On Forbidden Pairs Implying Hamilton-Connectedness

Jill R. Faudree[1], Ralph J. Faudree[2], Zdeněk Ryjáček[3,4], Petr Vrána[3]

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Abstract

Let X, Y be connected graphs. A graph G is (X, Y)-free if G contains a copy of neither X nor Y as an induced subgraph. Pairs of connected graphs X, Y such that every 3-connected (X, Y)-free graph is Hamilton connected have been investigated most recently in [8] and [5]. This paper improves those results. Specifically, it is shown that every 3-connected (X, Y)-free graph is Hamilton connected for $X = K_{1,3}$ and $Y = P_8$, $N_{1,1,3}$, or $N_{1,2,2}$ and the proof of this result uses a new closure technique developed by the third and fourth authors. A discussion of restrictions on the nature of the graph Y is also included.

1 Introduction

We will begin with some basic definitions, notation, and elementary results. We make every attempt to use standard definitions and notation, and generally follow the notation of [7]. With the exception of a couple of proof-specific terms we will place all the definitions in the first part of the introduction so that the reader may easily find them when needed.

Definitions and Notation

The word *graph* will generally mean a simple graph and we will use *multigraph* to indicate that multiple edges and loops are allowed. For $u, v \in V(G)$, a (u, v)*-walk* in G is a finite sequence of

[1]Department of Mathematics and Computer Science, University of Alaska at Fairbanks, Fairbanks, AK, USA, e-mail jrfaudree@alaska.edu.

[2]Department of Mathematical Sciences, University of Memphis, Memphis, TN, USA, e-mail rfaudree@memphis.edu.

[3]Department of Mathematics, University of West Bohemia, and Institute for Theoretical Computer Science (ITI), Charles University, P.O. Box 314, 306 14 Pilsen, Czech Republic, e-mail ryjacek@kma.zcu.cz, vranaxxpetr@o2active.cz.

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vertices and edges. A (u,v)-*trail* in G is a (u,v)-walk with no repeated edges. Given a trail T and an edge e in the multigraph G, we say e is *dominated* by T if e is incident to an interior vertex of T. Given $u,v \in V(G)$, we say T is a *maximal* (u,v)-*trail* if T dominates a maximum number of edges among all (u,v)-trails in G. A trail T in G is called a *dominating trail* if T dominates all edges in G.

Let V_1 and V_2 be nonempty subsets of $V(G)$. The *distance between* V_1 *and* V_2 is defined as the length of the shortest path in G beginning at a vertex in V_1 and ending at a vertex in V_2. If H_1 and H_2 are subgraphs of G, the distance between H_1 and H_2 is defined at the distance between $V(H_1)$ and $V(H_2)$.

Given $v \in V(G)$, the *neighborhood* of v, denoted $N(v)$, is the set of vertices adjacent to v in G. Given a graph G, the *local completion of* G *at* v, denoted G^*_v, is the graph obtained from G by adding all edges between the vertices in $N(v)$. Given $S \subseteq V(G)$, we use $\langle S \rangle$ to denote the subgraph of G induced by S. A vertex v in a graph G is *simplicial* if $\langle N(v) \rangle$ is complete. An edge e in G is called *pendant* if one of the end vertices of e has degree 1 in G. For (multi)graphs G_1 and G_2, we use $G_1 \simeq G_2$ to denote that G_1 and G_2 are isomorphic.

If H is a given graph, then a graph G is called H-*free* if G contains no induced subgraph isomorphic to H. In this case, the graph H is called a *forbidden* subgraph. Given graphs H_1, H_2, $\cdots, H_k$, we say G is $(H_1,\ H_2,\ \cdots, H_k)$-free if G contains no induced subgraph isomorphic to any of the graphs H_i, $i = 1, 2, \cdots, k$. In this paper and in the literature related to it, several specific forbidden subgraphs arise frequently. The *claw* is the graph $K_{1,3}$. The *center of a claw* refers to the vertex of degree 3 in $K_{1,3}$. A net, denoted $N_{i,j,k}$, consists of a triangle and three disjoint pendant paths (one on each vertex of the triangle) where i, j, and k denote the lengths of these paths. Others are $Z_i = N_{i,0,0}$ and $B_{i,j} = N_{i,j,0}$ and H_i, which denotes two triangles connected by a single path of length i.

The *line graph* of a given graph H, denoted by $L(H)$, is constructed with vertex set $E(H)$ and such that two vertices in $L(H)$ are adjacent if and only if the corresponding edges in H are incident to a common vertex in H. In [3], it was shown that for every graph H, $L(H)$ is claw-free. Given a line graph G, we define the *preimage* of G, denoted $L^{-1}(G)$ to be a graph H such that $L(H) = G$. An edge cut Y of a multigraph G is *essential* if $G - Y$ has at least two nontrivial components. For an integer $k > 0$, a multigraph G is *essentially* k-*edge-connected* if G every essential edge cut Y contains at least k edges. From the definitions it is easy to see that $G = L(H)$ is k-connected if and only if H is essentially k-edge-connected. Also, $G = L(H)$ contains the graph F as an induced subgraph if and only if H contains $L^{-1}(F)$ as a (not necessarily induced) subgraph.

Background

Pairs of forbidden subgraphs X, Y implying a 2-connected (X, Y)-free graph to be hamiltonian were characterized by Bedrossian [1] and the characterization was reconsidered by Faudree and

Gould [9]. There are similar characterizations for some other properties (see [9]). However, the corresponding question for Hamilton connectedness remains still open.

On the positive side, the following results were proved by Shepherd [14], Chen and Gould [8] and by Broersma et al. [5].

Theorem 1. (*i*) [14] *Every 3-connected* $(K_{1,3}, N_{1,1,1})$*-free graph is Hamilton connected.*

(*ii*) [8] *Let* G *be a 3-connected graph satisfying any of the following:*

(α) G *is* $(K_{1,3}, Z_3)$*-free,*

(β) G *is* $(K_{1,3}, P_6)$*-free,*

(γ) G *is* $(K_{1,3}, B_{1,2})$*-free.*

Then G *is Hamilton connected.*

(*iii*) [5] *Every 3-connected* $(K_{1,3}, H_1)$*-free graph is Hamilton connected.*

On the other hand, the following result appeared in [5].

Theorem 2. [5] *If* X, Y *is a pair of connected graphs such that* $X, Y \not\simeq P_3$ *and every 3-connected* (X, Y)*-free graph is Hamilton connected, then, up to a symmetry,* $X = K_{1,3}$ *and* Y *satisfies each of the following conditions:*

(*a*) $\Delta(Y) \leq 3$,

(*b*) *any longest induced path in* Y *has at most 9 vertices,*

(*c*) Y *contains no cycles of length at least 4,*

(*d*) *the distance between two distinct triangles in* Y *is either 1 or at least 3,*

(*e*) *There are at most two triangles in* Y*,*

(*f*) Y *is claw-free.*

In fact, we can show that item (d) can be strengthened to: *the distance between two distinct triangles in* Y *is either 1 or 3* which will follow from considering two examples G_1 and G_2 constructed below. The reader will see that both G_1 and G_2 are 3-connected, claw-free, and not Hamilton connected. Thus, in order for H_i to imply every 3-connected (C, H_i)-free graph is Hamilton connected, H_i must be an induced subgraph of both G_1 and G_2. It will be apparent that the only two such common induced subgraphs are H_1 and H_3, and the strengthened result follows. It is worth noting that the construction and analysis of these examples exploits (in much simplified form) one of the crucial techniques of the proof of Theorem 3: viewing the graph G as the line graph of a (multi)graph H, where the properties of G may be easier to analyze.

Now we construct G_1 and G_2. Let F be the graph obtained from the cycle C_8 with vertex set $\{x_1, ..., x_8\}$ by adding chords x_1x_5, x_2x_6, x_3x_7, x_4x_8. Let H_1 be the graph obtained as a subdivision of F. (That is, add one vertex of degree 2 into each edge.) Let H_2 be the graph obtained from F by adding a pendant edge to each vertex. Then, let $G_1 = L(H_1)$ and $G_2 = L(H_2)$. It is clear from this construction that G_1 and G_2 are 3-connected and claw-free. The graph G_1 has no Hamilton path connecting edges x_1y_1 and x_3y_3 where y_1 and y_3 are the vertices added to F to subdivide the chords at x_1 and x_3 respectively. The graph G_2 has no Hamilton path connecting edges x_1x_5 and x_3x_7.

Since H_1 is bipartite, any path between any two vertices of degree 3 has even length. Hence, in G_1, any path between any two triangles has odd length. Thus, the distance between two distinct triangles in Y must be odd.

In H_2, the preimage of a triangle is a claw $K_{1,3}$. The longest possible path between two such claw centers has length 5. Hence the longest induced path between two triangles in G_2 has length 4. But it must be odd by the previous observation. Hence, the only possible induced subgraphs H_i common to both G_1 and G_2 are when $i = 1$ or $i = 3$. This concludes the argument that the distance between two distinct triangles in Y is either 1 or 3.

