The rainbow connection number
of 2-connected graphs

Jan Ekstein[*]   Přemysl Holub[*]   Tomáš Kaiser[*]
Maria Koch[†]   Stephan Matos Camacho[†]   Zdeněk Ryjáček[*]
Ingo Schiermeyer[†]

Abstract

The rainbow connection number of a graph G is the least number of colours in a (not necessarily proper) edge-colouring of G such that every two vertices are joined by a path which contains no colour twice. Improving a result of Caro et al., we prove that the rainbow connection number of every 2-connected graph with n vertices is at most $\lceil n/2 \rceil$. The bound is optimal.

1   Introduction

We investigate a problem related to the concept of rainbow connection in graphs, introduced by Chartrand et al. [5]. Let G be an undirected graph with a colouring c of the edges, which is not assumed to be proper (that is, adjacent edges may get the same colour). A subgraph H of G is *rainbow* (with respect to c) if no two edges of H have the same colour under c. The edge-coloured graph (G, c) is *rainbow-connected* if every pair of vertices is joined by a rainbow path. The *rainbow connection number* of G, denoted by $rc(G)$, is the least number of colours in a colouring which makes G rainbow-connected.

If G has n vertices, then $rc(G) \leq n - 1$ as one may colour each edge of a spanning tree of G with a different colour, and use one of these colours for all the remaining edges. Chartrand et al. [5] determined the rainbow connection

number of several classes of graphs, such as the complete multipartite graphs. The rainbow connection number has been studied for further graph classes in [3] and for graphs with fixed minimum degree in [3, 7, 11, 12].

The computational complexity of rainbow connectivity has been studied in [2] where it is proved that determining the rainbow connection number is an NP-complete problem. Indeed, it is already NP-complete to decide whether $rc(G)$ equals two [2]. More generally, it was shown in [8] that for any fixed $k \geq 2$, deciding if $rc(G) = k$ is NP-complete.

Caro et al. [3] proved the following upper bound for the rainbow connection number of a 2-connected graph:

Theorem 1 ([3]). *If G is a 2-connected graph on n vertices, then*

$$rc(G) \leq \frac{n}{2} + O(\sqrt{n}).$$

In this paper, we improve this upper bound to an optimal one, which is attained, e.g., for all odd cycles:

Theorem 2. *For any 2-connected graph G with n vertices,*

$$rc(G) \leq \left\lceil \frac{n}{2} \right\rceil.$$

We remark that during the review process of this paper, an independent proof of Theorem 2 was published in [9]. Furthermore, the authors of [9] conjecture a positive answer to the question in our Problem 9 and prove several partial results in this direction.

The proof of Theorem 2 will be presented in Section 4. It is based on several lemmas which are established in the Sections 2 and 3.

In the remainder of this section, we fix the necessary notation and terminology. For the terms not defined here, as well as for broader background, the reader may wish to consult [1].

Our graphs are finite, undirected and simple; in particular, parallel edges are not allowed. The vertex and edge sets of G are denoted by $V(G)$ and $E(G)$. The number of vertices of a graph G is denoted by $|G|$. A path with endvertices x and y is referred to as an *xy-path*. For $H \subseteq G$, an *H-path* is a path disjoint from H except for its endvertices, which are contained in H. If P is a path in G and $u, v \in V(P)$, then uPv denotes the unique subpath of P with endvertices u and v. If Q is a path with $v, w \in V(Q)$, then $uPvQw$ denotes the concatenation of uPv and vQw. This may in general be a walk rather than a path, but this distinction is not too important since any rainbow uw-walk contains a rainbow uw-path.

Throughout this paper, the term *colouring* will be used as in this section — meaning an edge-colouring which is not necessarily proper. It is convenient to call a colouring c of G *rainbow-connecting* if (G, c) is rainbow-connected. Since

a colouring c is, formally, a function defined on $E(G)$, it makes sense to let *im c* denote the set of all colours used by c.

A somewhat technical strengthening of the concept of a rainbow-connecting colouring will be useful in our proofs. Let us call a rainbow path in (G, c) *blocking* if it uses all colours in *im c*. Given vertices $x, y \in V(G)$, we say that y is *blocked for x* if all rainbow xy-paths are blocking. A colouring is *safe* if for each vertex x, there is at most one blocked vertex y. A colouring which is both safe and rainbow-connecting is said to be *safely rainbow-connecting*.

Let A be a set of colours and let c_0 be a colouring of G. A subgraph $H \subseteq G$ is *A-free* in (G, c_0) if no colour from A is used by c_0 on an edge of H. To simplify the notation, we abbreviate, e.g., '$\{\alpha, \gamma\}$-free' to '$\alpha\gamma$-free'.

In the proofs in this paper, we use the symbol $\diamond$ to mark the end of the proof of a claim. The same symbol is used at the end of the discussion of each case in a case analysis.

2   A lemma on paths

In this section, we prove a lemma which is one of the key parts of our argument.

Let H be a subgraph of G. A subgraph H', $H \subseteq H' \subseteq G$, is a *k-extension* of H with *path sequence* $(P_1, \ldots, P_k)$ if each P_i is an $(H \cup P_1 \cup \ldots P_{i-1})$-path with at least one endvertex in $V(P_1 \cup \cdots \cup P_{i-1})$, and

$$H' = H \cup P_1 \cup \cdots \cup P_k.$$

If k is not important, we just say that H' is an extension of H. An extension is *even* (*odd*) if all the paths in the path sequence are even (odd, respectively). (Recall that even paths are those with an even number of edges.)

In the proof of the lemma, we will need a concept similar to that of an H-bridge as introduced by Tutte (see, e.g., [1, Section 9.4]). A *weak H-bridge of G* is any component B of $G - E(H)$ containing at least one edge. The vertices of $B \cap H$ are called the *attachment vertices* of B.

Lemma 3. *Let H be a connected subgraph of a 2-connected graph G such that the following holds:*

   (i) all 1-extensions of H are even,

   (ii) there is no even 2-extension of H.

