On 1-Hamilton-connected claw-free graphs

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Abstract

A graph G is k-Hamilton-connected (k-hamiltonian) if $G-X$ is Hamilton-connected (hamiltonian) for every set $X \subset V(G)$ with $|X| = k$. In the paper, we prove that

(i) every 5-connected claw-free graph with minimum degree at least 6 is 1-Hamilton-connected,

(ii) every 4-connected claw-free hourglass-free graph is 1-Hamilton-connected.

As a byproduct, we also show that every 5-connected line graph with minimum degree at least 6 is 3-hamiltonian.

1 Introduction

We follow the most common graph-theoretical terminology and for concepts and notations not defined here we refer e.g. to [2]. Specifically, by a *graph* we mean a finite undirected graph $G = (V(G), E(G))$; in general, we allow graphs to have multiple edges. We use $d_G(x)$ to denote the *degree* of a vertex x, and we set $V_i(G) = \{x \in V(G)|\ d_G(x) = i\}$, $V_{\leq i}(G) = \{x \in V(G)|\ d_G(x) \leq i\}$ and $V_{\geq i}(G) = \{x \in V(G)|\ d_G(x) \geq i\}$. The *weight* of an edge e is the number of edges incident with e and distinct from it.

For a set $M \subset V(G)$, $\langle M\rangle_G$ denotes the *induced subgraph* on M, and for a simple graph F, G is said to be F*-free* if G is a simple graph that does not contain an induced subgraph isomorphic to F. Specifically, for $F = K_{1,3}$ we say that G is *claw-free.* The *hourglass* Γ is the only graph with degree sequence $4, 2, 2, 2, 2$ (see Fig. 2), and for $F = \Gamma$ we say that G is *hourglass-free.*

The *neighborhood* of a vertex x, denoted $N_G(x)$, is the set of all neighbors of x, and a vertex $x \in V(G)$ is *simplicial* (*locally connected, locally disconnected, eligible*) if $\langle N_G(x)\rangle_G$ is a complete (connected, disconnected, connected noncomplete) subgraph of G. We will use $V_{EL}(G)$ to denote the set of all eligible vertices in G. The *closed neighborhood* of a vertex x is the set $N_G[x] = N_G(x) \cup \{x\}$, and an edge $e \in E(G)$ is *pendant* if one of its vertices is of degree 1.

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For $x \in V(G)$, $G - x$ is the graph obtained from G by removing x and all edges incident to it. If $x, y \in V(G)$ are such that $e = xy \notin E(G)$, then $G + e$ is the graph with $V(G + e) = V(G)$ and $E(G + e) = E(G) \cup \{e\}$, and, conversely, for $e = xy \in E(G)$ we denote $G - e$ the graph with $V(G - e) = V(G)$ and $E(G - e) = E(G) \setminus \{e\}$. Specifically, for $F \subset G$ and $e \in E(G)$, we set $F - e = F$ if $e \in E(G) \setminus E(F)$. We use $\omega(G)$ to denote the *number of components* of G.

A graph G is *hamiltonian* if G contains a *hamiltonian cycle*, i.e. a cycle of length $|V(G)|$, and G is *Hamilton-connected* if, for any $a, b \in V(G)$, G contains a *hamiltonian (a, b)-path*, i.e., an (a, b)-path P with $V(P) = V(G)$. For $k \geq 1$, G is *k-hamiltonian* if $G - X$ is hamiltonian for every set of vertices $X \subset V(G)$ with $|X| = k$, and *k-Hamilton-connected* if $G - X$ is Hamilton-connected for every set of vertices $X \subset V(G)$ with $|X| = k$. Note that a hamiltonian graph is necessarily 2-connected, a Hamilton-connected graph must be 3-connected, a k-hamiltonian graph must be $(k + 2)$-connected, and if G is k-Hamilton-connected, then G must be $(k + 3)$-connected. The *line graph* of a graph H is the simple graph $L(H)$ with vertex set $E(H)$, in which two vertices are adjacent if and only if the corresponding edges of H share a vertex, and a graph G is a *line graph* if there is a graph H such that $G = L(H)$. Note that every line graph is claw-free, and that the degree of a vertex in G equals the weight of the corresponding edge in H.

The main motivation of our research is the following conjecture by Thomassen.

Conjecture A [18]. *Every 4-connected line graph is hamiltonian.*

There are many known equivalent versions of the conjecture (see [3] for a survey on this topic). We mention here the following one, which is of importance for our results.

Theorem B [16]. *The following statements are equivalent:*
(i) *Every 4-connected line graph is hamiltonian.*
(ii) *Every 4-connected claw-free graph is 1-Hamilton-connected.*

In this paper, we prove the following two results giving a partial affirmative answer to the statement (ii) of Theorem B, i.e., equivalently, to Conjecture A:

- in Section 3, we show that every 4-connected claw-free hourglass-free graph is 1-Hamilton-connected,
- in Section 4, we show that every 5-connected claw-free graph with minimum degree at least 6 is 1-Hamilton-connected.

As a byproduct, in Section 5 we show that every 5-connected line graph with minimum degree at least 6 is 3-hamiltonian.

2 Preliminaries

In this section we summarize some background knowledge that will be needed for our results.

Let H be a graph and $G = L(H)$. It is well-known that if we allow H to be a multigraph, then (unlike in line graphs of simple graphs), for a line graph G, a graph H such that $G = L(H)$ is not uniquely determined. A simple example is the hourglass Γ in Fig. 2, where Γ is the line graph of all three graphs to the right. As shown in [15], this difficulty can be overcome by imposing an additional requirement that simplicial vertices in G correspond to pendant edges in H.

The *basic graph* of a multigraph is the simple graph with the same vertex set, in which every multiedge is replaced by a single edge. A *multitriangle* (*multistar*) is a multigraph such that its basic graph is a triangle (star). The *center* of a multistar S with m edges is the vertex $x \in V(S)$ with $d_S(x) = m$ (for $|V(S)| = 2$ we choose the center arbitrarily), and all other vertices of S are its *leaves*. An induced multistar S in H is *pendant* if none of its leaves has a neighbor in $V(H) \setminus V(S)$, and similarly a multitriangle T is pendant if exactly one of its vertices (called the *root*) has neighbors in $V(H) \setminus V(S)$. We will use the following operations, introduced in [20] (Operation B) and [15] (Operation C).

Operation B. Choose a pendant multitriangle in H with vertices $\{v, x, y\}$ and root v, delete all edges joining v and x, and add the same number of edges between v and y.

Operation C. Choose a pendant multistar in H and replace every leaf of degree $k \geq 2$ by k leaves of degree 1.

Now, for a multigraph H, $BC(H)$ denotes the multigraph obtained from H by recursively repeating operations B and C. It can be shown that $L(H) = L(BC(H))$, $BC(H)$ is uniquely determined and has the property that simplicial vertices in $L(H)$ correspond to pendant edges in H.

Proposition C [15]. *Let G be a connected line graph of a multigraph. Then there is, up to an isomorphism, a uniquely determined multigraph H such that a vertex $e \in V(G)$ is simplicial in G if and only if the corresponding edge $e \in E(H)$ is a pendant edge in H.*

For a line graph G, we will always consider its preimage to be the unique multigraph with the properties given in Proposition C; this preimage will be denoted $L^{-1}(G)$. Similarly, we will write $x = L(e)$ and $e = L^{-1}(x)$ if $e \in E(L^{-1}(G))$ is the edge corresponding to a vertex $x \in V(G)$. In accordance with our definitions, when working with a claw-free graph or with a line graph G, we always consider G to be a simple graph, while if G is a line graph, for its preimage $H = L^{-1}(G)$ we always admit H to be a multigraph, i.e. we always (even if we say "a graph H") allow H to have multiple edges.