Also, the graph G_2 is an important example because it is easy to check that it has the following properties:

(*i*) G_2 *is a 3-connected claw-free graph that is not Hamilton connected,*

(*ii*) G_2 *contains an induced* P_9 *and any induced* $N_{i,j,k}$ *with* $i + j + k \leq 7$,

(*iii*) G_2 *is* P_{10}*-free and* $N_{i,j,k}$*-free for any* i, j, k *with* $i + j + k \geq 8$.

Thus, the largest P_i that might imply a 3-connected (C, P_i)-free graph to be Hamilton connected is P_9, and the largest such $N_{i,j,k}$ is that for $i + j + k = 7$.

In this paper, we prove the following result.

Theorem 3. *If* G *is a 3-connected* (X, Y)*-free graph for* $X = K_{1,3}$ *and* $Y = P_8$, $N_{1,1,3}$, *or* $N_{1,2,2}$, *then* G *is Hamilton connected.*

2 Preliminary Results

The main tool for proving our result will be the concept of multigraph closure (or briefly M-closure) of a claw-free graph as introduced in [13].

A vertex $x \in V(G)$ is *k-eligible* ($k \geq 1$) if its neighborhood induces in G a k-connected noncomplete graph, and the *k-closure* of G, denoted $\mathrm{cl}_k(G)$, is the graph obtained from G by recursively performing the local completion operation at k-eligible vertices as long as this is possible. A graph G is *k-closed* if $G = \mathrm{cl}_k(G)$. The following properties of the k-closure will be important. (See [4], [12].)

Theorem 4. *For every claw-free graph* G,

(*i*) [4] $\mathrm{cl}_k(G)$ *is uniquely determined for any* $k \geq 1$,

(*ii*) [12] $\mathrm{cl}_2(G)$ *is Hamilton connected if and only if* G *is Hamilton connected.*

It is easy to see that, in general, $\mathrm{cl}_2(G)$ is not a line graph. Thus, another question is if a 2-closure of a claw-free graph is a line graph of a multigraph.

Line graphs of multigraphs were characterized by Bermond and Meyer [2] (see also Zverovich [15]) in terms of seven forbidden induced subgraphs. Using this characterization it is easy to show (see [12]) that, in general, $\mathrm{cl}_2(G)$ is not a line graph of a multigraph, since the graphs S_1 and S_2 of the characterization, shown in Figure 1, are 2-closed, i.e., they can be induced subgraphs in $\mathrm{cl}_2(G)$. However, it can be shown (see [12]) that a 2-closed graph cannot contain any of the remaining 5 forbidden subgraphs of the characterization.

Thus, a 2-closed claw-free graph can contain only induced S_1 and/or S_2, and an (S_1, S_2)-free 2-closed claw-free graph is a line graph of a multigraph. In the rest of the paper we will keep the notation of the graphs S_1, S_2 as shown in Figure 1.

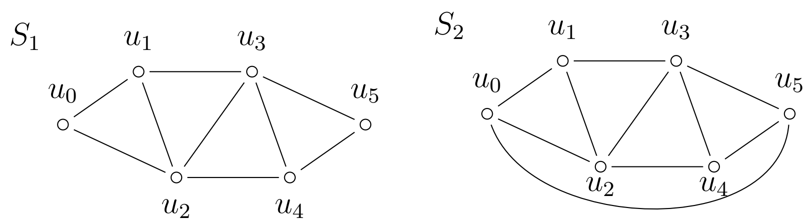


Figure 1: Two forbidden subgraphs for line graphs of multigraphs

Let $J = u_0u_1 \dots u_{k+1}$ be a walk in G. We say that J is *good in* G, if $k \geq 4$, $J^2 \subset G$ and for any i, $0 \leq i \leq k-4$, the subgraph induced by $\{u_i, u_{i+1}, \dots, u_{i+5}\}$ is isomorphic to S_1 or to S_2. The following lemma shows that good walks in a 2-closed graph have a very special structure.

Lemma 1. [13] *Let G be a connected 2-closed claw-free graph that is not the square of a cycle, and let $J = u_0u_1 \dots u_{k+1}$ be a good walk in G, $k \geq 5$. Then*

(i) $d_G(u_i) = 4$, $i = 3, \dots, k-2$,

(ii) $u_1 \dots u_k$ is a path.

Finally, a good walk J is *maximal* if, for every good walk J' in G, J being a subsequence of J' implies $J = J'$. It can be shown (see [13]) that if G is connected, 2-closed and is not the square of a cycle, then every good walk is contained in some maximal good walk, and maximal good walks are pairwise internally vertex-disjoint.

The M-closure of a claw-free graph G can be now defined as the graph $\mathrm{cl}^M(G)$ obtained by the following algorithm (see [13]).

Let G be a connected claw-free graph that is not the square of a cycle.

1. *Set $G_1 = \mathrm{cl}_2(G)$, $i := 1$.*
2. *If G_i contains a good walk, then*

   (a) choose a maximal good walk $J = u_0u_1 \dots u_{k+1}$,

(*b*) *set* $G_{i+1} = \mathrm{cl}_2((G_i)^*_{u_1u_k})$,
(*c*) $i := i + 1$ *and go to* (2).
3. *Set* $\mathrm{cl}^M(G) = G_i$.

A graph G is said to be M*-closed* if $G = \mathrm{cl}^M(G)$.

The following result from [13] summarizes basic properties of the M-closure.

Theorem 5. **[13]** *Let* G *be a connected claw-free graph and let* $\mathrm{cl}^M(G)$ *be the* M*-closure of* G*. Then*
(*i*) $\mathrm{cl}^M(G)$ *is uniquely determined,*
(*ii*) *there is a multigraph* H *such that* $\mathrm{cl}^M(G) = L(H)$,
(*iii*) $\mathrm{cl}^M(G)$ *is Hamilton-connected if and only if* G *is Hamilton-connected.*

A well-known drawback of line graphs of multigraphs is the fact that the preimage of a given line graph G is not uniquely determined, i.e., there can be multigraphs H_1, H_2 such that $H_1 \not\simeq H_2$ but $L(H_1) \simeq L(H_2)$. It was shown in [13] that this problem can be avoided under a very natural additional assumption that simplicial vertices in $G = L(H)$ correspond to pendant edges in H.

Theorem 6. **[13]** *Let* G *be a connected line graph of a multigraph. Then there is, up to an isomorphism, a uniquely determined multigraph* $H = L_M^{-1}(G)$ *such that a vertex* $e \in V(G)$ *is simplicial in* G *if and only if the corresponding edge* $e \in E(H)$ *is a pendant edge in* H*.*

Now it is easy to characterize all preimages of M-closed line graphs of multigraphs.

Theorem 7. **[13]** *Let* G *be a claw-free graph and let* T_1, T_2, T_3 *be the graphs shown in Figure 2. Then* G *is* M*-closed if and only if* G *is a line graph of a multigraph and* $L_M^{-1}(G)$ *does not contain a subgraph* S *(not necessarily induced) isomorphic to any of the graphs* T_1, T_2 *or* T_3.

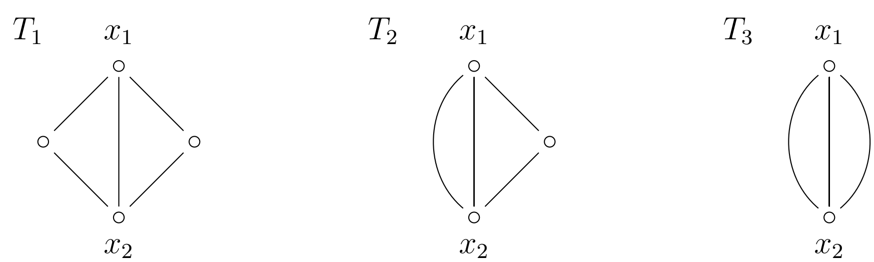


Figure 2: Forbidden subgraphs for preimages of M-closed graphs

Let $i, j, k \geq 1$. In [6], it was shown that if G is a $(K_{1,3}, P_i)$-free or $(K_{1,3}, N_{i,j,k})$-free graph, then so is G^*_x, for any 1-eligible vertex $x \in V(G)$. Since $\mathrm{cl}^M(G)$ is obtained from G by a sequence of local completions and a 2-eligible vertex is also 1-eligible, we immediately have the following facts.

Theorem 8. *Let G be a claw-free graph.*

(i) If G is P_i-free ($i \geq 1$), then $\mathrm{cl}^M(G)$ is also P_i-free.

(ii) If G is $N_{i,j,k}$-free ($i, j, k \geq 1$), then $\mathrm{cl}^M(G)$ is also $N_{i,j,k}$-free.

