Distance-locally disconnected graphs

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Abstract

For an integer $k \geq 1$, we say that a (finite simple undirected) graph G is k-distance-locally disconnected, or simply k-locally disconnected if, for any $x \in V(G)$, the set of vertices at distance at least 1 and at most k from x induces in G a disconnected graph. In this paper we study the asymptotic behavior of the number of edges of a k-locally disconnected graph on n vertices. For general graphs, we show that this number is $\Theta(n^2)$ for any fixed value of k and, in the special case of regular graphs, we show that this asymptotic rate of growth cannot be achieved. For regular graphs, we give a general upper bound and we show its asymptotic sharpness for some values of k. We also discuss some connections with cages.

1 Introduction

In this paper, we consider simple finite undirected graphs $G = (V(G), E(G))$; for notations and terminology not defined here we refer the reader e.g., to [1]. Specifically, we use $\mathrm{dist}^G(x, y)$ to denote the *distance* of x and y in G and $\mathrm{diam}(G)$ to denote the *diameter* of G; $d^G(x)$ stands for the *degree* of a vertex x in G, $\Delta(G)$ for the maximum degree of G and $g(G)$ for the *girth* (i.e., the length of a shortest cycle) of G. We use $H \subset G$ to denote that H is a subgraph of G and, for a set $M \subset V(G)$, we use $\langle M \rangle_G$ to denote the *induced subgraph* of G on M. A path with terminal vertices u, v will be referred to as a (u, v)-path. If $x \in V(G)$ is a cutvertex of G and B is a component of $G - x$ then the subgraph $\langle V(B) \cup \{x\} \rangle_G$ is called the *branch* of G at x (corresponding to B).

Let $f(n)$, $g(n)$ be two positive functions defined on the set of positive integers. We say that $f(n)$ is $O(g(n))$, denoted $f(n) \in O(g(n))$, if there are constants $K \geq 0$ and

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$n_0 \geq 0$ such that $f(n) \leq Kg(n)$, for every $n \geq n_0$. Similarly, $f(n)$ is $\Omega(g(n))$, denoted $f(n) \in \Omega(g(n))$, if there are constants $K' \geq 0$ and $n_0' \geq 0$ such that $f(n) \geq K'g(n)$ for every $n \geq n_0'$, and $f(n)$ is $\Theta(g(n))$, denoted $f(n) \in \Theta(g(n))$, if both $f(n) \in O(g(n))$ and $f(n) \in \Omega(g(n))$.

The *neighborhood* of a vertex x in G is the set $N^G(x) = \{y \in V(G) |\ xy \in E(G)\}$ and, more generally, for an integer $k \geq 1$, the set $N_k^G(x) = \{y \in V(G) |\ \mathrm{dist}^G(x, y) = k\}$ is called the *neighborhood at distance* k and the set $N_{[k]}^G(x) = \{y \in V(G) |\ 1 \leq \mathrm{dist}^G(x, y) \leq k\}$ is called the *neighborhood at distance at most* k (or simply the k*-neighborhood*) of x in G (thus, $N^G(x) = N_1^G(x) = N_{[1]}^G(x)$ and $N_{[k]}^G(x) = \cup_{j=1}^k N_j^G(x)$). We will also use the *closed neighborhood* and *closed* k*-neighborhood* (of x in G) defined as $N^G[x] = N^G(x) \cup \{x\}$ and $N_{[k]}^G[x] = N_{[k]}^G(x) \cup \{x\}$, respectively.

Finally, a graph G is *locally disconnected* if $\langle N^G(x)\rangle_G$ is a disconnected graph for every $x \in V(G)$ and, more generally, for $k \geq 1$, G is k*-distance-locally disconnected*, or simply k*-locally disconnected* if $\langle N_{[k]}^G(x)\rangle_G$ is disconnected for every $x \in V(G)$.

The problem of determining the maximum number of edges of a locally disconnected graph was originally posed by Bohdan Zelinka in 1985. In [7], Zelinka showed that this number cannot be expressed as a linear function of n and determined its exact value in the special case of planar graphs. In [6], it was shown that, surprisingly, this number can be, in a sense, "arbitrarily close" to the number of edges of a complete graph (more precisely, for any $n \geq 4$, there is a locally disconnected graph G_n on n vertices such that $\lim_{n\to\infty} \frac{|E(G_n)|}{\binom{n}{2}} = 1$). In [5], a similar question was studied in the case of edge-induced vertex neighborhoods.

