Closure, clique covering and degree conditions for Hamilton-connectedness in claw-free graphs

Roman Kužel[1,2], Zdeněk Ryjáček[1,2], Jakub Teska[1,2], Petr Vrána[1]

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Abstract

We strengthen the closure concept for Hamilton-connectedness in claw-free graphs, introduced by the second and fourth authors, such that the strong closure G^M of a claw-free graph G is the line graph of a multigraph containing at most two triangles or at most one double edge.

Using the concept of strong closure, we prove that a 3-connected claw-free graph G is Hamilton-connected if G satisfies one of the following: (i) G can be covered by at most 5 cliques, (ii) $\delta(G) \geq 4$ and G can be covered by at most 6 cliques, (iii) $\delta(G) \geq 6$ and G can be covered by at most 7 cliques.

Finally, by reconsidering the relation between degree conditions and clique coverings in the case of the strong closure G^M, we prove that every 3-connected claw-free graph G of minimum degree $\delta(G) \geq 24$ and minimum degree sum $\sigma_8(G) \geq n + 50$ (or, as a corollary, of order $n \geq 142$ and minimum degree $\delta(G) \geq \frac{n+50}{8}$) is Hamilton-connected.

We also show that our results are asymptotically sharp.

1 Notation and terminology

In this paper we follow the most common graph-theoretic terminology and notation and for notations and concepts not defined here we refer the reader to [3].

Specifically, by a *graph* we mean a finite simple undirected graph $G = (V(G), E(G))$; whenever we allow multiedges (multiple edges), we say that G is a *multigraph*. By a *multiedge* in a multigraph we mean an induced subgraph $X \subset G$ such that $|V(X)| = 2$ and $|E(X)| \geq 2$. More precisely, for an edge e_1e_2, we can define the induced subgraph $X \subset G$ with $V(X) = \{e_1, e_2\}$ and say that e_1e_2 is a single edge (multiedge) if $|E(X)| = 1$ ($|E(X)| \geq 2$), respectively. The number $|E(X)|$ will be also called the *multiplicity* of the

[1]Department of Mathematics, University of West Bohemia, and Institute for Theoretical Computer Science (ITI), Charles University, P.O. Box 314, 306 14 Pilsen, Czech Republic, e-mail {rkuzel,ryjacek,teska,vranap}@kma.zcu.cz

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edge e_1e_2. Thus a graph is a multigraph with all edges of multiplicity 1. By a *double edge* we mean an edge with multiplicity 2.

A *walk* in G is an alternating sequence $v_0e_0v_1e_1...e_{k-1}v_k$ of vertices and edges of G such that $e_i = v_iv_{i+1}$ for all $i = 0, 1, ..k-1$. A *trail* in G is a walk with no repeated edges. For $u, v \in V(G)$, a (u, v)-*walk* in G is a walk such that $u = v_0$, $v = v_k$. A (u, v)-*trail* in G is a trail such that $u = v_0$, $v = v_k$. A (u, v)-*path* in G is a (u, v)-trail with no repeated vertices. For $h, f \in E(G)$, an (h, f)-*trail* in G is a trail such that $e_0 = h$ and $e_{k-1} = f$.

Given a trail T and an edge e in a multigraph G, we say e is *dominated* (*internally dominated*) by T if e is incident to a vertex (to an interior vertex) of T, respectively. Given $u, v \in V(G)$, we say T is a *maximal* (u, v)-*trail* if T internally dominates a maximum number of edges among all (u, v) trails in G. A trail T in G is called an *internally dominating trail*, shortly *IDT*, if T internally dominates all the edges in G. A closed trail T in G is called a *dominating closed trail*, shortly *DCT*, if T dominates all edges in G. Note that in a *DCT* all the vertices are internal.

In a graph G, $d_G(x)$ denotes the degree of the vertex x and $N_G(x)$ denotes the *neighborhood* of x, i.e. the set of all the vertices adjacent to x. The induced subgraph by the set of vertices M is denoted $\langle M \rangle_G$. If the graph G is clear from the context, we omit the subscript and simply write $d(x)$, $N(x)$ or $\langle M \rangle$, respectively.

A vertex v in a graph G is *simplicial* if $\langle N(v) \rangle$ is complete. An edge e in G is called *pendant* if one of its vertices is of degree 1 in G; the other vertex of degree more than one is called the *root* of e. For graphs (multigraphs) G_1 and G_2, we use $G_1 \simeq G_2$ to denote that G_1 and G_2 are isomorphic.

We use $\delta(G)$ for the minimum degree of a graph G, $\alpha(G)$ for the independence number (i.e. the maximum size of an independent set) of G, $\nu(G)$ for the matching number (i.e. the maximum size of a matching) of G, and we set $\sigma_k(G) = min\{d(a_1) + ... + d(a_k) \mid \{a_1, ..., a_k\} \subset V(G)$ is an independent set$\}$. A *vertex cover* of a graph G is a set $M \subset V(G)$ such that every edge has at least one vertex in M, and the *vertex cover number* of G, denoted $\tau(G)$, is the minimum size of a vertex cover. A *clique* is a complete subgraph, not necessarily maximal, and a *clique covering* of a graph G is a set of cliques of G which covers all the vertices of G. The *clique covering number* of G, denoted $\vartheta(G)$, is the minimum number of cliques in a clique covering of G among all the cliques coverings of G.

If H is a given graph, then a graph G is called H-*free* if G contains no induced subgraph isomorphic to H. In this case, the graph H is called a *forbidden* subgraph. The *claw* is the graph $K_{1,3}$.

2 Introduction

In this section we summarize some background knowledge that will be needed for our results.

If H is a graph (multigraph), then the *line graph* of H, denoted $L(H)$, is the graph with $E(H)$ as vertex set, in which two vertices are adjacent if and only if the corresponding edges have a vertex in common. Recall that every line graph is claw-free.

It is well-known that if G is a line graph of a graph, then the graph H such that $G = L(H)$ is uniquely determined (with one exception of $G = K_3$). However, in line graphs of multigraphs this is, in general, not true, as can be seen from the graphs in Fig. 1, where $L(H_1) = L(H_2) = G$, i.e., in line graphs of multigraphs the "line graph preimage" is not unique. This difficulty can be avoided by introducing an additional requirement that, for any simplicial vertex in the line graph, the corresponding edge in the preimage is a pendant edge.

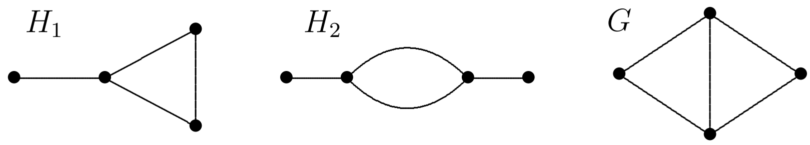

Figure 1

Proposition A [16]. *Let G be a connected line graph of a multigraph. Then there is, up to an isomorphism, a uniquely determined multigraph H such that a vertex $e \in V(G)$ is simplicial in G if and only if the corresponding edge $e \in E(H)$ is a pendant edge in H.*

For a given line graph G, its (unique) preimage with the properties given in Proposition A, will be denoted $L^{-1}(G)$ (note that if G is a line graph of a graph, then $L^{-1}(G)$ and the "obvious" line graph preimage can be different - see Fig. 1). If $H = L^{-1}(G)$, $a \in V(G)$ and $e \in E(H)$ is the edge of H corresponding to the vertex a, we will use the notation $e = L_G^{-1}(a)$ and $a = L_G(e)$ (or simply $e = L^{-1}(a)$ and $a = L(e)$ if the graph G is clear from the context).

We will need the following characterization of line graphs of multigraphs by Krausz [11].

Theorem B [11]. *A nonempty graph G is a line graph of a multigraph if and only if $V(G)$ can be covered by a system of cliques $\mathcal{K}$ such that every vertex of G is in exactly two cliques of $\mathcal{K}$ and every edge of G is in at least one clique of $\mathcal{K}$.*

A system of cliques $\mathcal{K} = \{K_1, ..., K_m\}$ with the properties given in Theorem B is called a *Krausz partition* of G. Also, if G is a line graph, then G has a Krausz partition $\mathcal{K}$ such that a vertex $x \in V(G)$ is simplicial if and only if one of the two cliques containing x is of order 1 (this can be easily seen from Proposition A), and then the preimage $L^{-1}(G)$ can be obtained from such a Krausz partition $\mathcal{K}$ as the intersection graph (multigraph) of the set system $\{V(K_1), ..., V(K_m)\}$, in which the number of vertices shared by two cliques equals the multiplicity of the (multi)edge joining the corresponding vertices of $L^{-1}(G)$.

The line graph preimage counterpart of hamiltonicity was established by Harary and Nash-Williams [9] who showed that a line graph G of order at least 3 is hamiltonian if and only if its preimage $H = L^{-1}(G)$ contains a DCT. A similar argument gives the following analogue for Hamilton-connectedness (see e.g. [12]).