An edge cut R of a graph H is *essential* if $H - R$ has at least two nontrivial components. For an integer $k > 0$, H is *essentially k-edge-connected* if every essential edge cut R of G contains at least k edges. Obviously, a line graph $G = L(H)$ of order at least $k + 1$ is k-connected if and only if the graph H is essentially k-edge-connected.

Given a trail T and an edge e in a multigraph G, we say that e is *dominated (internally dominated)* by T if e is incident to a vertex (to an interior vertex) of T, respectively. A trail T in G is called an *internally dominating trail*, shortly *IDT*, if T internally dominates all the edges in G. For $e_1, e_2 \in E(G)$, an IDT with terminal edges e_1, e_2 will be referred to as an (e_1, e_2)-*IDT*. If T is a closed trail, every vertex of T is considered to be an internal vertex, hence every dominated edge is internally dominated, and we simply say that T is a *dominating trail.* A classical result by Harary and Nash-Williams [5] shows that a line graph $G = L(H)$ of order at least 3 is hamiltonian if and only if H contains a dominating closed trail. The following result relates Hamilton paths in a line graph to internally dominating trails in its preimage.

Theorem D [9]. *A line graph G of order at least 3 is Hamilton-connected if and only if $H = L^{-1}(G)$ has an (e_1, e_2)-IDT for any pair of edges $e_1, e_2 \in E(H)$.*

For $x \in V(G)$, the *local completion of G at x* is the graph $G^*_x = (V(G), E(G) \cup \{y_1y_2 |\ y_1, y_2 \in N_G(x)\})$, i.e. the graph obtained from G by adding to $\langle N_G(x)\rangle_G$ all the missing edges. The *closure* $\mathrm{cl}(G)$ of a claw-free graph G is then defined [13] as the graph obtained from G by recursively performing the local completion operation at eligible vertices, as long as this is possible, and G is said to be *closed* if $G = \mathrm{cl}(G)$. It is well-known [13] that, for every claw-free graph G, $\mathrm{cl}(G)$ is uniquely determined, $\mathrm{cl}(G)$ is the line graph of a triangle-free simple graph, and $\mathrm{cl}(G)$ is hamiltonian if and only if G is hamiltonian. However, recall that the closure operation $\mathrm{cl}(G)$ does not preserve the (non-)Hamilton-connectedness of G [14, 1].

To handle Hamilton-connected graphs, the concept of SM-closure was developed in [7]. For a given claw-free graph G, we construct a graph G^M by the following construction.

- (i) If G is Hamilton-connected, we set $G^M = \mathrm{cl}(G)$.
- (ii) If G is not Hamilton-connected, we recursively perform the local completion operation at such eligible vertices for which the resulting graph is still not Hamilton-connected, as long as this is possible. We obtain a sequence of graphs $G_1, \ldots, G_k$ such that
  - $G_1 = G$,
  - $G_{i+1} = (G_i)^*_{x_i}$ for some $x_i \in V_{EL}(G_i)$, $i = 1, \ldots, k-1$,
  - G_k has no hamiltonian (a, b)-path for some $a, b \in V(G_k)$,
  - for any $x \in V_{EL}(G_k)$, $(G_k)^*_x$ is Hamilton-connected,

  and we set $G^M = G_k$.

A graph G^M obtained by the above construction is called a *strong multigraph closure* (or briefly an *SM-closure*) of the graph G, and a graph G equal to its SM-closure is said to be *SM-closed.*

The following theorem summarizes basic properties of the SM-closure operation.

Theorem E [7]. *Let G be a claw-free graph and let G^M be its SM-closure. Then G^M has the following properties:*

- (i) *$V(G) = V(G^M)$ and $E(G) \subset E(G^M)$,*
- (ii) *G^M is obtained from G by a sequence of local completions at eligible vertices,*
- (iii) *G is Hamilton-connected if and only if G^M is Hamilton-connected,*
- (iv) *if G is Hamilton-connected, then $G^M = \mathrm{cl}(G)$,*
- (v) *if G is not Hamilton-connected, then either*
  - (α) *$V_{EL}(G^M) = \emptyset$ and $G^M = \mathrm{cl}(G)$, or*
  - (β) *$V_{EL}(G^M) \neq \emptyset$ and $(G^M)^*_x$ is Hamilton-connected for any $x \in V_{EL}(G^M)$,*

(*vi*) $G^M = L(H)$, *where* H *contains either*

(*α*) *at most 2 triangles and no multiedge, or*

(*β*) *no triangle, at most one double edge and no other multiedge,*

(*vii*) *if* G *contains no hamiltonian* (a, b)*-path for some* $a, b \in V(G)$ *and*

(*α*) X *is a triangle in* H, *then* $E(X) \cap \{L^{-1}_{G^M}(a), L^{-1}_{G^M}(b)\} \neq \emptyset$,

(*β*) X *is a multiedge in* H, *then* $E(X) = \{L^{-1}_{G^M}(a), L^{-1}_{G^M}(b)\}$.

Note that in some cases (specifically, in cases (*iv*) and (*v*)(*α*) of Theorem E), we have $V_{EL}(G^M) = \emptyset$ and $G^M = \mathrm{cl}(G)$, implying that G^M is uniquely determined. However, if $V_{EL}(G^M) \neq \emptyset$, then, for a given graph G, its SM-closure G^M is in general not unique. It can be shown that if G is SM-closed and $H = L^{-1}(G)$, then H does not contain as a subgraph (not necessarily induced) any of the graphs shown in Fig. 1. The graph T_1 in Fig. 1 will be often referred to as the *diamond*.

Figure 1

For 1-Hamilton-connectedness, the second and third authors developed in [16] the concept of 1HC-closure. We do not give technical details of the construction here since these are not needed for our proofs. For details, we refer an interested reader to [16]. The following two results from [16] show that a 1HC-closure of a claw-free graph can be obtained by a sequence of local completions at eligible vertices, and is a line graph of a multigraph with a special structure.

Proposition F [16]. *Let* G *be a claw-free graph. Then there is a sequence of graphs* $G_0, \ldots, G_k$ *such that*

(*i*) $G_0 = G$,

(*ii*) $V(G_i) = V(G_{i+1})$ *and* $E(G_i) \subset E(G_{i+1}) \subset E((G_i)^*_{x_i})$ *for some* $x_i \in V(G_i)$ *eligible in* G_i,

(*iii*) G_k *is a 1HC-closure of* G.

Theorem G [16]. *Let* G *be a claw-free graph and let* $\overline{G}$ *be its 1HC-closure. Then*

(*i*) $\overline{G}$ *is a line graph,*

(*ii*) *for some* $x \in V(\overline{G})$, *the graph* $\overline{G} - x$ *is* SM*-closed,*

(*iii*) $\overline{G}$ *is 1-Hamilton-connected if and only if* G *is 1-Hamilton-connected.*

Let $\overline{G}$ be a 1HC-closure of G, let $H = L^{-1}(\overline{G})$ and, by Theorem G(*ii*), let $x \in V(G)$ be such that $\overline{G} - x$ is SM-closed. Then, for $e = L^{-1}(x)$, $L(H - e)$ is SM-closed, since clearly $L(H - e) = \overline{G} - x$. However, note that it is possible that the graph $H - e$ does not satisfy properties (*vi*) and (*vii*) of Theorem E, since possibly $H - e \neq L^{-1}(\overline{G} - x)$ (that is, $H - e$ can be another "multigraph preimage" of the graph $\overline{G} - x$). In order to obtain the graph $L^{-1}(\overline{G}-x)$, we have to apply the operations B, C to $H-e$; i.e., $L^{-1}(\overline{G}-x) = BC(H-e)$.