The line graph preimage counterpart of hamiltonicity was established by Harary and Nash-Williams [10] who showed that a line graph G of order at least 3 is Hamiltonian if its preimage $H = L^{-1}(G)$ contains a dominating eulerian subgraph (i.e., an Eulerian subgraph T such that every edge of H has at least one vertex on T). A similar argument gives the following analogue for Hamilton connectedness (see e.g. [11]).

Theorem 9. **[11]** *Let H be a multigraph with $|E(H)| \geq 3$. Then $G = L(H)$ is Hamilton connected if and only if for any pair of edges $e_1, e_2 \in E(H)$, H has a dominating (e_1, e_2)-trail.*

In [5] (and in a different form in [8]) the following result appears.

Lemma 2. **[5]** *For any pair of vertices x and y in a 3-connected claw-free graph G, there is a maximal (x, y)-path P such that $N(x) \subseteq V(P)$.*

The following lemma is a translation of Lemma 2 to a multigraph preimage in the special case of line graphs of multigraphs.

Lemma 3. *Let M be a multigraph such that $G = L(M)$ is 3-connected. Then for every pair of edges $e = e_1e_2$ and $f = f_1f_2$ in M, there exists a maximal (e, f)-trail T such that all edges incident to e_1 or e_2 are dominated by T.*

From Lemma 3 we have the following structural lemma. For ease of reference, given a trail T, we label as $I(T)$ the *interior trail of* T obtained by deleting the first and last edges from T.

Lemma 4. *Let M be a multigraph such that $G = L(M)$ is 3-connected and not Hamilton connected. Let e and f be edges in M such that there does not exist a dominating (e, f)-trail. Then there exists a maximal (e, f)-trail, T, such that for every edge $h = h_1h_2$ not dominated by T there exist at least two edge disjoint paths, Q_1 and Q_2, from h to $I(T)$ with the following properties:*

(a) for each i, the only vertex in $I(T) \cap Q_i$ is the last one on path Q_i,

(b) the last vertex on Q_1 and the last vertex on Q_2 are distinct,

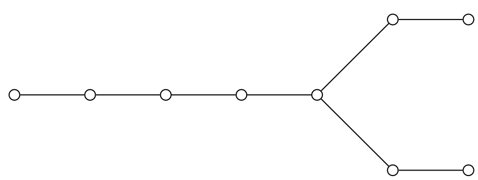
(c) neither Q_1 nor Q_2 use edges e or f.

Proof. Let e and f be edges and T a trail in M satisfying the conditions in Lemma 3. Let $h = h_1h_2$ be an edge not dominated by T. Since M is essentially 3-edge-connected there exist at least three edge disjoint paths from h to $I(T)$. Choose these paths to be as short as possible. This forces criterion (a). Criterion (b) follows from the observation that if any two paths end at the same vertex in $I(T)$, then it is possible to add to T a loop that dominates more edges, contradicting the maximality of T. Finally, to prove criterion (c), observe that since all edges incident to e_1 are dominated by T, given any edge $e_1v \in E(M)$ either e_1 or v is a vertex on $I(T)$. But, if e_1 appears on the interior of Q_i, say $Q_i = h_1 \cdots ve_1w \cdots x_i$, then we have a contradiction since either e_1 or v is a vertex in $I(T)$. Thus, e_1 cannot appear on any of the paths from h to $I(T)$ except (possibly) as the last vertex on that path. Thus, edge e cannot appear on any of the t paths. Hence criterion (c) holds. □

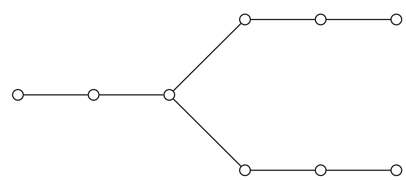
3 Proof of Theorem 3

Before beginning the formal proof, it is useful for the reader to know that all three parts of Theorem 3 will be proved simultaneously. Also, we make some elementary observations and introduce some notation.

Assume M is a multigraph and $G = L(M)$. Then G contains an induced P_8 if and only if M contains a (not necessarily induced) P_9. Similarly, G contains an induced $N_{1,1,3}$ (or $N_{1,2,2}$) if and only if M contains a subdivided $K_{1,3}$ where two edges are subdivided once and one edge is subdivided three times (or a $K_{1,3}$ where one edge is subdivided once and two edges are subdivided twice.) See the figures below.



$L^{-1}(N_{1,1,3})$



$L^{-1}(N_{1,2,2})$

We will refer to these preimages as $L^{-1}(P_8)$, $L^{-1}(N_{1,1,3})$, and $L^{-1}(N_{1,2,2})$ respectively. For graphs $L^{-1}(N_{1,1,3})$ and $L^{-1}(N_{1,2,2})$ we will refer to the unique vertex of degree three as the center vertex.

Also, as in the proof of Lemma 4, if T is a trail in M, then the interior of T, denoted $I(T)$, is the subtrail of T obtained by deleting the first and last edges of T.

Finally, assume we have an (e, f)-trail and an edge h that is not dominated by T. We will repeatedly refer to the existence of a *better trail*. This will, in all cases, mean a trail that includes all vertices of T plus at least one end vertex of h, thus implying that T is not a maximal (e, f)-trail.

Proof. Let G be a 3-connected (C, X)-free graph for $X = P_8$, $N_{1,1,3}$, or $N_{1,2,2}$ and assume G is not Hamilton connected. By Theorems 5 and 8 we can assume G is M-closed. Let M be a multigraph such that $L(M) = G$.

Since G is not Hamilton connected, let $e = e_1e_2$ and $f = f_1f_2$ be edges in M such that M does not contain a dominating (e, f)-trail. As M is connected we know there exists some (e, f)-trail T. We claim it is possible to choose an (e, f)-trail T in M such that all edges incident to any of $\{e_1, e_2, f_1, f_2\}$ are dominated by T.

If either e or f are pendant edges in M, apply Lemma 4 such that a pendant edge plays the role of edge f at the end of the trail and the claim follows. Thus we assume the degree of e_1, e_2, f_1, f_2 are all at least 2. Since M is essentially 3-edge-connected we know there are at least three edge disjoint paths from e to f.

It must be possible to choose these three paths such that one begins at vertex e_1 and a second begins at e_2. If this were not possible, all such paths would begin at, say, e_1. But this implies edge e forms an essential cut separating edge f from other edges incident to vertex e_2, a contradiction. A second application of this argument implies that one path must end at f_1 and a second must end at f_2.

Now the Pigeon-Hole Principle implies that, without loss of generality, path P_1 is an (e_1, f_1)-path and P_2 is an (e_2, f_2)-path. If P_3 is either an (e_1, f_2)-path or an (e_2, f_1)-path, then the trails $e_1e_2P_2f_2P_3e_1P_1f_1f_2$ or $e_2e_1P_1f_1P_3e_2P_2f_2f_1$ respectively are (e, f)-trails dominating all edges incident to any of e_1, e_2, f_1, and f_2. (See Figure 3.)

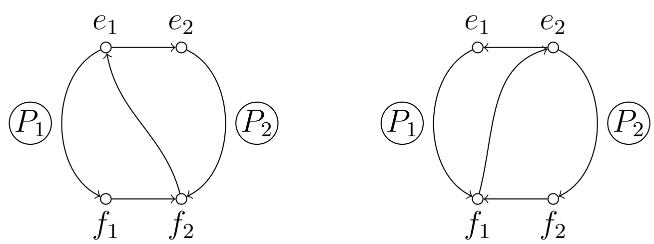


Figure 3:

Thus it remains to show that if P_3 is an (e_1, f_1)-path, an appropriate (e, f)-trail is possible.

Label the vertices of P_2 as $v_0 = e_2, v_1, v_2, \cdots, v_m = f_2$. Using again the essential 3-connectedness of M, edges e and f cannot form an essential edge cut separating $V(P_2)$ from $V(P_1 \cup P_3)$. Thus, there must exist a path between these two sets disjoint from e and f. Let $S \subseteq V(P_2)$ such that $v_i \in S$ if there exists a path from v_i to $V(P_1 \cup P_3)$ disjoint from e and f and edge-disjoint from P_2.

Assume $\deg(e_2) \geq 3$. (The case when $\deg(e_2) = 2$ proceeds in a similar fashion by observing that in this case $\deg(v_1) \geq 3$.) There is some smallest i such that $v_i \in S$. Note that $i \geq 1$ or the desired trail exists from a previous argument. Now the edges e and $v_{i-1}v_i$ cannot form an essential edge cut separating edges incident to e_2 from $V(P_1 \cup P_3)$. So there must exist a path P' from e_2 to $V(P_1 \cup P_3)$ disjoint from e and $v_{i-1}v_i$. Since $e_2 \notin S$, this path must use an edge on P_2. Let v_{i_0} be the first vertex that P' shares with P_2 other than v_0. (See Figure 4 (a).) If $i_0 \geq i$, the trail: $f_1, f_2, P_2, v_{i_0}, P', e_2, P_2, v_i, Q_i, f_1, P_1, e_1, e_2$ dominates the necessary edges, where Q_i is a path from v_i to f_1 that without loss of generality we assume uses P_3. (See Figure 4 (b).)