In the present paper, we will study the asymptotic behavior of the number of edges of a k-locally disconnected graph for $k \geq 2$. In Section 2, we will see that this maximum number is, for $k \geq 2$, of asymptotic order $\frac{n^2}{2k}$, i.e., asymptotically strictly less than $\binom{n}{2}$, but still $\Theta(n^2)$ for any fixed value of k, while in Section 3 we show that under the restriction to regular graphs the $\Theta(n^2)$ growth rate is not possible. For regular graphs we give a general upper bound and, for some values of k, we show its asymptotic sharpness. We also discuss some connections with cages.

2 Maximal k-locally disconnected graphs

It is easy to observe that, for any integers $k \geq 1$ and $n \geq 2k + 2$, there is a k-locally disconnected graph of order n (a cycle is an easy example). Thus, for $k \geq 1$ and $n \geq 2k+2$, we can define

$$\mathrm{ld}_k(n) = \max\{|E(G)| \mid G \text{ is } k\text{-locally disconnected}, |V(G)| = n\}.$$

We will also say that a k-locally disconnected graph G with $|V(G)| = n$ and $|E(G)| = \mathrm{ld}_k(n)$ is *maximal*. Note that any k-locally disconnected graph is also $(k-1)$-locally disconnected, hence, for any $k \geq 2$ and $n \geq 2k + 2$, we have

$$\mathrm{ld}_{k-1}(n) \geq \mathrm{ld}_k(n).$$

We begin with several structural observations.

Proposition 1. *Let G be a k-locally disconnected graph. Then* $\mathrm{diam}(G) \geq k+1$.

Proof. Suppose, to the contrary, that G is k-locally disconnected and $\mathrm{diam}(G) \leq k$, and let $x, y \in V(G)$ be such that $\mathrm{dist}^G(x,y) = \mathrm{diam}(G)$. Since $\mathrm{diam}(G) \leq k$, all vertices of G are at distance at most k from x, implying $\langle N_{[k]}(x)\rangle_G = G - x$. As G is k-locally disconnected, $G - x$ is disconnected, i.e., x is a cutvertex of G. But now, for a vertex z in the component $G - x$ not containing y, we have $\mathrm{dist}^G(z,y) = \mathrm{dist}^G(z,x) + \mathrm{dist}^G(x,y) > \mathrm{dist}^G(x,y)$, contradicting the assumption $\mathrm{dist}^G(x,y) = \mathrm{diam}(G)$. ∎

Note that e.g. the cycle C_{2k+2} is k-locally disconnected and $\mathrm{diam}(C_{2k+2}) = k+1$. Hence Proposition 1 is sharp.

Proposition 2. *Let G be a k-locally disconnected graph, and let $x \in V(G)$. Then every component of $\langle N^G_{[k]}(x)\rangle_G$ contains a vertex at distance k from x.*

Proof. Let, to the contrary, B be a component of $\langle N^G_{[k]}(x)\rangle_G$ with all vertices at distance at most $k-1$ from x. Then x is the only vertex in $G - B$ having a neighbor in B (for otherwise such a vertex would be at distance at most k from x, hence in B, contradicting its definition). Consequently, x is a cutvertex of G. Let B' be the branch of G at x corresponding to B and let $y \in V(B')$ be at maximum distance from x. Then all vertices in $G - B'$ that are at distance at most k from y are accessible from y only through x, hence all such vertices occur in the same component of $\langle N^G_{[k]}(y)\rangle_G$. As G is k-locally disconnected, $\langle N^{B'}_{[k]}(y)\rangle_{B'} = B' - y$ is disconnected, hence y is a cutvertex of B'. But then, similarly as before, for a vertex z in a component of $B' - y$ not containing x, we have $\mathrm{dist}^{B'}(z,y) = \mathrm{dist}^{B'}(z,x) + \mathrm{dist}^{B'}(x,y) > \mathrm{dist}^{B'}(x,y)$, contradicting the choice of y. ∎

We say that a k-locally disconnected graph is *critical* if, for any pair of nonadjacent vertices $x, y \in V(G)$, the graph $G + xy$ is not k-locally disconnected. Obviously, every maximal k-locally disconnected graph is also critical.

Theorem 3. *Let G be a critical k-locally disconnected graph. Then G is 2-connected.*

Proof. Suppose, to the contrary, that G is critical k-locally disconnected and x is a cutvertex of G. Let B be a branch of G at x and let $y \in V(B)$ be a vertex at maximum distance from x. Observe that all vertices in other branches of G at x are accessible from y only through x, hence those of them that are at distance at most k from y must occur in one component of $\langle N^G_{[k]}(y)\rangle_G$. Thus, if $\mathrm{dist}^G(y,x) \leq k$, all vertices of B are in $N^G_{[k]}(x)$, hence $\langle N^B_{[k]}(x)\rangle_B = B - y$ is disconnected, implying y is a cutvertex of B. But this, similarly as before, contradicts the maximality of $\mathrm{dist}^G(x,y)$.