Theorem C [12]. *Let H be a multigraph with $|E(H)| \geq 3$. Then $G = L(H)$ is Hamilton-connected if and only if for any pair of edges $e_1, e_2 \in E(H)$, H has an internally dominating (e_1, e_2)-trail.*

An edge cut Y of a multigraph G is *essential* if $G - Y$ has at least two nontrivial components. For an integer $k > 0$, a multigraph G is *essentially k-edge-connected* if every essential edge cut Y of G contains at least k edges. From the definitions it is easy to see that a line graph $G = L(H)$ with $\alpha(G) \geq 2$ is k-connected if and only if the graph H is essentially k-edge-connected. Also, $G = L(H)$ contains a graph F as an induced subgraph if and only if H contains $L^{-1}(F)$ as a (not necessarily induced) subgraph.

It is also easy to see that if $\delta(G) \geq k$, then there are no trivial edge-cuts of size less then k, hence G is k-edge-connected if and only if G is essentially k-edge-connected. Moreover, if G is cubic, then G is 3-edge-connected if and only if G is 3-connected. Thus, in cubic graphs, 3-connectedness, 3-edge-connectedness and essential 3-edge-connectedness are equivalent concepts.

For $x \in V(G)$, the *local completion of G at x* is the graph $G^*_x = (V(G), E(G) \cup \{y_1y_2 |\ y_1, y_2 \in N_G(x)\})$ (i.e., G^*_x is obtained from G by adding all the missing edges with both vertices in $N_G(x)$).

A vertex $x \in V(G)$ is *locally connected* (*eligible*), if $\langle N(x)\rangle$ is a connected (connected noncomplete) subgraph of G, respectively. The set of all eligible vertices in G will be denoted $V_{EL}(G)$. It is an easy observation that in the special case when G is a line graph and $H = L^{-1}(G)$, a vertex $x \in V(G)$ is locally connected if and only if the edge $e = L^{-1}_G(x)$ is in a triangle or in a multiedge in H, and $G^*_x = L(H|_e)$, where the graph $H|_e$ is obtained from H by contraction of e into a vertex and replacing the created loop(s) by pendant edge(s).

Based on the fact that if G is claw-free and $x \in V_{EL}(G)$, then G^*_x is hamiltonian if and only if G is hamiltonian, the *closure* $\mathrm{cl}(G)$ of a claw-free graph G was defined in [14] as the graph obtained from G by recursively performing the local completion operation at eligible vertices, as long as this is possible (more precisely: $\mathrm{cl}(G) = G_k$, where $G_1, \ldots, G_k$ is a sequence of graphs such that $G_1 = G$, $G_{i+1} = (G_i)^*_{x_i}$ for some $x_i \in V_{EL}(G)$, $i = 1, \ldots, k-1$, and $V_{EL}(G_k) = \emptyset$). We say that G is *closed* if $G = \mathrm{cl}(G)$.

The following result from [14] summarizes basic properties of the closure operation.

Theorem D [14]. *For every claw-free graph G:*

(i) $\mathrm{cl}(G)$ *is uniquely determined,*

(ii) $\mathrm{cl}(G)$ *is the line graph of a triangle-free graph,*

(iii) $\mathrm{cl}(G)$ *is hamiltonian if and only if G is hamiltonian.*

However, as observed in [4], the closure operation, in general, does not preserve the (non-)Hamilton-connectedness of G. This motivated the concept of k-closure as introduced in [2]: for an integer $k \geq 1$, a vertex x is *k-eligible* if $\langle N(x)\rangle$ is k-connected noncomplete, and the k-closure $\mathrm{cl}_k(G)$ is obtained analogously by recursively performing the local completion operation at k-eligible vertices, as long as this is possible. The resulting graph is again unique (see [2]). The following result was conjectured in [2] and proved in [15].

Theorem E [15]. *Let G be a claw-free graph. Then G is Hamilton-connected if and only if $\mathrm{cl}_2(G)$ is Hamilton-connected.*

It can be easily seen that, in general, $\mathrm{cl}_2(G)$ is not a line graph, and even not a line graph of a multigraph. To overcome this drawback, the second and fourth authors developed in [16] the concept of the *multigraph closure* (or briefly *M-closure*) $\mathrm{cl}^M(G)$ of a graph G: the graph $\mathrm{cl}^M(G)$ is obtained from $\mathrm{cl}_2(G)$ by performing local completions at some (but not all) eligible vertices, where these vertices are chosen in a special way such that the resulting graph is a line graph of a multigraph while still preserving the (non-)Hamilton-connectedness of G. We do not give technical details of the construction since these will not be needed for our proofs. We refer the interested reader to [15], [16]; we only note here that $\mathrm{cl}^M(G)$ can be constructed in polynomial time.

The following result summarizes basic properties of $\mathrm{cl}^M(G)$.

Theorem F [16]. *Let G be a claw-free graph and let $\mathrm{cl}^M(G)$ be the M-closure of G. Then*

(i) $\mathrm{cl}^M(G)$ *is uniquely determined,*

(ii) *there is a multigraph H such that $\mathrm{cl}^M(G) = L(H)$,*

(iii) $\mathrm{cl}^M(G)$ *is Hamilton-connected if and only if G is Hamilton-connected.*

We say that G is *M-closed* if $G = \mathrm{cl}^M(G)$. Consider the graphs T_1, T_2, T_3 in Fig. 2 (the graph T_1 will be often referred to as the *diamond* and T_2 as the *multitriangle*). It is easy

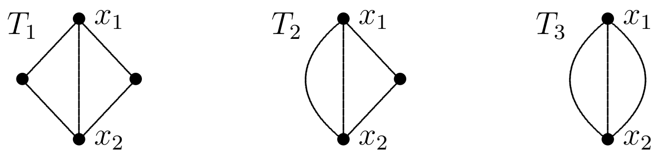

Figure 2

to observe that if $G = L(H)$ and $x \in V(G)$ is 2-eligible, then the edge $x_1x_2 = L_G^{-1}(x) \in E(H)$, corresponding to x, is contained in a copy of T_i for some i, $1 \le i \le 3$, such that $d_{T_i}(x_1) = d_{T_i}(x_2) = 3$. Although the converse is not true in general, it can be shown (see [16]) that it is true in the special case when $H = L^{-1}(G)$.

Proposition G [16]. *Let G be a claw-free graph and let T_1, T_2, T_3 be the graphs shown in Fig. 2. Then G is M-closed if and only if G is a line graph of a multigraph and $L^{-1}(G)$ does not contain a subgraph (not necessarily induced) isomorphic to any of the graphs T_1, T_2 or T_3.*

It is not difficult to observe that, roughly speaking, graphs that can be covered by few cliques are likely to have some hamiltonian properties and, similarly, graphs with high vertex degrees are likely to be coverable by few cliques. Using this approach, a relation between degree conditions and clique covering number was established and degree conditions for hamiltonicity in claw-free graphs (with exception classes) were obtained in [5], degree conditions for traceability and for the existence of a 2-factor with limited number of components were obtained in [8] and, finally, a general algorithm that generates all classes of 2-connected nonhamiltonian exceptions for a degree condition of type $\sigma_k(G) \geq n + \text{constant}$ (or, as a corollary, $\delta_k(G) \geq \frac{n+\text{constant}}{k}$) for arbitrary integer k was developed in [10], and performed (on a cluster of parallel workstations) for $k = 8$. In this paper, we will apply this approach to Hamilton-connectedness.

In Section 3 we strengthen the concept of M-closure such that the closure of a claw-free graph is the line graph of a multigraph with at most two triangles or at most one double edge.

In Section 4 we consider the relation between the clique covering number and Hamilton-connectedness. Among others, we prove that every 3-connected claw-free graph G with minimum degree $\delta(G) \geq 6$ and clique covering number $\vartheta(G) \leq 7$ is Hamilton-connected.

Finally, in Section 5 we reconsider the relation between degree conditions and clique covering number, developed in [5], in the case of the strengthened M-closure. As an application, we obtain the following asymptotically sharp degree conditions for Hamilton-connectedness in claw-free graphs (see Theorem 10 and Corollary 11):

> *If G is a 3-connected claw-free graph such that $\delta(G) \geq 24$ and $\sigma_8(G) \geq n+50$, then G is Hamilton-connected.*

> *If G is a 3-connected claw-free graph with $n \geq 142$ vertices and with minimum degree $\delta(G) \geq \frac{n+50}{8}$, then G is Hamilton-connected.*

These results extend the best known degree condition for Hamilton-connectedness in 3-connected claw-free graphs $\delta(G) \geq \frac{n+8}{5}$ proved in [13].

3 Strengthening the M-closure

In this section we further strengthen the concept of M-closure as introduced in [16] (see Theorem F) in such a way that the closure of a claw-free graph is the line graph of a multigraph with either at most two triangles and no multiedge, or with at most one double edge and no triangle.

For a given claw-free graph G, we construct a graph G^M by the following construction.

(i) If G is Hamilton-connected, we set $G^M = \text{cl}(G)$.

(ii) If G is not Hamilton-connected, we recursively perform the local completion operation at such eligible vertices for which the resulting graph is still not Hamilton-connected, as long as this is possible. We obtain a sequence of graphs $G_1, \dots, G_k$ such that

- $G_1 = G$,
- $G_{i+1} = (G_i)^*_{x_i}$ for some $x_i \in V_{EL}(G_i)$, $i = 1, \dots, k$,
- G_k has no hamiltonian (a, b)-path for some $a, b \in V(G_k)$,
- for any $x \in V_{EL}(G_k)$, $(G_k)^*_x$ is Hamilton-connected,

and we set $G^M = G_k$.

