP-complete in line graphs would disprove Thomassen's conjecture, unless P=NP.

Keywords: line graph, 4-connected, hamiltonian, Hamilton-connected, dominating cycle, Thomassen's conjecture, snark

1 Introduction.

By a *graph* we mean a finite undirected loopless graph $G = (V(G), E(G))$ allowing multiple edges. We follow the most common graph-theoretical notation and for notation and concepts not defined here we refer the reader e.g. to [2].

A graph G is said to be *hamiltonian* if G has a *hamiltonian cycle*, i.e. a cycle of length $|V(G)|$, and *Hamilton-connected* if, for any $x, y \in V(G)$, G has a *hamiltonian (x, y)-path*, i.e. an (x, y)-path P with $V(P) = V(G)$. Obviously, a hamiltonian graph must be 2-connected and a Hamilton-connected graph must be 3-conected. A graph G is *k-Hamilton-connected*

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if, for any $X \subset V(G)$ with $|X| = k$, the graph $G - X$ is Hamilton-connected. It is easy to see that a k-Hamilton-connected graph must be $(k+3)$-connected.

We will use $L(H)$ for the line graph of a graph H. Recall that every line graph is *claw-free*, i.e., does not contain an induced subgraph isomorphic to the claw $K_{1,3}$, and that a line graph $G = L(H)$ is k-connected if and only if H is *essentially k-edge-connected*, i.e., H has no edge-cutset $X \subset E(H)$ such that $|X| < k$ and at least two components of $G - X$ contain at least one edge (such an X will be referred to as an *essential edge-cutset*). Also recall that if an edge in a graph H is *pendant* (i.e. one of its vertices has degree 1), then the corresponding vertex in $G = L(H)$ is *simplicial*, i.e. its neighborhood induces a complete graph.

If a graph H has no edge-cutset $X \subset E(H)$ such that $|X| < k$ and at least two components of $G - X$ contain at least one cycle, we say that H is *cyclically k-edge-connected.* It is a well-known fact (see e.g. [5]) that a *cubic* (i.e. 3-regular) graph H is cyclically 4-edge-connected if and only if H is essentially 4-edge-connected. A cyclically 4-edge-connected cubic graph H of girth (length of shortest cycle) $g(H) \geq 5$ that is not 3-edge-colorable is called a *snark.*

A closed trail (i.e., an Eulerian subgraph) T in a graph H is said to be *dominating* if every edge of H has at least one vertex on T. It is a well-known fact (see [9]) that if G is a line graph of order at least 3 and $G = L(H)$, then G is hamiltonian if and only if H contains a dominating closed trail. For $a, b \in E(H)$, a trail T is said to be an *(a, b)-trail* if a is the first and b is the last edge of T. A trail T in a graph H is *internally dominating* if every edge of H has at least one vertex in the set of internal vertices of T. Let $G = L(H)$, $a, b \in V(G)$, and let $\bar{a}, \bar{b} \in E(H)$ be the edges of H that correspond to a, b. Analogously to [9] (see e.g. [14]), a line graph G of order at least 3 has a hamiltonian (a, b)-path if and only if H has an internally dominating $(\bar{a}, \bar{b})$-trail.

Thomassen [17] posed the following conjecture.

Conjecture A [17]. *Every 4-connected line graph is hamiltonian.*

Since then, many statements that are seemingly stronger or weaker than Conjecture A have been proved to be equivalent to it. Below we list some of them. The reference always refers to the paper in which the equivalence with Conjecture A was established.

Theorem B. *The following statements are equivalent with Conjecture A.*

(i) [15] *Every 4-connected claw-free graph is hamiltonian.*

(ii) [5] *Every essentially 4-edge-connected graph has a dominating closed trail.*

(iii) [5] *Every cyclically 4-edge-connected cubic graph has a dominating cycle.*

(iv) [11] *Every cyclically 4-edge-connected cubic graph that is not 3-edge-colorable has a dominating cycle.*

(v) [3] *Every snark has a dominating cycle.*

Statement (*iii*) of Theorem B was strengthened as follows.

Theorem C. *The following statements are equivalent with Conjecture A.*

(*i*) [7] *Any two independent edges of a cyclically 4-edge-connected cubic graph are contained in a dominating cycle.*

(*ii*) [6] *Any two edges of a cyclically 4-edge-connected cubic graph are contained in a dominating cycle.*

On the positive side, the strongest known results related to Conjecture A are the following.