Thus, every branch of G at x contains a vertex at distance at least $k+1$ from x. Let y_1, y_2 be two such vertices in different branches. Then the graph $G + y_1y_2$ is also k-locally disconnected, contradicting the criticality of G. ∎

Note that the graph G consisting of two cycles of length $2k+2$ sharing a vertex shows that the criticality assumption in Theorem 3 is essential.

The following technical proposition is crucial for the main result of this section, Theorem 6.

Proposition 4. *Let G be a k-locally disconnected graph, $k \geq 2$, $x \in V(G)$, $d^G(x) = d$. Then there are vertices $x_i^\ell \in V(G)$, $i = 1, \dots, d$, $\ell = 1, \dots, k$, such that*

(i) $\{x_1^\ell, \dots, x_d^\ell\} \subset N_\ell(x)$, $\ell = 1, \dots, k$;

(ii) $x, x_i^1, \dots, x_i^k$ *is an induced path in* G, $i = 1, \dots, d$;

(iii) for any i_1, i_2 *and* ℓ_1, ℓ_2, $1 \leq i_1, i_2 \leq d$, $i_1 \neq i_2$, $1 \leq \ell_1, \ell_2 \leq k$, *the vertices* $x_{i_1}^{\ell_1}$ *and* $x_{i_2}^{\ell_2}$ *are distinct and for* $\max\{\ell_1, \ell_2\} \geq 2$ *nonadjacent.*

Proof. Let $\{x_1^1, \dots, x_d^1\} = N_1(x)$, and consider $\langle N_{[k]}^G(x_1^1)\rangle_G$. As the vertex x and the vertices $x_2^1, \dots, x_d^1$ are at distance at most 2 from x_1^1, they are in one component of $\langle N_{[k]}^G(x_1^1)\rangle_G$. By Proposition 2, there are vertices $x_1^2, \dots, x_1^{k+1}$ in another component such that $\mathrm{dist}(x_1^1, x_1^{\ell+1}) = \ell$, $\ell = 1, \dots, k$ (for $d = 4$ and $k = 5$, see Fig. 1; note that some of the edges of the form $x_i^1 x_j^1$, $1 \leq i, j \leq d$, are possible in G).

By induction, for $i = 2, \dots, d$, some component of $\langle N_{[k]}^G(x_i^1)\rangle_G$ contains all of $N_1[x] \setminus \{x_i^1\}$ and all the vertices x_j^ℓ for $1 \leq j \leq i$ and $2 \leq \ell \leq k-1$, hence there are vertices $x_i^2, \dots, x_i^{k+1}$ in another component such that $\mathrm{dist}(x_i^1, x_i^{\ell+1}) = \ell$, $\ell = 1, \dots, k$. By the construction, it is straightforward to verify that the vertices x_i^ℓ, $i = 1, \dots, d$, $\ell = 1, \dots, k$ have the required properties (i), (ii) and (iii). ■

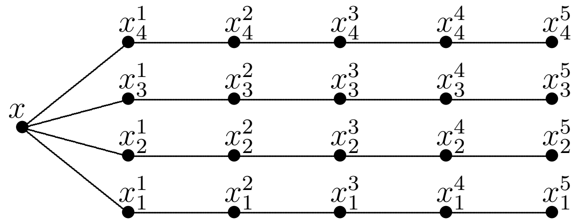

Figure 1

Theorem 5. *Let G be a k-locally disconnected graph of order n. Then*

$$\Delta(G) \leq \frac{n-2}{k}.$$

Proof. The statement is obvious for $k = 1$, thus let $k \geq 2$, and let $x \in V(G)$ be a vertex of degree $d = \Delta(G)$. By Proposition 4, $|N_\ell^G(x)| \geq d$ for $\ell = 1, \dots, k$ and clearly $|N_{k+1}(x)| \geq 1$. Hence we have $n \geq |N_{[k+1]}^G[x]| \geq 1 + \sum_{\ell=1}^k |N_\ell^G(x)| + 1 \geq kd + 2$, from which $d \leq \frac{n-2}{k}$. ■

Theorem 6. *Let G be a k-locally disconnected graph of order n. Then*

$$|E(G)| \leq \frac{1}{2k}(n^2 - 2n).$$

Proof. By Theorem 5, $|E(G)| = \frac{1}{2}\sum_{x\in V(G)} d^G(x) \leq \frac{1}{2}n\Delta(G) \leq \frac{1}{2k}n(n-2)$. ■

In Theorem 6 we have, for any $k \geq 1$, an $O(n^2)$ upper bound on the number of edges of a k-locally disconnected graph of order n. The next result shows that the quadratic growth rate is achievable.