A graph G^M obtained by the above construction will be called a *strong M-closure* (or briefly an *SM-closure*) of the graph G, and a graph G equal to its SM-closure will be said to be *SM-closed*.

The following theorem summarizes basic properties of the SM-closure operation.

Theorem 1. *Let G be a claw-free graph and let G^M be its SM-closure. Then G^M has the following properties:*

(i) *$V(G) = V(G^M)$ and $E(G) \subset E(G^M)$,*
(ii) *G^M is obtained from G by a sequence of local completions at eligible vertices,*
(iii) *G is Hamilton-connected if and only if G^M is Hamilton-connected,*
(iv) *if G is Hamilton-connected, then $G^M = \mathrm{cl}(G)$,*
(v) *if G is not Hamilton-connected, then either*
  (α) *$V_{EL}(G^M) = \emptyset$ and $G^M = \mathrm{cl}(G)$, or*
  (β) *$V_{EL}(G^M) \neq \emptyset$ and $(G^M)^*_x$ is Hamilton-connected for any $x \in V_{EL}(G^M)$,*
(vi) *$G^M = L(H)$, where H contains either*
  (α) *at most 2 triangles and no multiedge, or*
  (β) *no triangle, at most one double edge and no other multiedge,*
(vii) *if G contains no hamiltonian (a, b)-path for some $a, b \in V(G)$ and*
  (α) *X is a triangle in H, then $E(X) \cap \{L^{-1}_{G^M}(a), L^{-1}_{G^M}(b)\} \neq \emptyset$,*
  (β) *X is a multiedge in H, then $E(X) = \{L^{-1}_{G^M}(a), L^{-1}_{G^M}(b)\}$.*

Note that, by (vi), the structure of $L^{-1}(G^M)$ is very close to that of $L^{-1}(\mathrm{cl}(G))$ (only at most two triangles or at most one double edge). In some cases (specifically, in cases (iv) and $(v)(\alpha)$ of Theorem 1), we have $V_{EL}(G^M) = \emptyset$ and $G^M = \mathrm{cl}(G)$, implying that G^M is uniquely determined. However, if $V_{EL}(G^M) \neq \emptyset$, then, for a given graph G, its SM-closure G^M is in general not uniquely determined and, as will be seen from the proof, the construction of G^M requires knowledge of a pair of vertices a, b for which there is no hamiltonian (a, b)-path in G. Consequently, there is not much hope to construct G^M in polynomial time (unless P=NP). Nevertheless, the special structure of G^M will be very useful for our considerations in the next sections.

For the proof of Theorem 1 we will need the following result from [4].

Proposition H [4]. *Let x be an eligible vertex of a claw-free graph G, G^*_x the local completion of G at x, and a, b two distinct vertices of G. Then for every longest (a, b)-path $P'(a, b)$ in G^*_x there is a path P in G such that $V(P) = V(P')$ and P admits at least one of a, b as an endvertex. Moreover, there is an (a, b)-path $P(a, b)$ in G such that $V(P) = V(P')$ except perhaps in each of the following two situations (up to symmetry between a and b):*

(i) *There is an induced subgraph $F \subset G$ isomorphic to the graph S in Fig. 3 such that both a and x are vertices of degree 4 in F. In this case G contains a path P_b such that b is an endvertex of P and $V(P_b) = V(P')$. If, moreover, $b \in V(F)$, then G contains also a path P_a with endvertex a and with $V(P_a) = V(P')$.*

(*ii*) $x = a$ *and* $ab \in E(G)$. *In this case there is always both a path* P_a *in* G *with endvertex* a *and with* $V(P_a) = V(P')$ *and a path* P_b *in* G *with endvertex* b *and with* $V(P_b) = V(P')$.

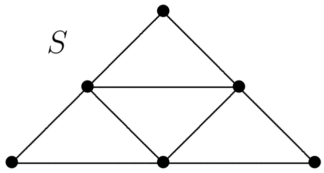

Figure 3

Proof of Theorem 1. Let G be a claw-free graph and let G^M be its SM-closure. Clearly, G^M satisfies (i), (ii), (iii), (iv) and, if $V_{EL}(G^M) \neq \emptyset$, then also $(v)(\beta)$. Suppose that G is such that $\mathrm{cl}(G)$ is not Hamilton-connected and, for some G^M, $V_{EL}(G^M) \neq \emptyset$. Then, by the definition of $\mathrm{cl}(G)$, for any $x \in V_{EL}(G)$, $(G^M)^*_x$ is a spanning subgraph of $\mathrm{cl}(G)$ and hence also not Hamilton-connected, a contradiction. Thus, if $\mathrm{cl}(G)$ is not Hamilton-connected, then $V_{EL}(G^M) = \emptyset$ for any G^M. By the uniqueness of $\mathrm{cl}(G)$, G^M satisfies also $(v)(\alpha)$.

Now, if some G^M is not a line graph (of a multigraph), then G^M is a proper subgraph of $\mathrm{cl}^M(G^M)$. However, the graph $\mathrm{cl}^M(G^M)$ is also not Hamilton-connected and was obtained from G^M by local completions at eligible vertices, contradicting $(v)(\beta)$. Hence every G^M is a line graph of a multigraph.

Let G^M be an SM-closure of G and set $H = L^{-1}(G^M)$, $e = L^{-1}_{G^M}(a)$ and $f = L^{-1}_{G^M}(b)$.

Claim 1. *Each triangle in* H *contains at least one of the edges* e, f.

Proof. Suppose that H contains a triangle T such that $\{e, f\} \cap E(T) = \emptyset$, and let $h \in E(T)$ and $x = L(h)$. Then $x \in V_{EL}(G^M)$. Suppose that G^M contains an induced subgraph F such that $F \simeq S$ (see Fig. 3) and x is a vertex of degree 4 in F. Since $L^{-1}(S)$ is the graph consisting of a triangle with a pendant edge at each vertex, $L^{-1}(S)$ contains a triangle containing h. By Proposition G, H contains no diamond (otherwise we have a 2-eligible vertex, contradicting the definition of G^M), hence $L^{-1}(S)$ contains T. Since $\{e, f\} \cap E(T) = \emptyset$, none of the vertices a, b is a vertex of degree 4 in F. By Proposition H(i), the graph $(G^M)^*_x$ has no hamiltonian (a, b)-path, contradicting the definition of G^M. $\square$

Claim 2. *If* H *contains a multiedge* X, *then* $E(X) = \{e, f\}$.

Proof. If X is a multiedge in H, then, by Proposition G, X is a double edge and no edge of X is in a triangle. Set $E(X) = \{h_1, h_2\}$, $x_i = L(h_i)$, $i = 1, 2$, and suppose that $h_1 \notin \{e, f\}$. Then $x_1x_2 \in E(G^M)$ and $x_i \in V_{EL}(G^M)$, $i = 1, 2$. Since $x_1 \notin \{a, b\}$, by Proposition H(ii), the graph $(G^M)^*_{x_1}$ has no hamiltonian (a, b)-path, contradicting the definition of G^M. $\square$

Now, the properties (vi) and (vii) of G^M follow immediately from Claims 1 and 2. ■

4 Graphs that can be covered by few cliques

In this section we prove that every 3-connected claw-free graph that can be covered by a small number of cliques is Hamilton-connected.

Theorem 2. *Let G be a 3-connected claw-free graph. If*
(i) $\vartheta(G) \leq 5$, *or*
(ii) $\vartheta(G) \leq 6$ *and* $\delta(G) \geq 4$, *or*
(iii) $\vartheta(G) \leq 7$ *and* $\delta(G) \geq 6$,
then G is Hamilton-connected.

Examples. (*i*) Consider the graph $G_1 = L(H_1)$, where H_1 is the left graph in Fig. 4 (in which the dots indicate that the number of pendant edges attached to the respective vertices can be arbitrarily large). The graph H_1 has no (e, f)-IDT (hence, by Theorem C, G_1 is not Hamilton-connected), but $\vartheta(G_1) = 6$ and $\delta(G_1) = 3$. This example shows that, in Theorem 2(*ii*), the condition $\delta(G) \geq 4$ is necessary.

(*ii*) Let $G_2 = L(H_2)$, where H_2 is the second graph in Fig. 4 (in which again the dots indicate an arbitrary number of pendant edges). Clearly, G_2 is 3-connected and $\vartheta(G_2) = 8$, but G_2 is not Hamilton-connected (since H_2 has e.g. no (u_1u_5, u_3u_7)-IDT). This example shows that Theorem 2 is sharp.

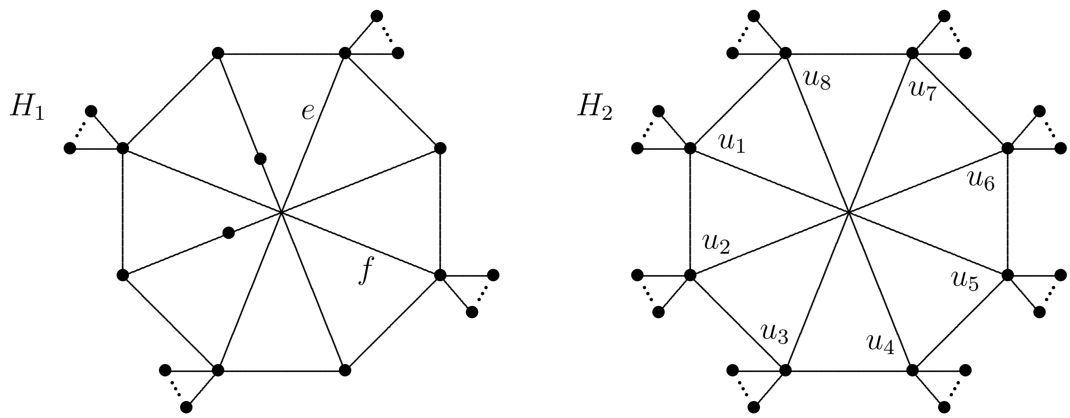

Figure 4

For the proof of Theorem 2 we will need several notations and auxiliary results.