Theorem D.

(*i*) [10] *Every 5-connected claw-free graph G with minimum degree $\delta(G) \geq 6$ is hamiltonian.*

(*ii*) [16] *Every 6-connected claw-free graph with at most 29 vertices of degree 6 is Hamilton-connected.*

2 Main result.

Set $E^+(G) = \{xy|\ x, y \in V(G)\}$, and for $X \subset E^+(G)$ set $G + X = (V(G), E(G) \cup X)$ (note that we admit $E(G) \cap X \neq \emptyset$). A graph G is said to be *k-edge-Hamilton-connected* if, for any $X \subset E^+(G)$ such that $|X| \leq k$ and X determines a path system, the graph $G + X$ has a hamiltonian cycle containing all edges of X (note that by a *path system* we mean a forest each component of which is a path).

The following facts are easy to observe.

Proposition 1. *Let G be a graph. Then*

(*i*) *G is 1-edge-Hamilton-connected if and only if G is Hamilton-connected,*

(*ii*) *G is 2-edge-Hamilton-connected if and only if*

- (α) *G is 1-Hamilton-connected, and*
- (β) *for any four distinct vertices $x_1, x_2, x_3, x_4 \in V(G)$, G has a path factor consisting of two paths P_1, P_2 such that both P_1 and P_2 have one endvertex in $\{x_1, x_2\}$ and one endvertex in $\{x_3, x_4\}$,*

(*iii*) *if G is k-edge-Hamilton-connected, then G is $(k + 2)$-connected.*

Proof. Parts (*i*) and (*ii*) follow immediately from the definitions. Let G be k-edge-Hamilton-connected and let $\{a_1, \ldots, a_\ell\} \subset V(G)$, $\ell \leq k + 1$, be a cutset of G. Then for $X = \{a_1a_2, a_2a_3, \ldots, a_{\ell-1}a_\ell\}$ the graph G has no hamiltonian cycle containing all edges of X. This contradiction proves part (*iii*). ■

Our main result, Theorem 2, shows that Conjecture A is equivalent to the statement(s) that every 4-connected line graph has any of the above mentioned properties. Note that the equivalence of (i) and (ii) was originally established in the unpublished paper [13].

Theorem 2. *The following statements are equivalent.*

(i) *Every 4-connected line graph is hamiltonian.*

(ii) *Every 4-connected line graph is Hamilton-connected.*

(iii) *Every 4-connected line graph is 1-Hamilton-connected.*

(iv) *Every 4-connected line graph is 2-edge-Hamilton-connected.*

Proof of Theorem 2 is postponed to Section 3.

We will now discuss complexity aspects of Theorem 2.

The problem to decide whether a given graph G has a hamiltonian (a, b)-path for given vertices a, b is one of the classical NP-complete problems (see [8]), and the hamiltonian problem remains NP-complete even when restricted to line graphs (see e.g. [1] for the hamiltonian path problem). The problem to decide whether G is Hamilton-connected is also known to be NP-complete [4]. The complexity of the corresponding Hamilton-connectedness problem in line graphs is not known, however, it is usually supposed to be NP-complete. We now consider the next step (we include the easy proof here since we are not aware of its being published).

1-HC
Instance: *A graph G.*
Question: *Is G 1-Hamilton-connected?*

Theorem 3. *1-HC is NP-complete.*

Proof. Obviously 1-HC $\in$ NP. We transform the Hamilton-connectedness problem to 1-HC. Given a graph G, take a vertex $w \notin V(G)$ and set $G' = (V(G) \cup \{w\}, E(G) \cup \{wx | \, x \in V(G)\})$. We show that G' is 1-Hamilton-connected if and only if G is Hamilton-connected. Suppose first that G is Hamilton-connected. We show that for any $x, y, u \in V(G')$, $G' - u$ has a hamiltonian (x, y)-path. Let P be a hamiltonian (x, y)-path in G. If $u \neq w$, then $P' = xPu^-wu^+Py$ is a hamiltonian (x, y)-path in $G' - u$, and for $u = w$ we simply set $P' = P$. Conversely, if G' is 1-Hamilton-connected, then $G = G' - w$ is Hamilton-connected by definition. ■

Thus, we can analogously define the following problems.