Theorem 7. *Let $k \geq 1$ and $t \geq 2$ be integers and let $n = t(k+1)$. Then there is a k-locally disconnected graph G with n vertices and*

$$|E(G)| = \frac{1}{(k+1)^2}(n^2 + (k^2-1)n)$$

edges.

Proof. For given $k \geq 1$ and $t \geq 2$, let H_1, H_2 be two copies of the complete graph K_t and let G be the graph obtained by joining the vertices of H_1 to the vertices of H_2 with t vertex-disjoint paths of length k (for $t = 4$ and $k = 3$, see Fig. 2). Then clearly G is

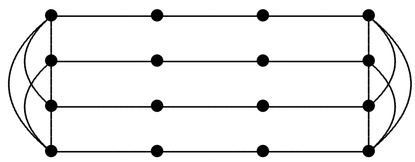

Figure 2

k-locally disconnected, $n = |V(G)| = (k+1)t$, and $|E(G)| = 2\binom{t}{2} + kt = t(t-1) + kt = t^2 + (k-1)t = (\frac{n}{k+1})^2 + (k-1)\frac{n}{k+1} = \frac{1}{(k+1)^2}(n^2 + (k^2-1)n)$, as required. ■

Now we are able to determine the asymptotic growth rate of the function $\mathrm{ld}_k(n)$.

Theorem 8. *For any fixed integer $k \geq 1$,*

$$\mathrm{ld}_k(n) \in \Theta(n^2).$$

Proof. We have $\mathrm{ld}_k(n) \in O(n^2)$ immediately by Theorem 6. To obtain $\mathrm{ld}_k(n) \in \Omega(n^2)$, we extend the construction from the proof of Theorem 7 in such a way that, for $n = t(k+1) + r$ with $1 \leq r \leq k$, we arbitrarily subdivide some of the t paths joining H_1 to H_2 with r vertices of degree 2. ■

3 Regular k-locally disconnected graphs

In the previous section we have seen that, for any fixed $k \geq 2$, the number of edges of a k-locally disconnected graph of order n can achieve the growth rate $\Theta(n^2)$. Here we will show that this is not possible under the additional restriction on G to be regular. Similarly to the general case, we set

$$\mathrm{ld}_k^R(n) = \max\{|E(G)| \mid G \text{ is regular and } k\text{-locally disconnected}, |V(G)| = n\}.$$

Furthermore, analogously, for any $k \geq 2$ and $n \geq 2k+2$ we have

$$\mathrm{ld}^R_{k-1}(n) \geq \mathrm{ld}^R_k(n).$$

We begin with a structural result that will be crucial for our proof of the main result of this section. Here, a *leaf* of a tree T is a vertex of degree 1 in T.

Proposition 9. *Let G be a k-locally disconnected ($k \geq 1$) d-regular graph and let $x \in V(G)$. Then G contains a tree T such that*

(i) *$V(T) \subset N^G_{[k]}[x]$;*

(ii) *all leaves of T are in $N^G_k(x)$;*

(iii) *for any $y \in V(T)$, $dist^T(x,y) = dist^G(x,y)$;*

(iv) *for any t, $1 \leq t \leq k$,*

$$|N^T_t(x)| \geq \begin{cases} d(d-1)^{\frac{t-1}{2}} & \text{for } t \text{ odd,} \\ d(d-1)^{\frac{t-2}{2}} & \text{for } t \text{ even.} \end{cases}$$

Proof. We construct a sequence of trees $\{T_t\}_{t=1}^k$ such that, for any t, $1 \leq t \leq k$, we have

(i) $V(T_t) \subset N^G_{[t]}[x]$;

(ii) all leaves of T_t are in $N^G_t(x)$;

(iii) for any $y \in V(T_t)$, $\mathrm{dist}^{T_t}(x,y) = dist^G(x,y)$;

(iv) $|N^{T_t}_t(x)| \geq \begin{cases} d(d-1)^{\frac{t-1}{2}} & \text{for } t \text{ odd,} \\ d(d-1)^{\frac{t-2}{2}} & \text{for } t \text{ even;} \end{cases}$

(v) for $t \geq 2$, $T_{t-1} \subset T_t$.

We proceed by induction on t.

1. For $t = 1$, we set $V(T_1) = N^G_1[x]$ and $E(T_1) = \{xy |\ y \in N^G_1(x)\}$. Then clearly T_1 satisfies $(i) - (iv)$ for $t = 1$.