But $i_0 < i$ implies edges e and $v_{i_0}v_{i_0+1}$ form an essential edge cut unless there exists another path from the vertices $\{v_1, v_2, \cdots v_{i_0}\}$ to $\{v_{i_0+1}, \cdots v_m\}$ disjoint from e, $v_{i_0}v_{i_0+1}$, and P_2. Let $v_j \in \{v_1, v_2, \cdots v_{i_0}\}$ be the first vertex on this path P'' and $v_{i_j} \in \{v_{i_0+1}, \cdots v_m\}$ be the last. (See Figure

4 (c).) If $i_j \geq i$, the trail: $f_1, f_2, P_2, v_{i_j}, P'', v_j, P_2, e_2, P', v_{i_0}, P_2, v_i, Q_i, f_1, P_1, e_1, e_2$ dominates the necessary edges. (See Figure 4 (d).)

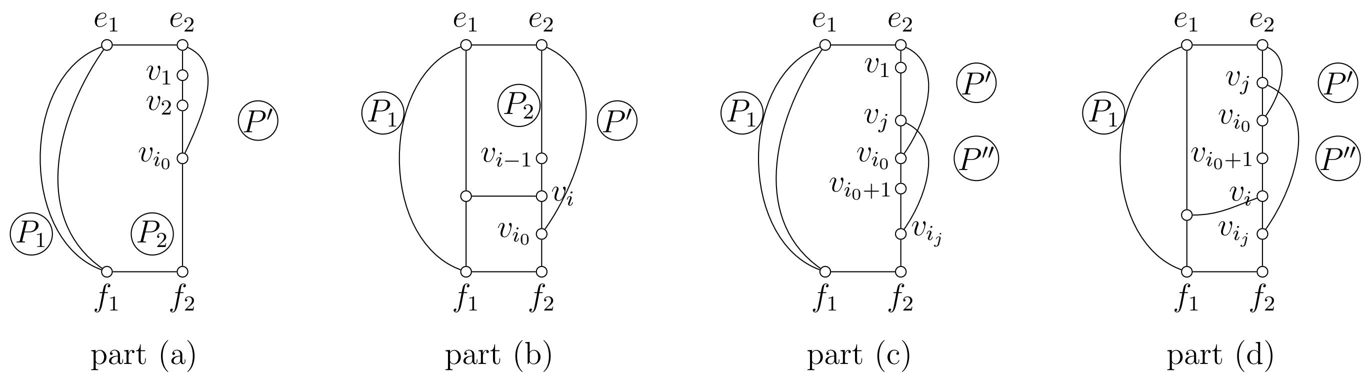


Figure 4:

If $i_j < i$, then e and $v_{i_j}v_{i_j+1}$ cannot form an essential edge cut, and the same process begins again – there must exist a path from the set $\{v_{i_0+1}, \cdots v_{i_j}\}$ to $\{v_{i_j+1}, \cdots v_m\}$. But as m is finite, this process must terminate in a desired trail.

Thus, we know it is possible to find an e, f-trail in M such that all edges incident to any of $\{e_1, e_2, f_1, f_2\}$ are dominated. Among all trails with this property, we choose a trail T that dominates the most edges of M.

Now, assume $h = h_1h_2$ is an edge not dominated by T. Let x_1, x_2, and x_3 be end vertices of three shortest edge disjoint paths from h to T. Note that our choice of T implies that x_1, x_2, and x_3 are distinct vertices on $I(T)$. Thus, h_1 and h_2 are distinct from x_1, x_2, and x_3. Without loss of generality, assume that the first appearance of vertex x_1 on T occurs before that of x_2, which in turns appears before x_3.

Define $T_{1,2}$ to be the subtrail of T between the first occurrence of x_1 and the first occurrence of x_2. Let P_1 be an (x_1, x_2)-path constructed from $T_{1,2}$ by deleting any loops. Specifically, if the vertices of $T_{1,2}$ are listed as they occur: $x_1 = v_0, v_1, \cdots, v_n, v_{n+1} = x_2$, and vertex v appears in positions v_i and v_j for $i < j$, then delete the loop $v_iv_{i+1}...v_{j-1}v_j$. Continue this process until no vertices are repeated to obtain P_1. The deleted subgraph will be the union of Euler circuits each of which contains at least one vertex on P_1. If a particular circuit contains vertex v from P_1, we say the edges of that circuit are *accessible* from v. Of all the ways to construct P_1 we choose the construction such that P_1 is as short as possible. Second, if any Euler circuit is accessible from a vertex of $I(T)$ outside $T_{1,2}$, we move it to the alternate vertex. Finally, we define the interior of P_1, denoted $I(P_1)$, to be the subpath of P_1 obtained by deleting the first and last edges of P_1.

We define $T_{2,3}$ and choose P_2 in the same fashion. We do not assert that $P_1 \cup P_2$ necessarily form a path. Let $i_1 = |V(I(P_1))|$ and $i_2 = |V(I(P_2))|$. Clearly $i_1, i_2 \geq 1$ since otherwise T is not maximal.

Figure 5 below illustrates an instance of $P_1 \cup P_2$ along with paths from h. Note that in this illustration $P_1 = x_1 z_{1,1} \cdots x_2$ and $P_2 = x_2 z_{2,1} \cdots x_3$. It is possible that $\{z_{1,1}, \cdots z_{1,i_1}\} \cap \{z_{2,1}, \cdots z_{2,i_2}\}$ may not be empty, though we can assume $x_1 \neq x_2 \neq x_3$, $x_2 \neq z_{i,j}$, $x_3 \neq z_{i,j}$, and $x_1 \neq z_{1,j}$. In particular, it is possible for $x_1 \in \{z_{2,1}, \cdots, z_{2,i_2}\}$. Since x_i is in $I(T)$ for all i, we do know there exists a vertex on T prior to x_1, denoted x_1^- and a vertex following x_3, denoted x_3^+. It is possible for $x_1^- x_1 = e$ and $x_3 x_3^+ = f$. Finally, paths from h to T may be more complicated than the illustration suggests. Specifically, these paths could be long and intersecting and may use both end vertices of h. However, we know each path consists of at least one edge and the limiting case is always the instance in which the non-dominated edge h has exactly three edges to T all using a single vertex of h. That is, if this case is proved, all others easily follow.

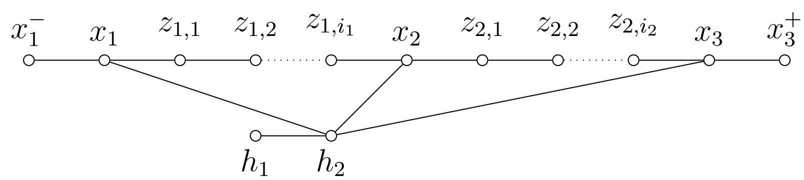


Figure 5: structure of worst case in general

Next, for $v \in I(P_1) \cup I(P_2)$, we define a *crucial edge* incident to v as an edge vw such that (1) $w \notin V(I(T))$ or (2) edge vw is on an Euler circuit accessible only from vertices of P_i for some i or (3) vw is an edge of P_i and neither v nor w appears on $I(T)$ outside $T_{i,i+1}$. Intuitively, crucial edges are edges that force vertex v to be on T. Clearly, for all i, P_i must contain a vertex that appears nowhere on $I(T)$ outside $T_{i,i+1}$ and is incident to a crucial edge. Observe that crucial edges of type (1) or (2) lie off the paths P_1 and P_2. These will be used frequently in the proof and thus, for ease of reference, we will call crucial edges of type (1) or (2) *green edges.*

The remainder of the proof will be split into cases according to the lengths of P_1 and P_2 and the nature of their intersections if they exist.

Claim A: For $j = 1, 2$, $i_j \leq 3$.

Assume $i_j \geq 4$ for some $j = 1$ or 2. Then P_j along with h and the paths from h to x_j and x_{j+1} must form an induced cycle, C_s, where $s \geq 7$. See Figure 6.

Note that M must have an edge not dominated by C_s since the other path, P_{j+1} (where $j+1$ is considered modulo 2), must have a crucial edge. But now C_s along with any edge a distance 1 away from C_s contains $L^{-1}(P_8)$, $L^{-1}(N_{1,1,3})$, and $L^{-1}(N_{1,2,2})$. This concludes the proof of Claim A.

Claim B: If $i_j = 1$ for $j = 1$ or 2, then $z_{j,1}$ is incident to a green edge, $z_{j,1} \neq x_1^-$, and $P_1 P_2$ is a path.