1-HCL
Instance: *A line graph G.*
Question: *Is G 1-Hamilton-connected?*

2-E-HCL
Instance: *A line graph G.*
Question: *Is G 2-edge-Hamilton-connected?*

Note that, with respect to the above mentioned facts, a common expectation would probably be that both these problems are NP-complete.

If Conjecture A is true, then, by Theorem 2, we have that every 4-connected line graph is 2-edge-Hamilton-connected (hence also 1-Hamilton-connected). Conversely, by Proposition 1(iii), every 2-edge-Hamilton-connected graph is 4-connected and, similarly, every 1-Hamilton-connected graph is 4-connected. From this we observe that if Conjecture A is true, then

(i) a line graph G is 1-Hamilton-connected if and only if G is 4-connected,

(ii) a line graph G is 2-edge-Hamilton-connected if and only if G is 4-connected.

Consequently, Conjecture A, if true, would imply polynomiality of both 1-HCL and 2-E-HCL. We thus have the following consequence.

Theorem 4. *At least one of the following is true:*

(i) *Both 1-HCL and 2-E-HCL are polynomial.*

(ii) *Conjecture A fails.* ■

Remark. Note that Theorem 4 means that proving NP-completeness of 1-HCL or 2-E-HCL would imply the existence of a 4-connected nonhamiltonian line graph (and also e.g. the existence of a snark with no dominating cycle etc.), unless P=NP.

3 Proof of Theorem 2.

We first mention several results that will be needed for our proof.

Set $V_i(H) = \{x \in V(H)|\ d_H(x) = i\}$ and let H be a graph with $\delta(H) = 2$ and $|V_2(H)| = 4$. Then H is said to be $V_2(H)$-*dominated* if for any two edges $e_1 = u_1v_1, e_2 = u_2v_2 \in E^+(H)$ with $\{u_1, v_1, u_2, v_2\} = V_2(H)$ the graph $H + \{e_1, e_2\}$ has a dominating closed trail containing e_1 and e_2, and H is said to be *strongly* $V_2(H)$-*dominated* if H is $V_2(H)$-dominated and for any $e = uv \in E^+(H)$ with $u, v \in V_2(H)$, the graph $H + \{e\}$ has a dominating closed trail containing e. Note that in the special case of a cubic graph a dominating closed trail becomes a dominating cycle.

The following was proved in [12].

Theorem E [12]. *Conjecture A is equivalent to the statement that any subgraph H of an essentially 4-edge-connected cubic graph with $\delta(H) = 2$ and $|V_2(H)| = 4$ is $V_2(H)$-dominated.*

We will need the following slight strengthening of Theorem E.

Theorem 5. *Conjecture A is equivalent to the statement that any subgraph H of an essentially 4-edge-connected cubic graph with $\delta(H) = 2$ and $|V_2(H)| = 4$ is strongly $V_2(H)$-dominated.*

Proof. Suppose that Conjecture A is true, let H be a subgraph of an essentially 4-edge-connected cubic graph with $\delta(H) = 2$ and $|V_2(H)| = 4$, let $V_2(H) = \{a, b, c, d\}$, set $e = ab$ and suppose that $H + \{e\}$ has no dominating cycle containing e.

Let H_i, $i = 1, 2, 3, 4$ be four vertex-disjoint copies of H, denote $V_2(H_i) = \{a_i, b_i, c_i, d_i\}$, $i = 1, 2, 3, 4$, and let F' be the graph with $V(F') = \cup_{i=1}^{4} V(H_i)$ and $E(F') = (\cup_{i=1}^{4} E(H_i)) \cup \{a_1a_2, b_1b_2, a_3a_4, b_3b_4, c_1d_3, c_2d_4, d_1c_4, d_2c_3\}$. Finally, let F be the graph obtained from F' by subdividing the following edges with new vertices: c_1d_3 with a vertex x, c_2d_4 with a vertex y, c_3d_2 with a vertex z and c_4d_1 with a vertex w, and set $e_1 = xy$ and $e_2 = zw$ (see Figure 1).