In Section 4, we will also need the following result (see Corollary 3.2 of [8]).

Theorem H [8]. *Let H be a 4-edge-connected graph and let $G = L(H)$. Then G is 2-Hamilton-connected if and only if G is 5-connected.*

3 4-connected claw-free hourglass-free graphs

Our first main result is a strengthening of the main result of [10] and gives a partial solution to the statement (*ii*) of Theorem B, i.e., equivalently, to Thomassen's conjecture.

Here the *hourglass* is the unique graph Γ with degree sequence $4, 2, 2, 2, 2$. The vertex $x \in V(\Gamma)$ of degree 4 is called the *center* of Γ and we also say that Γ is *centered at x*. Note that Γ is a line graph and, in multigraphs, it has three nonisomorphic preimages (see Fig. 2).

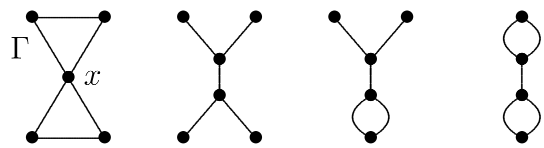

Figure 2

Theorem 1. *Every 4-connected claw-free hourglass-free graph is 1-Hamilton-connected.*

As already noted, a 1-Hamilton-connected graph is necessarily 4-connected. Thus, by Theorem 1, we observe that a claw-free hourglass-free graph G is 1-Hamilton-connected if and only if G is 4-connected. This immediately implies the following fact.

Corollary 2. *1-Hamilton-connectedness is polynomial-time decidable in the class of claw-free hourglass-free graphs.*

Note than an analogous result is known to be true in planar graphs (an easy consequence of [17], page 342).

For the proof of Theorem 1, we will need several auxiliary results.

Lemma 3. *Let G be a claw-free graph such that every induced hourglass in G is centered at an eligible vertex and let $\overline{G}$ be a 1HC-closure of G satisfying the statement of Proposition F. Then every induced hourglass in $\overline{G}$ is centered at an eligible vertex.*

Proof. Let $G_0, \ldots, G_k$ be a sequence of graphs with the properties given in Proposition F, let $\overline{G} = G_k$, and let i, $0 \le i \le k-1$, be the smallest integer such that G_{i+1} contains an induced hourglass Γ centered at a locally disconnected vertex. Denote $V(\Gamma) = \{u_0, u_1, u_2, u_3, u_4\}$ such that $E(\Gamma) = \{u_0u_1, u_0u_2, u_0u_3, u_0u_4, u_1u_2, u_3u_4\}$ (i.e., u_0 is the center of Γ). By the choice of i, $E(\Gamma) \not\subset E(G_i)$. If G_i contains all the edges of Γ containing u_0, then u_0 centers a claw in G_i; hence we can choose the notation such that $u_0u_1 \notin E(G_i)$. By Proposition F, there is a vertex v eligible in G_i such that $u_0, u_1 \in N_{G_i}(v)$. Let u_5 be the first vertex of a (u_0, u_1)-path in $\langle N_{G_i}(v)\rangle_{G_i}$. Then $\langle\{u_0, v, u_5, u_3, u_4\}\rangle_{G_i}$ is an induced hourglass in G_i, centered at u_0. This contradicts the choice of i since u_0 is locally disconnected in G_i. ■

Lemma 4. *Let G be a 4-connected claw-free hourglass-free graph. Then there is a 1HC-closure $\overline{G}$ of G such that $L^{-1}(\overline{G})$ has at most three vertices of degree three.*

Proof. Let $\overline{G}$ be a 1HC-closure of G with the properties given in Proposition F and let $H = L^{-1}(\overline{G})$. Recall that H is essentially 4-edge-connected and that a vertex of $\overline{G}$ is eligible if and only if the corresponding edge of H is in a triangle or in a multiedge.

Claim 1. *Let $x \in V(H)$ be of degree 3 in H. Then there is a subgraph $T \subset H$ such that T is isomorphic to the graph T_1 or T_2 of Fig. 1 and $d_T(x) = 3$.*

Proof. Let $N_H(x) = \{u, v, w\}$. We distinguish two possibilities.

First suppose that u, v, w are distinct. Since H is essentially 4-edge-connected, we have $d_H(w) \ge 3$, and since the vertex $L(xw)$ does not center in $\overline{G}$ an hourglass with a locally disconnected center, xw is in a triangle. Since $d_H(x) = 3$, we have, up to a symmetry, $uw \in E(H)$. The same idea, applied to the edge xv, implies $vw \in E(H)$. But then x, w, u, v are vertices of a T_1 in H.

Secondly, let $u = v$. Similarly as before, the edge xw is in a triangle, implying $uw \in E(H)$ and then x, u, w are vertices of a T_2 in H. □

Let now $x \in V(H)$ be of degree 3 in H. We distinguish two cases.

Case 1: *All vertices of degree 3 in H are in $N_H[x]$.*

If x is in a T_2, then $|N_H[x]| = 3$ and we are done. Thus, suppose x is in a $T_1 \subset H$. If all vertices of T_1 are of degree 3, then either T_1 is connected to $H - T_1$ with exactly two edges, in which case H is not essentially 4-edge-connected, or H contains a K_4, but then the removal of any edge from the K_4 yields a diamond, contradicting the fact that $\overline{G}$ contains a vertex whose removal yields an SM-closed graph (recall that a preimage of an SM-closed graph cannot contain a diamond). Hence H contains at most three vertices of degree 3.

Case 2: *There is $y \in V(H)$ such that $d_H(y) = 3$ and $xy \notin E(H)$.*

By Claim 1, there are subgraphs T_x and T_y of H (not necessarily induced) such that $d_{T_x}(x) = d_{T_y}(y) = 3$ and each of T_x, T_y is isomorphic to T_1 or to T_2. By the properties of the 1HC-closure, there is an edge $e \in E(H)$ such that $L(H - e)$ is SM-closed, i.e., the graph $H' = BC(H - e)$ contains at most two triangles or at most one double edge.

Suppose now that $T_x - e$ (not excluding the possibility $e \notin E(T_x)$, in which case $T_x - e = T_x$) is a pendant multistar or a pendant multitriangle in $H - e$. By the connectivity assumption and since $d_{T_x}(x) = 3$, we easily observe that T_x is isomorphic to T_2 such that, denoting $V(T_x) = \{x, x_1, x_2\}$ with x_2 having no neighbors in $H - T_x$, x_1 is connected to $\{x, x_2\}$ with at least three edges (two between x, x_1 and at least one between x_2, x_1). Now, if $e \neq xx_2$, then $BC(H - e)$ contains a multiedge with multiplicity at least 3, a contradiction. Hence $T_x - e$, and, symmetrically, also $T_y - e$, is neither a pendant multistar nor a pendant multitriangle in $H - e$, implying that both $T_x - e$ and $T_y - e$ is a subgraph of $H' = BC(H - e)$. Since H' contains at most two triangles or at most one double edge, e is an edge of both T_x and T_y and, since x, y are nonadjacent, e contains neither x nor y. Now, if one of T_x, T_y is a T_2, then the removal of any edge leaves in H' two double edges or a double edge and a triangle, which is not possible. Hence both T_x and T_y are the diamond T_1.

Let $e = wz$, and let u and v be the fourth vertex in T_x and T_y, respectively. Then we have, up to a symmetry, the following two possibilities (see Fig. 3):

(a) $d_{T_x}(w) = d_{T_y}(w) = 3$ (implying $d_{T_x}(z) = d_{T_y}(z) = 2$),

(b) $d_{T_x}(w) = d_{T_y}(z) = 3$ (implying $d_{T_x}(z) = d_{T_y}(w) = 2$).

We consider these possibilites separately.