First, observe that $z_{j,1}$ must have some crucial edge incident to it or again there exists a better

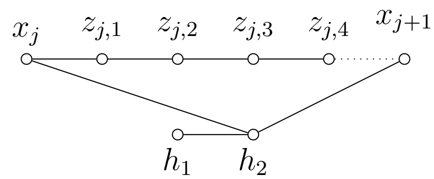


Figure 6: Claim A

trail and since $z_{j,1}$ is the only vertex on $I(P_j)$ this crucial edge must be green. Second, observe that if $z_{j,1} = x_1^-$, then a better trail exists.

Finally, to show P_1P_2 is a path, we need to show $z_{j,1} \notin V(I(P_{j+1}))$ and $x_1 \notin V(I(P_2))$. If $z_{j,1} \in V(I(P_{j+1}))$, then a better trail exists. If $x_1 = z_{2,1}$ or $x_1 = z_{2,i_2}$, then a better trail exists. If $i_2 = 3$ and $x_1 = z_{2,2}$, then $z_{2,3}$ must be incident to a green edge or a better trail exists. But now, the graph containing P_1P_2, h, paths from h to P_1P_2 and the two green edges at $z_{1,1}$ and $z_{2,3}$ must contain $L^{-1}(P_8)$, $L^{-1}(N_{1,1,3})$, and $L^{-1}(N_{1,2,2})$. Thus, $x_1 \notin V(I(P_2))$.

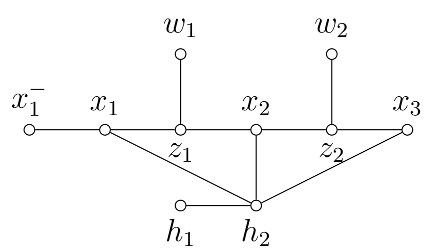
This completes the proof of Claim B.

Claim C: $i_1 \geq 2$ and $i_2 \geq 2$

Assume there exists a path, P_j, such that $|I(P_j)| = 1$. We will split the argument into three cases.

Case 1: $i_1 = i_2 = 1$

Let z_1 and z_2 be the internal vertices on each path. Let z_1w_1 and z_2w_2 be the green edges incident to these internal vertices. Let x_1^- be the vertex that precedes x_1 on T. Then Claim B implies $x_1^-x_1z_1x_2z_2x_3$ must be a path. If $w_1 = w_2$ we have a better trail. Thus all the vertices $x_1^-, x_1, z_1, x_2, z_2, x_3, w_1, w_2$ are distinct. Define $S = \langle x_1^-, x_1, z_1, w_1, x_2, z_2, w_2, x_3 \rangle$. Now consider S along with the three edge disjoint paths from h. See the figure below.



This structure always contains $L^{-1}(N_{1,2,2})$. If any of the paths from h are more than a single edge, this structure also contains $L^{-1}(P_8)$ and $L^{-1}(N_{1,1,3})$. Furthermore, if there exist two independent edges from h to S the structure contains $L^{-1}(P_8)$ and $L^{-1}(N_{1,1,3})$. So, we assume that all three paths from h to S are edges incident to vertex h_2 of edge h.

Let $H = \langle x_1^-, x_1, z_1, w_1, x_2, z_2, w_2, x_3, h_1, h_2 \rangle$. Observe that if x_1^- is adjacent to a vertex other than those in H, then M contains $L^{-1}(P_8)$ and $L^{-1}(N_{1,1,3})$. So, any edges in M incident to x_1^- must be incident to other vertices of H. But in every case (except x_1) the existence of these edges implies M contains a better trail. So x_1^- may have a second edge to x_1 but can have no other edges in M whatsoever.

But M is essentially 3-edge connected. So there must exist an additional edge disjoint path from x_1^- or x_1 to $H - \{x_1^-, x_1\}$. But the addition of any path from $\{x_1^-, x_1\}$ to $H - \{x_1^-, x_1\}$ produces a better trail since such a path must use x_1 and have length at most two such that the middle vertex has no additional neighbors. This completes the proof of Case 1.

Case 2: $i_1 + i_2 = 3$

There are two possibilities: $x_1^- x_1 z_{1,1} x_2 z_{2,1} z_{2,2} x_3$ or $x_1^- x_1 z_{1,1} z_{1,2} x_2 z_{2,1} x_3$. If path $P_{i,i+1}$ has only one interior vertex, $z_{i,1}$, then that vertex must be incident to at least one green edge. Call this edge $z_{i,1} w_i$. See Figure 7.

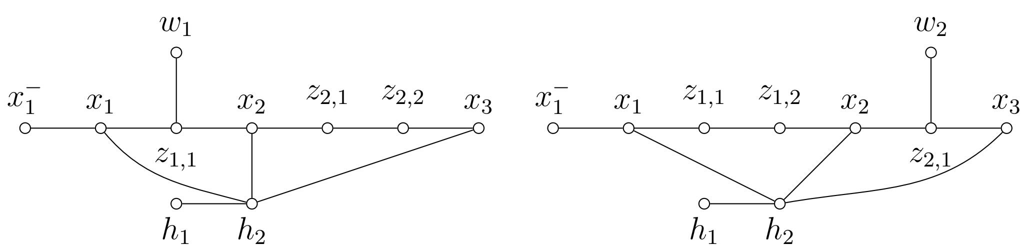


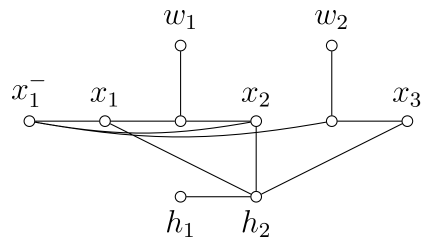
Figure 7: Case C.2

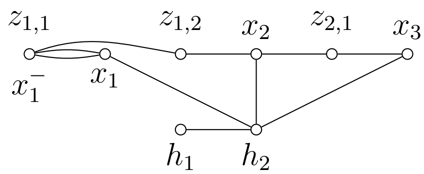
First, we will show that $x_1^- P_1 P_2$ is not a path. If $x_1^- P_1 P_2$ is a path, then every configuration of $x_1^- P_1 P_2$ including w_i and edge h includes $L^{-1}(P_8)$ and $L^{-1}(N_{1,2,2})$. Furthermore, if $i_1 = 2$ and $i_2 = 1$, then the graph contains $L^{-1}(N_{1,1,3})$. So the only remaining case is when $i_1 = 1$ and $i_2 = 2$.

But in this arrangement, edge $x_1^- x_1$ must have at least one additional path to $T - \{x_1^-, x_1\}$. If x_1^- has an edge to a vertex not shown or to $z_{2,1}$ the graph contains $L^{-1}(N_{1,1,3})$. If x_1^- has an edge to any of the other vertices, there exists a better trail. A similar argument applies to vertex x_1. Specifically, if there exists a path of length 2 starting at x_1 independent of $H - \{x_1^-, x_1\}$, then the graph contains $L^{-1}(N_{1,1,3})$. If there exists an edge (or path) from x_1 to any vertex of H other than $z_{2,1}$, then there exists a better trail. If there exists an edge (or path) from x_1 to $z_{2,1}$, then the graph contains $L^{-1}(N_{1,1,3})$. Thus, $x_1^- P_1 P_2$ cannot be a path.

However, if $x_1^- P_1 P_2$ is not a path, Claim B implies that the only vertices that could appear more than once are x_1^- and the interior vertices on the P_j for which $i_j = 2$. Thus, the only possible repetitions are: (i) $x_1^- = z_{1,1}$, (ii) $x_1^- = z_{1,2}$, (iii) $x_1^- = z_{2,1}$, and (iv) $x_1^- = z_{2,2}$. Instances (ii) and (iv) immediately lead to better trails. In arrangement (iii), there is immediately a better trail, unless vertex $z_{2,2}$ is incident to a green edge. But this implies the graph contains a $L^{-1}(P_8)$, $L^{-1}(N_{1,2,2})$ (center h_2), and $L^{-1}(N_{1,1,3})$ (center x_2). See figure below.

Finally, in the case $x_1^- = z_{1,1}$, observe that the graph is not presently essentially 3-edge connected. See figures below.





Specifically, there must be another path from $\{x_1^-, x_1\}$ to $M - \{x_1, x_1^-\}$. Note that since M contains a $C_7 = x_1 x_1^- z_{1,2} x_2 z_{2,1} x_3 h_2 x_1$, this cycle must dominate all edges of M. So any path from the vertices $\{x_1^-, x_1\}$ to $H - \{x_1^-, x_1\}$ is either an edge or a path of length two whose interior vertex has no neighbors. But every such path produces a better trail.

This completes the proof of Case 2.

Case 3: $i_1 + i_2 = 4$

In this case P_1P_2 either has structure $x_1 z_{1,1} x_2 z_{2,1} z_{2,2} z_{2,3} x_3$ or $x_1 z_{1,1} z_{1,2} z_{1,3} x_2 z_{2,1} x_3$. See Figure 8.