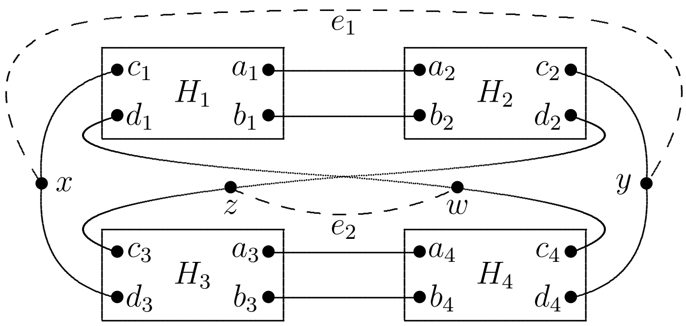

Figure 1: The graph F

By Theorem E, the graph $F + \{e_1, e_2\}$ has a dominating cycle C with $e_1, e_2 \in E(C)$. As $\{w, x, y, z\}$ separates $H_1 \cup H_2$ from $H_3 \cup H_4$, both e_1 and e_2 must be incident to edges on C to both $H_1 \cup H_2$ and $H_3 \cup H_4$. But no matter how we pick these edges, two of w, x, y, z are adjacent on C to some c_i, d_i, contradicting that $H_j + a_jb_j$ has no dominating cycle containing a_jb_j for $j \in \{1, 2, 3, 4\} \cap \{3 - i, 7 - i\}$.

Conversely, if every subgraph H of an essentially 4-edge-connected cubic graph with $\delta(H) = 2$ and $|V_2(H)| = 4$ is strongly $V_2(H)$-dominated, then clearly every such H is $V_2(H)$-dominated and Conjecture A is true by Theorem E. ■

We will also need the following operation (see [5]). Let H be a graph, $z \in V(H)$ a vertex of degree $d \geq 4$, and let $u_1, u_2, \ldots, u_d$ be an ordering of neighbors of z (we allow repetition in case of parallel edges). Then the graph H_z, obtained from the disjoint union of $G - z$ and the cycle $C_z = z_1, z_2, \ldots, z_d z_1$ by adding the edges u_iz_i, $i = 1, \ldots, d$, is called an *inflation of H at z*. If $\delta(H) \geq 3$, then, by successively taking an inflation at each vertex of degree greater than 3 we can obtain a cubic graph H^I, called a *cubic inflation of H*. The inflation of a graph at a vertex is not unique (since it depends on the ordering of neighbors of z) and it is possible that the operation decreases the edge-connectivity of the graph. However, the following was proved in [5].

Lemma F [5]. *Let H be an essentially 4-edge-connected graph with minimum degree $\delta(H) \geq 3$. Then some cubic inflation of H is essentially 4-edge-connected.*

Let H' be a cubic inflation of a graph H and for any $z \in V(H)$ set $I(z) = V(C_z)$ if $d_H(z) > 3$ and $I(z) = \{z\}$ otherwise. Observing that a dominating cycle in H' must contain at least one vertex in $I(z)$ for each $z \in V(H)$ with $d_H(z) \geq 4$, we immediately have the following fact (which is implicit in [5]).

Lemma G [5]. *Let H be a graph with $\delta(H) \geq 3$ and let H^I be a cubic inflation of H. Let C be a dominating cycle in H^I. Then H has a dominating closed trail T such that*

(i) T contains all vertices of degree at least 4,

(ii) if $uv \in E(C)$ and $u \in I(x)$, $v \in I(y)$ for some $x, y \in V(H)$, $x \neq y$, then $xy \in E(T)$.

Proof of Theorem 2. It is sufficient to prove that (i) implies (iv). Thus, suppose that Conjecture A is true and let G be a minimum counterexample to the statement (iv) of Theorem 2, i.e. G is a 4-connected line graph that is not 2-edge-Hamilton-connected but every 4-connected line graph G' with $|V(G')| < |V(G)|$ is 2-edge-Hamilton-connected. Let $Y \subset E^+(G)$ be such that $|Y| \leq 2$ and $G+Y$ has no hamiltonian cycle containing all edges of Y.

If $|Y| = 1$, then denote $Y = \{e_1\}$, choose an arbitrary $e_2 \in E(G)$ such that e_1, e_2 have no vertex in common, and set $X = \{e_1, e_2\}$. If $|Y| = 2$, then denote $Y = \{e_1, e_2\}$ and set $X = Y$. Denote $e_1 = ab$, $e_2 = cd$, and choose the notation such that possibly $b = d$. With a slight abuse of notation, we will use X also for the subgraph determined by e_1, e_2. To reach a contradiction, it is sufficient to show that $G+X$ has a hamiltonian cycle containing all edges of X.

