A closure for 1-Hamilton-connectedness in claw-free graphs

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April 4, 2013

Abstract

A graph G is 1-Hamilton-connected if $G - x$ is Hamilton-connected for every vertex $x \in V(G)$. In the paper we introduce a closure concept for 1-Hamilton-connectedness in claw-free graphs. If $\overline{G}$ is a (new) closure of a claw-free graph G, then $\overline{G}$ is 1-Hamilton-connected if and only if G is 1-Hamilton-connected, $\overline{G}$ is the line graph of a multigraph, and for some $x \in V(G)$, $\overline{G} - x$ is the line graph of a multigraph with at most two triangles or at most one double edge. As applications, we prove that Thomassen's Conjecture (every 4-connected line graph is hamiltonian) is equivalent to the statement that every 4-connected claw-free graph is 1-Hamilton-connected, and we present results showing that every 5-connected claw-free graph with minimum degree at least 6 is 1-Hamilton-connected and that every 4-connected claw-free and hourglass-free graph is 1-Hamilton-connected.

1 Introduction

A well-known concept in Hamiltonian graph theory is the closure operation $\mathrm{cl}(G)$ for claw-free graphs, introduced in [21]. The closure operation turns a claw-free graph into the line graph of a triangle-free graph while preserving the hamiltonicity of the graph. While $\mathrm{cl}(G)$ also preserves many weaker graph properties (such as traceability or the existence of a 2-factor), stronger properties, such as Hamilton-connectedness, turn out not to be preserved [4], [22]. The first attempt to develop a closure for Hamilton-connectedness was by Brandt [3], the technique was further developed in [23] and [13]. In the present paper, we further strengthen these techniques to the property of 1-Hamilton-connectedness (where a graph G is k-Hamilton-connected if $G - M$ is Hamilton-connected for any set of vertices $M \subset V(G)$ with $|M| = k$).

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[5]Research supported by project P202/12/G061 of the Czech Science Foundation and by the European Regional Development Fund (ERDF), project NTIS - New Technologies for Information Society, European Centre of Excellence, CZ.1.05/1.1.00/02.0090.

The concept of k-Hamilton-connectedness was introduced already in 1970 by Lick [19], and since then studied in many papers (see e.g. [16], [10]). The property of 1-Hamilton-connectedness is closely related to a well-known conjecture by Thomassen [25] which states that every 4-connected line graph is hamiltonian, since it was recently shown [12] that Thomassen's Conjecture is equivalent with the statement that every 4-connected line graph is 1-Hamilton-connected. Having in mind that 4-connectedness is a necessary condition for a graph to be 1-Hamilton-connected, it was observed in [12] that Thomassen's Conjecture, if true, would imply that a line graph is 1-Hamilton-connected if and only if it is 4-connected, which means that 1-Hamilton-connectedness would be polynomial in line graphs (and, as a corollary of our main result, also in claw-free graphs).

Note that Lai and Shao recently proved that, for $s \geq 5$, a line graph G is s-hamiltonian (i.e., $G - X$ is hamiltonian for any $X \subset V(G)$ with $|X| = s$), if and only if G is $(s+2)$-connected [15]. A similar result is also known to be true in planar graphs, where every 4-connected planar graph is 1-Hamilton-connected (an easy consequence of [24], page 342), implying polynomiality of 1-Hamilton-connectedness. Also, there are results indicating that Tutte cycles [26] seem to behave similarly in planar graphs and in claw-free graphs [6]. This potential connection to planar graphs is also one of the motivations of our research.

Also note that there are many further known equivalent versions of Thomassen's Conjecture (see [5] for a survey on this topic).

In the present paper, we

- in Section 3, develop a closure concept for 1-Hamilton-connectedness in claw-free graphs,
- in Section 4, show three applications of the closure: we prove that (i) Thomassen's Conjecture is equivalent with the statement that every 4-connected claw-free graph is 1-Hamilton-connected, and we present results showing that (ii) every 5-connected claw-free graph with minimum degree at least 6 is 1-Hamilton-connected, and (iii) every 4-connected claw-free hourglass-free graph is 1-Hamilton-connected. The last two results require nontrivial proofs and hence their proofs are published in a separate paper [8].

We follow the most common graph-theoretical terminology and for concepts and notations not defined here we refer e.g. to [2]. Specifically, by a *graph* we mean a finite undirected graph $G = (V(G), E(G))$; in general, we allow a graph to have multiple edges. The precise way of using (simple) graphs and multigraphs will be specified later in Section 2. Even if not explicitly stated, we assume all graphs under consideration to be connected. We use $d_G(x)$ to denote the *degree* of a vertex x, and we set $V_i(G) = \{x \in V(G) |\ d_G(x) = i\}$. The *neighborhood* of a vertex x, denoted $N_G(x)$, is the set of all neighbors of x, and we define the *closed neighborhood* of x as $N_G[x] = N_G(x) \cup \{x\}$. For a set $M \subset V(G)$, $\langle M \rangle_G$ denotes the *induced subgraph* on M, and for a graph F, G is said to be *F-free* if G does not contain an induced subgraph isomorphic to F. Specifically, for $F = K_{1,3}$ we say that G is *claw-free*.

If $\{x, y\} \subset V(G)$ is a vertex-cut of G and G_1, G_2 are components of $G - \{x, y\}$; then the subgraphs $\langle V(G_1) \cup \{x, y\}\rangle_G$ and $\langle V(G_2) \cup \{x, y\}\rangle_G$ are called the *bicomponents* (of G at $\{x, y\}$).

For $x \in V(G)$, $G - x$ is the graph obtained from G by removing x and all edges incident with x. If $x, y \in V(G)$ are such that $e = xy \notin E(G)$, then $G + e$ is the graph with $V(G + e) = V(G)$ and $E(G + e) = E(G) \cup \{e\}$, and, conversely, for $e = xy \in E(G)$ we denote by $G - e$ the graph with $V(G - e) = V(G)$ and $E(G - e) = E(G) \setminus \{e\}$.

We use $\alpha(G)$ to denote the *independence number* of G, $\nu(G)$ to denote the *matching number* of G (i.e., the size of a largest matching in G), and $\omega(G)$ stands for the *number of components* of G. A complete subgraph $K \subset G$ will be called a *clique* and, when no confusion can arise, we will use K also for the vertex set of a clique (thus, for cliques K^1, K^2, instead of $V(K^1) \cap V(K^2)$, we will simply write $K^1 \cap K^2$). A vertex $x \in V(G)$ is *simplicial* if $\langle N_G(x)\rangle_G$ is a clique, and an edge $e \in E(G)$ is *pendant* if one of its vertices is of degree 1.

A graph G is *hamiltonian* if G contains a *hamiltonian cycle*, i.e. a cycle of length $|V(G)|$, and G is Hamilton-connected if, for any $a, b \in V(G)$, G contains a *hamiltonian (a, b)-path*, i.e., an (a, b)-path P with $V(P) = V(G)$. For $k \geq 1$, G is *k-Hamilton-connected* if $G - X$ is Hamilton-connected for every set of vertices $X \subset V(G)$ with $|X| = k$. Note that a hamiltonian graph is necessarily 2-connected, a Hamilton-connected graph is 3-connected and if G is k-Hamilton-connected, then G is $(k + 3)$-connected.

2 Preliminary results

In this section we summarize some background knowledge that will be needed for our results.

The *line graph* of a graph (multigraph) H, denoted $L(H)$, is the graph with $E(H)$ as its vertex set, in which two vertices are adjacent if and only if the corresponding edges have a vertex in common. Recall that every line graph is claw-free.

It is well-known that if G is a line graph of a connected simple graph, then the graph H such that $G = L(H)$ (called the *preimage* of G) is uniquely determined, with one exception, namely $G = K_3$. However, for line graphs of multigraphs this is, in general, not true – an easy example is the graph T_1 in Fig. 1 which is the line graph of two nonisomorphic graphs: the unique (simple) graph H_1 with degree sequence $3, 2, 2, 1$, and the unique multigraph H_2 with degree sequence $3, 3, 1, 1$. This difficulty can be overcome by imposing an additional requirement that simplicial vertices in the line graph correspond to pendant edges.

Proposition A [23]. *Let G be a connected line graph of a multigraph. Then there is, up to isomorphism, a uniquely determined multigraph H such that $G = L(H)$ and a vertex $e \in V(G)$ is simplicial in G if and only if the corresponding edge $e \in E(H)$ is a pendant edge in H.*

For a line graph G, we will always consider its preimage to be the unique multigraph with the properties given in Proposition A; this preimage will be denoted $L^{-1}(G)$. This means that, throughout the paper, when working with a claw-free graph or with a line graph G, we always consider G to be a simple graph, while if G is a line graph, for its preimage $H = L^{-1}(G)$ we always admit H to be a multigraph, i.e. we always allow H to have multiple edges.