2. Let $t \geq 2$ and suppose that we have already constructed a tree T_{t-1} satisfying $(i) - (v)$ with $t := t-1$ (hence also its subtrees $T_{t'}$ for all t', $1 \leq t' \leq t-1$). Note that, by the induction hypothesis, we have

$$|N^{T_{t-1}}_{t-1}(x)| \geq \begin{cases} d(d-1)^{\frac{t-3}{2}} & \text{for } t \text{ odd,} \\ d(d-1)^{\frac{t-2}{2}} & \text{for } t \text{ even} \end{cases}$$

(since $t-1$ is even/odd for t odd/even, respectively).

(a) Suppose first that t is even and let y be a leaf of T_{t-1}. Then all vertices in T_1 are at distance at most t from y and, since $t \leq k$, T_1 is (together with the (y,x)-path in T_{t-1}) a subgraph of one component of $\langle N^G_{[k]}(y)\rangle_G$. Thus, y has a neighbor y^+ that is in another component of $\langle N^G_{[k]}(y)\rangle_G$ (not containing T_1). Choose such a vertex y^+ for every leaf y of T_{t-1}. Then all these vertices are distinct and nonadjacent in G, for if e.g. $y_1^+ y_2^+ \in E(G)$ for some two leaves y_1, y_2 of T_{t-1}, then y_1, y_2 and all vertices of the (y_2,x)-path in T_{t-1} would be at distance at most $t \leq k$ from y_1, implying y_1^+ is in the same component of $\langle N^G_{[k]}(y_1)\rangle_G$ as T_1, a contradiction (the case $y_1 = y_2$ is similar). Thus, adding to T_{t-1} the vertices y^+ and the edges yy^+ for all leaves y of T_{t-1}, we obtain the desired tree T_t (note that if t is even then the lower bound (iv) is the same for t and for $t-1$).

(b) If t is odd (implying $t \geq 3$), we construct a desired tree T_t by attaching to every leaf of T_{t-2} a tree T_y rooted at y such that T_y contains the edge yy^+ (where y^+ is the leaf of T_{t-1} defined in the previous step) and T_y has $d-1$ leaves at distance 2 from y (hence at distance t from x).

Thus, let y be a leaf of T_{t-2} and y^+ the corresponding leaf of T_{t-1}. Since G is d-regular, y^+ has, besides y, $d-1$ other neighbors $y'_1, \ldots, y'_{d-1}$. Choose the notation such that, for some s, $0 \leq s \leq d-1$, we have $yy'_i \notin E(G)$ for $1 \leq i \leq s$ and $yy'_i \in E(G)$ for $s+1 \leq i \leq d-1$. First observe that all y'_i, $i = 1, \ldots, s$, are at distance t from x (in G), for, if some y'_{i_0} is at distance at most $t-1$ from x, then, since $yy'_i \notin E(G)$, there is a path from y'_{i_0} to some vertex in T_1 of length at most $t-2$ avoiding y, hence y^+ is in the same component of $\langle N^G_{[k]}(y)\rangle_G$ as T_1, a contradiction.

Now let $s+1 \leq i \leq d-1$. Then y'_i is adjacent to both y and y^+, implying $\mathrm{dist}^G(x, y'_i) \leq t-1$. Similarly as before, $\mathrm{dist}^G(x, y'_i) = t-1$ and y'_i is nonadjacent to any vertex in T_{t-1}, for otherwise y^+ is in the same component of $\langle N^G_{[k]}(y)\rangle_G$ as T_1, a contradiction. But now T_1, all vertices of the (y^+, x)-path in T_{t-1}, all vertices y'_j for $j \neq i$, and all their neighbors are in the same component of $\langle N^G_{[k]}(y'_i)\rangle_G$. Thus, y'_i has a neighbor y''_i in another component of $\langle N^G_{[k]}(y'_i)\rangle_G$, and clearly y''_i is at distance t from x. By their definition, all the vertices y''_i, $i = s+1, \ldots, d-1$, are distinct and nonadjacent.

Now, we define T_y as the tree containing the vertices $y, y^+, y'_1, \ldots, y'_{d-1}$ and $y''_{s+1}, \ldots, y''_{d-1}$, the edge yy^+, the edges $y^+y'_i$ for $1 \leq i \leq s$, and the paths $yy'_iy''_i$ for $s+1 \leq i \leq d-1$. Then T_y has $d-1$ leaves $y'_1, \ldots, y'_s, y''_{s+1}, \ldots, y''_{d-1}$ at distance 2 from y (see Fig. 3, where the edges of the tree T_y appear as thick lines).