Then for any $k \geq 1$ and any extension of H with path sequence $(P_1, \ldots, P_k)$, P_1 is even and all the other paths P_i are odd.

Proof. Suppose $H' = H \cup P_1 \cup \cdots \cup P_k$ is a k-extension of H, and proceed by induction on k. The case $k \leq 2$ follows from the assumptions, so we assume

3

that $k > 2$, and that the assertion holds for j-extensions of H with $j < k$. In particular, each P_i $(2 \leq i < k)$ is odd, and therefore is not an H-path by condition (i). Observe also that we may assume $|H| \geq 2$, since otherwise there exists no H-path and the statement is trivially true.

For the sake of a contradiction, suppose that P_k is an even $(H \cup P_1 \cup \cdots \cup P_{k-1})$-path with at least one endvertex in $P_1 \cup \cdots \cup P_{k-1}$. We will find either an even 2-extension of H (contradicting (ii)), or an extension of H with a shorter path sequence terminating with P_k (contradicting the induction hypothesis).

Let G/H be the multigraph obtained by contracting all the edges of H. Since H is connected, the contraction merges all the vertices of H into one vertex $*_H$.

Claim 1. *The graph G/H is bipartite.*

Suppose, for the sake of a contradiction, that C_* is an odd cycle in G/H. If $*_H \notin V(C_*)$, then C_* is also a subgraph of G, and we can find two vertex-disjoint paths, each joining a vertex of H to a vertex of C_* and having no internal vertices in $H \cup C_*$. Combining the paths with a suitable subpath of C_*, we obtain an odd H-path, in violation of (i).

Thus, C_* must contain $*_H$. The subgraph of G corresponding to C_* is either a path or a cycle. If it is a path, then it is an odd H-path violating condition (i). Hence, G contains an odd cycle C containing exactly one vertex u_1 of H. Using the 2-connectedness of G and the assumption that $|H| \geq 2$, we can find a path which joins a vertex $u_2 \neq u_1$ of H to a vertex of C, and has no internal vertices in $H \cup C$. The concatenation of this path with a suitable subpath of C is an odd H-path, a contradiction. $\diamond$

In view of Claim 1, we can let b_* be a 2-colouring of the vertices of G/H which is proper (adjacent vertices get different colours). Consider the corresponding 2-colouring b of $G - E(H)$ obtained by assigning each vertex w the colour $b_*(w)$ if $w \notin V(H)$ and $b_*(*_H)$ otherwise.

Let B be the weak $H \cup P_1$-bridge of H' containing P_k. We let A denote the set of attachment vertices of B which are contained in P_1 (that is, $A = B \cap P_1$). Furthermore, we let $J \subseteq \{2, \ldots, k\}$ be the set of indices i such that P_i is contained in B. We write $J = \{i_1, \ldots, i_\ell\}$ with $i_1 < \cdots < i_\ell$ and note that $i_\ell = k$.

Clearly, if $\ell = 1$ (that is, if $B = P_k$), then P_k is odd as we would otherwise obtain an even 2-extension of H. In the sequel, we will therefore assume that $\ell \geq 2$.

Claim 2. *Any two vertices of A have different colours under b.*

Let $x, y \in A$. Suppose that $b(x) = b(y)$. Since B is connected, it contains an xy-path P. As P is edge-disjoint from H, the colours of the vertices along P alternate and hence P is even. Adding P to P_1, we obtain an even 2-extension of H, a contradiction with (ii). $\diamond$

By Claim 2, A contains at most two vertices.

4

Claim 3. *The size of A is at most one.*

For the sake of a contradiction, suppose that $A = \{x, y\}$. We have $b(x) \neq b(y)$, so if $B \cap H$ contains any other vertex z, then the colour of z matches that of x or y, and consequently B contains either an even xz-path or an even yz-path. This provides us again with an even 2-extension of H, contradicting (ii).

Thus, x and y are the only two attachment vertices of B. Necessarily, P_{i_1} is an xy-path. Let Q be the unique H-path obtained by concatenating P_{i_1} with subpaths of P_1. Note that for any j ($2 \leq j \leq \ell$) and any endvertex z of P_{i_j} with $z \in V(P_1)$, we also have $z \in V(Q)$. Therefore, $H \cup Q \cup B$ can be obtained as the ℓ-extension of H with path sequence

$$(Q, P_{i_2}, \ldots, P_{i_\ell}).$$

Since $2 \leq \ell \leq k - 1$, the induction hypothesis implies that the path P_{i_ℓ} is odd. Furthermore (again since we assume that $\ell \geq 2$), P_{i_ℓ} coincides with P_k. This is a contradiction with the assumption that P_k is even. $\diamond$

We cannot have $A = \emptyset$, since at least one endvertex of P_{i_1} is required to lie on $P_1 \cup \cdots \cup P_{i_1 - 1}$ and thus (by the choice of i_1) it must actually be contained in P_1.

Hence, A contains a single vertex, say $A = \{x\}$. The argument for this case is similar to that in the proof of Claim 3. We note that one endvertex of P_{i_1} is x and the other endvertex is in H. Let Q be an H-path obtained by concatenating P_{i_1} with a subpath R of P_1; of the two possibilities for R, we choose one where the endvertex of R in H is different from x. As before, if $z \in V(P_{i_j} \cap P_1)$ ($2 \leq j \leq \ell$), then $z \in V(Q)$. Consequently, $H \cup Q \cup B$ is an ℓ-extension of H with path sequence

$$(Q, P_{i_2}, \ldots, P_{i_\ell}).$$

Again, we infer from the induction hypothesis that P_{i_ℓ} is odd, which contradicts the assumption that $P_k = P_{i_\ell}$ is even. $\square$

3 Extending the colourings

Throughout this section, let H be a connected subgraph of a graph G, $|H| \geq 3$, and let c be a rainbow-connecting colouring of H. We introduce 'standard' ways to extend c to an odd 1-extension of H and to an even 2-extension of H. In this and the following section, the symbol γ will denote a fixed colour which is assumed to be contained in $im\ c$. We make the assumption that $|im\ c| \geq 2$.