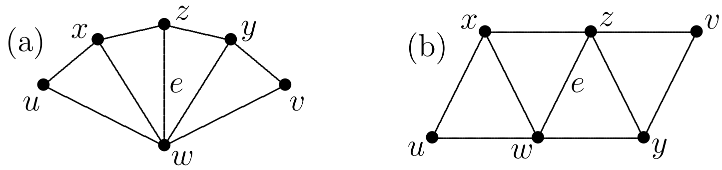

Figure 3

(a) Let first $d_{T_x}(w) = d_{T_y}(w) = 3$. If $u = v$, then $H - e$, hence also H', contains a diamond, a contradiction. Thus, $u \neq v$. If $d_H(u) = 3$, then, by the previous observations, u is a vertex of degree 3 of a diamond T_u. This implies either $uu' \in E(H)$ and $u'w \in E(H)$ for some other vertex u', or $uz \in E(T_u)$, but then, in both cases, T_u is a diamond also in $H - e$ and H', a contradiction. If $d_H(u) = 2$, then $\{uw, xw, xz\}$ is an edge-cut separating the edge ux, a contradiction. Hence $d_H(u) > 3$ and, symmetrically, $d_H(v) > 3$. Thus, among the vertices in $V(T_x) \cup V(T_y)$, only x, y and possibly z are of degree 3. If H contains another vertex t of degree 3, then t is adjacent to neither x nor y and, by Claim 1, t is in a diamond T_1. But then, for any edge $f \in E(H)$, $H - f$ contains at least three triangles (none of them being pendant), a contradiction.

(b) Secondly, let $d_{T_x}(w) = d_{T_y}(z) = 3$. For $u = v$ immediately $d_H(u) = d_H(v) > 3$; for $u \neq v$, similarly as before, $d_H(u) = 3$ implies that H' contains a diamond, and $d_H(u) = 2$ contradicts the connectivity assumption. Thus, in both cases, we have $d_H(u) > 3$ and, symmetrically, $d_H(v) > 3$. Hence x and y are the only vertices of degree 3 in $V(T_x) \cup V(T_y)$. Similarly, if $d_H(t) = 3$ for some other $t \in V(H)$, then t is adjacent to neither x nor y, t is in a diamond and, for any $f \in E(H)$, $H - f$ contains at least three nonpendant triangles, a contradiction. ■

The *core* of a graph H, denoted $\mathrm{co}(H)$, is the graph obtained from H by deleting all vertices of degree 1 and suppressing all vertices of degree 2 (i.e., contracting exactly one of the edges xy, yz for each path xyz with $d_H(y) = 2$). Note that, by the definition of the core, all vertices of degree one or two are deleted or suppressed, hence $\delta(\mathrm{co}(H)) \geq 3$.

For the proof of Theorem 1, we will need two more results.

Theorem I [8]. *Let H be a graph such that $co(H)$ has two edge-disjoint spanning trees and $G = L(H)$ is 3-connected. Then, for any pair of edges $e_1, e_2 \in E(H)$, H has an internally dominating (e_1, e_2)-trail.*

Theorem J [12], [19]. *A graph G has k edge-disjoint spanning trees if and only if*

$$|E_0| \geq k(\omega(G - E_0) - 1)$$

for each subset E_0 of the edge set $E(G)$.

Proof of Theorem 1. Let G be a 4-connected claw-free hourglass-free graph and, by Lemma 4, let $\overline{G}$ be a 1HC-closure of G such that $H = L^{-1}(\overline{G})$ has at most three vertices of degree 3. Recall that H is essentially 4-edge-connected.

By Theorem D, we need to show that for any $f, e_1, e_2 \in E(H)$, the graph $H - f$ has an (e_1, e_2)-IDT. Since the graph $L(H - f) = \overline{G} - x$ (where $x = L(f)$) is clearly 3-connected, by Theorem I, it is sufficient to show that the graph $\mathrm{co}(H - f)$ has two edge-disjoint spanning trees.

<u>Claim 1.</u> *The graph $\mathrm{co}(H) - f$ has two edge-disjoint spanning trees.*

<u>Proof.</u> First note that possibly $f \notin E(\mathrm{co}(H))$ if f is a pendant edge of H; in this case $\mathrm{co}(H) - f = \mathrm{co}(H)$. Obviously, $\mathrm{co}(H)$ is essentially 4-edge-connected (since so is H) and has at most three vertices of degree 3 (since, by the connectivity assumption, pendant edges in H can be incident only to vertices of degree at least 4 in $\mathrm{co}(H)$). Hence, for any set $E \subset E(\mathrm{co}(H))$, every component C of $\mathrm{co}(H) - E$ is connected to $(\mathrm{co}(H) - E) - C$ by at least 4 edges, except for the case when C is a trivial component consisting of one of the at most three vertices of degree 3. This implies $2|E| \geq 4(\omega(\mathrm{co}(H) - E) - 3) + 3 \cdot 3 = 4\omega(\mathrm{co}(H) - E) - 3$, from which, by parity, $2|E| \geq 4\omega(\mathrm{co}(H) - E) - 2$, i.e., $|E| \geq 2\omega(\mathrm{co}(H) - E) - 1$.

Now, set $H' = \mathrm{co}(H) - f$ and let $E_0 \subset E(H')$. Set $E = E_0 \cup \{f\}$ if $f \in E(\mathrm{co}(H))$ and $E = E_0$ otherwise . Then clearly $|E_0| \leq |E| \leq |E_0| + 1$, $E \subset E(\mathrm{co}(H))$ and $\omega(\mathrm{co}(H) - E) = \omega(H' - E_0)$. Hence $|E_0| \geq |E| - 1 \geq 2\omega(\mathrm{co}(H) - E) - 2 = 2\omega(H' - E_0) - 2$. By Theorem J, H' has two edge-disjoint spanning trees. □

<u>Claim 2.</u> *The graph $\mathrm{co}(\mathrm{co}(H) - f)$ has two edge-disjoint spanning trees.*

<u>Proof.</u> The claim follows from the well-known fact that if a graph has k edge-disjoint spanning trees, then so does any of its contractions (see e.g. [11], Lemma 2.1(*iii*)). □

Claim 3. $\operatorname{co}(H-f)=\operatorname{co}(\operatorname{co}(H)-f)$.

Proof. The claim is trivially true if f is a pendant edge of H, so suppose f is nonpendant. As already noted, we have $V_1(\operatorname{co}(H)-f)=\emptyset$ and $V_2(\operatorname{co}(H)-f)=V_3(\operatorname{co}(H))\cap V(f)$, from which $V(\operatorname{co}(\operatorname{co}(H)-f))=V(H)\setminus[V_1(H)\cup V_2(H)\cup(V_3(H)\cap V(f))]$. On the other hand, $V_1(H-f)=V_1(H)\cup(V_2(H)\cap V(f))$ (note that $V_2(H)$ is an independent set by the connectivity assumption) and $V_2(H-f)=(V_2(H)\setminus V(f))\cup(V_3(H)\cap V(f))$, from which $V_1(H-f)\cup V_2(H-f)=V_1(H)\cup V_2(H)\cup(V_3(H)\cap V(f))$, implying $V(\operatorname{co}(H-f))=V(H)\setminus[V_1(H)\cup V_2(H)\cup(V_3(H)\cap V(f))]$. Thus, $\operatorname{co}(H-f)$ and $\operatorname{co}(\operatorname{co}(H)-f)$ are graphs on the same vertex set.