We will also use the notation $e = L^{-1}(a)$ and $a = L(e)$ in situations when $H = L^{-1}(G)$, $a \in V(G)$ and $e \in E(H)$ is the edge of H corresponding to the vertex a. Note that our special choice of the line graph preimage already implies some restrictions on its structure: for example, it is not difficult to observe that $H = L^{-1}(G)$ can never contain a triangle with two vertices of degree 2, for if $\langle\{x_1, x_2, x_3\}\rangle_H$ is such a triangle with $d_H(x_1) = d_H(x_2) = 2$, then $L(x_1x_2)$ is a simplicial vertex in G, but x_1x_2 is not a pendant edge in H (see the graphs H_1 and G in the example prior to Proposition A). More generally, if $\langle\{x_1, x_2\}\rangle_H$ is a multiedge in $H = L^{-1}(G)$, then both x_1 and x_2 must have a neighbor outside the set $\{x_1, x_2\}$, and if $\langle\{x_1, x_2, x_3\}\rangle_H$ is a triangle or a *multitriangle* (a triangle with some multiple edges) in H, then at most one of the vertices x_1, x_2, x_3 can have no neighbor outside the set $\{x_1, x_2, x_3\}$ (for otherwise G contains a simplicial vertex corresponding to a nonpendant edge of H).

We will need the following characterization of line graphs of multigraphs by Krausz [11].

Theorem B [11]. *A graph G of order at least 1 is a line graph of a multigraph if and only if $V(G)$ can be covered by a system of cliques $\mathcal{K}$ such that every vertex of G is in exactly two cliques of $\mathcal{K}$ and every edge of G is in at least one clique of $\mathcal{K}$.*

If G is a line graph and $\mathcal{K} = \{K^1, ..., K^m\}$ is a partition with the properties given in Theorem B, then a graph H such that $G = L(H)$ can be obtained from $\mathcal{K}$ as the intersection graph (multigraph) of the set system $\{V(K^1), ..., V(K^m)\}$, in which the number of vertices shared by two cliques equals the multiplicity of the (multi)edge joining the corresponding vertices of H. A system of cliques $\mathcal{K} = \{K^1, ..., K^m\}$ with the properties given in Theorem B is called a *Krausz partition* of G, and its elements are called *Krausz cliques*. Note that not every clique (and even not every maximal clique) in a line graph G has to be a Krausz clique. If $G = L(H)$, then such non-Krausz cliques in G can correspond to (some of the) triangles, multiple edges or multitriangles.

In general, for a given line graph G, a Krausz partition is not uniquely determined, but every such partition uniquely determines a graph H with the property $G = L(H)$ as its intersection graph. However, by Proposition A, every line graph G has a unique Krausz partition $\mathcal{K}$ such that a vertex $x \in V(G)$ is simplicial if and only if one of the two cliques containing x is of order 1. Thus, whenever we will be working with Krausz cliques and Krausz partitions, we will be always using this particular uniquely determined partition (which gives the unique preimage $L^{-1}(G)$).

Harary and Nash-Williams [7] showed that a line graph G of order at least 3 is hamiltonian if and only if $H = L^{-1}(G)$ contains a *dominating closed trail*, i.e. a closed trail (eulerian subgraph) T such that every edge of H has at least one vertex on T. A similar argument gives the following analogue for Hamilton-connectedness (see e.g. [17]). Here an *internally dominating trail* (abbreviated IDT) is a trail T such that every edge in

$E(H) \setminus E(T)$ has at least one vertex on T as its internal vertex, and, for $e_1, e_2 \in E(H)$, an (e_1, e_2)-IDT is an IDT having e_1 and e_2 as terminal edges.

Theorem C [17]. *A line graph G of order at least 3 is Hamilton-connected if and only if $H = L^{-1}(G)$ has an (e_1, e_2)-IDT for any pair of edges $e_1, e_2 \in E(H)$.*

An edge cut R of a graph H is *essential* if $H - R$ has at least two nontrivial components. For an integer $k > 0$, H is *essentially k-edge-connected* if every essential edge cut R of G contains at least k edges. Obviously, a line graph $G = L(H)$ with $\alpha(G) \geq 2$ is k-connected if and only if the graph H is essentially k-edge-connected.

A vertex $x \in V(G)$ is *locally connected* (*eligible*), if $\langle N(x) \rangle$ is a connected (connected noncomplete) subgraph of G, respectively. The set of all eligible vertices in G will be denoted $V_{EL}(G)$.

For $x \in V(G)$, the *local completion of G at x* is the graph $G_x^* = (V(G), E(G) \cup \{y_1y_2 | \; y_1, y_2 \in N_G(x)\})$, i.e. the graph obtained from G by turning $\langle N_G(x) \rangle_G$ into a clique. It is an easy observation that in the special case when G is a line graph and $H = L^{-1}(G)$, a vertex $x \in V(G)$ is locally connected if and only if the edge $e = L_G^{-1}(x)$ is in a triangle or in a multiedge in H, and $G_x^* = L(H|_e)$, where the graph $H|_e$ is obtained from H by contraction of e into a vertex and replacing the created loop(s) by pendant edge(s).

As shown in [21], if G is claw-free and $x \in V_{EL}(G)$, then G_x^* is hamiltonian if and only if G is hamiltonian. The *closure* $\text{cl}(G)$ of a claw-free graph G is then defined [21] as the graph obtained from G by recursively performing the local completion operation at eligible vertices, as long as this is possible. We say that G is *closed* if $G = \text{cl}(G)$. It is well-known [21] that, for every claw-free graph G, (*i*) $\text{cl}(G)$ is uniquely determined, (*ii*) $\text{cl}(G)$ is the line graph of a triangle-free graph, and (*iii*) $\text{cl}(G)$ is hamiltonian if and only if G is hamiltonian.

Recall that the closure operation $\text{cl}(G)$ does not preserve the Hamilton-connectedness of G [22], [4]. Thus, more generally, for $k \geq 1$, we say that a vertex x is *k-eligible* if $\langle N(x) \rangle$ is k-connected noncomplete. The following fact was conjectured in [1] and proved in [22].

Proposition D [22]. *If G is claw-free and $x \in V(G)$ is 2-eligible, then G is Hamilton-connected if and only if G_x^* is Hamilton-connected.*

We will often use the following observation. Let T_1, T_2 be the graphs shown in Fig. 1 (the graph T_1 will be referred to as the *diamond*). Let $G = L(H)$, suppose that H contains

Figure 1

a subgraph F isomorphic to T_1 or T_2 (in case of T_2 such that at least one vertex incident with e has a neighbor outside F), and set $x = L(e)$. Then it is easy to see that x is 2-eligible in G and, consequently, by Proposition D, $G = L(H)$ is Hamilton-connected if and only if $G^*_x = L(H|_e)$ is Hamilton-connected (or, equivalently, H has an (f_1, f_2)-IDT for any $f_1, f_2 \in E(H)$ if and only if $H|_e$ has an (f_1, f_2)-IDT for any $f_1, f_2 \in E(H|_e)$).

By recursively performing the local completion operation at k-eligible vertices, we can define [1] the k-closure $\mathrm{cl}_k(G)$ of G, which is uniquely determined [1] and, if G is claw-free, $\mathrm{cl}_2(G)$ is Hamilton-connected if and only if so is G [22].

It can be easily seen that, in general, $\mathrm{cl}_2(G)$ is not a line graph, and even not a line graph of a multigraph. To overcome this drawback, the authors developed in [23] the concept of the *multigraph closure* (or briefly *M-closure*) $\mathrm{cl}^M(G)$ of a graph G: the graph $\mathrm{cl}^M(G)$ is obtained from $\mathrm{cl}_2(G)$ by performing local completions at some (but not all) 1-eligible vertices, where these vertices are chosen in a special way such that the resulting graph is a line graph of a multigraph while still preserving the (non-)Hamilton-connectedness of G. We do not give technical details of the construction since these will not be needed for our proofs; we refer the interested reader to [22], [23].

The concept of M-closure was further strengthened in [13] in such a way that the closure of a claw-free graph is the line graph of a multigraph with either at most two triangles and no multiedge, or with at most one double edge and no triangle.

For a given claw-free graph G, we construct a graph G^M by the following construction.