Claim 1. *None of the vertices a, b, c, d is simplicial.*

Proof of Claim 1. Suppose that $u \in \{a, b, c, d\}$ is simplicial.

Case 1: $d_X(u) = 1$. Without loss of generality suppose $u = a$, and set $G' = G - u$. Then G' is a 4-connected line graph with $|V(G')| < |V(G)|$, hence G' is 2-edge-Hamilton-connected. Choose $a' \in N_G(u)$ such that $a' \notin \{b, c, d\}$ (this is always possible since $d_G(u) \geq 4$) and set $e_1' = a'b$ and $X' = \{e_1', e_2\}$. Let C' be a hamiltonian cycle in $G' + X'$ containing e_1' and e_2. Then $C = a'ae_1bC'a'$ is a hamiltonian cycle in G containing e_1 and e_2, a contradiction.

Case 2: $d_X(u) = 2$. Then, by the choice of notation, $u = b = d$. Similarly as before, $G' = G - u$ is 2-edge-Hamilton-connected. Set $e' = ac$, $X' = \{e'\}$ and let C' be a hamiltonian cycle in G' containing X'. Then $C = aucC'a$ is a hamiltonian cycle in G containing X, a contradiction. □

Let now H be a graph such that $L(H) = G$, and let $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ be the edges corresponding to the vertices $a, b, c, d \in V(G)$, respectively. By Claim 1, none of the edges $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ is pendant.

We now distinguish two cases.

Case 1: $\{a, b\} \cap \{c, d\} = \emptyset$. We define a graph H_4 by the following construction.

- H' is a graph obtained from H by subdividing each of the edges $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ with a new vertex a', b', c', d', respectively,
- H_1 is a graph obtained from H' by adding a new vertex u and edges ua', ub', uc', ud',
- H_2 is obtained from H_1 by removing vertices of degree 1 and suppressing vertices of degree 2.

Then H_2 is essentially 4-edge-connected with minimum degree $\delta(H_2) \geq 3$ and, by Lemma F, H_2 has an essentially 4-edge-connected cubic inflation H_3. Finally, let H_4 be obtained from H_3 by removing $I(u)$ (i.e. the vertices of the cycle that corresponds to the vertex u of H_2).

Then H_4 satisfies the assumptions of Theorem 5, hence $H_4 + \{a'b', c'd'\}$ has a dominating cycle containing $a'b'$ and $c'd'$.

By Lemma G, $(H_2 - u) + \{a'b', c'd'\}$ has a dominating closed trail T containing the edges $a'b'$, $c'd'$ and all vertices of degree at least 4. The graph H is essentially 4-edge-connected, hence for every vertex of H of degree 1 or 2, all its neighbors are of degree at least 4. Thus, T is a dominating closed trail also in $H' + \{a'b', c'd'\}$. Since T contains the edges $a'b'$ and $c'd'$, $G + X$ has a hamiltonian cycle containing the edges e_1 and e_2, a contradiction.

Case 2: $\{a, b\} \cap \{c, d\} \neq \emptyset$. By the choice of notation, we have $b = d$ and the vertices a, b, c are distinct. By the assumption, G is not 2-edge-Hamilton-connected, hence $G - b$ has no hamiltonian (a, c)-path, implying that $H - \bar{b}$ has no internally dominating $(\bar{a}, \bar{c})$-trail.

Claim 2. *Neither $\bar{a}$ and $\bar{b}$ nor $\bar{b}$ and $\bar{c}$ share a vertex of degree 2.*

Proof of Claim 2. By symmetry, suppose that $\bar{a}$ and $\bar{b}$ share a vertex v of degree 2. Then $ab \in E(G)$. Let K denote the subgraph of G induced by $N_G(a) \setminus \{b, c\}$. Since $d_H(v) = 2$, K is a clique of order at least 2.