In the construction of $\operatorname{co}(H-f)$, each of the vertices in $V_1(H-f)=V_1(H)\cup(V_2(H)\cap V(f))$ was removed together with a pendant edge; in $\operatorname{co}(\operatorname{co}(H)-f)$, in the construction of $\operatorname{co}(H)$, the set $V_1(H)$ was removed, and in the step from $\operatorname{co}(H)-f$ to $\operatorname{co}(\operatorname{co}(H)-f)$, $V_2(H)\cap V(f)$ was removed. Thus, in the construction of both graphs, the sets of removed vertices are the same. Consequently, the sets of suppressed vertices are also the same and the claim follows. □

Now, $\operatorname{co}(H-f)$ has two edge-disjoint spanning trees by Claims 3 and 2. ■

4 5-connected claw-free graphs

Our second main result strengthens the main result of [6] and can be considered as another partial solution to the statement (*ii*) of Theorem B, i.e., equivalently, to Thomassen's conjecture.

Theorem 5. *Every 5-connected claw-free graph with minimum degree at least 6 is 1-Hamilton-connected.*

In our proof, we will use the hypergraph technique developed in [6], in which vertices of degree 3 are replaced with 3-hyperedges. For the sake of completeness, we repeat here some essential parts from [6]. We include here only basic definitions and facts that are needed for our proof; for more details we refer the reader to the original paper [6].

A *hypergraph* is a pair $\mathcal{H}=(V(\mathcal{H}),E(\mathcal{H}))$, where $V(\mathcal{H})$ is a set of vertices and $E(\mathcal{H})$ is a (multi)set of subsets of $V(\mathcal{H})$ that are called the *hyperedges* of $\mathcal{H}$. A hyperedge of cardinality k is called a *k-hyperedge*. We consider only *3-hypergraphs*, i.e., hypergraphs in which each hyperedge is a 2-hyperedge or a 3-hyperedge. Multiple copies of the same hyperedge are allowed. Throughout the rest of this chapter, the symbol $\mathcal{H}$ will always stand for a 3-hypergraph.

In our argument, 3-hypergraphs are obtained from graphs by replacing vertices of degree 3 by hyperedges consisting of their neighbors. Conversely, to every 3-hypergraph $\mathcal{H}$ we assign a graph $Gr(\mathcal{H})$ obtained such that for each 3-hyperedge e of $\mathcal{H}$ we add a vertex v_e and replace e by three edges joining v_e to each vertex of e.

A hypergraph $\mathcal{H}$ is *connected* if for every nonempty proper subset $X\subset V(\mathcal{H})$, there is a hyperedge of $\mathcal{H}$ intersecting both X and $V(\mathcal{H})\setminus X$. If $\mathcal{H}$ is connected, then an *edge-cut* in $\mathcal{H}$ is an inclusionwise minimal set of hyperedges F such that $\mathcal{H} - F$ is disconnected. For an integer k, $\mathcal{H}$ is k-edge-connected if it is connected and contains no edge-cuts of cardinality less than k. The *degree* of a vertex v is the number of hyperedges incident with v.

For $X \subset V$, we define $\mathcal{H}[X]$ (the *induced subhypergraph of* $\mathcal{H}$ *on* X) as the hypergraph with vertex set X and hyperedge set $E(\mathcal{H}[X]) = \{e \cap X : e \in E(\mathcal{H}) \text{ and } |e \cap X| \geq 2\}$. If $e \cap X = f \cap X$ for distinct hyperedges e, f, we include this hyperedge in multiple copies. Furthermore, we assume a canonical assignment of hyperedges of $\mathcal{H}$ to hyperedges of $\mathcal{H}[X]$. To stress this fact, we always write the hyperedges of $\mathcal{H}[X]$ as $e \cap X$, where $e \in E(\mathcal{H})$.

A *quasigraph* in $\mathcal{H}$ is a pair $(\mathcal{H}, \pi)$, where π is a function assigning to each hyperedge e of $\mathcal{H}$ a set $\pi(e) \subset e$ which is either empty or has cardinality 2. The value $\pi(e)$ is called the representation of e under π. When the underlying hypergraph is clear from the context, we simply speak about a quasigraph π. Quasigraphs will be denoted by lowercase Greek letters. Considering all the nonempty sets $\pi(e)$ as graph edges, we obtain a graph π^* on $V(\mathcal{H})$. We say that hyperedges e with $\pi(e) \neq \emptyset$ are *used* by π; the set of all such hyperedges of $\mathcal{H}$ is denoted by $E(\pi)$, and the edges of the graph π^* are denoted by $E(\pi^*)$.

A quasigraph π is *acyclic* if π^* is a forest and π is a *quasitree* if π^* is a tree. If e is a hyperedge of $\mathcal{H}$, then $\pi - e$ is the quasigraph obtained from π by changing the value at e to $\emptyset$. The *complement* $\overline{\pi}$ of π is the spanning subhypergraph of $\mathcal{H}$ consisting of all the hyperedges of $\mathcal{H}$ not used by π. Note that $\overline{\pi}$ is not a quasigraph, and since π includes the information about its underlying hypergraph $\mathcal{H}$, we can speak about $\overline{\pi}$ without specifying $\mathcal{H}$.

For $X \subset V(\mathcal{H})$, the π-*section of* $\mathcal{H}$ *at* X is the hypergraph $\mathcal{H}[X]^\pi$ with $V(\mathcal{H}[X]^\pi) = X$ and $E(\mathcal{H}[X]^\pi) = \{e \cap X : e \in E(\mathcal{H}) \text{ is such that } |e \cap X| \geq 2 \text{ and } \pi(e) \subset X\}$. The quasigraph π in $\mathcal{H}$ naturally determines a quasigraph $\pi[X]$ in $\mathcal{H}[X]^\pi$, defined by $(\pi[X])(e \cap X) = \pi(e)$, where $e \in E(\mathcal{H})$ and $e \cap X$ is any hyperedge of $\mathcal{H}[X]^\pi$. The quasigraph $\pi[X]$ is called the *quasigraph induced by* π *on* X. Note that whenever we speak about the complement of $\pi[X]$, it is, in accordance with the definition, its complement in $\mathcal{H}[X]^\pi$.

A quasigraph π has *tight complement* (in $\mathcal{H}$) if π satisfies one of the following:

(a) $\overline{\pi}$ is connected, or

(b) there is a partition $V(\mathcal{H}) = X_1 \cup X_2$ such that for $i = 1, 2$, X_i is nonempty and $\pi[X_i]$ has tight complement (in $\mathcal{H}[X_i]^\pi$); furthermore, there is a hyperedge $e \in E(\pi)$ such that $\pi(e) \subset X_1$ and $e \cap X_2 \neq \emptyset$.

A set $X \subset V(\mathcal{H})$ is π-*solid* (in $\mathcal{H}$), if $\pi[X]$ is a quasitree with tight complement in $\mathcal{H}[X]^\pi$.

Let $\mathcal{P}$ be a partition of $V(\mathcal{H})$. An edge $e \in E(\mathcal{H})$ is $\mathcal{P}$-*crossing* if e intersects at least two classes of $\mathcal{P}$ and, for a $\mathcal{P}$-crossing edge e, $e/\mathcal{P}$ is the set of all classes $P \in \mathcal{P}$ with $e \cap P \neq \emptyset$.

The *contraction of* $\mathcal{P}$ is the operation resulting in the hypergraph $\mathcal{H}/\mathcal{P}$ with $V(\mathcal{H}/\mathcal{P}) = \mathcal{P}$ and $E(\mathcal{H}/\mathcal{P}) = \{e/\mathcal{P} : e \text{ is } \mathcal{P}\text{-crossing}\}$. Thus, $\mathcal{H}/\mathcal{P}$ is a 3-hypergraph, possibly with multiple hyperedges.