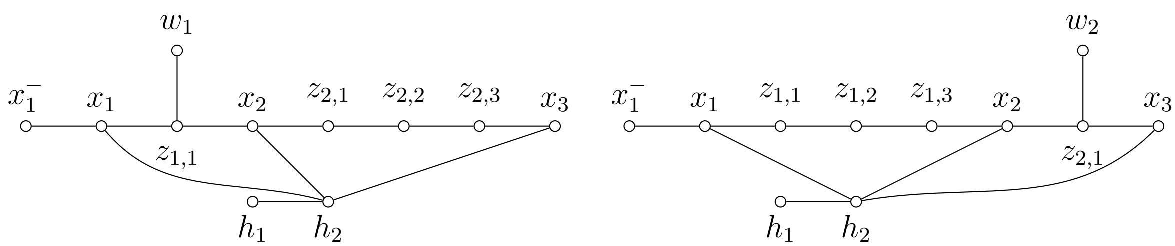


Figure 8: Case C.3

Again, the P_i containing a single interior vertex is incident to a green edge. Now the graph containing P_1P_2 (ignoring x_1^-) along with h and its paths to P_1P_2 always contains $L^{-1}(P_8)$, $L^{-1}(N_{1,2,2})$, and $L^{-1}(N_{1,1,3})$.

This completes the proof of Case 3 and the proof of Claim C.

Now we conclude that $I(P_j)$ contains either 2 or 3 vertices, producing 4 distinct cases. We will make a few general observations and then proceed by cases.

First observe that $x_1^- P_1 P_2$ cannot be a path as in that case M would contain $L^{-1}(P_8)$, $L^{-1}(N_{1,1,3})$, and $L^{-1}(N_{1,2,2})$. Also, observe that certain repetitions of vertices immediately give rise to a better trail. These are: $x_1^- = z_{1,i_1}$, $x_1^- = z_{2,i_2}$ $z_{1,1} = z_{2,1}$, $z_{1,i_1} = z_{2,i_2}$, $x_1 = z_{2,1}$, and $x_1 = z_{2,i_2}$.

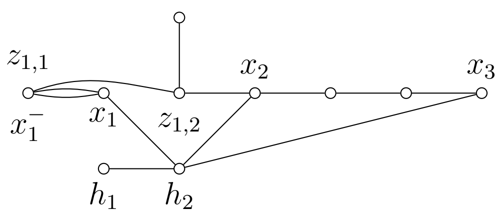
In every case, let $S = x_1^- x_1 z_{1,1} z_{1,2} (z_{1,3}) x_2 z_{2,1} z_{2,2} (z_{2,3}) x_3$.

Case 1: $i_1 = i_2 = 2$

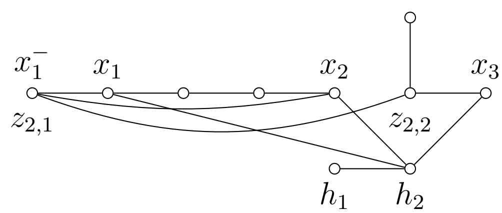
Subcase 1.1: $|V(S)| = 7$ (That is, one vertex on S is repeated.)

Then the possibilities that do not immediately lead to a better trail are: $x_1^- = z_{1,1}$, $x_1^- = z_{2,1}$, $z_{1,1} = z_{2,2}$, and $z_{1,2} = z_{2,1}$. In each of these instances, the trail can be extended to dominate h unless particular vertices on S are incident to pendant green edges. These particular vertices are (respectively) $z_{1,2}$, $z_{2,2}$, $z_{2,1}$, and, in the last case, both $z_{1,1}$ and $z_{2,2}$. Once these pendant edges are added, each graph contains $L^{-1}(P_8)$, $L^{-1}(N_{1,1,3})$, and $L^{-1}(N_{1,2,2})$. See the figures below.

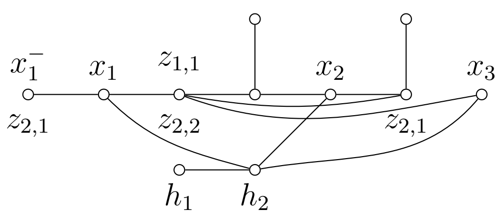
$x_1^- = z_{1,1}$



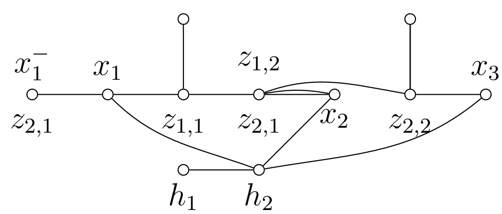
$x_1^- = z_{2,1}$



$z_{1,1} = z_{2,2}$



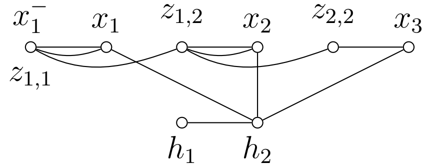
$z_{1,2} = z_{2,1}$



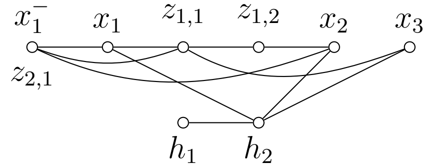
Subcase 1.2: $|V(S)| = 6$ (That is, either one vertex is used three times on S or two vertices each appears twice.)

The only vertex that could appear three times is x_1^-. Since $x_1^- \neq z_{j,2}$, the only possibility is $x_1^- = z_{1,1} = z_{2,1}$; but the second equality immediately produces a better trail. So there must be two vertices each appearing twice. The possibilities are: $x_1^- = z_{1,1}$ and $z_{1,2} = z_{2,1}$ or $x_1^- = z_{2,1}$ and $z_{1,1} = z_{2,2}$ (see figures below) and both produce better trails.

$x_1^- = z_{1,1}$ and $z_{1,2} = z_{2,1}$



$x_1^- = z_{2,1}$ and $z_{1,1} = z_{2,2}$



This concludes the proof of Case 1.

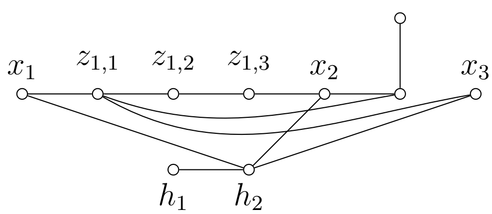
Case 2: $i_1 = 3$, $i_2 = 2$

Note that if $S - x_1^-$ is a path, then M contains $L^{-1}(P_8)$, $L^{-1}(N_{1,1,3})$, and $L^{-1}(N_{1,2,2})$.

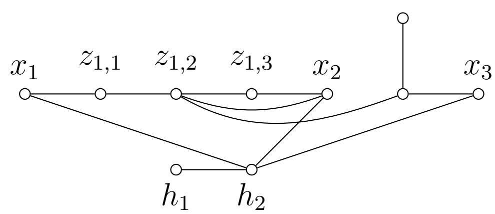
Subcase 2.1: $|V(S - x_1^-)| = 7$ (That is, there is one repeated vertex.)

The possibilities are: $z_{1,1} = z_{2,2}$, $z_{1,2} = z_{2,1}$, $z_{1,2} = z_{2,2}$, and $z_{1,3} = z_{2,1}$. In each case, a better trail exists unless a specific vertex is incident to a green edge. The specific vertex is (respectively) $z_{2,1}$, $z_{2,2}$, $z_{2,1}$, $z_{2,2}$. In all cases except the last, the multigraph contains $L^{-1}(P_8)$, $L^{-1}(N_{1,1,3})$, and $L^{-1}(N_{1,2,2})$. In the last instance, the multigraph contains $L^{-1}(P_8)$ and $L^{-1}(N_{1,2,2})$. See figures below.

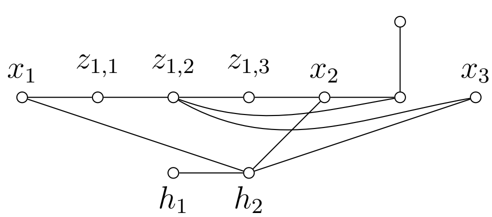
$z_{1,1} = z_{2,2}$



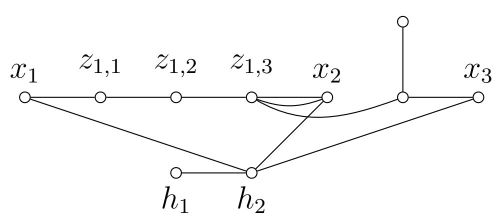
$z_{1,2} = z_{2,1}$



$z_{1,2} = z_{2,2}$



$z_{1,3} = z_{2,1}$

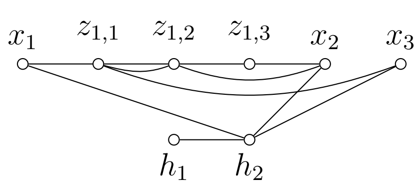


To finish this last case, we consider vertex x_1^-. If x_1^- is the same as $z_{1,2}$, $z_{1,3} = z_{2,1}$, or $z_{2,2}$, a better trail exists. If x_1^- is independent of the other vertices of S, then S contains $L^{-1}(N_{1,1,3})$. If $x_1^- = z_{1,1}$ then $z_{1,2}$ has a pendant green edge and the multigraph contains $L^{-1}(N_{1,1,3})$.