Suppose first that P is an odd H-path, say of length $2k + 1$. A *continuation* of c to $H \cup P$ is any colouring c' which agrees with c on H and assigns to the edges of P, in some direction, the colours

$$a_1, a_2, \ldots, a_k, \gamma, a_1, a_2, \ldots, a_k,$$

where the a_i $(1 \leq i \leq k)$ are some distinct colours not contained in $im\ c$. Thus, a continuation of c to $H \cup P$ is not uniquely determined, but any two such continuations are isomorphic in the obvious sense, so we may regard them as identical. Note that c' uses k colours not contained in $im\ c$, which is half the number of vertices in $V(P) \setminus V(H)$.

Lemma 4. *Let c be a safely rainbow-connecting colouring of H. If P is an H-path of odd length, then any continuation of c to $H \cup P$ is safely rainbow-connecting.*

Proof. Let c' be a continuation of c to $H \cup P$. Suppose that P is an H-path of length $2k + 1$ with endvertices u and v. We may assume that $k \geq 1$ since the statement is trivially true for $k = 0$.

To see that c' is rainbow-connecting, we need to exhibit a rainbow xy-path R_{xy} for each pair $x, y \in V(H \cup P)$. If $x, y \in V(H)$, we define R_{xy} as a rainbow xy-path in (H, c) which exists by the assumption, and we choose a path which is non-blocking under c if possible. Note that R_{xy} is rainbow and non-blocking under c'.

From now on, we assume that $x \in V(P)$. Suppose first that $y \notin V(P)$. Since P is odd, the subpaths xPu, xPv are not of the same length; without loss of generality, let xPu be the shorter one. Note that on xPu, c' uses no colour from $im\ c$. Thus, if we let $R_{xy} = xPuR_{uy}y$ (where R_{uy} has been defined above because $u \in V(H)$), then R_{xy} is rainbow in $H \cup P$.

Assume next that $y \in V(P)$. We may assume that $y \in V(xPv)$. If xPy is not rainbow, then it includes a pair of edges with the same colour, in which case each colour used by c' on P (including γ) must appear on xPy. But then c' is rainbow on $uPx \cup vPy$ and uses no colour from $im\ c$. Thus, $R_{xy} := xPuR_{uv}vPy$ is rainbow.

It remains to show that c' is safe. Let x be a vertex of $H \cup P$; we show that at most one vertex is blocked for x under c'. Since $k > 1$, we may choose a colour $\varepsilon \neq \gamma$ used by c' on P.

Suppose that $x \in V(H)$. Since ε is not used by c' on H, no $y \in V(H)$ is blocked for x under c'. As for $y \in V(P)$, y will only be blocked if $R_{xy} \cap P$ includes all the colours used by c' on P, possibly except γ. This happens only if y is incident with the central edge e of P. Let u' and v' be the endvertices of e, where u' is closer to u than to v. Note that the path $R_{xu'}$ contains R_{xu} as a subpath, and similarly $R_{xv} \subseteq R_{xv'}$. Since only one of u and v can be blocked for x under c, we may assume that R_{xu} is not blocking under c. Therefore, $R_{xu'}$ is not blocking under c' since no colour from $im\ c$ is used on the complementary subpath uPu' of $R_{xu'}$. Hence, v' is the only vertex which may be blocked for x under c'.

The last case to consider is that both x and y are internal vertices of P. If $R_{xy} \subseteq P$, then the only colour it uses from $im\ c$ (if any) is γ. By the assumption that $|im\ c| \geq 2$ (made at the beginning of this section), R_{xy} is not blocking. Thus, we may assume that $R_{xy} = xPuR_{uv}Py$. There is only one vertex y for

which xPu and vPy cover $im\ c' \setminus im\ c$, namely the other vertex of P which is incident with edges of the same colours as x. Thus, we have shown that c' is safe, and the proof is complete. $\qquad\square$

Next, we define a continuation of c to an even 2-extension $H \cup Q \cup Q'$ of H, where the length of Q is 2ℓ and the length of Q' is $2\ell'$. Suppose that the vertices of Q are $u_0, \ldots, u_{2\ell}$ and the vertices of Q' are $u'_0, \ldots, u'_{2\ell'}$. Let us write $u = u_0$, $v = u_{2\ell}$, $u' = u'_0$ and $v' = u'_{2\ell'}$. We may assume that the distance of u' from H in $H \cup Q$ is greater than or equal to the distance of v' from H. In particular, $u' \in V(Q)$. We may also assume that $u' = u_k$ with $k \leq \ell$, and if $u' = u_\ell$, then $v' \in V(u_\ell Q v \cup H)$.

We colour the edges of Q, in order from u to v, by

$$a_1, a_2, \ldots, a_{\ell-1}, a_\ell, \gamma, a_1, a_2, \ldots, a_{\ell-1}.$$

The edges of Q', in order from u' to v', will be coloured by

$$a_{\ell+1}, a_{\ell+2}, \ldots, a_{\ell+\ell'-1}, \gamma, a_\ell, a_{\ell+1}, a_{\ell+2}, \ldots, a_{\ell+\ell'-1}.$$

Here, a_i $(1 \leq i \leq \ell+\ell'-1)$ are again some distinct colours not contained in $im\ c$. Any colouring c' obtained in this way from c is said to be a *continuation of c to $H \cup Q \cup Q'$*. Note that c' uses $\ell + \ell' - 1$ colours not contained in $im\ c$, which is half the number of vertices in $V(Q \cup Q') \setminus V(H)$.