If π is a quasigraph in $\mathcal{H}$, we define $\pi/\mathcal{P}$ as the quasigraph in $\mathcal{H}/\mathcal{P}$ consisting of the hyperedges $e/\mathcal{P}$ such that $\pi(e)$ is $\mathcal{P}$-crossing; the representation is defined by

$(\pi/\mathcal{P})(e/\mathcal{P}) = \pi(e)/\mathcal{P}$. Obviously, the complement of $\pi/\mathcal{P}$ in $\mathcal{H}/\mathcal{P}$ is denoted by $\overline{\pi/\mathcal{P}}$.

Finally, if π is a quasigraph in $\mathcal{H}$, then a partition $\mathcal{P}$ of $V(\mathcal{H})$ is said to be π*-skeletal* if every $X \in \mathcal{P}$ is π-solid and the complement of $\pi/\mathcal{P}$ in $\mathcal{H}/\mathcal{P}$ is acyclic.

The following lemma is a special case of the Skeletal Lemma (Lemma 17 of [6]).

Lemma K [6]. *Every 3-hypergraph $\mathcal{H}$ contains an acyclic quasigraph σ such that there is a σ-skeletal partition $\mathcal{S}$ of $V(\mathcal{H})$.*

Proof of Theorem 5. If G is a counterexample to Theorem 5 and $\overline{G}$ is a 1HC-closure of G, then $\overline{G}$ is also a counterexample to Theorem 5; hence it is sufficient to prove Theorem 5 for line graphs (of multigraphs). Thus, suppose that G is a line graph and let $G = L(H)$. If H is 4-edge-connected, then the statement follows from Theorem H, hence it remains to prove the theorem in the case when $G = L(H)$ and H is not 4-edge-connected. By Theorem D, we need to show that, for any $e_1, e_2, e_3 \in E(H)$, the graph $H - e_3$ has an (e_1, e_2)-IDT.

By the minimum degree assumption on G, every edge of H is of weight at least 6, and by the connectivity assumption, H is essentially 5-edge-connected. By the assumption that H is not 4-edge-connected, H must contain vertices of degree at most 3, and since H is essentially 5-edge-connected, $V_{\leq 3}(H)$ is an independent set in H. Thus, it is sufficient to find in $H - e_3$ an (e_1, e_2)-trail spanning all vertices in $V_{\geq 4}(H)$. For finding such a trail, we use the concept of an X-join: for $X \subset V(H)$, an X*-join in* H is a subgraph H_0 of H such that a vertex of H is in X if and only if its degree in H_0 is odd (in particular, $\emptyset$-joins are eulerian subgraphs).

For each edge e of H, fix its vertex u_e of degree at least 4 in H (which exists since $V_{\leq 3}(H)$ is independent), and for $e_1, e_2 \in E(H)$ set

$$X(e_1, e_2) = \begin{cases} \{u_{e_1}, u_{e_2}\} & \text{if } u_{e_1} \neq u_{e_2}, \\ \emptyset & \text{otherwise.} \end{cases}$$

Now, if the graph $H - e_1 - e_2 - e_3$ contains a connected $X(e_1, e_2)$-join J spanning all of $V_{\geq 4}(H)$, then, by the classical observation of Euler, all the edges of J can be arranged in a trail T_J with first edge incident with u_{e_1} and last edge incident with u_{e_2}. Adding e_1 and e_2, we obtain a desired (e_1, e_2)-trail T in $H - e_3$ spanning $V_{\geq 4}(H)$ (if $u_{e_1} = u_{e_2}$, we use the fact that u_{e_1} is incident with an edge of T_J). Thus, our task is reduced to find a connected $X(e_1, e_2)$-join in $H - e_1 - e_2 - e_3$ spanning all vertices in $V_{\geq 4}(H)$. For finding such a join we use the hypergraph technique.

For this, we need the following observation. Suppose that there is a vertex $u \in V_{\leq 3}(H)$ having at most 2 neighbors, and let H' be the graph obtained by removing u if u has 1 neighbor, or by suppressing u (i.e., removing u and adding an edge joining its neighbors) if u has 2 neighbors, respectively. By the connectivity and minimum degree assumptions, $V_{\geq 4}(H') = V_{\geq 4}(H)$ and if, for any $f_1, f_2, f_3 \in E(H')$, $H' - f_3$ contains an (f_1, f_2)-trail spanning $V_{\geq 4}(H')$, we can easily find a desired (e_1, e_2)-trail in $H - e_3$ spanning $V_{\geq 4}(H)$. Thus, we can suppose that $V_1(H) = V_2(H) = \emptyset$ and every vertex in $V_3(H)$ has 3 neighbors.

We can now define the 3-hypergraph $\mathcal{H}$ with vertex set $V(\mathcal{H}) = V_{\geq 4}(H)$; the hyperedges of $\mathcal{H}$ are of two types:

- the edges of H with both endvertices in $V(\mathcal{H})$,
- 3-hyperedges consisting of the neighbors of any vertex in $V_3(H)$.

Recall that every vertex in $V_3(H)$ has three distinct neighbors and these are in $V(\mathcal{H}) = V_{\geq 4}(H)$ since $V_3(H)$ is independent. Also note that clearly $\mathrm{co}(H) = Gr(\mathcal{H})$.

We show that $\mathcal{H}$ has properties that will be of importance for us (Claims 1 and 2 are proved in [6]; for the sake of completeness, we include their proofs here as well).

Claim 1. *The hypergraph $\mathcal{H}$ is 4-edge-connected.*

Proof. Suppose that this is not the case, let F be an inclusionwise minimal edge-cut in $\mathcal{H}$ with $|F| \leq 3$ and let A be the vertex set of a component of $\mathcal{H} - F$. Let $e \in F$. By the minimality of F, $|e - A| \geq 1$. We assign to e an edge e' of H, defined as follows:

- if $|e| = 2$, then $e' = e$,
- if $|e| = 3$ and $e \cap A = \{u\}$, then $e' = uv_e$,
- if $|e| = 3, |e \cap A| = 2$ and $e - A = \{u\}$, then $e' = uv_e$.

Then $F' := \{e' : e \in F\}$ is an edge-cut in H and since H is essentially 5-edge-connected, F' is a trivial edge-cut. Since $|F'| \leq 3$, A contains a vertex of degree at most 3 (in H), a contradiction. □

Claim 2. *No 3-hyperedge of $\mathcal{H}$ is included in an edge-cut of size 4 in $\mathcal{H}$.*

Proof. Let F be an edge-cut of size 4 in $\mathcal{H}$. As in the proof of Claim 1, we consider the corresponding edge-cut F' in H. Since H is essentially 5-edge-connected, one component of $H - F'$ consists of a single vertex w whose degree in H is 4. Assuming that F includes a 3-hyperedge e, we find that in H, w has a neighbor v of degree 3. Thus, the weight of the edge vw is 5, a contradiction. □

Let $e_1, e_2, e_3 \in E(H)$ be the given edges, and let w_i, $i = 1, 2, 3$, be the vertex of e_i distinct from u_{e_i}. We define a 3-hypergraph $\mathcal{H}^{\{e_1,e_2,e_3\}}$ by the following construction.

(1) If some two (possibly all three) of e_1, e_2, e_3 have a common vertex of degree 3, i.e., $w_i = w_j$ for some $i, j \in \{1, 2, 3\}$, then let $\mathcal{H}_1$ be the hypergraph obtained from $\mathcal{H}$ by removing the 3-hyperedge corresponding to the vertex $w_i = w_j$;
otherwise set $\mathcal{H}_1 = \mathcal{H}$
(note that, after this step, $|\{w_i : 1 \leq i \leq 3 \text{ and } w_i \in Gr(\mathcal{H}_1)\}| \in \{0, 1, 3\}$).