- (i) If G is Hamilton-connected, we set $G^M = \mathrm{cl}(G)$.
- (ii) If G is not Hamilton-connected, we recursively perform the local completion operation at such eligible vertices for which the resulting graph is still not Hamilton-connected, as long as this is possible. We obtain a sequence of graphs $G_1, \ldots, G_k$ such that
  - $G_1 = G$,
  - $G_{i+1} = (G_i)^*_{x_i}$ for some $x_i \in V_{EL}(G_i)$, $i = 1, \ldots, k-1$,
  - G_k has no hamiltonian (a, b)-path for some $a, b \in V(G_k)$,
  - for any $x \in V_{EL}(G_k)$, $(G_k)^*_x$ is Hamilton-connected,

  and we set $G^M = G_k$.

A graph G^M obtained by the above construction will be called a *strong M-closure* (or briefly an *SM-closure*) of the graph G, and a graph G equal to its SM-closure will be said to be *SM-closed*.

The following theorem summarizes basic properties of the SM-closure operation.

Theorem E [13]. *Let G be a claw-free graph and let G^M be its SM-closure. Then G^M has the following properties:*

- (i) *$V(G) = V(G^M)$ and $E(G) \subset E(G^M)$,*
- (ii) *G^M is obtained from G by a sequence of local completions at eligible vertices,*
- (iii) *G is Hamilton-connected if and only if G^M is Hamilton-connected,*
- (iv) *if G is Hamilton-connected, then $G^M = \mathrm{cl}(G)$,*
- (v) *if G is not Hamilton-connected, then either*
  - (α) *$V_{EL}(G^M) = \emptyset$ and $G^M = \mathrm{cl}(G)$, or*
  - (β) *$V_{EL}(G^M) \neq \emptyset$ and $(G^M)^*_x$ is Hamilton-connected for any $x \in V_{EL}(G^M)$,*

(*vi*) $G^M = L(H)$, *where* H *contains either*

(α) *at most 2 triangles and no multiedge, or*

(β) *no triangle, at most one double edge and no other multiedge,*

(*vii*) *if* G *contains no hamiltonian* (a, b)*-path for some* $a, b \in V(G)$ *and*

(α) X *is a triangle in* H*, then* $E(X) \cap \{L^{-1}_{G^M}(a), L^{-1}_{G^M}(b)\} \neq \emptyset$,

(β) X *is a multiedge in* H*, then* $E(X) = \{L^{-1}_{G^M}(a), L^{-1}_{G^M}(b)\}$.

Note that in some cases (specifically, in cases (*iv*) and (*v*)(α) of Theorem E), we have $V_{EL}(G^M) = \emptyset$ and $G^M = \mathrm{cl}(G)$, implying that G^M is uniquely determined. However, if $V_{EL}(G^M) \neq \emptyset$, then, for a given graph G, its SM-closure G^M is in general not uniquely determined and its construction requires knowledge of a pair of vertices a, b for which there is no hamiltonian (a, b)-path in G. Consequently, there is not much hope to construct G^M in polynomial time (unless P=NP).

3 Closure for 1-Hamilton-connectedness

Let G be a claw-free graph and let $x \in V(G)$ be such that $G-x$ is not Hamilton-connected. Let $\widetilde{G}_x$ be a graph obtained by the following construction.

(1) Set $G_0 := G$, $i := 0$.

(2) If there is a $u_i \in V(G_i - x)$ such that u_i is eligible in $G_i - x$ and $(G_i)^*_{u_i} - x$ is not Hamilton-connected, then set $G_{i+1} = (G_i)^*_{u_i}$ and go to (3),

otherwise set $\widetilde{G}_x := G_i$ and stop.

(3) Set $i := i + 1$ and go to (2).

Then we say that $\widetilde{G}_x$ is a *partial* x*-closure* of the graph G.

The following proposition summarizes the main properties of a partial x-closure of a claw-free graph. Here the *5-wheel*, denoted W_5, is the graph consisting of a 5-cycle C_5 and a vertex (the *center* of the W_5) adjacent to all vertices of the C_5.

Proposition 1. *Let* G *be a claw-free graph, let* $x \in V(G)$ *be such that* $G - x$ *is not Hamilton-connected, and let* $\widetilde{G}_x$ *be a partial* x*-closure of* G*. Then* $\widetilde{G}_x - x$ *is an* SM*-closed line graph and* $\widetilde{G}_x$ *satisfies one of the following:*

(*i*) $\widetilde{G}_x$ *is a line graph;*

(*ii*) x *is a center of an induced* W_5*, and there are* $u_1, u_2 \in N_{\widetilde{G}_x}(x)$ *such that*

(α) $\{u_1, u_2\}$ *is a cut set of* $\widetilde{G}_x - x$,

(β) *one of the bicomponents of* $\widetilde{G}_x - x$ *at* $\{u_1, u_2\}$ *is isomorphic to* $K_3 - e$,

(γ) *the graph* $(\widetilde{G}_x + \{u_1, u_2\}) - x$ *contains no induced* W_5 *with center at* x,

(δ) *the graph* $(\widetilde{G}_x + \{u_1, u_2\}) - x$ *is* SM*-closed;*

(*iii*) *there are Krausz cliques* K^x_1, K^x_2 *in* $\widetilde{G}_x - x$ *such that*

(α) $N_{\widetilde{G}_x}(x) \subset K^x_1 \cup K^x_2$,

(β) *the graph* $(V(\widetilde{G}_x), E(\widetilde{G}_x) \cup \{xv|\ v \in K^x_1 \cup K^x_2\})$ *is a line graph.*

Proof of Proposition 1 is postponed to Section 5. ■

Note that if G is such that $\widetilde{G}_x$ satisfies (ii) of Proposition 1, then the graph $\widetilde{G}_x + uv$ contains no induced W_5 with center at x, hence $\widetilde{G}_x + uv$ satisfies (i) or (iii) of Proposition 1.

It is also easy to see that, in case (ii), $\{L^{-1}(u_1), L^{-1}(u_2)\}$ is a 2-element edge cut of $H = L^{-1}(\widetilde{G}_x - x)$ separating a single edge from the rest of H.

Let now G be a claw-free graph, and let $\overline{G}$ be a graph obtained by the following construction:

(1) If G is 1-Hamilton-connected, set $\overline{G} = \mathrm{cl}(G)$.

(2) If G is not 1-Hamilton-connected, choose a vertex $x \in V(G)$ such that $G - x$ is not Hamilton-connected and a partial x-closure $\widetilde{G}_x$ of G.

(3) If $\widetilde{G}_x$ satisfies (ii) of Proposition 1 (i.e., x is a center of an induced W_5 in $\widetilde{G}_x$), choose a cut set $\{u_1, u_2\}$ of $\widetilde{G}_x - x$, add the edge u_1u_2 to $\widetilde{G}_x$ (i.e., set $\widetilde{G}_x := \widetilde{G}_x + u_1u_2$), and proceed to (4).

(4) If $\widetilde{G}_x$ is a line graph, set $\overline{G} = \widetilde{G}_x$.

Otherwise, $\widetilde{G}_x$ satisfies (iii) of Proposition 1, i.e. some two Krausz cliques K_1^x, K_2^x in $\widetilde{G}_x - x$ cover all vertices in $N_{\overline{G}}(x)$, and then set $\overline{G} = (V(\widetilde{G}_x), E(\widetilde{G}_x) \cup \{xv|\ v \in (K_1^x \cup K_2^x)\})$.

Then we say that the resulting graph $\overline{G}$ is a *1HC-closure* of the graph G.

The following result summarizes basic properties of a 1HC-closure of a graph G.

Theorem 2. *Let G be a claw-free graph and let $\overline{G}$ be its 1HC-closure. Then*

(i) *$\overline{G}$ is a line graph,*

(ii) *for some $x \in V(\overline{G})$, the graph $\overline{G} - x$ is SM-closed,*

(iii) *$\overline{G}$ is 1-Hamilton-connected if and only if G is 1-Hamilton-connected.*

Proof. Properties (i) and (ii) follow immediately by the definition of $\overline{G}$. Also clearly $\overline{G}$ is 1-Hamilton-connected if so is G, and if G is not 1-Hamilton-connected, then neither is $\widetilde{G}_x$ (for some $x \in V(G)$ which is used in the construction). It remains to show that $\overline{G}$ is not 1-Hamilton-connected if $\widetilde{G}_x$ is not. This is clear if $\widetilde{G}_x$ satisfies (i) or (iii) of Proposition 1. Finally, if $\widetilde{G}_x$ satisfies (ii), then $\overline{G}$ is not 1-Hamilton-connected since neither $\widetilde{G}_x$ nor $\overline{G}$ is 4-connected. ■

Note that (ii) is equivalent to the statement that $H = L^{-1}(\overline{G})$ contains an edge $e \in E(H)$ such that $L(H - e)$ is SM-closed.

Also note that, for a given claw-free graph G, its 1-Hamilton-connected closure is not uniquely determined.

We finish this section with a result which shows that steps (3) and (4) in the definition of a 1HC-closure of a graph can be also accomplished by adding (some) edges in neighborhoods of eligible vertices.