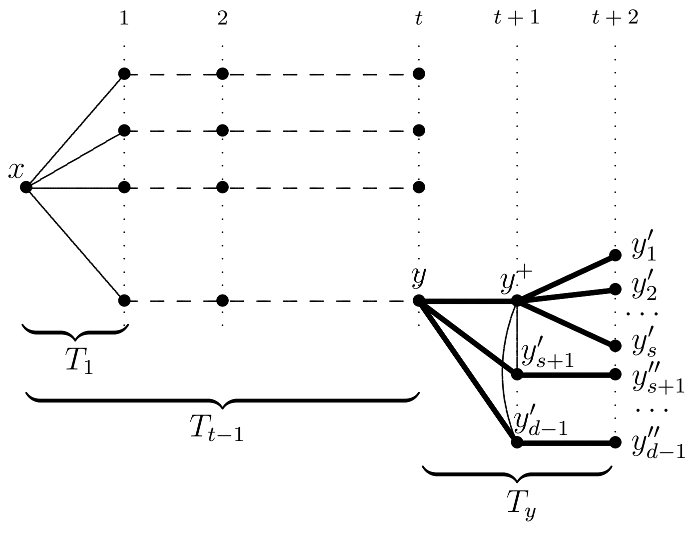

Figure 3

Now it is again straightforward to verify that if y, v are two leaves of T_{t-2} and T_y, T_v are the corresponding trees, then all vertices of T_y and T_v are distinct and nonadjacent, for otherwise we have a contradiction with the definition of y^+, v^+ or of some of y''_i or v''_i. Thus, for the tree T_t, obtained from T_{t-2} by attaching T_y to y for any leaf y of T_{t-2}, we have

$$|N^{T_t}_t(x)| \geq |N^{T_{t-2}}_{t-2}(x)|(d-1) \geq d(d-1)^{\frac{t-3}{2}}(d-1) = d(d-1)^{\frac{t-1}{2}},$$

as requested. ■

The following result is a counterpart to Theorem 5 for regular graphs.

Theorem 10. *Let $k \geq 1$ be odd and let G be a d-regular k-locally disconnected graph of order n. Then*

$$d \leq n^{\frac{2}{k+1}} + 1.$$

Proof. Choose a vertex $x \in V(G)$ and let T be the tree given in Proposition 9. Then we have

$n \geq |V(T)| + 1 = 1 + \sum_{i=1}^{k} |N_i^T(x)| + 1 \geq$
$1 + d + d + d(d-1) + d(d-1) + \ldots + d(d-1)^{\frac{k-3}{2}} + d(d-1)^{\frac{k-3}{2}} + d(d-1)^{\frac{k-1}{2}} + 1 =$
$2 + 2d[1 + (d-1) + \ldots + (d-1)^{\frac{k-3}{2}}] + d(d-1)^{\frac{k-1}{2}} = 2 + 2d\frac{(d-1)^{\frac{k-1}{2}}-1}{(d-1)-1} + d(d-1)^{\frac{k-1}{2}} =$
$2 + 2\frac{d}{d-2}[(d-1)^{\frac{k-1}{2}} - 1] + d(d-1)^{\frac{k-1}{2}} \geq 2 + 2[(d-1)^{\frac{k-1}{2}} - 1] + d(d-1)^{\frac{k-1}{2}} =$
$(d+2)(d-1)^{\frac{k-1}{2}} \geq (d-1)^{\frac{k+1}{2}}.$

Thus, we have $n \geq (d-1)^{\frac{k+1}{2}}$, from which $d \leq n^{\frac{2}{k+1}} + 1$. ■

Now we are able to give an upper bound on the function $\mathrm{ld}^R(n)$. We will show that, unlike in the general case (cf. Theorem 8), $\Theta(n^2)$ growth rate is not possible in the case of regular graphs.

Theorem 11. *Let G be a regular k-locally disconnected graph of order n. Then*

$$|E(G)| \leq \begin{cases} \frac{n}{2}(1 + n^{\frac{2}{k+1}}) & \text{for } k \text{ odd,} \\ \frac{n}{2}(1 + n^{\frac{2}{k}}) & \text{for } k \text{ even.} \end{cases}$$

Proof. If k is even, then $k-1$ is odd and the upper bound for $k-1$ equals the upper bound for k. Since $\mathrm{ld}_{k-1}^R(n) \geq \mathrm{ld}_k^R(n)$, it is sufficient to prove the bound for k odd. If G is d-regular, then $d \leq n^{\frac{2}{k+1}} + 1$ by Theorem 10. From this we have