Let H be a graph, $u \in V(H)$ a vertex of degree 2, and let v_1, v_2 be the neighbors of u. Then $H|_{(u)}$ denotes the graph obtained from H by suppressing the vertex u (i.e., by replacing the path v_1, u, v_2 by the edge v_1v_2) and by adding two pendant edges f_1 and f_2 such that f_1 is incident with v_1 and f_2 is incident with v_2.

Lemma 3. *Let H be a graph, $u \in V(H)$ a vertex of degree 2, and let v_1, v_2 be the neighbors of u. Set $H' = H|_{(u)}$, $h = v_1v_2 \in E(H')$, and let $f_1, f_2 \in E(H') \setminus E(H)$ be the two pendant edges attached to v_1 and v_2, respectively.*

(*i*) *If* $L(H)$ *is Hamilton-connected, then* $L(H')$ *has a hamiltonian* (x, y)*-path for every* $x, y \in V(L(H'))$ *for which either* $L(h) \notin \{x, y\}$*, or* $L(h) \in \{x, y\}$ *and* $\{x, y\} \cap \{L(f_1), L(f_2)\} \neq \emptyset$*.*

(*ii*) *If* $L(H')$ *is Hamilton-connected, then* $L(H)$ *has a hamiltonian* (x, y)*-path for every* $x, y \in V(L(H))$ *for which* $\{x, y\} \neq \{L(uv_1), L(uv_2)\}$*.*

Proof. Suppose first that $L(H)$ is Hamilton-connected, i.e. H contains an (e, f)-IDT for any $e, f \in E(H)$. For given $e', f' \in E(H')$, we construct an (e', f')-IDT in H'. Up to a symmetry, we have the following possibilities.

(a) If $\{e', f'\} = \{f_1, f_2\}$, we take a (uv_1, uv_2)-IDT in H and replace the edges uv_1 and uv_2 with f_1 and f_2, respectively. The resulting trail is an (f_1, f_2)-IDT in H'.

(b) If $e' = f_1$ and $f' = h$, we similarly take a (uv_1, uv_2)-IDT in H and, replacing uv_1 and uv_2 with f_1 and h, we get an (f_1, h)-IDT in H'.

(c) Suppose that $e' = f_1$ and $f' \notin \{f_1, f_2, h\}$. Let $f \in E(H)$ be the edge corresponding to f', and let T be a (uv_1, f)-IDT in H. If u is not an internal vertex of T, we replace uv_1 with f_1; otherwise (i.e. if u is an internal vertex of T), we replace v_1u and uv_2 with f_1 and h. In both cases we get an (f_1, f')-IDT in H' (note that if u is not an internal vertex of T, then $v_2 \in V(T)$, since otherwise the edge uv_2 would not be dominated).

(d) Finally, let $\{e', f'\} \cap \{f_1, f_2, h\} = \emptyset$ and let $e, f \in E(H)$ be the edges corresponding to $e', f' \in E(H')$. Then any (e, f)-IDT in H corresponds to an (e', f')-IDT in H'.

In all cases, we have constructed an (e', f')-IDT in H'.

Conversely, suppose that $L(H')$ is Hamilton-connected, i.e. H' has an (e', f')-IDT for any $e', f' \in E(H')$. For given $e, f \in E(H)$, we construct an (e, f)-IDT in H.

(a) If $\{e, f\} \cap \{uv_1, uv_2\} = \emptyset$, then, for $e', f' \in E(H')$ corresponding to $e, f \in E(H)$, any (e', f')-IDT in H' corresponds to an (e, f)-IDT in H.

(b) Let $e = uv_1$ and $f \neq uv_2$, let $f' \in E(H')$ be corresponding to f, and let T' be an (h, f')-IDT in H'. To obtain a (uv_1, f)-IDT in H, we replace the edge h with either the edge uv_1 if v_1 is the first interior vertex on T', or with the path v_1uv_2 if v_2 is the first interior vertex on T'.

In both cases, we have constructed an (e, f)-IDT in H. ■

Corollary 4. *Let* G *be an* SM*-closed graph that is not Hamilton-connected and suppose that the graph* $H = L^{-1}(G)$ *contains a vertex* $u \in V(H)$ *of degree 2 and a triangle not containing* u*. Then the graph* $L(H|_{(u)})$ *is not Hamilton-connected.*

Proof. Let v_1 and v_2 be the two neighbors of u in H and let T be a triangle in H not containing u. Since $L(H)$ is SM-closed, there are $e, f \in E(H)$ such that at least one of the edges e, f is in T and H has no (e, f)-IDT (see Theorem 1$(v)(\alpha)$). Clearly $\{e, f\} \neq \{uv_1, uv_2\}$. If $L(H|_{(u)})$ is Hamilton-connected, then, by Lemma 3(ii), H has an (e, f)-IDT, a contradiction. ■

We will also need the following operation (see [7]). Let H be a graph, $z \in V(H)$ a vertex of degree $d \geq 4$, and let $u_1, u_2, \ldots, u_d$ be an ordering of neighbors of z (allowing repetition in case of parallel edges). Then the graph H_z, obtained from the disjoint union of $G - z$ and the cycle $C_z = z_1, z_2, \ldots, z_d z_1$ by adding the edges $u_i z_i$, $i = 1, \ldots, d$, is called an *inflation of H at z*. If $\delta(H) \geq 3$, then, by successively taking an inflation at each vertex of degree greater than 3 we can obtain a cubic graph H^I, called a *cubic inflation of H*. The inflation of a graph at a vertex is not unique (since it depends on the ordering of neighbors of z) and it is possible that the operation decreases the edge-connectivity of the graph; however, it can be shown that with a proper choice of the ordering of neighbors, the connectivity can be preserved. This was shown in [7] for essential edge-connectivity 4, and the following proposition is an analogue for essential edge-connectivity 3. Its proof is implicit in the proof of Lemma 2 of [6].

Also recall that, in cubic graphs, 3-connectedness, 3-edge-connectedness and essential 3-edge-connectedness are equivalent concepts; we state the result here in a form in which it will be needed for our proof.

Proposition I [6]. *Let H be an essentially 3-edge-connected graph with $\delta(H) \geq 3$ and let $z \in V(H)$ be a vertex of degree $d(z) \geq 4$. Then there exists an inflation H_z of H at z which is essentially 3-edge-connected.*

For the proof of Theorem 2 we will also need the following result by Bau and Holton [1].

Proposition J [1]. *Let G be a 3-connected cubic graph, $M \subset V(G)$ such that $|M| \leq 7$ and $e \in E(G)$. Then there exists a cycle C in G, such that $M \subset V(C)$ and $e \in E(C)$.*

Now we are ready to prove the main result of this section.

Proof of Theorem 2. Let G be a graph satisfying the assumptions of the theorem and suppose, to the contrary, that G is not Hamilton-connected. Let G^M be an SM-closure of G. Clearly, if G can be covered by ϑ cliques, then so can be G^M, hence $\vartheta(G^M) \leq \vartheta(G)$. Obviously, G^M is 3-connected and $\delta(G^M) \geq \delta(G)$, hence G^M also satisfies the assumptions of the theorem. Thus, we can suppose that G is SM-closed. Set $H = L^{-1}(G)$.

Let $\mathcal{K} = \{K_1, \ldots, K_{\vartheta(G)}\}$ be a minimum clique covering of G.

In a clique in G, all the vertices are pairwise adjacent and therefore the corresponding edges in H are also pairwise adjacent. Hence, the cliques in $\mathcal{K}$ correspond in H either to stars or to triangles. If $L^{-1}(K_i)$ is a star, then its center will be refereed to as a *black vertex*, and if $L^{-1}(K_i)$ is a triangle, we say that $L^{-1}(K_i)$ is a *black triangle* in H. Edges of black triangles are called *black edges*, and all the other edges are said to be *white edges*.

We will use the following notation:

B_V denotes the set of black vertices in H (i.e. $B_V \subset V(H)$) and $\beta_V = |B_V|$,
B_T denotes the set of black triangles in H and $\beta_T = |B_T|$,
$W = V(H) \setminus B_V$; the vertices in W we will called *white vertices*,
$B_2 = \{b \in B_V \mid d_H(b) = 2\}$ and $\beta_2 = |B_2|$,
$Y = \{y \in V(H) \mid yb \in E(H) \text{ for some } b \in B_2\}$ and $\eta = |Y|$,
$\beta = \beta_V + \beta_T$ (i.e., $\beta = \vartheta(G)$).

We choose the graph G and the clique covering $\mathcal{K}$ of G such that

(i) G is SM-closed and not Hamilton-connected,

(ii) subject to (i), $|\mathcal{K}|$ is minimum,

(iii) subject to (i) and (ii), β_2 is minimum.