Proposition 3. *Let G be a claw-free graph. Then there is a sequence of graphs $G_0, \dots, G_k$ such that*

(i) $G_0 = G$,

(ii) $V(G_i) = V(G_{i+1})$ *and* $E(G_i) \subset E(G_{i+1}) \subset E((G_i)^*_{x_i})$ *for some* $x_i \in V(G_i)$ *eligible in* G_i,

(iii) G_k *is a 1HC-closure of* G.

Proof. Steps (1) and (2) of the definition of a 1HC-closure clearly satisfy the conditions of the proposition, and so does step (3), since the added edge has both vertices in $N_G(x)$ and x is eligible. It remains to verify the statement in step (4). Suppose, to the contrary, that, in step (4), for some Krausz clique K_i^x in $\widetilde{G}_x - x$, adding the edges joining K_i^x to x does not satisfy the conditions.

If $|K_i^x \cap N_G(x)| \geq 2$, then K_i^x and $\langle N_G(x)\rangle_{\widetilde{G}_x}$ share an edge, say, v_1v_2, but then v_1 is eligible, a contradiction. Hence $|K_i^x \cap N_G(x)| = 1$. Let $K_i^x \cap N_G(x) = \{u\}$. By the properties of the Krausz partition, u is, besides K_i^x, in some other Krausz clique K_j^x. If $\langle N_G(x)\rangle_{\widetilde{G}_x}$ is disconnected, then u is a simplicial vertex in $G - x$ (otherwise u centers a claw in G) and, since simplicial vertices in $G - x$ correspond to pendant edges in $H = L^{-1}(G)$, one of K_i^x, K_j^x (say, K_j^x) is of size 1. But then, extending K_j^x to x adds no new edge to $\widetilde{G}_x$.

Finally, if $\langle N_G(x)\rangle_{\widetilde{G}_x}$ is connected, then there is an edge e in $\langle N_G(x)\rangle_{\widetilde{G}_x}$ containing u, and necessarily e is in K_j^x. But then, for the clique K_j^x, we have $|K_j^x \cap N_G(x)| \geq 2$ and we are in the previous case. ■

4 Applications of the closure

In this section we show three applications of the 1HC-closure, related to Thomassen's Conjecture. As already mentioned, there are many known equivalent versions of the conjecture. As our first application, we show the following equivalence.

Theorem 4. *The following statements are equivalent:*

(i) *Every 4-connected line graph is hamiltonian.*

(ii) *Every 4-connected claw-free graph is 1-Hamilton-connected.*

Proof. Obviously, (ii) implies (i). Conversely, first recall that, by a recent result [12], (i) is equivalent to the statement that every 4-connected line graph is 1-Hamilton-connected. Thus, if G be a counterexample to (ii), then its 1HC-closure provides a counterexample to (i). ■

Secondly, Kaiser and Vrána [9] proved that every 5-connected line graph with minimum degree at least 6 is Hamilton-connected. Extending the argument of the proof of this result, and applying the 1HC-closure, it is possible to obtain the following result.

Theorem 5. *Every 5-connected claw-free graph with minimum degree at least 6 is 1-Hamilton-connected.*

Finally, we mention here a theorem on hourglass-free graphs, which is a strengthening of the main result of [18] and can be considered as another partial solution to the statement (ii) of Theorem 4, i.e., equivalently, to Thomassen's Conjecture. Here the *hourglass* is the unique graph with degree sequence $4, 2, 2, 2, 2$.

Theorem 6. *Every 4-connected claw-free and hourglass-free graph is 1-Hamilton-connected.*

As already mentioned, the (nontrivial) proofs of Theorems 5 and 6 will be published in a separate paper [8].

5 Proof of Proposition 1

For our proof we will need four lemmas describing subgraphs that cannot occur in the preimage of an SM-closed graph.

Lemma 7. *Let G be an SM-closed graph and let $H = L^{-1}(G)$. Then H does not contain a triangle with a vertex of degree 2 in H.*

For the proof of Lemma 7, we will need the following proposition from [4].

Proposition F [4]. *Let x be an eligible vertex of a claw-free graph G, G^*_x the local completion of G at x, and a, b two distinct vertices of G. Then for every longest (a, b)-path $P'(a, b)$ in G^*_x there is a path P in G such that $V(P) = V(P')$ and P admits at least one of a, b as an endvertex. Moreover, there is an (a, b)-path $P(a, b)$ in G such that $V(P) = V(P')$ except perhaps in each of the following two situations (up to symmetry between a and b):*

(i) There is an induced subgraph $F \subset G$ isomorphic to the graph S in Fig. 2 such that both a and x are vertices of degree 4 in F. In this case G contains a path P_b such that b is an endvertex of P and $V(P_b) = V(P')$. If, moreover, $b \in V(F)$, then G contains also a path P_a with endvertex a and with $V(P_a) = V(P')$.

(ii) $x = a$ and $ab \in E(G)$. In this case there is always both a path P_a in G with endvertex a and with $V(P_a) = V(P')$ and a path P_b in G with endvertex b and with $V(P_b) = V(P')$.

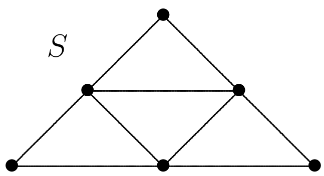

Figure 2

Proof of Lemma 7. Let G be an SM-closed graph. If G is Hamilton-connected, the lemma is obvious since $H = L^{-1}(G)$ is triangle-free by the definition of the SM-closure. Thus, suppose that G is not Hamilton-connected. Let, to the contrary, $T = \langle\{v_1, v_2, v_2\}\rangle_H$ be a triangle in H with $d_H(v_1) = 2$, and set $x_i = L(v_iv_{i+1})$, $i = 1, 2, 3$ (indices mod 3).

Observe that $L^{-1}(S)$ (where S is the graph in Fig. 2) is isomorphic to the net N, i.e. the graph obtained by attaching a pendant edge to each vertex of a triangle. Since $d_H(v_1) = 2$, T is not contained in a copy of N, hence the triangle $L(T) = \langle\{x_1, x_2, x_3\}\rangle_G$ is not contained in an induced subgraph of $G = L(H)$ isomorphic to $S = L(N)$.

Since the edge $L^{-1}(x_2) = v_2v_3$ is in the triangle T, and T cannot have two vertices of degree 2 by the definition of the preimage L^{-1}, x_2 is eligible in G and, by the definition of the SM-closure, $G^*_{x_2}$ is not Hamilton-connected, i.e., there is no hamiltonian (a, b)-path in $G^*_{x_2}$ for some $a, b \in V(G)$ for which there is no hamiltonian (a, b)-path in G. By Proposition F(ii), for every such hamiltonian (a, b)-path in $G^*_{x_2}$, one of a, b is x_2 (say, $a = x_2$), and $b \in N(x_2)$.

Now, x_1 is also eligible in G, and since $N_G(x_1) \subset N_G(x_2)$ (this follows easily from $d_h(v_1) = 2$), also $G^*_{x_1} \subset G^*_{x_2}$, hence every hamiltonian path in $G^*_{x_1}$ is also a hamiltonian path in $G^*_{x_2}$. We already know that every such (a, b)-path satisfies $a = x_1$, and, applying Proposition F(ii) to x_1, we have $b = x_1$.

Thus, we conclude that the only possible vertices for which there is a hamiltonian path in $G^*_{x_2}$ but not in G are the vertices x_1 and x_2. However, x_3 is also eligible in G and $N_G(x_3) \subset N_G(x_2)$, thus, by a symmetric argument, we obtain the same conclusion for x_3 and x_2, a contradiction. ∎

In the proof of the next three lemmas we will need the following slight extension of a technical lemma from [13].

For a graph H, $u \in V(H)$ with $d_H(u) = 2$ and $N_H(u) = \{v_1, v_2\}$, $H|_{(u)}$ denotes the graph obtained from H by suppressing the vertex u (i.e., by replacing the path v_1uv_2 by the edge v_1v_2) and by adding one pendant edge to each of v_1 and v_2.