Based on the position of the endvertex v' of Q' relative to Q, we distinguish three possible types of the 2-extension $H \cup Q \cup Q'$. As shown in Figure 1, we may have $v' \in V(u'Qv)$ (Type I), $v' \in V(H) \setminus V(Q)$ (Type II) or $v' \in V(uQu')$ (Type III). Note that Types I and III include the possibility that v' coincides with v or u, respectively. Observe also that if $u' = u_\ell$, then the 2-extension is of Type I or II.

Lemma 5. *Let c be a safely rainbow-connecting colouring of H. If $H' = H \cup Q \cup Q'$ is an even 2-extension of H, then any continuation of c to H' is safely rainbow-connecting.*

Proof. Let c' be a continuation of c to H'. We use the same notation as in the definition of c'. We define A as the set of colours used by c' on $H \cup Q$.

First, we show that c' is rainbow-connecting. Let $x, y \in V(H')$. We are looking for a rainbow xy-path R_{xy}. If $x, y \in V(H \cup Q)$, then the argument is similar to that used in the proof of Lemma 4. For $x, y \in V(H)$, R_{xy} is a rainbow xy-path in (H, c). If $x \in V(Q)$ and $y \in V(H)$, then consider a γ-free subpath of Q from x to a vertex $w \in \{u, v\}$ and define $R_{xy} = xQwR_{wy}y$ (where R_{wy} has been defined before as $w \in V(H)$). Finally, if both x and y are vertices of Q, then we may assume that $x \in V(uQv)$; the path R_{xy} is defined as xQy if this path is rainbow, and $xQuR_{uv}vQy$ otherwise.
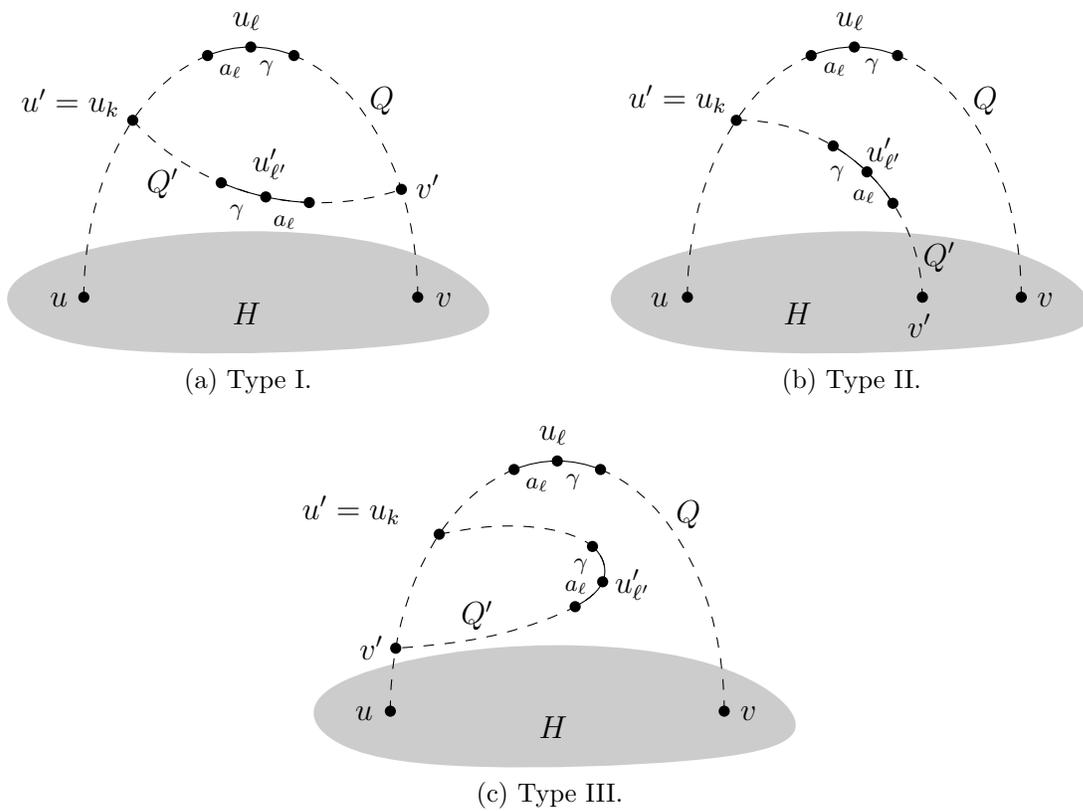
7

(a) Type I.

(b) Type II.

(c) Type III.

Figure 1: The possible types of the 2-extension $H \cup Q \cup Q'$. Dashed and solid lines represent paths and edges, respectively.

For later use, note that in all the cases considered up to now, R_{xy} is either a_ℓ-free or disjoint from H, with one exception, namely if $x = u_\ell$ and $y \in V(H)$.

It remains to discuss the case that x or y is in Q'. By symmetry, we may assume that $x \in V(Q')$.

Case 1. $y \in V(Q')$.

If the path $xQ'y$ is not rainbow, then without loss of generality, we can write $x = u'_i$, $y = u'_j$, where $j \geq i + \ell' + 2$. In particular, $i \leq \ell' - 2$ and $j \geq \ell' + 2$, so c' uses colours $a_{\ell+1}, \ldots, a_{\ell+i}$ on the path $xQ'u'$ and a subset of the colours $a_{\ell+i+2}, \ldots, a_{\ell+\ell'-1}$ on $v'Q'y$. It follows that the path $xQ'u'R_{u'v'}v'Q'y'$ is rainbow. ◇

Case 2. $y \in V(H \cup Q)$ and $x \neq u'_{\ell'}$.

If $x \neq u'_{\ell'}$, then there is an A-free path S from x to a vertex w of $H \cup Q$ (just take a suitable subpath of Q'), and the path $xSwR_{wy}y$ is rainbow. ◇

We are left with the following last case:

Case 3. $y \in V(H \cup Q)$ and $x = u'_{\ell'}$.