From the definitions we immediately see the following properties of B_V, Y and W:

- every white edge has at least one vertex in B_V,
- $Y \subset W$ (otherwise there is a black vertex incident with a black vertex u of degree 2, but now we can lower β_2 by coloring u white and its neighbors black),
- the vertices in W (and hence also in Y) can be connected only by black edges (note that a white edge is contained only in a star in H which corresponds to a clique in G),
- every vertex in Y has degree at least three (otherwise we have a contradiction with the 3-connectedness of G).

We denote $B_2 = \{x_1, x_2, ..., x_{\beta_2}\}$ and, for any $x_i \in B_2$, we set $N(x_i) = \{y_i^1, y_i^2\}$, $i = 1, ..., \beta_2$. Now we present several claims concerning the vertices in B_2.

Claim 1. *For every $i = 1, ..., \beta_2$, $y_i^1 y_i^2 \notin E(H)$.*

Proof. Let, to the contrary, $y_i^1 y_i^2 \in E(H)$. Since $Y \subset W$, $y_i^1 y_i^2$ is an edge of a black triangle T. If $T = x_i y_i^1 y_i^2$, then we can color x_i with white color, thus lower β_2, a contradiction. Therefore $T = z y_i^1 y_i^2$ and $z \neq x_i$. But now $z y_i^1 y_i^2 x_i$ is a diamond, which is also a contradiction (see Proposition G). $\square$

Consider the bipartite graph $F = (B_2, Y)$. There is no cycle in F, otherwise we could switch colors of the vertices along this cycle and lower β_2. Recall that the vertices in B_2 are of degree two, thus F is a subdivision of a forest. This immediately implies the following fact.

Claim 2. *If $\beta_2 > 0$, then $\beta_2 + 1 \leq \eta \leq 2\beta_2$.* $\square$

Let $e_1, e_2 \in E(H)$ be two edges such that there is no (e_1, e_2)-IDT in H.

Claim 3. *Let $x_i \in B_2$ and $N(x_i) = \{y_i^1, y_i^2\}$.*

(i) If H contains a multiedge or two triangles, then $N(y_i^1) \cap N(y_i^2) = \{x_i\}$.

(ii) If $\{e_1, e_2\} \neq \{x_i y_i^1, x_i y_i^2\}$, then $|N(y_i^1) \cap N(y_i^2)| \leq 2\}$.

Proof. Suppose first that H contains a multiedge or two triangles. If H contains two triangles T_1, T_2, then, since $y_i^1 y_i^2 \notin E(H)$ (by Claim 1) and x_i has degree two, neither of the edges $x_i y_i^1$, $x_i y_1^2$ is contained in T_1 or T_2, hence neither of them is e_1 or e_2. If H contains a multiedge X, then $E(X) = \{e_1, e_2\}$ (by Theorem $1(v)(\beta)$), and neither of $x_i y_i^1$, $x_i y_1^2$ is in X since x_i has degree two. Thus, in both cases, $\{x_i y_i^1, x_i y_i^2\} \cap \{e_1, e_2\} = \emptyset$.

We consider the graph $H' = H|_{(x_i)}$. By Lemma $3(ii)$, $L(H')$ is not Hamilton-connected. Moreover, if H contains triangles T_1, T_2 (a multiedge X), then T_1, T_2 (or X) are triangles (a multiedge) also in H'.

Suppose that there is a vertex $z \in N_H(y_i^1) \cap N_H(y_i^2)$, $z \neq x_i$. Then $\langle\{z, y_i^1, y_i^2\}\rangle_{H'}$ is a triangle in H'. We show that neither of the edges $y_i^1 z$, $y_i^2 z$ can be in one of T_1, T_2 or in X.

Let first T_1, T_2 be triangles in H' and let, say, $y_i^1 z \in E(T_1)$. Then $\langle\{y_i^1, y_i^2, z, w\}\rangle_{H'}$ (where w is the third vertex of T_1) is a diamond (see Fig. 2). Hence the vertex $u = L(y_i^1 z) \in V(L(H'))$ is 2-eligible in $L(H')$, implying $L(H')_u^* = L(H'|_{y_i^1 z})$ is not Hamilton-connected. However, coloring the vertices $y_i^1 = z$ and y_i^2 of $H'|_{y_i^1 z}$ black, we reduce β_2, a contradiction. By symmetry, we conclude that $(E(T_1) \cup E(T_2)) \cap \{y_i^1 z, y_i^2 z\} = \emptyset$. Similarly, if X is a multiedge in H' and, say, $y_i^1 z \in E(X)$, then the graph T with $V(T) = \{y_i^1, y_i^2, z\}$ and $E(T) = E(X) \cup \{y_i^1 y_i^2, z y_i^2\}$ is a multitriangle (see Fig. 2) in H', and contracting one of the edges of X we have a similar contradiction. Thus, in both cases, neither of $y_i^1 z$, $y_i^2 z$ can be in one of T_1, T_2 or in X. This specifically implies that $z \in B_V$ and $\{y_i^1 z, y_i^2 z\} \cap \{e_1, e_2\} = \emptyset$.

Hence none of the edges e_1, e_2 is in the triangle $\langle\{z, y_i^1, y_i^2\}\rangle_{H'}$. Let H'' be the graph obtained from H' by contracting the triangle $\langle\{z, y_i^1, y_i^2\}\rangle_{H'}$ (note that this corresponds to two local completions in $L(H')$). Then, by Proposition H, $L(H'')$ is not Hamilton-connected. However, $L(H'')$ can be covered by $\vartheta(G) - 1$ cliques, a contradiction.

Now suppose that $\{e_1, e_2\} \neq \{x_i y_i^1, x_i y_i^2\}$ and, to the contrary, $\{x_i, z_1, z_2\} \subset N(y_i^1) \cap N(y_i^2)$. If, say, $z_1 \in W$, then, since $Y \subset W$, the edges $z y_i^1$, $z y_i^2$ are edges of black triangles; by Claim 1, these triangles are distinct and we are in the previous case. Hence $z_1, z_2 \in B_V$. Set again $H' = L(H|_{(x_i)})$. By Corollary 4, $L(H')$ is not Hamilton-connected. However, $\langle\{y_i^1, y_i^2, z_1, z_2\}\rangle_{H'}$ is a diamond in H', and contracting the edge $y_i^1 y_i^2$ and coloring the contracted vertex black we again reduce β_2, a contradiction. □

Claim 4. *If $\beta_2 > 0$ and $\{e_1, e_2\} \neq \{x_i y_i^1, x_i y_i^2\}$ for every $x_i \in B_2$, then $\eta - \beta_2 \leq 7 - \vartheta(G^M)$.*

Proof. We distinguish three cases.

Case 1: $\vartheta(G) \leq 5$. We need to show that $\eta - \beta_2 \leq 2$. If $\beta_2 \leq 2$, then, by Claim 2, $\eta - \beta_2 \leq \beta_2 \leq 2$ and we are done. Thus, let $\beta_2 \geq 3$ and assume, to the contrary, that $\eta - \beta_2 \geq 3$ (i.e, the forest F has at least three components). Then $\eta \geq 3 + \beta_2 \geq 6$. This means that at least six vertices in Y are connected using some edges from black triangles or some edges ending in black vertices outside F such that the resulting graph is essentially 3-edge-connected. Recall that any edge not in a black triangle must have at least one vertex black. There are at least six vertices in F of degree one in F (at least 2 in each component of F) and, since $\delta(G) \geq 3$, every such vertex is incident to at least two edges outside F. Thus $\beta - \beta_2 \geq 2$. Since $\vartheta(G) = \beta \leq 5$, we have $\beta - \beta_2 = 2$ and $\beta_2 = 3$. Since $\eta \geq 6$ and every vertex in B_2 has 2 neighbors in Y, $\eta = 6$ and all the vertices in Y are of degree one in F.

If some vertex in Y, say, y_1^1, is not in a black triangle, then, since $\delta(G) \geq 3$, y_1^1 has two black neighbors z_1, z_2 outside F, implying $B_v \setminus B_2 = \{z_1, z_2\}$ and $\beta_T = 0$, but then all vertices in Y have to be adjacent to both z_1 and z_2, contradicting Claim 3. Hence all vertices in Y are in black triangles, i.e. $\beta_T = 2$. By Claim 1, the only possibility is the graph H in Fig 5. But then the graph $G = L(H)$ is Hamilton-connected, a contradiction.

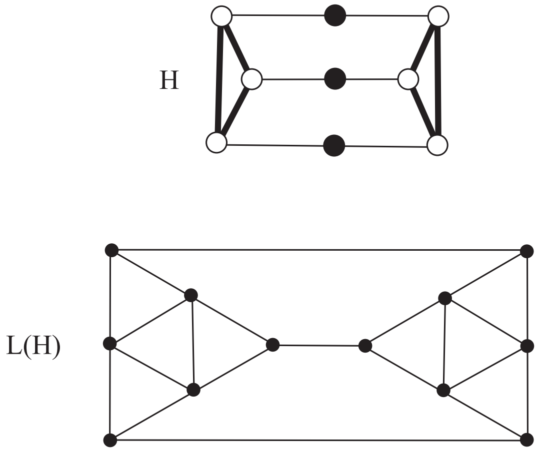

Figure 5

Case 2: $\vartheta(G) = 6$ and $\delta(G) \geq 4$. We have to show that $\eta - \beta_2 \leq 1$. If $\beta_2 \leq 1$, then again $\eta - \beta_2 \leq \beta_2 \leq 1$ and we are done, hence let $\beta_2 \geq 2$ and assume, to the contrary, that $\eta - \beta_2 \geq 2$. Then $\eta \geq 2 + \beta_2 = 4$, the forest F has at least two components and there are at least four vertices in Y of degree one in F.