Lemma G [13]. *Let H be a graph and $u \in V(H)$ of degree 2 with $N_H(u) = \{v_1, v_2\}$ and $h_i = uv_i$, $i = 1, 2$. Set $H' = H|_{(u)}$, $h = v_1v_2 \in E(H')$, and let $f_1, f_2 \in E(H') \setminus E(H)$ be the two pendant edges attached to v_1 and v_2, respectively.*

- (i) *If $L(H)$ is Hamilton-connected, then H' has an (e_1, e_2)-IDT for every $e_1, e_2 \in E(H')$ such that either*
  - (α) *$h \notin \{e_1, e_2\}$, or*
  - (β) *$h \in \{e_1, e_2\}$ and $\{e_1, e_2\} \cap \{f_1, f_2\} \neq \emptyset$.*
- (ii) *If $L(H')$ is Hamilton-connected, then H has an (e_1, e_2)-IDT for every $e_1, e_2 \in E(H)$ such that $\{e_1, e_2\} \neq \{h_1, h_2\}$.*
- (iii) *If moreover H contains a pendant edge attached to v_1 and H has an (h_1, e)-IDT for every $e \in E(H)$, then H' has an (h, e')-IDT for every $e' \in E(H')$*

Proof. Parts (i) and (ii) are a reformulation of Lemma 3 from [13]. We prove (iii). Thus, for any $e' \in E(H')$, we construct an (h, e')-IDT in H'. Let f denote the pendant edge at v_1 in H. If $e' \in \{f, f_1, f_2\}$, then, for any (h_1, h_2)-IDT in H, an appropriate replacement of h_1 and h_2 with h and e' gives the desired (h, e')-IDT in H'. Thus, let $e' \notin \{f, f_1, f_2\}$. Let $e \in E(H)$ be the edge corresponding to e', and let T be an (h_1, e)-IDT in H. If $h_2 \in E(T)$, then necessarily $v_1 \in V(T)$ (otherwise f is not dominated), and then T' obtained from T by replacing h_1, h_2 with h is an (h, e')-IDT in H'. Similarly, if $h_2 \notin E(T)$, then necessarily $v_2 \in V(T)$ (otherwise h_2 is not dominated), and then T' obtained from T by replacing h_1 with h is a desired (h, e')-IDT in H'. ■

Lemma 8. *Let G be an SM-closed graph and let $H = L^{-1}(G)$. Then H does not contain a subgraph $\overline{H}$ isomorphic to a cycle C_5 with a vertex of degree 2 in H and with a chord.*

Proof. If G is Hamilton-connected, the lemma is obvious. Thus, suppose that G is not Hamilton-connected and let, to the contrary, $\overline{H} \subset H$ be a graph consisting of a cycle $C = v_1v_2v_3v_4v_5v_1$ with a chord e, and choose the notation such that $d_H(v_4) = 2$. If $e = v_3v_5$, we have a contradiction with Lemma 7, hence without loss of generality suppose that $e = v_2v_5$. First observe that e is the only chord of C in H, for otherwise H contains a diamond, a contradiction. Denote $v_iv_{i+1} = h_{i+1}$, $i = 1, \ldots, 5$ (indices mod 5) and set $H_1 = H|_{(v_4)}$. Then $L(H_1)$ is not Hamilton-connected by Lemma G(ii). It is straightforward to see that in $L(H_1)$, the neighborhood of the vertex $L(e)$ is 2-connected. By Proposition D, the graph $(L(H_1))^*_{L(e)} = L(H_1|_e)$ is not Hamilton-connected. Set $H_2 = H_1|_e$ (denoting v_2 the vertex obtained by merging $v_2, v_5 \in V(H_1)$). Now, the subgraph of H_2 corresponding to $\overline{H} \subset H$ consists of three vertices v_1, v_2, v_3, a double edge h_1, h_2 joining v_1 and v_2, a double edge h_3, h_4 joining v_1 and v_2, two pendant edges at v_2 and one pendant edge at v_3.

Now we return back the suppressed vertex v_4: let H_3 be the graph obtained from H_2 by subdividing the edge h_4 with a vertex v_4 (denoting $h_5 = v_4v_2$) and removing a pendant edge from each of v_2, v_3. If $L(H_3)$ is Hamilton-connected, then H_2 has, for $e_1, e_2 \in E(H_2)$, an (e_1, e_2)-IDT for $e_1, e_2 \neq h_4$ by Lemma G(i), and for $h_4 \in \{e_1, e_2\}$ by Lemma G(iii), hence $L(H_2)$ is Hamilton-connected, a contradiction. Thus, $L(H_3)$ is not Hamilton-connected. But H_3 can be alternatively obtained from H by contracting the chord e, i.e., $H_3 = H|_e$, or, equivalently, $L(H_3) = G^*_{L(e)}$. As $L(H_3)$ is not Hamilton-connected and $L(e)$ is eligible in G (since e is in a triangle in H), we have a contradiction with the fact that G is SM-closed. ■

Lemma 9. *Let G be an SM-closed graph and let $H = L^{-1}(G)$. Then H does not contain a cycle C of length 5 such that some two vertices of C are of degree 2 in H and some edge of C is in a double edge or in a triangle in H.*

Proof. If G is Hamilton-connected, the lemma is obvious. Thus, suppose that G is not Hamilton-connected, let $C = v_1v_2v_3v_3v_5v_1 \subset H$ and let v_j, v_k, $j < k$, be of degree 2 in H. Set $v_iv_{i+1} = h_{i+1}$, $i = 1, \ldots, 5$ (indices mod 5).

Suppose first that v_j, v_k are consecutive on C, say, $j = 1$, $k = 2$. Then $R = \{h_1, h_2\}$ is an essential edge-cut separating h_2 from the rest of H. By the assumptions, some of h_4, h_5 (say, h_4), is in a triangle or in a double edge, implying $L(h_4)$ is eligible in G. But R is an essential edge-cut also in $H|_{h_4} = L^{-1}(G^*_{L(h_4)})$, hence $G^*_{L(h_4)}$ is not Hamilton-connected, contradicting the definition of SM-closure. Thus, v_j, v_k are not consecutive on C.

Choose the notation such that $j = 3$ and $k = 5$, i.e., $d_H(v_3) = d_H(v_5) = 2$. Then the only possible chords of C are the edges v_1v_4 and v_2v_4, but if some of them is present, we have a contradiction with Lemma 7. Thus, C is chordless. This implies that either

(i) h_2 is in a double edge, or

(ii) h_2 is in a triangle $T = v_1v_2z$ with $z \in V(H) \setminus V(C)$.

In case (i), we use h_2' to denote the edge parallel with h_2 and $\overline{H}$ to denote the graph with $V(\overline{H}) = V(C)$ and $E(\overline{H}) = E(C) \cup \{h_2'\}$; in case ($ii$) we set $h_2' = zv_1$, $h_2'' = zv_2$, $V(\overline{H}) = V(C) \cup \{z\}$ and $E(\overline{H}) = E(C) \cup \{h_2', h_2''\}$. Recall that in both cases $d_H(v_3) = d_H(v_5) = 2$.

By the properties of the SM-closure, for each pair $e, fE(H)$, for which there is no (e, f)-IDT in H, we have $\{e, f\} = \{h_2, h_2'\}$ in case (i), or $\{e, f\} \cap \{h_2, h_2', h_2''\}$ in case (ii), respectively. Thus, by Lemma G(ii), for the graph $H_1 = H|_{(v_5)}$ (in which we denote $v_1v_4 = h_1$), $L(H_1)$ is not Hamilton-connected. Similarly, the graph $L(H_2)$, where $H_2 = H_1|_{(v_3)}$ (in which we set $v_2v_4 = h_3$) is also not Hamilton-connected. But now $\langle\{v_1, v_2, v_4\}\rangle_{H_2}$ is a triangle with a double edge h_2, h_2' in case (i), or $\langle\{v_1, v_2, v_4, z\}\rangle_{H_2}$ is a diamond in case (ii). In both cases, it is straightforward to verify that, in $L(H_2)$, the neighborhood of the vertex $x_2 = L(h_2)$ is 2-connected. Thus, setting $H_3 = H_2|_{h_2}$, we obviously have $L(H_3) = (L(H_2))^*_{x_2}$ and, by Proposition D, $L(H_3)$ is also not Hamilton-connected. Note that in H_3 the subgraph corresponding to $\overline{H}$ consists of: in case (i) two vertices v_1, v_4 joined by h_1 and h_3, 4 pendant edges at v_1 and 2 pendant edges at v_4, or in case (ii) three vertices z, v_1, v_4, where z, v_1 are joined by h_2', h_2'' and v_1, v_4 are joined by h_1, h_3, and there are 3 pendant edges at v_1 and 2 pendant edges at v_4.