$|E(G)| = \frac{1}{2} \sum_{x \in V(G)} d^G(x) = \frac{1}{2}nd \leq \frac{1}{2}(n^{1+\frac{2}{k+1}} + n) = \frac{n}{2}(1 + n^{\frac{2}{k+1}}),$

as requested. ■

Corollary 12. *For any fixed integer $k \geq 1$,*

$$\mathrm{ld}_k^R(n) \in \begin{cases} O(n^{1+\frac{2}{k+1}}) & \text{for } k \text{ odd,} \\ O(n^{1+\frac{2}{k}}) & \text{for } k \text{ even.} \end{cases}$$

Proof follows immediately from Theorem 11. ■

Specifically, we have:

$$\mathrm{ld}_k^R(n) \in \begin{cases} O(n^2) & \text{for } k = 1, 2, \\ O(n^{\frac{3}{2}}) & \text{for } k = 3, 4, \\ O(n^{\frac{4}{3}}) & \text{for } k = 5, 6, \\ O(n^{\frac{5}{4}}) & \text{for } k = 7, 8, \end{cases}$$

etc. We finish with examples of infinite families of regular locally disconnected graphs showing that, for $1 \le k \le 5$, $k = 7$ and $k = 11$ these asymptotic growth rates can really be achieved. We do not know similar constructions for $k \ge 12$; for these values of k we only give some general observations.

Since $\mathrm{ld}_1^R(n) \ge \mathrm{ld}_2^R(n)$ and $\mathrm{ld}_3^R(n) \ge \mathrm{ld}_4^R(n)$, it is not necessary to give the constructions for $k = 1, 3$; constructions for $k = 2, 4$ are sufficient.

Example 1: $k = 2$**.** Let H_0, H_1, H_2 be three copies of the complete bipartite graph $K_{t,t}$, $t \ge 2$, with vertices colored black and white, and let G be the graph obtained by joining black vertices in H_i to white vertices in H_{i+1} with a matching, $i = 0, 1, 2$ (indices modulo 3). For $t = 3$, see Fig. 4. Then $n = |V(G)| = 6t$, i.e., $t = \frac{n}{6}$, and

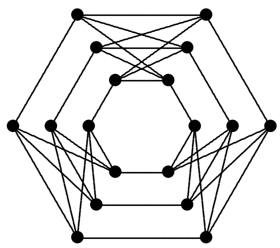

Figure 4

$|E(G)| = 3t^2 + 3t = 3(\frac{n}{6})^2 + 3\frac{n}{6}$, i.e. $|E(G)| = \frac{1}{12}(n^2 + 6n)$. Thus $|E(G)| \in \Omega(n^2)$.

Example 2: $k = 4$**.** Let $t \ge 2$ and let $H_{i,j}$, $i = 0, \ldots, 4$, $j = 0, \ldots, t-1$, be $5t$ copies of the graph $K_{t,t}$ and let $w_{i,j}^0, \ldots, w_{i,j}^{t-1}$ ($b_{i,j}^0, \ldots, b_{i,j}^{t-1}$) denote the white (black) vertices of $H_{i,j}$, respectively. The graph G is obtained by joining $b_{i,j}^k$ to $w_{i+1,k}^j$ for all $j, k = 0, \ldots, t-1$ and $i = 0, \ldots, 4$ (index i modulo 5). For $t = 2$, see Fig. 5. Then $n = |V(G)| = 5t \cdot 2t = 10t^2$,

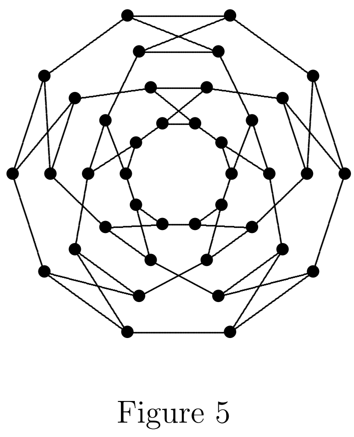

Figure 5

i.e., $t = \sqrt{\frac{n}{10}}$, and $|E(G)| = 5t \cdot t^2 + 5t^2 = 5t^3 + 5t^2 = \frac{1}{2\sqrt{10}}n^{\frac{3}{2}} + \frac{1}{2}n$, hence we have $|E(G)| \in \Omega(n^{\frac{3}{2}})$.

For our next examples, we will need some definitions and observations. Given integers d, g, a (d,g)-*graph* is a d-regular graph of girth g, and a (d,g)-graph of minimum order (number of vertices) is called a (d,g)-*cage*. For a survey paper on cages, see [2]. Since a graph of girth g is clearly $(\lfloor \frac{g}{2} \rfloor - 1)$-locally disconnected and cages are such graphs of minimum order, cages are candidates for "good" locally disconnected graphs. While it can be seen that cages themselves are not dense enough to provide a sharpness example, we show that they can be used as "building blocks" for such a construction.