The path $xQ'u'Qu$ is rainbow as the colours from A used on it are, in the order from x to u,

$$\gamma, a_{\ell'+\ell-1}, a_{\ell'+\ell-2}, \ldots, a_{\ell+1}, a_k, a_{k-1}, \ldots, a_1.$$

If y is contained in this path, then the appropriate subpath is a rainbow xy-path. Similarly, if $y \in V(u_kQu_\ell)$, then there is a rainbow xy-path as $xQ'u'Qu_\ell$ is rainbow.

The path $xQ'v'$ is rainbow and c' uses no colours from A on it except a_ℓ. It follows that if $y \in V(H)$, then we can append either $v'QvR_{vy}y$ (for Type I), $v'R_{v'y}y$ (for Type II) or $v'QuR_{uy}y$ (for Type III), and get a rainbow xy-path.

It remains to consider the case that $y \in V(u_{\ell+1}Qv)$. Observe that the following subgraphs are rainbow and A-free under c': $xQ'v' \cup u_{\ell+1}Qv$ (Types I and II) and $xQ'v'Qu \cup u_{\ell+1}Qv$ (Type III). Adding the rainbow path $R_{v'v}$ (Type II) or R_{uv} (Type III) if necessary, we obtain a rainbow xy-path R_{xy} in each of the cases. ◇

It remains to check that c' is safe. Observe first that whenever the above-defined path R_{xy} is either a_ℓ-free or edge-disjoint from H, then it is non-blocking (in the latter case, this is because $|im\ c| \geq 2$, and only one colour from $im\ c$ is used on $Q \cup Q'$ by c').

By directly inspecting the above construction, we can readily check that the following cases are (up to symmetry) the only ones where R_{xy} is neither a_ℓ-free nor edge-disjoint from H:

(a) $x = u_\ell$ and $y \in V(H)$,

(b) $x = u'_{\ell'}$ and $y \in V(H)$,

(c) $x = u'_{\ell'}$, $y \in V(u_{\ell+1}Qv)$ and the 2-extension $H \cup Q \cup Q'$ is of Type II or III.

We will now show that only at most one vertex is blocked for u_ℓ. If y is such a vertex, then we are in case (a) above. By the construction, $R_{xy} = xQuR_{uy}y$, so R_{uy} must be blocking in (H, c). By the assumption that c is safe, there is at most one such vertex y as claimed.

Next, we consider the vertex $x = u'_{\ell'}$. Suppose that some vertex $y \in V(u_{\ell+1}Qv)$ is blocked for x (case (c)). Since the 2-extension must be of Type II or III, we have $y = u_{\ell+1}$, for otherwise the colour a_1 is not used on R_{xy}. Furthermore, no vertex $y' \in V(H)$ is blocked for y as a_1 is not used on $R_{xy'}$. Thus, $u_{\ell+1}$ is the only blocked vertex for $u'_{\ell'}$.

For this choice of x, it remains to consider the case that no vertex of $u_{\ell+1}Qv$ is blocked for x. Suppose that $y \in V(H)$ is blocked for x (case (b)). By the construction, R_{xy} contains as a subpath the path R_{vy} (for Type I), $R_{v'y}$ (Type II) or R_{uy} (Type III). In addition, all the colours from $im\ c$ are used on this subpath. It follows that in (H, c), y is blocked for v, v' or u depending on the type, so y is uniquely determined since c is safe.

We have shown that for $x \in \{u_\ell, u'_{\ell'}\}$, there is at most one y which is blocked for x. Considering the cases (a)–(c) above, c' will be proved safe if we show that no $y \in V(H)$ is blocked for both u_ℓ and $u'_{\ell'}$. Suppose that $y \in V(H)$ is blocked for both of these vertices. Since $R_{u_\ell y} = u_\ell QuR_{uy}y$, all the colours from $im\ c$ must be used on R_{uy}, so u is blocked for y in (H, c). By similarly considering $R_{u'_{\ell'}y}$, we find that for Type I or II, the vertex v or v', respectively, would also be blocked for y in (H, c), which is impossible as c is safe.

Hence, the 2-extension must be of Type III. In this case, as observed in the definition of types, $u' \neq u_\ell$ and hence $v' = u_i$ with $i \leq \ell - 2$. Consequently, the colour $a_{\ell-1}$ is not used by c' on $R_{u'_{\ell'}y}$, contradicting the assumption that y is blocked for $u'_{\ell'}$. The proof is complete. $\qquad\square$

4   Proof of Theorem 2

In this section, we prove Theorem 2. Let G be a 2-connected graph with n vertices.

If G is an odd cycle of order n, then the colouring of its edges by

$$1, 2, \ldots, \lfloor n/2 \rfloor, \lceil n/2 \rceil, 1, 2, \ldots, \lfloor n/2 \rfloor$$

is a rainbow colouring with $\lceil n/2 \rceil$ colours. Thus, we may assume that G is not an odd cycle.

We claim that G contains an even cycle. If not, let Z be an odd cycle in G (recall that G is 2-connected) and let v be a vertex not contained in Z. Taking two internally disjoint paths from v to distinct vertices z_1, z_2 on Z and concatenating

them with a $z_1 z_2$-subpath of Z with the appropriate parity, we obtain an even cycle, a contradiction.

Thus, let H_0 be an even cycle in G, say of length $2k$. We construct a subgraph H^* of G by means of a sequence $H_0, H_1, \ldots$ of subgraphs of G. To construct H_{i+1} ($i \geq 0$), we proceed as follows:

- if there is an odd H_i-path P_i, we set $H_{i+1} = H_i \cup P_i$ (making an arbitrary choice if there are more such paths),

- otherwise, if there is a 2-extension $H_i \cup Q_i \cup Q_i'$ of H_i with both Q_i and Q_i' even, we set $H_{i+1} = H_i \cup Q_i \cup Q_i'$,

- if there is neither an odd H_i-path nor an even 2-extension of H_i, we stop and set $H^* = H_i$.