Now we return back the suppressed vertices of degree 2: H_4 is obtained from H_3 by subdividing h_3 with v_3 (denoting $v_3v_4 = h_4$) and removing a pendant edge from each of v_1, v_4, and, similarly, H_5 is obtained from H_4 by subdividing h_1 with v_5 (denoting $v_4v_5 = h_5$), and removing a pendant edge from each of v_1, v_4. If $L(H_4)$ is Hamilton-connected, then H_3 has, for $e, f \in E(H_3)$, an (e, f)-IDT for $e, f \neq h_34$ by Lemma G(i), and for $h_3 \in \{e, f\}$ by Lemma G(iii), hence $L(H_3)$ is Hamilton-connected, a contradiction. Thus, $L(H_4)$ is not Hamilton-connected. By a similar argument, $L(H_5)$ is also not Hamilton-connected. But now we observe that $H_5 = H|_{h_2}$, or, equivalently, $L(H_5) = G^*_{x_2}$. As h_2 is in a double edge or in a triangle, x_2 is eligible in G and we have a contradiction with the fact that G is SM-closed. ■

Lemma 10. *Let G be an SM-closed graph, let $H = L^{-1}(G)$ and let F be the graph with $V(F) = \{v_1, v_2, v_3, v_4, v_5, z\}$ and $E(F) = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_1, v_3v_5, zv_1, zv_2\}$ (see Fig. 3). Then H does not contain a subgraph $\overline{H}$ isomorphic to the graph F such that $N_H(\{v_1, v_2, v_3, v_5\}) \subset V(\overline{H})$.*

Proof. If G is Hamilton-connected, the lemma is obvious. Thus, suppose that G is not Hamilton-connected and let $\overline{H}$ be a subgraph of H with the properties given in the lemma. Let $h_1, \ldots, h_8$ denote the edges of $\overline{H}$ as shown in Fig. 3 and denote $T_1 = v_1v_2zv_1$ and $T_2 = v_3v_4v_5v_3$ the two triangles in $\overline{H}$. Observe that H contains no multiple edge since H already contains two triangles, and that neither of the vertices v_1, v_2, v_3, v_5 can have another neighbor in $\overline{H}$ for otherwise H contains a diamond, a contradiction. Thus, $\overline{H}$ is either induced, or $\langle V(\overline{H})\rangle_H = \overline{H}+zv_4$. Moreover, if $zv_4 \notin E(H)$, then, by the connectivity

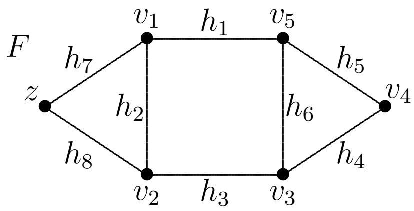

Figure 3

assumption, the graph $H-\{v_1,v_2,v_3,v_5\}$ contains a (z,v_4)-path (since otherwise $\{h_1,h_3\}$ is an essential edge-cut of size 2 in H). Specifically, we have $d_H(z)\geq 3$ and $d_H(v_4)\geq 3$.

Since h_2 is in a triangle, $x_2=L(h_2)$ is eligible in G, implying $G^*_{x_2}=L(H|_{h_2})$ is Hamilton-connected since G is SM-closed. Thus, the graph $H_1=H|_{h_2}$ has an (e_1,e_2)-IDT for any $e_1,e_2\in E(H_1)$. We will show that H has an (f_1,f_2)-IDT for any $f_1\in E(T_1)$ and $f_2\in E(T_2)$, contradicting the fact that $G=L(H)$ is SM-closed.

Thus, choose any $e_1,e_2\in E(H_1)$, let T' be an (e_1,e_2)-IDT in H_1 and let T be the part of T' that is outside $\overline{H}|_{h_2}$ (in the special case when $V(\overline{H})$ dominates all edges of H and $T'\subset\overline{H}|_{h_2}$, necessarily $zv_4\in E(H)$ or $zw,wv_4\in E(H)$ for some $w\in V(H)\setminus V(\overline{H})$, and we choose $T=zv_4$ or $T=zwv_4$, respectively).

Then T is also a trail in $H-\overline{H}$, with initial and terminal edges incident to z and/or v_4 and dominating all edges in $H-\overline{H}$. We distinguish two possibilities:

(α) both $d_T(z)$ and $d_T(v_4)$ is odd,

(β) both $d_T(z)$ and $d_T(v_4)$ is even (possibly zero).

In the case (β), only one of $d_T(z)$, $d_T(v_4)$ can be zero and, by symmetry, we choose the notation such that $d_T(z)\neq 0$. Up to a symmetry, we have the following possibilities for $f_1\in E(T_1)$ and $f_2\in E(T_2)$. In each of them, we find an (f_1,f_2)-IDT in H for both possibilities (α) and (β).

Case	f_1	f_2	An (f_1,f_2)-IDT for the possibility	
			(α)	(β)
1	h_7	h_5	$zv_1v_2zTv_4v_3v_5v_4$	$v_1zTzv_2v_1v_5v_3v_4v_5$
2	h_7	h_4	$zv_1v_2zTv_4v_5v_3v_4$	$v_1zTzv_2v_1v_5v_4v_3$
3	h_7	h_6	$zv_1v_2zTv_4v_3v_5$	$v_1zTzv_2v_1v_5v_4v_3v_5$
4	h_2	h_4	Symmetric to 3(α)	$v_2v_1zTzv_2v_3v_5v_4v_3$
5	h_2	h_6	$v_2v_1zTv_4v_3v_5$	$v_2v_1zTzv_2v_3v_4v_5v_3$

■

Proof of Proposition 1. Let G_0 be a claw-free graph and $x\in V(G_0)$ such that G_0-x is not Hamilton-connected, and let $\widetilde{(G_0)}_x$ be a partial x-closure of G_0. In the rest of the proof, we will simply denote $G:=\widetilde{(G_0)}_x$.

Immediately by the construction of G, G is claw-free and $G-x$ is SM-closed. Thus, it remains to show that G satisfies (i), (ii) or (iii).

We introduce the following notation:

$N_G(x)=\{x_1,\ldots,x_d\}$ (i.e., $d_G(x)=d$),

$\mathcal{K}$ – Krausz partition of $G-x$,

$K'_1, \dots, K'_k$ – all cliques in $\mathcal{K}$ with $K'_i \cap N_G(x) \neq \emptyset$, $i = 1, \dots, k$,
$H' = L^{-1}(G - x)$,
$K''_i = K'_i \cap N_G(x)$, $i = 1, \dots, k$.
The cliques $K''_1, \dots, K''_k \subset \langle N_G(x)\rangle_G$ satisfy the conditions of Theorem B (applied on $\langle N_G(x)\rangle_G$), and we use H to denote the intersection graph of the system $\{K''_1, \dots, K''_k\}$. Then we have $H \subset H'$ and $L(H) = \langle N_G(x)\rangle_G$. However, note that not necessarily $H = L^{-1}(\langle N_G(x)\rangle_G)$ (since the graph H can be another "preimage" of $\langle N_G(x)\rangle_G$, see e.g. the example prior to Proposition A.

Using the correspondence between a line graph and its preimage, we will identify Krausz cliques in $G - x$ with the vertices of H' (the centers of the stars in H' that correspond to the cliques in $\mathcal{K}$). Thus, $\{K'_1, \dots, K'_k\} \subset V(H')$ and $\{K''_1, \dots, K''_k\} = V(H)$.

Note that if $N_G(x)$ can be covered by two Krausz cliques, then at most two cliques from $\mathcal{K}$ have at least two vertices in $N_G(x)$ (hence at least one edge in $\langle N_G(x)\rangle_G$), and extending these cliques to x we get a Krausz partition of G. Thus, to show that G satisfies (iii), it is sufficient to show that $N_G(x)$ can be covered by two Krausz cliques.

Suppose first that $\langle N_G(x)\rangle_G$ is disconnected, and let F_1, F_2 be its components. Then both F_1 and F_2 are cliques since G is claw-free. If F_1, F_2 are subcliques of Krausz cliques in $G - x$, we are done; so, suppose that, say, F_1 is not. Then, as noted in Section 2, $L^{-1}(F_1)$ is a (multi)triangle or a multiedge in $H' = L^{-1}(G-x)$; since $G-x$ is SM-closed, $L^{-1}(F_1)$ is a triangle or a double edge.

If $L^{-1}(F_1)$ is a double edge, then $L^{-1}(F_1) = \langle\{K'_a, K'_b\}\rangle_H$ for some $a, b \in \{1, \dots, k\}$, and since F_1 is a clique, one of K'_a, K'_b, say, K'_b, has no neighbor w with $L(K'_b w) \in N_G(x)$, but then F_1 is a subclique of $K'_a \in \mathcal{K}$, a contradiction. So, suppose that $L^{-1}(F_1)$ is a triangle, set $L^{-1}(F_1) = \langle\{K'_a, K'_b, K'_c\}\rangle_H$ (where $a, b, c \in \{1, \dots, k\}$), and let $z \in V(F_2)$ be arbitrary. By the properties of the preimage L^{-1} (see Section 2), at least two of the vertices K'_a, K'_b, K'_c have a neighbor outside $\{K'_a, K'_b, K'_c\}$. Let, say, $K'_a w_1, K'_b w_2 \in E(H')$, where $w_1, w_2 \in V(H') \setminus \{K'_a, K'_b, K'_c\}$. Then $w_1 \neq w_2$ (otherwise H' contains a diamond), both $L(K'_a w_1) \notin N_G(x)$ and $L(K'_b w_2) \notin N_G(x)$ (for if e.g. $L(K'_a w_1) \in N_G(x)$, then $\langle\{x, L(K'_a w_1), L(K'_b K'_c), z\}\rangle_G$ is a claw), but then $\langle\{L(K'_a K'_b), L(K'_a w_1), L(K'_b w_2), z\}\rangle_G$ is a claw, a contradiction again.