First suppose that some vertex in Y of degree one in F, say, y_1^1, is not in a black triangle. We can assume that y_1^1 is not in a multiedge for otherwise $\beta_T = 0$ and we choose a different vertex in Y of degree one in F. Since $\delta(G) \geq 4$, y_1^1 has three neighbors $z_1, z_2, z_3 \in B_V \setminus V(F)$, implying $\beta_V - \beta_2 \geq 3$. Since $\vartheta(G) = \beta = 6$, we have $\beta_2 \leq 3$.

If $\beta_2 = 3$, then $\beta_T = 0$, some component of F has only one black vertex and both its neighbors have to be adjacent to each of z_1, z_2, z_3, contradicting Claim 3. Hence $\beta_2 = 2$ and $\eta = 4$. This means that the forest F has two components isomorphic to P_3 and four vertices in Y of degree one in F. We already have that $\beta_V \geq 5$, thus $\beta_T \leq 1$. Label the vertices such that $\{x_1y_1^1, x_1y_1^2\} \neq \{e_1, e_2\}$ and none of the vertices y_1^1, y_1^2 is incident with a multiedge (this is always possible). If $\beta_T = 0$, then y_1^2 has at least two neighbors among the vertices z_1, z_2, z_3, contradicting Claim 3, hence $\beta_T = 1$. Let T be the black triangle. If only one of x_1, x_2 has a neighbor in T, then we get the same contradiction for the other vertex. So, both x_1 and x_2 have a neighbor in T. Choose the notation such that $y_1^2, y_2^2 \in V(T)$. Since $\delta(G) \geq 4$, the vertices y_1^2, y_2^2 must have at least one other black neighbor outside F. Since $\vartheta(G) = 6$, both y_1^2 and y_2^2 is adjacent to some of z_1, z_2, z_3. If y_1^2, y_2^2 are adjacent to the same vertex, we get a diamond, a contradiction. So, without loss of generality suppose that we have the edges $z_1y_1^2$ and $z_3y_2^2$.

Consider the graphs $H_1 = H|_{(x_1)}$ and $H' = H_1|_{(x_2)}$. Since one of the edges e_1, e_2 is in T, by Corollary 4, $L(H')$ is not Hamilton-connected. But $L(H')$ is not SM-closed, and as one of the edges e_1, e_2 is in T, we can contract one of the triangles $\langle\{y_1^1, y_1^2, z_1\}\rangle_{H'}$, $\langle\{y_2^1, y_2^2, z_3\}\rangle_{H'}$. Let H'' denote the resulting graph. Then clearly $L(H'')$ is not Hamilton-connected and has a clique covering with fewer cliques of size two, contradicting the choice of G (note that contracting a triangle in H' corresponds to local completions in $L(H')$).

Hence all vertices in Y of degree one in F are in black triangles. Since F has at least two components, at least 4 vertices in Y are of degree 1 in F. Hence $\beta_T = 2$ and $\beta_2 \leq 4$ (and of course $\eta = 6$).

If $\beta_2 = 4$, we can recolor $E(B_T)$ white, $V(B_T)$ black and B_2 white (see Fig. 6) and reduce β_2, a contradiction.

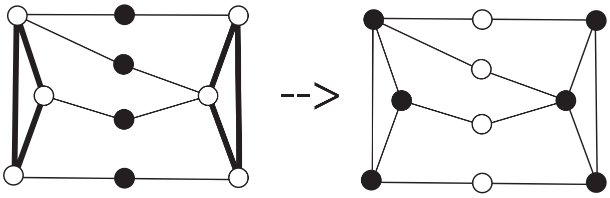

Figure 6

If $\beta_2 = 3$, then $5 \leq \eta \leq 6$. If $\eta = 5$, we can similarly recolor $E(B_T)$ white, Y black and B_2 white (see Fig 7) and reduce β_2, a contradiction. If $\eta = 6$, then F has three components isomorphic to P_3. By Claim 1, we can label the vertices in Y such that $B_T = \{T_1, T_2\}$, where $T_1 = \langle\{y_1^1, y_2^1, y_3^1\}\rangle_H$ and $T_2 = \langle\{y_1^2, y_2^2, y_3^2\}\rangle_H$. Since $\delta(G) \geq 4$, the vertices y_i^j must have a common black neighbor outside F, but then we have a diamond, a contradiction.

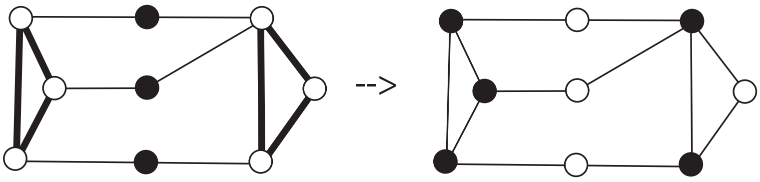

Figure 7

Finally, if $\beta_2 = 2$, then $\eta = 4$ and F has two components isomorphic to P_3. We can again label the vertices in Y such that $B_T = \{T_1, T_2\}$, where $y_1^1, y_2^1 \in T_1$ and $y_1^2, y_2^2 \in T_2$. A recoloring similar to that in the previous cases then again reduces β_2, a contradiction.

Case 3: $\vartheta(G) = 7$ and $\delta(G) \geq 6$. We need to show that $\beta_2 = 0$. Let, to the contrary, $\beta_2 \geq 1$, thus $\eta \geq 2$. Let $y_1^1 \in Y$ be of degree one in F. Since $\delta(G) \geq 6$, there are at least 5 edges joining y_1^1 with a vertex outside F, and there are at least $6 - \beta_2$ edges joining y_1^2 with a vertex outside F. Together we have at least $11 - \beta_2$ outgoing edges from F containing one of y_1^1, y_1^2. Since $Y \subset W$, each of these edges must either be black, or must contain a black vertex. However, there are only $7 - \beta_2$ elements in $B \setminus B_2$.

If $\beta_T = 0$, then one of the vertices in $B_v \setminus B_2$ can be in at most three such edges (with a possible multiedge), and each of the remaining vertices in $B_v \setminus B_2$ is in at most two such edges. Then y_1^1, y_1^2 have at least two common neighbors, contradicting Claim 3.

If $\beta_T > 0$, then each of the black triangles contains at most two edges from Y, otherwise we have a diamond. Similarly as in the previous case, y_1^1 and y_1^2 have at least two common neighbors, contradicting Claim 3 again. $\square$

Claim 5. $\beta_2 \leq 9 - \vartheta(G)$.

Proof. Due to the Claim 4 it is impossible that $\beta_2 = 0$ or $\beta_2 > 0$ and $\{e_1, e_2\} \neq \{x_i y_i^1, x_i y_i^2\}$ for every $x_i \in B_2$. So, choose the notation such that $\{e_1, e_2\} = \{x_1 y_1^1, x_1 y_1^2\}$. By Claim 1, H has no triangle or multiedge.

- (i) If $\vartheta(G) \leq 5$, it is enough to show that $\beta_2 \leq 4$. Let, to the contrary, $\beta_2 \geq 5$. Then clearly $\beta_2 = 5$ since $5 \leq \beta_2 \leq \beta = \vartheta \leq 5$. Thus $H = F$ and H is not essentially 3-edge-connected because F is a forest.
- (ii) If $\vartheta(G) = 6$ and $\delta(G) \geq 4$, we have to show that $\beta_2 \leq 3$. Let, to the contrary, $\beta_2 \geq 4$. Then $\beta - \beta_2 \leq 2$, but since $\delta(G) \geq 4$, every vertex in Y of degree one in F has at least three neighbors outside F. Hence $\beta - \beta_2 \geq 3$, a contradiction.
- (iii) If $\vartheta(G) = 7$ and $\delta(G) \geq 5$, we have to show that $\beta_2 \leq 2$. Let, to the contrary, $\beta_2 \geq 3$. Since F is a forest, we can choose the notation such that the vertex $y_2^1 \in N_H(x_2)$ is of degree one in F and $y_2^1 x_1 \notin E(H)$. Since $\delta(G) \geq 5$, y_2^1 has at least 4 black neighbors outside F, thus $\beta - \beta_2 \geq 4$. Therefore $3 \leq \beta_2 \leq \beta - 4 \leq \vartheta - 4 = 3$, from which $\beta_2 = 3$ and $\beta - \beta_2 = 4$. The vertex y_2^2 has at most three neighbors in F and, since $\delta(G) \geq 5$, y_2^2 has at least two black neighbors outside F. But then y_2^1 and y_2^2 have at least two common neighbors in $B \setminus B_2$, contradicting Claim 3. $\square$

Now we can continue with the main proof. We define a graph H^+ obtained from H by specifying an (in most cases new) edge $h = u_1 u_2$ and by (in most cases) recoloring black vertices. We use a notation $B(H)$, $B(H^+)$, $\beta(H)$ etc. to distinguish black etc. vertices in H and in H^+. Also note that, by the properties of $B(H)$, the root of a pendant edge in H has always to be black in H.