In the rest of this section, the symbol H^* will denote the subgraph of G just constructed. Observe that in the above sequence, each subgraph H_i has an even number of vertices. Thus, $|H^*|$ is even. The following proposition describes the weak H^*-bridges.

Proposition 6. *Let B be a weak H^*-bridge. Then the following holds:*

(i) $H^* \cup B$ is an extension of H^* by a path sequence $(P_1, P_2, \ldots, P_\ell)$, where P_1 is even and the other paths are odd,

(ii) $|B|$ is odd.

Proof. (i) Let M be an inclusionwise maximal extension of H^* contained in $H^* \cup B$. Choose a path sequence $(Q_1, \ldots, Q_s)$ for M. Note that by the construction of H^* and Lemma 3, Q_1 must be even and all the other paths Q_i must be odd.

We claim that $M = H^* \cup B$. Suppose that this is not the case and choose a vertex $w \in V(B) \setminus V(M)$. By the 2-connectedness of G, there are internally disjoint paths R_1, R_2 from w to distinct vertices of M. The concatenation of R_1 and R_2 is an M-path of length at least 2 which can be added to M and provides a contradiction with the maximality of M. This proves part (i).

Part (ii) is a direct consequence of (i). $\qquad\square$

On each H_i, we define a safely rainbow-connecting colouring c_i by $|H_i|/2$ colours. We begin with a colouring c_0 of the even cycle H_0 with values

$$1, 2, \ldots, k, 1, 2, \ldots, k.$$

It is easy to check that this colouring is safely rainbow-connecting. Given c_i, the colouring c_{i+1} is constructed as a continuation to $H_i \cup P_i$ or $H_i \cup Q_i \cup Q_i'$, respectively. By Lemmas 4 and 5, we eventually obtain a safely rainbow-connecting colouring c^* of H^*. By the construction, c^* uses $|H^*|/2$ colours.

At this point, we fix two more 'special' colours in addition to γ, namely α and β. The choice is such that neither α nor β is contained in $im\ c^*$.

Let $u, v \in V(H^*)$ and let H be a subgraph of G such that $H^* \subseteq H$. A colouring b of H is *grounded in* (u, v) if b coincides with c^* on H^*, and for each vertex $x \in V(H) \setminus V(H^*)$, both of the following conditions hold:

(A1) $(H - E(H^*), b)$ contains either a rainbow $\alpha\beta\gamma$-free path from x to H^*, or both a rainbow $\beta\gamma$-free xu-path and a rainbow $\alpha\gamma$-free xv-path,

(A2) for every vertex $y \in V(H) \setminus V(H^*)$, there is a rainbow xy-path in H which is either edge-disjoint from H^*, or β-free.

Note that the definition of a grounded colouring is always related to the same subgraph H^* of G defined above. If the pair (u, v) is not essential, then we just say that b is *grounded*.

The following lemma shows that a continuation of a grounded colouring to an odd 1-extension is grounded.

Lemma 7. *Let $u, v \in V(H^*)$ and let c be a colouring of H, $H^* \subseteq H \subseteq G$, which is grounded in (u, v). If P is an H-path of odd length, then any continuation of c to $H \cup P$ is grounded in (u, v).*

Proof. Let c' be a continuation of c to $H \cup P$ and let x be a vertex in $V(H \cup P) \setminus V(H^*)$. It suffices to verify properties (A1) and (A2) for $x \in V(P)$ since for any other x they follow from the assumption.

We begin with (A1). Let P' be the shorter of the two subpaths of P with one endvertex x and the other endvertex in H. Let the latter endvertex be denoted by w. Observe that P' is rainbow and $\alpha\beta\gamma$-free in $(H \cup P, c')$. By the assumption on c, (H, c) contains either a rainbow $\alpha\beta\gamma$-free path R from w to a vertex $z \in V(H)$, or a $\beta\gamma$-free wu-path R_1 and an $\alpha\gamma$-free wv-path R_2, both rainbow. In the former case, the path $xP'wRz$ is rainbow and $\alpha\beta\gamma$-free, because P' is $\alpha\beta\gamma$-free and no colour from $im\ c$ is used on P'. In the latter case, we similarly obtain paths with the desired properties by prepending P' to R_1 and R_2, respectively.

To verify (A2), let y be a vertex in $V(H \cup P) \setminus V(H^*)$. If $y \notin V(P)$, then we can utilise the path P' as above and concatenate it with a rainbow wy-path in (H, c) satisfying (A2); the resulting xy-path satisfies (A2) as well, because P' is $\beta\gamma$-free and edge-disjoint from H^*.

It remains to discuss the case that $y \in V(P)$. If xPy is rainbow, then we are done as it is β-free. Otherwise, let the endvertices of P be denoted by x' and y' in such a way that $x \in V(x'Py)$. Observe that $xPx' \cup yPy'$ is rainbow and $\alpha\beta\gamma$-free.

If the vertices x', y' are not in H^*, then by the assumption that c is grounded, a rainbow $x'y'$-path S in (H, c) is either edge-disjoint from H^*, or β-free. It follows that the xy-path $xPx'Sy'Py$ is edge-disjoint from H^* or β-free as well.

Since γ may be used on S, it is important that $xPx' \cup yPy'$ is γ-free. This makes the xy-path rainbow.

We may therefore assume that $y' \in V(H^*)$. If $x' \in V(H^*)$, then since c^* is rainbow-connecting, there is a rainbow $x'y'$-path S_1 in (H^*, c^*) which is β-free as $\beta \notin im\ c^*$. Consequently, the xy-path $xPx'Sy'Py$ is rainbow and β-free.