Thus, we can suppose that $\langle N_G(x)\rangle_G$ (and therefore also H) is connected.

<u>Claim 1.</u> *If H contains a triangle and does not contain a C_5, then $L(H) = \langle N_G(x)\rangle_G$ can be covered by two Krausz cliques.*

<u>Proof.</u> Let, say, $T = \langle\{K'_1, K'_2, K'_3\}\rangle_H$ be a triangle in H and denote $h_1 = K'_1 K'_3$, $h_2 = K'_1 K'_2$, $h_3 = K'_2 K'_3$. By Lemma 7, $d_{H'}(K'_i) \geq 3$, $i = 1, 2, 3$. Let $e_i \in E(H') \setminus E(T)$ be an edge incident to K'_i, and set $y_i = L(e_i)$ and $x_i = L(h_i)$, $i = 1, 2, 3$. Since H' does not contain a diamond, the edges e_1, e_2, e_3 have no vertex in common, i.e., $\{e_1, e_2, e_3\}$ is a matching in H'. Hence the vertices y_1, y_2, y_3 are independent in $G - x$.

Now, if all y_i, $i = 1.2.3$, are in $N_G(x)$, then $\langle\{x, y_1, y_2, y_3\}\rangle_G$ is a claw in G, and if, say, $y_1, y_2 \in V(G) \setminus N_G(x)$, then $\langle\{x_2, y_1, y_2, x\}\rangle_G$ is a claw in G, a contradiction. Hence exactly two x_i's are in $N_G(x)$. Choose the notation such that $x_1, x_2 \in N_G(x)$ and $x_3 \in V(G) \setminus N_G(x)$. Then, since the edge e_3 was chosen arbitrarily, we have $d_H(K'_3) = 2$.

If all other edges of H are incident to K'_1 or K'_2, then $E(H)$ can be covered by two stars centered at K'_1, K'_2, hence $\langle N_G(x)\rangle_G$ can be covered by two cliques and we are done. Hence suppose that there is an $f \in E(H)$ that is incident to none of K'_1, K'_2, K'_3. since H is connected, we can choose f such that f has a common vertex with, say, e_1. Set $L(f) = z$.

But now, if f has a common vertex with e_2, then e_1, h_1, h_3, e_2, f determine a C_5 in H, contradicting the assumption, and if f does not share a vertex with e_2, then $\{f, h_1, e_2\}$ is a matching in H, implying $\langle\{x, z, x_1, y_2\}\rangle_G$ is a claw in G, a contradiction again. □

We now distinguish two cases.

<u>Case 1:</u> *$\langle N_G(x)\rangle_G$ does not contain an induced cycle of length 5.*

Then, equivalently, H does not contain a cycle C_5 (not necessarily induced).

First observe that $\alpha(\langle N_G(x)\rangle_G) = \nu(H) \leq 2$, for otherwise x is a center of an induced claw in G, This immediately implies that H does not contain a cycle C_ℓ of length $\ell \geq 6$, since such a cycle contains a matching of size 3. If H contains a triangle, then $\langle N_G(x)\rangle_G$ can be covered by two cliques by Claim 1 and we are done. Thus, the only possible cycles in H are of length 4.

Let $C = x_1x_2x_3x_4x_1$ be a cycle of length 4 in H. Since H is triangle-free, C is chordless. If $V(H) = V(C)$, then H can be covered by two stars (hence $\langle N_G(x)\rangle_G$ can be covered by two cliques) and we are done; if H contains an edge $e = uv$ with $\{u, v\} \cap V(C) = \emptyset$, then e together with two edges from $E(C)$ form a matching of size 3 in H, a contradiction. Hence every edge in $E(H) \setminus E(C)$ has exactly one vertex in $V(C)$.

Now, if some two consecutive vertices of C have a neighbor outside C, say, $x_1y_1 \in E(H)$ and $x_2y_2 \in E(H)$ for some $y_1, y_2 \in V(H) \setminus V(C)$, then $y_1 \neq y_2$ (since H is triangle-free) and $\{x_1y_1, x_2y_2, x_3x_4\}$ is a matching in H, a contradiction. Hence all edges in $E(H)\setminus E(C)$ are incident to some pair of nonconsecutive vertices of C, implying H can be covered by two stars.

Thus, it remains to consider the case when H is a tree. Let $D = \{d_1, \ldots, d_\gamma\}$ be a minimum dominating set in H. By the minimality of D, for every i, $1 \leq i \leq \gamma$, there is a vertex $w_i \in V(H) \setminus D$ such that d_i is the only neighbor of w_i in D. If $\gamma \geq 3$, then $\{d_1w_1, d_2w_2, d_3w_3\}$ is a matching in H, hence $\gamma \leq 2$, implying $\{d_1\}$ (if $\gamma = 1$) or $\{d_1, d_2\}$ (if $\gamma = 2$) are centers of stars covering all edges of H.

<u>Case 2:</u> *$\langle N_G(x)\rangle_G$ contains an induced cycle of length 5.*

Let C be an induced cycle of length 5 in $\langle N_G(x)\rangle_G$. Then $L^{-1}(C)$ is a C_5 (not necessarily induced) in H. If $k \geq 6$, then there is an edge $e \in E(H) \setminus E(C)$ with at least one vertex outside C, but then e together with two edges of C form a matching of size 3 in H, a contradiction. Hence $k = 5$ and $N_G(x) = V(C)$.

We choose the notation such that $C = x_1x_2x_3x_4x_5x_1$ and $x_ix_{i+1} \in E(K'_i)$ (i.e., $x_{i+1} \in K'_i \cap K'_{i+1}$), $i = 1, \ldots, 5$ (indices mod 5). Then $C_H = K'_1K'_2K'_3K'_4K'_5K'_1$ is the corresponding 5-cycle in $H' = L^{-1}(G - x)$, and we denote its edges $h_i = L^{-1}(x_i)$ (i.e., $h_{i+1} = K'_iK'_{i+1}$), $i = 1, \ldots, 5$ (indices mod 5).

<u>Claim 2.</u> *For any $y \in N_G(x)$, $y \in K''_i \cap K''_j$ for some $i, j = 1, \ldots, 5$, $i \neq j$.*

Proof. If e.g. $y \in K_1'' \setminus (\cup_{i=2}^{5} K_i'')$ for some $y \in N_G(x)$, then $y \in K'$ for some other $K' \in \mathcal{K}$ (since every vertex is in 2 Krausz cliques), implying $k \geq 6$, a contradiction. □

We introduce the following notation:
$\mathcal{K}_x = \{K_1', \ldots, K_5'\}$,
$K_x = \cup_{i=1}^{5} K_i'$,
$R = V(G) \setminus (\{x\} \cup K_x)$,
$\mathcal{K}_R = \mathcal{K} \setminus \mathcal{K}_x$,
$I(K_i') = K_i' \setminus (\cup_{j \in (\{1,\ldots,5\} \setminus \{i\})} K_j')$, $i = 1, \ldots, 5$.
The vertices in $I(K_i')$ will be referred to as the *internal vertices* of the clique K_i'. Note that, by Claim 2, $I(K_i') \cap N_G(x) = \emptyset$, $i = 1, \ldots, 5$.

Claim 3. *If $y \in K_x$ has a neighbor in R, then $y \in I(K_i')$ for some $i = 1, \ldots, 5$.*

Proof. By the properties of the Krausz cliques and by Claim 2, only vertices in $I(K_i')$ can have a neighbor in R, since if a vertex $y \in K_i' \cap K_j'$ (for some $i, j \in i, \ldots, 5$) has a neighbor in R, then y is in three Krausz cliques, a contradiction. □

Claim 4. *If $y_1 \in I(K_i')$ and $y_2 \in I(K_{i+1}')$ for some $i = 1, \ldots, 5$, then*
(*i*) $y_1 y_2 \in E(G)$,
(*ii*) $y_1 y_2 \in E(\langle K \rangle_G)$ *for some* $K \in \mathcal{K}_R$,
(*iii*) $\langle \{K_i', K_{i+1}', K\} \rangle_G$ *is a traingle in* $H' = L^{-1}(G - x)$,
(*iv*) $|I(K_i')| = |I(K_{i+1}')| = 1$.