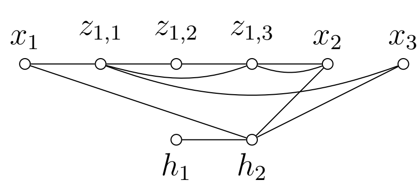
Subcase 2.2: $|V(S - x_1^-)| = 6$

Note that it is not possible for a single vertex in $S - x_1^-$ to be repeated twice. The possibilities are $z_{1,1} = z_{2,2}$ and $z_{1,2} = z_{2,1}$ or $z_{1,1} = z_{2,2}$ and $z_{1,3} = z_{2,1}$ or $z_{1,2} = z_{2,2}$ and $z_{1,3} = z_{2,1}$. In each case a better trail is possible.

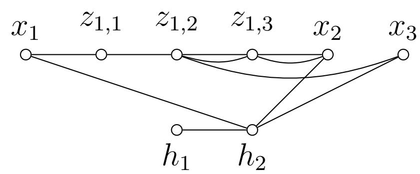
$z_{1,1} = z_{2,2}$ and $z_{1,2} = z_{2,1}$



$z_{1,1} = z_{2,2}$ and $z_{1,3} = z_{2,1}$



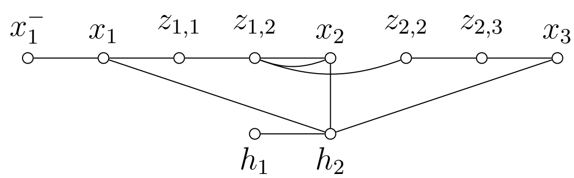
$z_{1,2} = z_{2,2}$ and $z_{1,3} = z_{2,1}$



This completes the proof of Case 2.

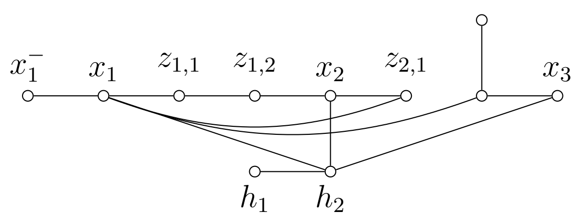
Case 3: $i_1 = 2$, $i_2 = 3$

Note that other than two instances, this case is completely symmetric to that of Case 2. The first instance to consider is the one configuration from Case 2 in which vertex x_1^- is used. Recall that this is the case where $z_{1,2} = z_{2,1}$ and the configuration (without x_1^-) contains $L^{-1}(P_8)$ and $L^{-1}(N_{1,2,2})$, but not $L^{-1}(N_{1,1,3})$. See figure below.



In this case, we observe that if x_1^- is independent of the remaining vertices of S or $x_1^- = z_{2,2}$, then M contains $L^{-1}(N_{1,1,3})$. If x_1^- is the same as $z_{1,1}, z_{1,2} = z_{2,1}, z_{2,3}$, then a better trail exists.

The second instance which is special to this case is when $x_1 = z_{2,2}$. In this case, $z_{2,1}$ must be incident to a green edge or a better trail exists. Now, $S - x_1^-$ along with the green edge contains $L^{-1}(P_8)$, $L^{-1}(N_{1,2,2})$, and $L^{-1}(N_{1,1,3})$. See figure below.



This concludes the proof of Case 3.

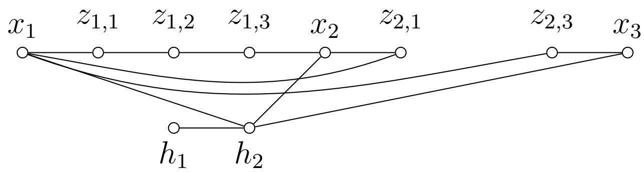
Case 4: $i_1 = i_2 = 3$

Similarly, if $S - x_1^-$ is a path, then M contains $L^{-1}(P_8)$, $L^{-1}(N_{1,1,3})$, and $L^{-1}(N_{1,2,2})$.

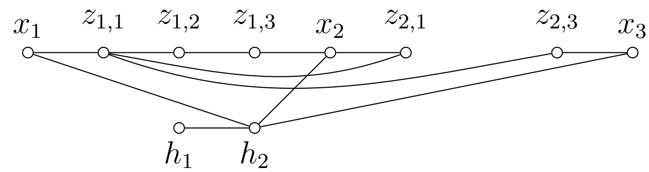
Subcase 4.1: Assume $|V(S - x_1^-)| = 8$ (That is, there exists one vertex repeated twice.)

The possibilities are: $x_1 = z_{2,2}$, $z_{1,1} = z_{2,2}$, $z_{1,1} = z_{2,3}$, $z_{1,2} = z_{2,1}$, $z_{1,2} = z_{2,2}$, $z_{1,2} = z_{2,3}$, $z_{1,3} = z_{2,1}$, and $z_{1,3} = z_{2,2}$. In each case, M contains $L^{-1}(P_8)$, $L^{-1}(N_{1,1,3})$, and $L^{-1}(N_{1,2,2})$. See figures below.

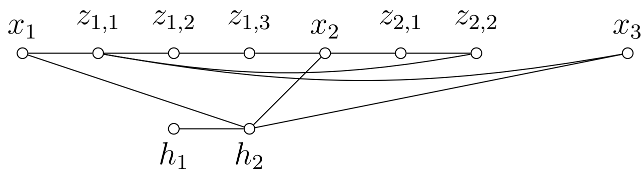
$x_1 = z_{2,2}$



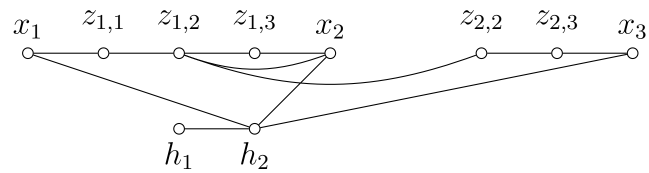
$z_{1,1} = z_{2,2}$



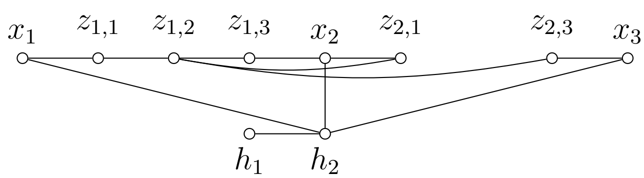
$z_{1,1} = z_{2,3}$



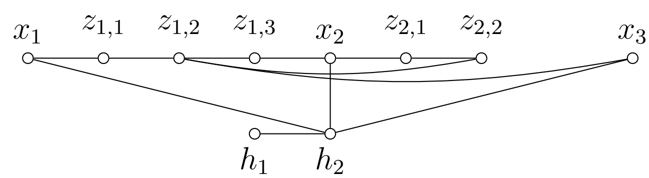
$z_{1,2} = z_{2,1}$



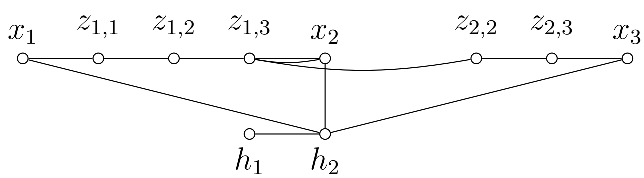
$z_{1,2} = z_{2,2}$



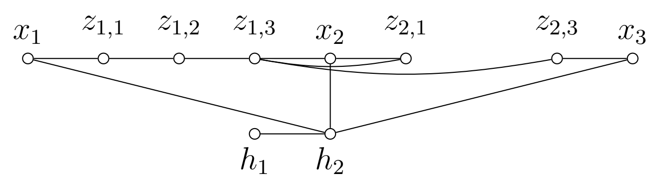
$z_{1,2} = z_{2,3}$



$z_{1,3} = z_{2,1}$



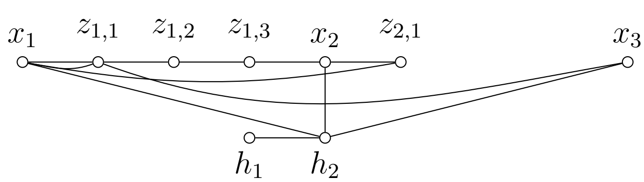
$z_{1,3} = z_{2,2}$



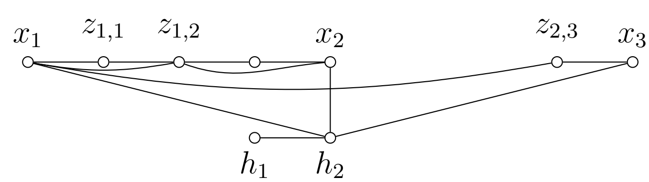
Subcase 4.2: Assume $|V(S - x_1^-)| = 7$

Again, it is not possible for a single vertex to be used three times, so assume two vertices are each used exactly twice. There are 15 possibilities listed below.