Lastly, if $x' \in V(H) \setminus V(H^*)$ and $y' \in V(H^*)$, then by the assumption that c is grounded, $H - E(H^*)$ contains a rainbow $\beta\gamma$-free path S_2 from x' to a vertex $z \in V(H^*)$. Furthermore, there is a rainbow (necessarily β-free) zy' path S_3 in (H^*, c^*). The path $xPx'S_2zS_3y'Py$ is then β-free and rainbow. $\qquad\square$

Let $B^1, \ldots, B^r$ be all the weak H^*-bridges in G. Fix j, $1 \le j \le r$. We use Proposition 6 to choose a path sequence $(P_1^j, \ldots, P_{\ell^j}^j)$ for $H^* \cup B^j$. The only even path in this sequence, P_1^j, will be called the *base* of B^j. We let the length of P_i^j be denoted by $2k_i^j$ or $2k_i^j + 1$, according to whether it is even or odd. Furthermore, the endvertices of the base of B^j will be denoted by u^j and v^j.

We now extend c^* to a suitable colouring c^j of $H^* \cup B^j$. We first colour the base of B^j, in the direction from u^j to v^j, by

$$a_1, a_2, \ldots, a_{k_1^j - 1}, \alpha, \beta, a_1, a_2, \ldots, a_{k_1^j - 1},$$

where the a_i $(1 \le i \le k_1^j - 1)$ are distinct colours not contained in $im\ c^* \cup \{\alpha, \beta\}$ nor used for the colouring of any other weak H^*-bridge. The colours α and β are used for all the bases. Observe that the colouring is grounded in (u^j, v^j).

We extend the colouring to all of $H^* \cup B^j$ by successively taking continuations to odd 1-extensions in the above path sequence for $H^* \cup B^j$ (recall that all the paths P_i^j with $i \ge 2$ are odd). By a repeated use of Lemmas 4 and 7, we obtain a safely rainbow-connecting colouring c^j which is grounded in (u^j, v^j). Let x^j be the number of colours used by c^j on $B^j - E(H^*)$. We compare x^j to the number of vertices in $V(B^j) \setminus V(H^*)$. For $2 \le i \le \ell^j$, the path P_i^j has length $2k_i^j + 1$, and therefore $2k_i^j$ internal vertices. The number of vertices in $V(B^j) \setminus V(H^*)$ is thus $2k_1^j - 1 + 2k_2^j + \cdots + 2k_{\ell^j}^j$. On the other hand,

$$x^j = k_1^j + 1 + \sum_{i=2}^{\ell^j} k_i^j = \frac{|V(B^j) \setminus V(H^*)|}{2} + \frac{3}{2}. \qquad (1)$$

Since each c^j extends c^* and the weak H^*-bridges are pairwise edge-disjoint, we can combine all the colourings c^j, $1 \le j \le r$, to a colouring $\tilde{c}$ of G. We assert that $(G, \tilde{c})$ is rainbow-connected. To check this, it is enough to show that any two vertices contained in different H^*-bridges are joined by a rainbow path. Let us say that $x \in V(B^1) \setminus V(H^*)$ and $y \in V(B^2) \setminus V(H^*)$. By condition (A1) in the definition of a grounded colouring, $(H^* \cup B^1, c^1)$ contains an $\alpha\gamma$-free rainbow path from x to a vertex w_1 in H^*, and $(H^* \cup B^2, c^2)$ contains a $\beta\gamma$-free rainbow path from y to a vertex w_2 in H^*. Since no colour from $im\ c^*$ is used on these paths

by $\tilde{c}$, and each of α and β is used on at most one of them, we can concatenate the paths with a rainbow w_1w_2-path in (H^*, c^*) and obtain a rainbow xy-path in $(G, \tilde{c})$. This completes the proof that $(G, \tilde{c})$ is rainbow.

The number of colours used by $\tilde{c}$ can be obtained by using (1) and making a correction to account for the fact that the same colours α, β are used in all B^j:

$$|im\ \tilde{c}| = |im\ c^*| + \left(\sum_{j=1}^{r} x^j\right) - 2(r-1)$$

$$= \frac{|H^*|}{2} + \left(\sum_{j=1}^{r} \frac{|V(B^j) \setminus V(H^*)|}{2}\right) - \frac{r}{2} + 2 = \frac{|G|}{2} - \frac{r}{2} + 2. \qquad (2)$$

Note that for $r \geq 3$, we have $|im\ \tilde{c}| \leq \lceil |G|/2 \rceil$, which implies the statement in Theorem 2. Similarly, if $r = 0$, then $H^* = G$ and we are done as well, because (G, c^*) is then rainbow-connected and c^* uses $|G|/2$ colours. Consequently, we may assume that

$$1 \leq r \leq 2.$$

We will perform a simple recolouring to reduce the number of colours by one and obtain a colouring satisfying the bound in Theorem 2.

The case $r = 1$ is simple. Since $\tilde{c}$ is grounded, every vertex in $V(G) \setminus V(H^*)$ is joined to H^* by a $\beta\gamma$-free path. Thus, if we recolour the unique edge coloured by β to a colour $\gamma' \in im\ c^* \setminus \{\gamma\}$, G is still rainbow-connected with respect to the resulting colouring. The number of colours used after this reduction is exactly $\lceil |G|/2 \rceil$.

It thus remains to consider the case $r = 2$. This case is the reason why we need to restrict ourselves to safe colourings. For $j = 1, 2$, let e_α^j and e_β^j denote the edge of P_1^j coloured by α and β, respectively.

Since c^* is safe, there is a rainbow path P from u^1 to either u^2 or v^2, and a colour $\delta \in im\ c^*$ which is not used on P. (We allow $\delta = \gamma$.) As the reversal of the colouring on P_1^2 affects neither the rainbow-connectedness of G nor the groundedness of the colouring, we may actually assume that the endvertices of P are u^1 and v^2.

We recolour the edges e_β^1 and e_β^2 to δ, and we argue that with the resulting colouring $\bar{c}$, G is still rainbow-connected. Let x, y be vertices of G. Since no rainbow paths inside H^* are affected by the recolouring, we may assume that $x \in V(B^1) \setminus V(H^*)$.

We distinguish several cases based on the location of y.