Proof. Let e.g. $y_1 \in I(K_1')$ and $y_2 \in I(K_2')$.
(*i*) If $y_1 y_2 \notin E(G)$, then $\langle \{x_2, x, y_1, y_2\} \rangle_G$ is a claw in G.
(*ii*) If $y_1 y_2 \in E(\langle K_i' \rangle_G)$ for some $i = 2, \ldots, 5$, then $y_1 \in K_1' \cap K_i'$, contradicting the assumption $y_1 \in I(K_i')$. Hence $y_1 y_2 \in E(\langle K \rangle_G)$ for some $K \in \mathcal{K}_R$,
(*iii*) Follows immediately by the structure of K_1', K_2' and K.
(*iv*) If e.g. $y_1, y_1' \in I(K_1')$, $y_1 \neq y_1'$, then $y_1, y_1' \in K_i \cap K$, implying H' contains a triangle and a double edge, a contradiction. □

Claim 5. *There is no j, $1 \leq j \leq 5$, such that $I(K_i') \neq \emptyset$ for $i = j, j+1, j+2$.*

Proof. Let e.g. $I(K_i') \neq \emptyset$ for $i = 1, 2, 3$. By Claim 4, the edge $y_1 y_2$ is in some clique $K^1 \in \mathcal{K}_R$, and $y_2 y_3$ is in some $K^2 \in \mathcal{K}_R$. Since y_2 cannot be in three Krausz cliques, we have $K^1 = K^2$, implying that $y_1 y_3 \in E(G)$ and $y_1 y_3$ is also in K^1. Then we have $y_1 \in K^1 \cap K_1'$, $y_2 \in K^1 \cap K_2'$, $y_3 \in K^1 \cap K_3'$, $x_2 \in K_1' \cap K_2'$ and $x_3 \in K_2' \cap K_3'$, implying that K^1, K_1', K_2', K_3' are vertices of a diamond in H', a contradiction. □

Claim 6. $|\{i| \ 1 \leq i \leq 5, I(K_i') \neq \emptyset\}| \leq 3$.

Proof. Otherwise we have $I(K_i') \neq \emptyset$ for some three consecutive cliques K_i', contradicting Claim 5. □

Claim 7. $|K_i' \cap K_{i+1}'| = 1$, $i = 1, \ldots, 5$.

Proof. Let, to the contrary, e.g. $|K_1' \cap K_2'| \geq 2$. Then $\langle\{K_1', K_2'\}\rangle_{H'}$ is a multiedge, implying $|K_1' \cap K_2'| = 2$ and $|K_i' \cap K_{i+1}'| = 1$ for $i = 2, 3, 4, 5$. Moreover, there is no i, $1 \leq i \leq 5$, such that both $I(K_i') \neq \emptyset$ and $I(K_{i+1}') \neq \emptyset$, for otherwise, by Claim 4, H' contains a triangle, contradicting the fact that H' already contains a double edge. Hence $|\{i|\ 1 \leq i \leq 5, I(K_i') \neq \emptyset\}| \leq 2$, and the vertices K_i' with $I(K_i') \neq \emptyset$ are nonconsecutive on the 5-cycle $C_H = K_1'K_2'K_3'K_4'K_5'K_1'$ in H'. Moreover, if $I(K_i') \neq \emptyset$ and $I(K_j') \neq \emptyset$ for some i, j, then $K_i' \cap K_j' = \emptyset$, for otherwise the edge $K_i'K_j' \in E(H')$ is a chord in C_H, contradicting again the properties of SM-closed graphs.

This means that the 5-cycle C_H is chordless, $\langle\{K_1', K_2'\}\rangle_{H'}$ is the only double edge, at most two vertices of C_H can have a neighbor outside C_H (namely, those for which the corresponding clique in $G - x$ has some internal vertices), and these verties are nonconsecutive.

Now, if $I(K_1') = I(K_2') = \emptyset$, then $\{K_1'K_5', K_2'K_3'\}$ is an essential edge-cut in both H' and $H|_{K_1'K_2'}$, implying that neither $G - x = L(H')$ nor $(G-x)^*_{x_2} = L(H'|_{K_1'K_2'})$ is Hamilton-connected, contradicting the fact that $G - x$ is SM-closed (note that x_2 is eligible since $x_2 = L^{-1}(K_1'K_2')$ and $K_1'K_2'$ is in a double edge). Thus, we can suppose $I(K_1') \neq \emptyset$. But then at least two vertices of C_H are of degree 2 in H' and we have a contradiction with Lemma 9. □

Now we can finish the proof of Proposition 1. Clearly, $I(K_i') \neq \emptyset$ for at least one i, $1 \leq i \leq 5$, for otherwise $V(G) = N_G(x)$ and there is nothing to do. Thus, by Claim 6, one, two or three cliques K_i' have $I(K_i') \neq \emptyset$. We consider these possibilities separately.

Subcase 2.1: $|\{i|\ 1 \leq i \leq 5, I(K_i') \neq \emptyset\}| = 3$.

By Claim 5, we have $I(K_i') \neq \emptyset$ for at most two consecutive cliques K_i'. Thus, without loss of generality let $I(K_i') \neq \emptyset$ for $i = 1, 2, 4$ (i.e., $I(K_3') = I(K_5') = \emptyset$). By Claim 4, there is a vertex $y \in V(H') \setminus V(H)$ such that $\langle\{K_1', K_2', y\}\rangle_{H'}$ is a triangle. If $d_{H'}(K_3') = d_{H'}(K_5') = 2$, we have a contradiction by Lemma 9. Thus we have, say, $d_{H'}(K_3') \geq 3$, i.e., besides K_2' and K_4', K_3' has at least one more neighbor, say, z. Then $z \in \{K_1', K_2', K_4', K_5'\}$ since $I(K_3') = \emptyset$, and the only possibility that does not create a double edge or a diamond (recall that H' already contains a triangle) is $z = K_5'$ and $d_{H'}(K_3') = d_{H'}(K_5') = 3$. Set $\overline{H} = \langle\{K_1', K_2', K_3', K_4', K_5', y\}\rangle_{H'}$ and note that $T_1 = \langle\{K_1', K_2', y\}\rangle_{H'}$ and $T_2 = \langle\{K_3', K_4', K_5'\}\rangle_{H'}$ are two triangles in $\overline{H}$ (hence also in H') and, by Claim 4(iv), y and K_4' are the only vertices of $\overline{H}$ that can have adjacencies outside $\overline{H}$. But then $\overline{H}$ (or possibly $\overline{H} - yK_4'$, if $yK_4' \in E(H')$), has the structure shown in Fig. 3 and we have a contradiction by Lemma 10.

Subcase 2.2: $|\{i|\ 1 \leq i \leq 5, I(K_i') \neq \emptyset\}| = 2$.

By symmetry, we can choose the notation such that $I(K_1') \neq \emptyset$ and either $I(K_2') \neq \emptyset$ or $I(K_3') \neq \emptyset$.

Let first $I(K_1') \neq \emptyset$, $I(K_2') \neq \emptyset$. By Claim 4, there is a vertex $y \in V(H') \setminus V(H)$ such that $\langle\{y, K_1', K_2'\}\rangle_{H'}$ is a triangle and y is the only neighbor of K_1' and K_2' outside H. If the cycle C_H is cordless, we have a contradiction by Lemma 9, and if C_H has a chord, we have a contradiction by Lemma 7.

Thus, suppose that $I(K'_1) \neq \emptyset$, $I(K'_3) \neq \emptyset$. By Claim 7 and by the properties of SM-closed graphs, C_H has no multiedge and at most one chord, but if C_H has a chord, we have a contradiction with Lemma 8. Hence C_H is chordless. Then $\{h_1, h_4\}$ is an essential edge-cut in H', separating h_5 from the rest of H', hence $\{x_1, x_4\}$ is a vertex-cut in $G - x$, separating x_5 from the rest of $G - x$. The graph $(G - x) + x_1x_4$ is SM-closed, since it is the line graph of a graph obtained from H' by contracting the edge h_5 and adding a pendant edge to the contracted vertex, and this operation creates neither a triangle nor a multiedge. Thus, the graph $G - x$ satisfies all conditions of part (ii) of Proposition 1.

Subcase 2.3: $|\{i|\ 1 \leq i \leq 5, I(K'_i) \neq \emptyset\}| = 1$.

If C_H has a chord, we have a contradiction with Lemma 8, hence C_H is chordless. But then again, e.g. $\{h_1, h_4\}$ is an edge-cut in H' and we can add the edge x_1x_4 to $G - x$ to satisfy all conditions of part (ii) of Proposition 1. ■

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