An *inflation* of a graph H is the graph G obtained from H by

(i) replacing each vertex $x \in V(H)$ with a clique K_x of order $d^H(x)$; and

(ii) replacing each edge $xy \in E(H)$ with an edge joining a vertex in K_x to a vertex in K_y in such a way that the edges of G corresponding to edges of H form a perfect matching in G.

Obviously, an inflation of a (d,g)-graph (hence also of a (d,g)-cage) is a d-regular $(g-1)$-locally disconnected graph. We will use known families of cages of girths $g = 6, 8$ and 12 to construct examples showing asymptotic sharpness for $k = 5, 7$ and 11.

Example 3: $k = 5$**.** The incidence graph of a projective plane of order q is a cage of degree $d' = q+1$ and girth $g = 6$, has $n' = 2(q^2+q+1)$ vertices (see [2], Section 2.2.1), and its inflation is a 5-locally disconnected graph. Let G be such an inflation. Then G is d-regular with $d = d' = q+1$ and has $n = |V(G)| = dn' = 2(q+1)(q^2+q+1) \leq 2(q+1)^3$ vertices, from which $q+1 \geq (\frac{n}{2})^{\frac{1}{3}}$. Since G is d-regular, we have $|E(G)| = \frac{1}{2}\sum_{x \in V(G)} d^G(x) = \frac{1}{2}dn = \frac{1}{2}(q+1)n \geq \frac{1}{2\sqrt[3]{2}}n^{\frac{4}{3}}$.

Example 4: $k = 7$**.** The incidence graph of a generalized quadrangle of order (q,q) is a cage of degree $d' = q+1$ and girth $g = 8$, and has $n' = 2(q+1)(q^2+1)$ vertices (see [2], Section 2.2.2). Let G be its inflation. Then G is 7-locally disconnected, d-regular with $d = d' = q+1$, and has $n = dn' = 2(q+1)^2(q^2+1) \leq 2(q+1)^4$ vertices, from which $q+1 \geq (\frac{n}{2})^{\frac{1}{4}}$. As G is d-regular, we have $|E(G)| = \frac{1}{2}dn = \frac{1}{2}(q+1)n \geq \frac{1}{2\sqrt[4]{2}}n^{\frac{5}{4}}$.

Example 5: $k = 11$**.** Similarly, the incidence graph of a generalized hexagon of order (q,q) is a cage of degree $d' = q+1$ and girth $g = 12$, and has $n' = 2(q^3+1)(q^2+q+1)$ vertices (see [2], Section 2.2.3). Again, its inflation G is 11-locally disconnected, d-regular with $d = d' = q+1$, and has $n = dn' = 2(q+1)(q^3+1)(q^2+q+1) \leq 2(q+1)^6$ vertices, from which $q+1 \geq (\frac{n}{2})^{\frac{1}{6}}$ and hence $|E(G)| = \frac{1}{2}dn = \frac{1}{2}(q+1)n \geq \frac{1}{2\sqrt[6]{2}}n^{\frac{7}{6}}$.

For $k \geq 12$, no infinite families of cages of girth $g > 12$ are known. Thus, to obtain similar constructions based on inflations, instead of cages we can only use "good" families of (d,g)-graphs. The best known such families have (see [3], or Theorem 12 in [2]) $n' \leq 2dq^{\frac{3}{4}g-4}$ vertices, where q denotes the smallest odd prime power for which $d \leq q$. By a well-known result (proved by Chebychev in the mid of 19th century, see e.g. [4], page 96), for any integer $a \geq 2$, there is a prime between a and $2a$, hence certainly $q \leq 2d$, which gives $n' \leq (2d)^{\frac{3}{4}g-3}$. For the inflation G we then have $|V(G)| = n = dn' \leq 2^{\frac{3}{4}g-3}d^{\frac{3}{4}g-2}$, from which, for fixed g, we have $d \geq c_1 n^{\frac{4}{3g-8}}$, where c_1 is a suitable constant, and $|E(G)| = \frac{1}{2}dn \geq c_2 n^{1+\frac{4}{3g-8}} = c_2 n^{\frac{3g-4}{3g-8}}$, where again c_2 is a suitable constant.

Similarly to before, G is k-locally disconnected, where $k = g - 1$, and hence $|E(G)| \geq c_2 n^{\frac{3(k+1)-4}{3(k+1)-8}} = c_2 n^{1+\frac{4}{3k-5}}$. Thus, we have $|E(G)| \in \Omega(n^{1+\frac{4}{3k-5}})$, which is noticeably less than the upper bound of Corollary 12.

Of course, there could possibly be a better special construction (not based on the inflation of a (d, g)-graph), however, this remains an open question.

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