Let H' be obtained from H by suppressing the vertex v, i.e., $\bar{a}$ and $\bar{b}$ coincide in H' into an edge $\bar{w}$. Set $G' = L(H')$. Then G' is obtained from G by contraction of the edge ab into a vertex w. Clearly, G' is 4-connected, hence, by the minimality of G, G' is 2-edge-Hamilton-connected. Let a_1 be an arbitrary vertex in K, set $e_1' = wa_1$ and $e_2' = wc$, and let C' be a hamiltonian cycle in $G' + \{e_1', e_2'\}$ containing e_1' and e_2'. Then $C = a_1abcC'a_1$ is a hamiltonian cycle in $G + X$ containing e_1 and e_2, a contradiction. $\square$

Let H_1 be the graph obtained from H by removing vertices of degree 1 and suppressing vertices of degree 2. Then H_1 is essentially 4-edge-connected. Let a^*, b^*, c^* denote the edges of H_1 that correspond to the edges $\bar{a}, \bar{b}, \bar{c}$ of H. Note that possibly $a^* = c^*$ (if $\bar{a}$ and $\bar{c}$ share a vertex of degree 2), but, by Claim 2, $a^* \neq b^*$ and $b^* \neq c^*$.

Let H_2 be an essentially 4-edge-connected cubic inflation of H_1 and, with a slight abuse of notation, let a^*, b^*, c^* denote the edges of H_2 that correspond to these edges of H_1. Set $a^* = a_1a_2$, $b^* = b_1b_2$, $c^* = c_1c_2$.

Claim 3. *The edges a^*, b^*, c^* (and hence also the edges $\bar{a}, \bar{b}, \bar{c}$) do not share a vertex of degree 3.*

Proof of Claim 3. Let, to the contrary, $w = a_1 = b_1 = c_1$ be of degree 3. If $\bar{a} = wa_1'$ for some $a_1' \neq a_2$, then, by the construction of H_1, a_1' is of degree 2 in H and $\{a_1'a_2, b_2w, c_2w\}$ is an essential edge-cutset separating the edge $a_1'w$ from the rest of H, a contradiction. Hence $a^* = \bar{a}$ and, similarly, $b^* = \bar{b}$ and $c^* = \bar{c}$.

By Theorem C(ii), H_2 has a dominating cycle C containing a^* and c^*. Since w is of degree 3, C does not contain b^*. By Lemma G and since H is essentially 4-edge-connected, H has a dominating closed trail T containing $\bar{a}$ and $\bar{c}$ and not containing $\bar{b}$. But then T is an internally dominating $(\bar{a}, \bar{c})$-trail in $H - \bar{b}$, a contradiction. □

By Claim 3, we either have $a^* = c^*$, or either a^*, c^* or a^*, b^* have no common vertex. Let H_3 and H_4 be the graphs obtained from H_2 as follows:

- (i) if a^*, c^* have no vertex in common, then H_3 is obtained from H_2 by subdividing each of the edges a^*, c^* with a new vertex a', c', respectively, and by adding the edge $a'c'$, and H_4 is obtained from H_3 by deleting the edges $a'c'$ and b^* (but keeping the vertices a', c', b_1, b_2);
- (ii) if $a^* = c^*$, then $H_3 = H_2$ and H_4 is obtained from H_3 by deleting the edges a^*, b^* (but keeping the vertices a_1, a_2, b_1, b_2), and, for consistence, by relabeling $a_1 := a'$ and $a_2 := c'$;
- (iii) if a^*, b^* have no vertex in common, then H_3 is obtained from H_2 by subdividing a^* and b^* with a new vertex a' and b' and adding the edge $a'b'$ and then subdividing $a'b'$ and c^* with a new vertex d' and c' and adding the edge $d'c'$, and H_4 is obtained from H_3 by deleting the vertices b' and d'.

It is an easy observation that an essentially 4-edge-connected cubic graph remains essentially 4-edge-connected if we subdivide two independent edges and connect the new vertices with a new edge. Hence, in all three cases, the graph H_3 is essentially 4-edge-connected. Since H_4 is a subgraph of H_3 with $\delta(H_4) = 2$ and $|V_2(H_4)| = 4$, H_4 satisfies the assumptions of Theorem 5. Then the graph $H_4 + \{a'c'\}$ has a dominating cycle containing the edge $a'c'$, implying that $H - \bar{b}$ has an internally dominating $(\bar{a}, \bar{c})$-trail, a contradiction. ■

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