The construction of H^+ is as follows:

- (i) if e_1, e_2 are pendant edges with a common root w, we set $H^+ = H$, $B(H^+) = B(H)$, and we choose $h = u_1 u_2$ to be an arbitrary non-pendant edge of H;
- (ii) if e_1, e_2 share a vertex w of degree 2 in H, we denote $e_1 = u_1 w$, $e_2 = u_2 w$ and we set $h = u_1 u_2$, $V(H^+) = V(H) \setminus \{w\}$, $E(H^+) = (E(H) \setminus \{u_1 w, u_2 w\}) \cup \{h\}$ (i.e., H^+ is obtained from H by suppressing the vertex w), and we set $B(H^+) = B(H) \setminus \{u_1, u_2\}$ (i.e., u_1, u_2 are white in H^+, whatever was their color in H);
- (iii) otherwise we set for $i = 1, 2$:
  - (α) if e_i is pendant, then u_i is the root of e_i,
  - (β) if e_i is not pendant, then u_i is a new vertex subdividing e_i,
  - (γ) $H^+ = H + h$, where $h = u_1 u_2$,
  - (δ) $B(H^+) = B(H) \setminus \{u_1, u_2\}$.

  Moreover, if some e_i ($i \in \{1, 2\}$) is black in H, say, $e_i = a_i b_i$, where $\langle \{a_i, b_i, c_i\} \rangle_H \in B_T(H)$, we color the edges $c_i a_i$, $c_i b_i$ white in H^+ (since $\langle \{a_i, b_i, c_i\} \rangle_{H^+}$ is no more a triangle), and we set $c_i \in B_V(H^+)$.

Note that:

- in case (ii), if $f = u_1 u_2 \in E(H)$, then $\langle \{u_1, u_2, w\} \rangle_H$ is a triangle in H (possibly even black), and then f and h are parallel edges in H^+,

- in case (iii), if both e_1 and e_2 are pendant with adjacent roots (i.e., $f = u_1u_2 \in E(H)$), then similarly f and h are parallel edges in H^+,
- if e_1, e_2 are parallel edges in H, then case (iii) applies, and H^+ contains a diamond.

Claim 6. *The graph H^+ has the following properties:*
(i) *if X is a triangle or a multiedge in H^+, then $h \in E(X)$,*
(ii) $d_{H^+}(u_i) \geq 3$ *for* $i = 1, 2$.
(iii) *H^+ has no black triangles,*
(iv) *$B(H^+) \cup \{u_1, u_2\}$ dominates all edges in H^+,*
(v) $\beta(H^+) \leq \beta(H) = \vartheta(G)$,
(vi) *if e_1, e_2 share a vertex of degree two in H, then $\beta(H^+) \leq \beta(H) - 1$,*
(vii) *H^+ has no DCT containing the edge h and all vertices in $B(H^+)$.*

Proof follows immediately from the construction of H^+ (see also Theorem 1(v)). □

Now we construct a graph H^+_R from H^+ by removing all pendant edges, suppressing all white vertices of degree two, and suppressing all black vertices of degree two but recoloring their neighbors in H^+ black in H^+_R.

Claim 7. *The graph H^+_R has the following properties:*
(i) $\delta(H^+_R) \geq 3$,
(ii) *H^+_R is essentially 3-edge-connected,*
(iii) *H^+_R has no DCT containing the edge h and all vertices in $B(H^+_R)$,*
(iv) $\beta(H^+_R) \leq 7$.

Proof of (i), (ii) and (iii) is immediate from the construction of H^+_R.

We prove (iv). From the construction of H^+_R we immediately have $\beta(H^+_R) = \beta(H^+) + \beta_2(H^+)$. If $\{e_1, e_2\} \neq \{x_iy_i^1, x_iy_i^2\}$ for every $x_i \in B_2(H)$, then immediately from the construction we have $\beta(H^+) \leq \beta(H)$, $\beta_2(H^+) = \beta_2(H)$ and $\beta(H^+_R) = \beta(H^+) + (\eta(H^+) - \beta_2(H^+))$. By Claim 4, $\beta(H^+_R) \leq \beta(H) + (\eta(H) - \beta_2(H)) \leq \vartheta(G^M) + (7 - \vartheta(G^M)) = 7$. If there is an $x_i \in B_2(H)$ such that $\{e_1, e_2\} = \{x_iy_i^1, x_iy_i^2\}$, then, by the construction, $\beta(H^+) \leq \beta(H) - 1$, $\beta_2(H^+) = \beta_2(H) - 1$ and $\beta(H^+_R) \leq \beta(H^+) + \beta_2(H^+)$. Using Claim 5 we further have $\beta(H^+_R) \leq (\beta(H) - 1) + (\beta_2(H) - 1) = \vartheta(G^M) + \beta_2(H) - 2 \leq \vartheta(G^M) + (9 - \vartheta(G^M)) - 2 = 7$. □

Since $\delta(H^+_R) \geq 3$ and H^+_R is essentially 3-edge-connected, we can construct a 3-connected cubic inflation H^I of the graph H^+_R. In the graph H^I we define black vertices as follows:

- vertices of degree three in H^+_R have the same color in H^I as in H^+_R,
- a white vertex x of degree at least four in H^+_R corresponds to a white cycle C_x in H^I (i.e., all vertices on C_x are white),
- a black vertex x of degree at least four in H^+_R corresponds to a cycle C_x in which one arbitrary vertex is black and all other vertices are white.

It is obvious that $\beta(H^I) = \beta(H_R^+) \leq 7$. By Proposition J, there exists a cycle in H^I containing the edge h and all vertices in $B(H^I)$. But then contracting all cycles C_x we obtain a DCT in H_R^+ containing the edge h and all vertices in $B(H_R^+)$, which contradicts Claim 7(iii). ■

5 Degree conditions for Hamilton-connectedness

In this section we prove a σ_8-condition and, as a corollary, a minimum degree condition for Hamilton-connectedness in 3-connected claw-free graphs. The best known result in this direction is by MingChu Li [13] who proved that every 3-connected claw free graph G with $\delta(G) \geq \frac{n+8}{5}$ is Hamilton-connected. We improve this result by showing that a 3-connected claw-free graph such that $\delta(G) \geq 24$ and $\sigma_8(G) \geq n + 50$ (or, as a corollary, $n \geq 142$ and $\delta(G) \geq \frac{n+50}{8}$) is Hamilton-connected. We also show that our results are asymptotically sharp. We start with some useful lemmas.

The following three lemmas were originally proved in [5] for closed graphs; we will prove here their analogues for SM-closed graphs.

Lemma 5. *Let G be an SM-closed graph and let $A = \{a_1, ..., a_t\} \subset V(G)$ be an independent set. Then:*

(i) $|N(a_i) \cap N(a_j)| \leq 2$ *for all* $i, j \in \{1, ..., t\}$ *except possibly for one pair* a_{i_0}, a_{j_0}, *for which* $|N(a_{i_0}) \cap N(a_{j_0})| \leq 3$,

(ii) $\sum_{i=1}^{t} d(a_i) \leq n + t^2 - 2t + 1$.

Proof. Let G be SM-closed, $A = \{a_1, ..., a_t\} \subset V(G)$ independent, let $\mathcal{K}$ be a Krausz partition of G, and set $H = L^{-1}(G)$.

We first show that any two vertices $a_i, a_j \in A$ can have at most three common neighbors. Assume, to the contrary, that some $a_i, a_j \in A$ have four common neighbors $z_1, ..., z_4$. By the properties of $\mathcal{K}$, the edges a_iz_1, a_iz_2, a_iz_3, a_iz_4 (and, symmetrically, also the edges a_jz_1, a_jz_2, a_jz_3, a_jz_4) can be covered by at most two cliques from $\mathcal{K}$ (see Theorem B).

Suppose first that, say, $\langle\{a_i, z_1, z_2, z_3\}\rangle_G$ is a clique. Then, symmetrically, either $\langle\{a_j, z_1, z_2, z_3\}\rangle_G$ is a clique, or both $\langle\{a_j, z_1, z_2\}\rangle_G$ and $\langle\{a_j, z_3\}\rangle_G$ are cliques. But then in the first case H contains a multiedge with multiplicity three and in the second case H contains a multitriangle, both contradicting Proposition G. By symmetry, we conclude that there is no clique that covers any of a_i, a_j with any three of $z_1, ..., z_4$.

Thus, by symmetry, we can suppose that $\langle\{a_i, z_1, z_2, \}\rangle_G$ and $\langle\{a_i, z_3, z_4\}\rangle_G$ are cliques. Then either $\langle\{a_j, z_1, z_2\}\rangle_G$ and $\langle\{a_j, z_3, z_4\}\rangle_G$ are cliques, implying H contains two multiedges, or $\langle\{a_j, z_1, z_3\}\rangle_G$ and $\langle\{a_j, z_2, z_4\}\rangle_G$ are cliques, implying H contains three triangles. In both cases, we have a contradiction with Theorem 1.