Case 1. $y \in V(B^1) \setminus V(H^*)$.

Since c^1 is grounded, condition (A2) says that in $(H^* \cup B^1, c^1)$ there is a rainbow xy-path which is either β-free, or edge-disjoint from H^*. In either case, the colouring $\bar{c}$ only uses the colour δ at most once on this path, which is therefore rainbow. ◇

14

Case 2. $y \in V(H^*)$.

Since c^1 is grounded, (B^1, c^1) contains a $\beta\gamma$-free rainbow path Q_1 from x to a vertex $w \in V(H^*)$. If we let Q_2 be a rainbow wy-path in (H^*, c^*), then xQ_1wQ_2y is a rainbow xy-path. $\diamond$

Case 3. $y \in V(B^2) \setminus V(H^*)$.

As c^1 is grounded, there is a rainbow path R_1 in (B^1, c^1) from y to a vertex $w_1 \in V(H^*)$, where R_1 is either $\alpha\beta\gamma$-free, or it is $\beta\gamma$-free and $w_1 = u^1$.

As a first subcase, assume that R_1 is $\alpha\beta\gamma$-free, and choose a rainbow $\beta\gamma$-free path R_2 in (B^2, c^2) from y to a vertex $w_2 \in V(H^*)$ using condition (A1) in the definition of a grounded colouring. Thus, $R_1 \cup R_2$ is $\beta\gamma$-free with respect to $\tilde{c}$. Since in $(G, \tilde{c})$, the sets of colours used to colour B^1 and B^2 are disjoint except for α, β, γ, $R_1 \cup R_2$ is rainbow with respect to $\tilde{c}$.

Let R_0 be a rainbow w_1w_2-path in (H^*, c^*), and define $R = xR_1w_1Rw_2R_2y$. Since γ is not used by $\tilde{c}$ on $R_1 \cup R_2$, R is rainbow in $(G, \tilde{c})$. Furthermore, it remains rainbow after the recolouring of e_β^1 and e_β^2 to δ, since $R_1 \cup R_2$ is β-free. Thus, R is rainbow in $(G, \bar{c})$.

A symmetric situation occurs when (B^2, c^2) contains an $\alpha\beta\gamma$-free rainbow path from y to H^*. We may therefore assume that R_1 is a $\beta\gamma$-free xu^1-path, and choose an $\alpha\gamma$-free rainbow yv^2-path R_2' in $(H^* \cup B^2, c^2)$. Let P be the δ-free rainbow u^1v^2-path in (H^*, c^*) defined above, and $R' = xR_1u^1Pv^2R_2'y$.

We claim that no colour is repeated on R' in $(G, \bar{c})$: α may only used on R_1 as R_2' is α-free and $\alpha \notin im\, c^*$, γ is not used on $R_1 \cup R_2'$, and δ may only be used on R_2', since P is δ-free and so is R_1 (being chosen to be β-free with respect to c^1). This concludes the discussion of this case. $\diamond$

We have just shown that $(G, \bar{c})$ is rainbow-connected. As for $|im\, \bar{c}|$, by eliminating the color β we decreased the value in (2) by one, and we get

$$|im\, \bar{c}| = \frac{|G|}{2}.$$

The proof of Theorem 2 is now complete.

5 Higher connectivity

In view of Theorem 2, it is natural to ask whether one can further improve the bound for graphs of higher connectivity. The question on the relation between connectivity and the rainbow connection number was previously asked by H. J. Broersma (cf. [10, Problem 2.22]). One estimate follows from a result of Chandran et al. [4], improving a bound of Schiermeyer [12]:
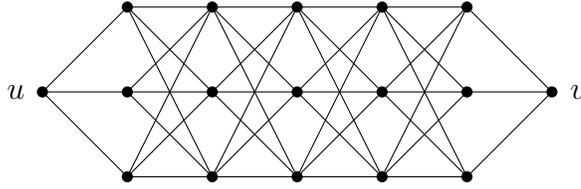
Figure 2: A k-connected graph G with $k(\ell-1)+2$ vertices and $rc(G) \geq \ell$. (Shown for $k = 3$ and $\ell = 6$.)

Theorem 8 ([4]). *A connected graph G with n vertices and minimum degree $\delta(G)$ has*

$$rc(G) \leq \frac{3n}{\delta(G)+1} + 3.$$

Since the minimum degree of a graph is greater than or equal to its connectivity, the theorem implies a bound for k-connected graphs.

We show that for every k, there are k-connected graphs G with n vertices and

$$rc(G) \geq \frac{n-2}{k} + 1.$$

For fixed k, ℓ, let P be a path of length ℓ with endvertices u_0 and v_0, and let I be a graph consisting of k independent vertices. Let G_0 be the lexicographic product of P and I. Thus, G_0 has vertex set $V(P) \times V(I)$ and vertices (x, y) and (x', y') are joined by an edge whenever x and x' are adjacent in P. Let G be the graph obtained from G_0 by identifying all vertices of the form (u_0, y) $(y \in V(I))$ into one vertex u, and all vertices of the form (v_0, y) $(y \in V(I))$ into another vertex v. See Figure 2 for an illustration.

It is easy to see that G is k-connected and has $n := k(\ell - 1) + 2$ vertices. Since shortest uv-paths have length ℓ, we have

$$rc(G) \geq \ell = \frac{n-2}{k} + 1$$

as claimed above. We propose the following question:

Problem 9. *Is there a constant $C = C(k)$ such that every k-connected graph G with n vertices satisfies*

$$rc(G) \leq \frac{n}{k} + C?$$

As remarked in Section 1, a positive answer to the above question was independently conjectured in [9]. The same paper proves the conjecture for graphs of high girth, and asymptotically improves the estimate obtained from Theorem 8 (for k-connected graphs of order n) to $rc(G) \leq (2 + \varepsilon)n/k + 23/\varepsilon^2$, where ε is an arbitrary real number in the interval $(0, 1)$.

16

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