(2) Let $\mathcal{H}^{\{e_1,e_2,e_3\}}$ be the hypergraph obtained from $\mathcal{H}_1$ by performing the following for every $w_i \in Gr(\mathcal{H}_1)$:
   (2a) if w_i has degree 3 in $Gr(\mathcal{H}_1)$, then the 3-hyperedge e_{w_i} of $\mathcal{H}_1$ corresponding to w_i is replaced by the 2-hyperedge $e_{w_i} - \{u_{e_i}\}$,
   (2b) otherwise, the 2-hyperedge e_i of $\mathcal{H}_1$ is deleted

(note that, unlike $\mathcal{H}$, the hypergraph $\mathcal{H}^{\{e_1,e_2,e_3\}}$ can contain vertices of degree 3).

Note that $Gr(\mathcal{H}^{\{e_1,e_2,e_3\}})$ is the graph obtained from $H - e_1 - e_2 - e_3$ by deleting those its vertices of degree 1 and suppressing those its vertices of degree 2, which are not in $V(\mathcal{H})$ (i.e., deleting $V_1(H-e_1-e_2-e_3)\setminus V(\mathcal{H})$ and suppressing $V_2(H-e_1-e_2-e_3)\setminus V(\mathcal{H})$).

Thus, our task is reduced to finding a connected $X(e_1,e_2)$-join in $Gr(\mathcal{H}^{\{e_1,e_2,e_3\}})$ spanning all vertices in $V(\mathcal{H})$. If $\mathcal{H}^{\{e_1,e_2,e_3\}}$ has a quasitree with tight complement, then the existence of such a join is guaranteed by the following result (Lemma 28 of [6]).

Lemma L [6]. *Let $\mathcal{H}$ be a 3-hypergraph containing a quasitree π with tight complement, and let $X \subset V(\mathcal{H})$. Then there is a quasigraph τ such that $E(\pi)$ and $E(\tau)$ are disjoint, and $\pi^* + \tau^*$ is a connected X-join in $Gr(\mathcal{H})$ spanning all vertices in $V(\mathcal{H})$.*

It should be noted here that, by Claims 1 and 2, $\mathcal{H}$ satisfies the assumptions of the following result (Theorem 5 of [6]), which therefore guarantees the existence of a quasitree with tight complement in $\mathcal{H}$.

Theorem M [6]. *Let $\mathcal{H}$ be a 4-edge-connected 3-hypergraph. If no 3-hyperedge in $\mathcal{H}$ is included in any edge-cut of size 4, then $\mathcal{H}$ contains a quasitree with tight complement.*

However, in $\mathcal{H}^{\{e_1,e_2,e_3\}}$, a quasitree with tight complement does not have to exist due to the fact that $\mathcal{H}^{\{e_1,e_2,e_3\}}$ is obtained from $\mathcal{H}$ by removing some hyperedges, hence reducing degrees of some vertices. In such case, the following result gives the existence of a quasigraph and a skeletal partition $\mathcal{S}$ of $V(\mathcal{H})$ with a special structure, and we can use $\mathcal{S}$ for constructing the desired join.

Theorem 6. *Let $\mathcal{H}$ be a 4-edge-connected 3-hypergraph with at least one 3-hyperedge such that no 3-hyperedge of $\mathcal{H}$ is included in an edge-cut of size 4 and $Gr(\mathcal{H})$ is essentially 5-edge-connected. Let $e_1, e_2, e_3 \in E(Gr(\mathcal{H}))$ and set $\mathcal{H}' = \mathcal{H}^{\{e_1,e_2,e_3\}}$. If $\mathcal{H}'$ has no quasitree with tight complement, then there is an acyclic quasigraph π in $\mathcal{H}'$ and a π-skeletal partition $\mathcal{S}$ of $V(\mathcal{H}')$ such that:*

- *(i) one of the classes of $\mathcal{S}$ is a trivial class $\{x\}$,*
- *(ii) the degree of x in $\mathcal{H}$ is 4,*
- *(iii) e_1, e_2, e_3 are 2-hyperedges in $\mathcal{H}$ and each of e_1, e_2, e_3 is incident (in $\mathcal{H}$) with x.*

Proof. The proof of Theorem 6 basically follows the proof of the main result of [6] (Sections 8 and 11), where the calculations are adapted to our assumptions.

By Lemma K, $\mathcal{H}'$ contains an acyclic quasigraph π and a π-skeletal partition $\mathcal{P}$. By the assumption, $\mathcal{H}'$ has no quasitree with tight complement, hence $\mathcal{P}$ is nontrivial. Assume that $\mathcal{H}/\mathcal{P}$ has n vertices (i.e., $|\mathcal{P}| = n$) and m_3 3-hyperedges. Let m' denote the number of hyperedges in $\overline{\mathcal{H}'/\mathcal{P}}$, m'_k the number of k-hyperedges of $\pi/\mathcal{P}$ and $\overline{m'_k}$ the number of k-hyperedges of $\overline{\pi/\mathcal{P}}$, $k \in \{2,3\}$. Thus, $m' = m'_2 + m'_3 + \overline{m'_2} + \overline{m'_3}$.

Since $\overline{\pi/\mathcal{P}}$ is acyclic, the graph $Gr(\overline{\pi/\mathcal{P}})$ is a forest. As $Gr(\overline{\pi/\mathcal{P}})$ has $n + \overline{m'_3}$ vertices and $\overline{m'_2} + 3\overline{m'_3}$ edges, we find that

$$\overline{m'_2} + 2\overline{m'_3} \leq n - 1. \tag{1}$$

Since $\mathcal{P}$ is π-solid and π is an acyclic quasigraph, we know that $m'_2 + m'_3 \leq n - 1$. Moreover, by the assumption that π is not a quasitree with a tight complement, either this inequality or (1) is strict. Summing the two, we obtain

$$m' + \overline{m'_3} \leq 2n - 3. \tag{2}$$

For an arbitrary hypergraph $\mathcal{H}^*$, let $s(\mathcal{H}^*)$ denote the sum of degrees of all its vertices. By the construction of $\mathcal{H}'$ from $\mathcal{H}$, the operations (1), (2a) and (2b) can decrease the degree sum by at most 6 (if all the edges e_1, e_2, e_3 are hyperedges of size 2; otherwise the decrease is less than 6). Hence we have

$$s(\mathcal{H}'/\mathcal{P}) \geq s(\mathcal{H}/\mathcal{P}) - 6. \tag{3}$$

Set $n_4 = |V_4(\mathcal{H}/\mathcal{P})|$ and $n_{5^+} = |V_{\geq 5}(\mathcal{H}/\mathcal{P})|$. Since $\mathcal{H}$ is 4-edge-connected, we have $n_4 + n_{5^+} = n$ and

$$s(\mathcal{H}/\mathcal{P}) \geq 4n_4 + 5n_{5^+}. \tag{4}$$

By simple counting,

$$s(\mathcal{H}'/\mathcal{P}) = 2(m'_2 + \overline{m'_2}) + 3(m'_3 + \overline{m'_3}) = 2m' + m'_3 + \overline{m'_3}. \tag{5}$$