To finish the proof of (i), it suffices to show that if two vertices a_i, a_j have three common neighbors, then some two of them that are connected by an edge correspond to a multiedge in H (since then, by Theorem 1, there can be only one such pair of vertices in G). So, let z_1, z_2, z_3 be common neighbors of a_i, a_j. If $z_1z_2 \in E(G)$ and $\langle\{a_i, z_1, z_2\}\rangle_G$ and $\langle\{a_j, z_1, z_2\}\rangle_G$ are cliques, then in H we have a multiedge and we are done. Thus, every edge between z_1, z_2, z_3 is in exactly one clique. Then, up to a symmetry, the only possible partition is $\langle\{a_i, z_1, z_2\}\rangle_G$, $\langle\{a_i, z_3\}\rangle_G$, $\langle\{a_j, z_1\}\rangle_G$, $\langle\{a_j, z_2, z_3, \}\rangle_G$, but then we have a diamond in H, a contradiction.

To prove (ii), we observe that, by (i), $\sum_{i=1}^{t} d(a_i) \leq (n-t)+2\binom{t}{2}+1 = n+t^2-2t+1$. ■

Lemma 6. *Let G be an SM-closed graph and let $H = L^{-1}(G)$. If $\nu(H) < \tau(H)$, then there is an edge $xy \in E(H)$ such that $d(x) + d(y) \leq \nu(H) + \tau(H) + 2$.*

Proof. Let $T \subset V(H)$ be a minimum vertex cover and let M be a maximum matching such that $|V(M) \cap T|$ is smallest possible. Note that $V(H) \setminus T$ is independent since T is a vertex cover and $V(H) \setminus V(M)$ is also independent since M is maximal.

We first show that there is a vertex $x \in T$ such that $N(x) \subset V(M)$. If $T \not\subset V(M)$, we choose an $x \in T \setminus V(M)$ and then clearly $N(x) \subset V(M)$ for otherwise we can extend M. Thus let $T \subset V(M)$. Since $\nu(H) < \tau(H)$, there is an edge $xx' \in M$ such that $x, x' \in T$. Then for any $w \in N(x)$ either $w \in T$ and then $w \in V(M)$ since $T \subset V(M)$, or $w \notin T$ and then also $w \in M$ for otherwise we can modify M by replacing in M the edge xx' with the edge xw and lower $|V(M) \cap T|$, contradicting the choice of M. So, in both cases we have $N(x) \subset V(M)$.

If H contains a multiedge, then x has at least $d(x)-1$ neighbors in H and no edge from M has both ends in $N(x)$, from which $d(x) - 1 \leq |M| = \nu(H)$ and $d(x) \leq \nu(H)+1$. If G contains no multiedge, then x is in at most two triangles, hence x has $d(x)$ neighbors and at most two edges from M have both ends in $N(x)$, from which $d(x) - 2 \leq |M| = \nu(H)$ and $d(x) \leq \nu(H) + 2$. So we have $d(x) \leq \nu(H) + 1$ if H contains a multiedge and $d(x) \leq \nu(H) + 2$ otherwise.

Since $x \in T$ and T is minimum, there is a $y \in N(x)$ which is not in T (otherwise x can be removed from T). But since T is a vertex cover, all neighbors of y are in T. Now, if G contains a multiedge, then y has at least $d(y) - 1$ neighbors, hence $d(y) \leq \tau(H) + 1$ and $d(x)+d(y) \leq (\nu(H)+1)+(\tau(H)+1) = \nu(H)+\tau(H)+2$; if G contains no multiedge, then y has $d(y)$ neighbors and $d(y) \leq \tau(H)$, from which also $d(x)+d(y) \leq (\nu(H)+2)+\tau(H) = \nu(H) + \tau(H) + 2$. ■

Lemma 7. *Let G be an SM-closed graph and let $\alpha(G) < \vartheta(G)$. Then $\delta(G) \leq \alpha(G) + \vartheta(G) + 2$.*

Proof. Let $H = L^{-1}(G)$. We first show that $\vartheta(G) \leq \tau(H) \leq \vartheta(G) + 2$.

First, assume to the contrary that $\tau(H) > \vartheta(G)$ and let $\{b_1, ..., b_t\}$ be a vertex cover in H with $t = \tau(H)$ vertices. Then the system of stars with centers in $\{b_1, ..., b_t\}$ determines in G a clique covering with $t = \tau(H) < \vartheta(G)$ cliques, a contradiction. Hence $\tau(H) \geq \vartheta(G)$.

Now we show that $\tau(H) \leq \vartheta(G) + 2$. Let $K_1, ..., K_S \subset G$ be cliques in G and let $H_1, ..., H_S \subset H$ be their preimages (i.e., $L(H_i) = K_i$), and choose $K_1, ..., K_S$ such that the number of triangles among the graphs H_i is smallest possible. Since H has at most two triangles, at most two H_i, say, H_1 and H_2, are triangles. Let $V(H_1) = \{u, v, w\}$, and let H_1', H_1'' denote the stars with centers at u and v. Then the system $\{H_1', H_1'', H_2, ..., H_S\}$ does not contain the triangle H_1 and corresponds to a clique covering of G with at most $S+1$ cliques. If H_2 is a triangle, we proceed analogously. By this construction, we get a vertex cover of H such that the corresponding clique covering of G has at most $S+2$ cliques. Hence $\tau(H) \leq \vartheta(G) + 2$.

Now, since $\alpha(G) < \vartheta(G)$, we have $\nu(H) = \alpha(G) < \vartheta(G) \leq \tau(H)$. Therefore $\nu(H) < \tau(H)$ and, by Lemma 6, there is an edge $xy \in E(H)$ such that $d(x) + d(y) \leq \nu(H) + \tau(H) + 2$. Let $u \in V(G)$ be the vertex corresponding to xy. Then $d(u) = d(x) + d(y) - 2 \leq (\nu(H) + \tau(H) + 2) - 2 = \nu(H) + \tau(H) \leq \alpha(G) + \vartheta(G) + 2$. ■

Lemma 8. *Let G be an SM-closed graph. Then $\vartheta(G) \leq 2\alpha(G)$.*

Proof. In a line graph, the neighborhood of every vertex can be covered by at most two cliques, and since any maximal independent set is also dominating, any line graph can be covered by at most $2\alpha(G)$ cliques. ■

Proposition 9. *Let G be a claw-free graph, let G^M be an SM-closure of G and let $k \geq 2$ be an integer such that $\delta(G) \geq 3k$ and $\sigma_k(G) \geq n + k^2 - 2k + 2$. Then $\vartheta(G^M) \leq k-1$.*

Proof. Clearly, if G satisfies the assumptions, then so does G^M, hence we can assume that G is SM-closed. Let, to the contrary, $\vartheta(G) \geq k$. If $\alpha(G) \geq k$, then G contains an independent set of size k and, by Lemma 5, we have $\sigma_k(G) \leq n + k^2 - 2k + 1$, a contradiction. If $\alpha(G) \leq k - 1$, then $\alpha(G) < \vartheta(G)$ and by Lemma 7 and Lemma 8 we have $\delta(G) \leq \alpha(G) + \vartheta(G) + 2 \leq (k-1) + 2(k-1) + 2 = 3k - 1$, contradicting the assumption $\delta(G) \geq 3k$. ■

Theorem 10. *Let G be a 3-connected claw-free graph such that $\delta(G) \geq 24$ and $\sigma_8(G) \geq n + 50$. Then G is Hamilton-connected.*

Proof. Clearly, if G satisfies the assumptions of the theorem, then so does G^M, hence we can assume that G is SM-closed. Then G satisfies the assumptions of Proposition 9 with $k = 8$, hence $\vartheta(G) \leq 7$. By Theorem 2, G is Hamilton-connected. ■

Corollary 11. *Let G be a 3-connected claw-free graph of order $n \geq 142$ and minimum degree $\delta(G) \geq \frac{n+50}{8}$. Then G is Hamilton-connected.*

Proof. Under the assumptions of the corollary, $\delta(G) \geq \frac{n+50}{8} \geq \frac{142+50}{8} = 24$ and $\sigma_8(G) \geq 8 \cdot \delta(G) \geq n + 50$, hence G is Hamilton-connected by Theorem 10. ■

Example. Let H_ℓ be a copy of the graph H_2 of Fig. 4 in which there are ℓ pendant edges attached to every vertex, and let $G_\ell = L(H_\ell)$. For every vertex u_i choose a neighbor v_i of degree one and let $w_i \in V(G_\ell)$ be the vertex corresponding to the edge $u_iv_i \in E(H_\ell)$, $i = 1, ..., 8$. Then $\delta(G_\ell) = d_{G_\ell}(w_i) = d_{H_\ell}(v_i) + d_{H_\ell}(u_i) - 2 = 1 + (\ell + 3) - 2 = \ell + 2$, thus $\ell = \delta(G_\ell) - 2$. Since $n = |V(G_\ell)| = |E(H_\ell)| = 8\ell + 12 = 8(\delta(G_\ell) - 2) + 12 = 8\delta(G_\ell) - 4$, we have $\delta(G_\ell) = \frac{n+4}{8}$. Moreover, $\{w_1, ..., w_8\}$ is an independent set in G_ℓ and hence $\sigma_8(G_\ell) = \sum_{i=1}^{8} d(w_i) = 8 \cdot \frac{n+4}{8} = n + 4$. However, the graph G_ℓ is not Hamilton-connected. Therefore Theorem 10 and Corollary 11 are asymptotically sharp.

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