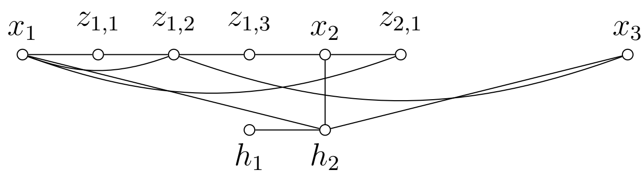
$x_1 = z_{2,2}$ and $z_{1,1} = z_{2,3}$



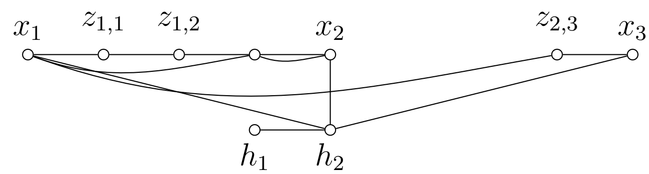
$x_1 = z_{2,2}$ and $z_{1,2} = z_{2,1}$



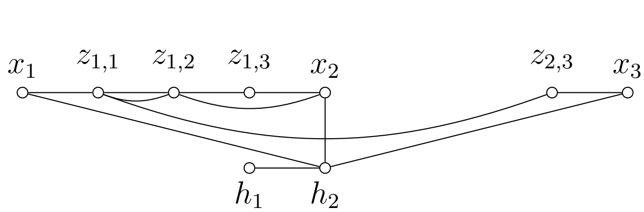
$x_1 = z_{2,2}$ and $z_{1,2} = z_{2,3}$



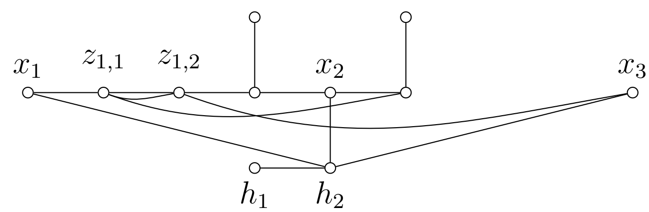
$x_1 = z_{2,2}$ and $z_{1,3} = z_{2,1}$



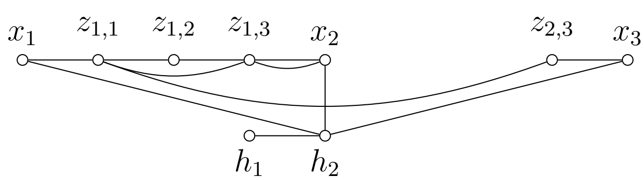
$z_{1,1} = z_{2,2}$ and $z_{1,2} = z_{2,1}$



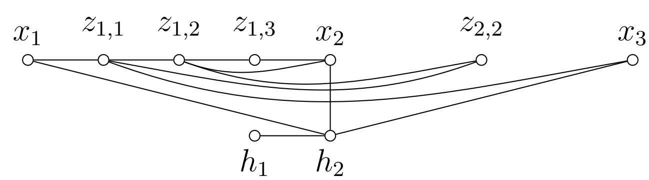
$z_{1,1} = z_{2,2}$ and $z_{1,2} = z_{2,3}$



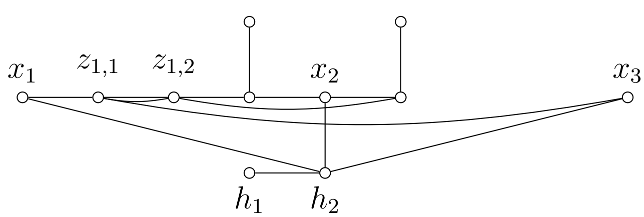
$z_{1,1} = z_{2,2}$ and $z_{1,3} = z_{2,1}$



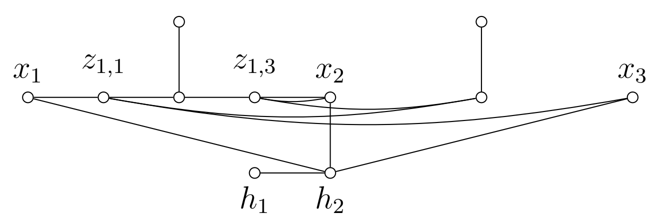
$z_{1,1} = z_{2,3}$ and $z_{1,2} = z_{2,1}$



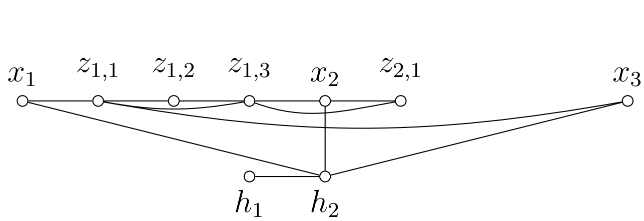
$z_{1,1} = z_{2,3}$ and $z_{1,2} = z_{2,2}$



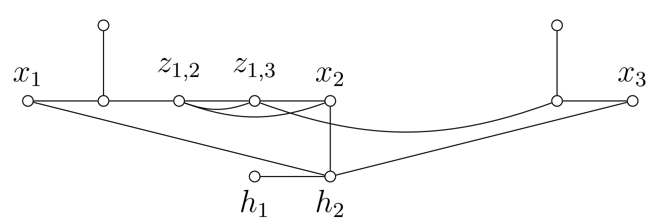
$z_{1,1} = z_{2,3}$ and $z_{1,3} = z_{2,1}$



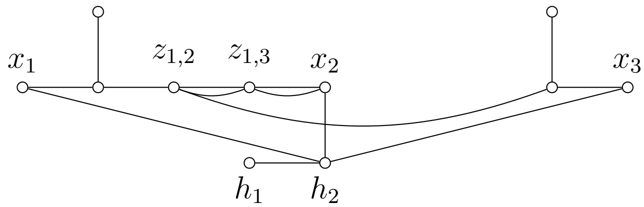
$z_{1,1} = z_{2,3}$ and $z_{1,3} = z_{2,2}$



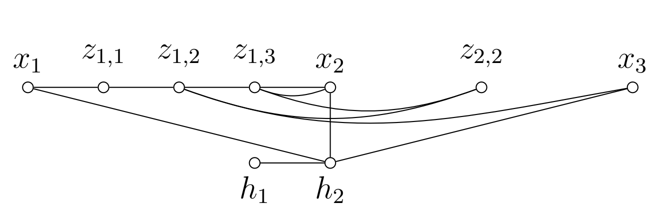
$z_{1,2} = z_{2,1}$ and $z_{1,3} = z_{2,2}$



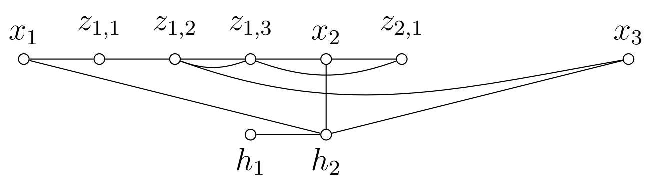
$z_{1,2} = z_{2,2}$ and $z_{1,3} = z_{2,1}$



$z_{1,2} = z_{2,3}$ and $z_{1,3} = z_{2,1}$



$z_{1,2} = z_{2,3}$ and $z_{1,3} = z_{2,2}$



In all cases, the remaining vertices must be distinct. Other than five exceptional cases, all configurations immediately produce a better trail. In those five exceptional cases, there exist two vertices which must be adjacent to green edges or a better trail exists. Using the green edges, all five configurations each contain $L^{-1}(P_8)$, $L^{-1}(N_{1,1,3})$ and $L^{-1}(N_{1,2,2})$.

This concludes this case and the proof of Theorem 3.

4 Concluding Remarks

If we now return to the discussion in Section 1, we can describe several remaining questions. The last example in this section shows that the pairs (C, Y) for $Y = P_9$ and $Y = N_{i,j,k}$ with $6 \leq i+j+k \leq 7$ remain open. As for pairs of the form (C, Y) for $Y = Z_i$ and $Y = B_{i,j}$, we have no improvement; the reason is that the (C, Z_i)-free and $(C, B_{i,j})$-free classes are not stable (i.e. the analogue of Theorem 8 fails) under the 2-closure. Attacking these pairs will require the development of new techniques. One potential pair is (C, H_3). In fact, there are a variety of potential subgraphs obtained by attaching paths to vertices of degree 2 in H_1 or in H_3.

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