Combining (3), (4) and (5), we have

$$4n_4 + 5n_{5^+} - 6 \leq s(\mathcal{H}/\mathcal{P}) - 6 \leq s(\mathcal{H}'/\mathcal{P}) = 2m' + m'_3 + \overline{m'_3},$$

from which

$$4n_4 + 5n_{5^+} - 6 \leq 2m' + m'_3 + \overline{m'_3}. \tag{6}$$

From (2) we have

$$2m' + 2\overline{m'_3} \leq 4(n_4 + n_{5^+}) - 6,$$

which, using (6), gives

$$2m' + 2\overline{m'_3} \leq 2m' + m'_3 + \overline{m'_3} - n_{5^+},$$

or, equivalently,

$$\overline{m'_3} + n_{5^+} \leq m'_3. \tag{7}$$

Suppose that $m'_3 > 0$. Let $T' = (\pi/\mathcal{P})^*$ be the forest on $\mathcal{P}$ which represents $\pi/\mathcal{P}$. In each component of T', containing an edge corresponding to a 3-hyperedge of $(\pi/\mathcal{P})$, choose a root in that edge and direct the edges of T' away from it. To each 3-hyperedge $e \in E(\pi/\mathcal{P})$, assign the head $h(e)$ of the arc $\pi(e)$. By the assumptions of the theorem, no edge-cut of size 4 contains a 3-hyperedge, so $h(e)$ is a vertex of degree at least 5 and, by the same argument, the root is also of degree at least 5. At the same time, since each vertex is the head of at most one arc in the directed forest, it gets assigned to at most one hyperedge. From this we have

$$n_{5^+} \geq m'_3 + 1. \tag{8}$$

Combining (7) and (8), we obtain $\overline{m'_3} + n_{5^+} \leq m'_3 \leq n_{5^+} - 1$, implying $\overline{m'_3} + 1 \leq 0$, a contradiction.

Hence we have $m'_3 = 0$, and from (7) we have $n_{5^+} = 0$. Since every vertex of a 3-hyperedge is of degree at least 5, we have also $m_3 = 0$. Thus, $\mathcal{H}/\mathcal{P}$ is 4-regular and all its hyperedges are of size 2.

By the assumption, $\mathcal{H}$ has at least one 3-hyperedge, hence at least one element of $\mathcal{P}$ is nontrivial. If there are two nontrivial elements of $\mathcal{P}$, say, P_1 and P_2, then the edges connecting P_1 to the rest of $\mathcal{H}$ form an essential edge-cut of size 4 in $Gr(\mathcal{H})$, contradicting the assumption that $Gr(\mathcal{H})$ is essentially 5-edge-connected. Hence exactly one element of $\mathcal{P}$, say, P_1, is nontrivial. Similarly, by the essential 5-edge-connectivity assumption, $Gr(\mathcal{H}) - \mathcal{P}_1$ is independent. Since $\mathcal{H}/\mathcal{P}$ is 4-regular and $\mathcal{P}$ has at least 2 elements, $|\mathcal{P}| = 2$ and the second element of $\mathcal{P}$, P_2, is trivial. Set $P_2 = \{x\}$.

If x is connected in $\mathcal{H}'$ to $\mathcal{H}'[P_1]$ with at least two edges, we easily extend $\pi[P_1]$ to a quasitree with tight complement in $\mathcal{H}'$, a contradiction. Hence x is connected to $\mathcal{H}'[P_1]$ in $\mathcal{H}'$ with exactly one edge. Since x is of degree 4 in $\mathcal{H}$, x is incident with each of e_1, e_2, e_3. ■

Now we can complete the proof of Theorem 5. Let u_{e_i} be the vertex of e_i different from x, $i = 1, 2$. By the structure of $\mathcal{H}$ described in Theorem 6, an $X(e_1, e_2)$-join in $Gr(\mathcal{H}'[P_1])$ (which exists by Lemma L and since P_1 is π-solid) has the required properties (see the last paragraph before Lemma L). ■

5 Concluding remarks

1. By a slight modification of the proof of Theorem 5, we can also obtain the following result.

Theorem 7. *Every 5-connected line graph with minimum degree at least 6 is 3-hamiltonian.*

Proof of Theorem 7 is similar to the proof of Theorem 5 with the only difference that, in the notation of the proof of Theorem 5, instead of proving the existence of an $X(e_1, e_2)$-join, we find in $Gr(\mathcal{H}'[P_1])$ a $\emptyset$-join. If $\text{co}(H)$ is not 4-edge-connected, then the existence of a $\emptyset$-join in $Gr(\mathcal{H}'[P_1])$ follows by Theorem 6 in the same way as in Section 4; the case when $\text{co}(H)$ is 4-edge-connected has to be treated in a slightly different way. For this, we will need to recall some more concepts and facts.

A graph G is *collapsible* if, for any even subset $R \subset V(G)$, G has a spanning connected subgraph F such that $O(F) = R$, where $O(F)$ denotes the set of vertices of odd degree in F. The *reduction* of G is the graph obtained from G by contracting every maximal collapsible subgraph of G to a distinct vertex. Clearly, if a graph G is collapsible, then G has an X-join for any $X \subset V(G)$. For a graph G, let $f(G)$ denote the minimum number of edges that have to be added to G so that the resulting graph has two edge-disjoint spanning trees. We will need the following fact by Catlin et al. [4].

Theorem N [4]. *Let G be a connected graph with $f(G) \leq 2$. Then G is collapsible or the reduction of G is either K_2 or a $K_{2,t}$ for some $t \geq 1$.*

Thus, suppose now that $\mathrm{co}(H)$ is 4-edge-connected, and for $i = 1, 2, 3$, let $f_i \in E(\mathrm{co}(H))$ be the edge corresponding to e_i (in the special cases when some e_j is a pendant edge, we set $f_j = \emptyset$, and if some e_{j_1}, e_{j_2} share a vertex of degree 2, we set $f_{j_1} = f_{j_2}$). Set $H' = \mathrm{co}(H) - f_1$ and $H'' = H' - f_2 - f_3 = \mathrm{co}(H) - f_1 - f_2 - f_3$. If H' has two edge-disjoint spanning trees, then, by Theorem N, either H'' is collapsible and we are done, or the reduction of H'' is either K_2 or a $K_{2,t}$ for some $t \geq 1$. However, the second case is impossible since adding three edges to a $K_{2,t}$ can never create a reduction of a 4-edge-connected graph, and if the reduction of H'' is K_2, we find a desired $\emptyset$-join in the same way as in the proof of Theorem 5.

Thus, it remains to show that $H' = \mathrm{co}(H) - f_1$ has 2 edge-disjoint spanning trees. By Theorem J, we need to show that $|E_0| \geq 2(\omega(H' - E_0) - 1)$ for any $E_0 \subset E(H')$. Since $\mathrm{co}(H)$ is 4-edge-connected, every component of $\mathrm{co}(H) - E_0$ is connected to the rest of $\mathrm{co}(H) - E_0$ by at least 4 edges, implying $2|E_0| \geq 4\omega(\mathrm{co}(H) - E_0)$, from which $|E_0| \geq 2\omega(\mathrm{co}(H) - E_0)$ (for any $E_0 \subset E(\mathrm{co}(H))$, hence also for any $E_0 \subset E(H')$). Since $H' = \mathrm{co}(H) - f_1$, we have $\omega(\mathrm{co}(H) - E_0) \geq \omega(H' - E_0) - 1$, implying $|E_0| \geq 2\omega(\mathrm{co}(H) - E_0) \geq 2(\omega(H' - E_0) - 1)$, as required. ■

We are not able to extend Theorem 7 to claw-free graphs since a closure concept that would make this possible is not known so far.

2. We also note here that some authors define k-hamiltonicity in a slightly stronger way, requiring $G - X$ to be hamiltonian for every $X \subset V(G)$ with $|X| \leq k$ (instead of our condition $|X| = k$). Thus, a graph G is k-hamiltonian in this stronger sense if and only if, in our sense, G is s-hamiltonian for all s with $0 \leq s \leq k$ (where of course "0-hamiltonian" means "hamiltonian"). However, it is not difficult to observe that a graph G, satisfying the assumptions of Theorem 7, is 3-hamiltonian even in this stronger sense: for $k = 0$ and $k = 1$ this fact follows immediately from Theorem 5, and the statement for $k = 2$ can be obtained by the same modification of the proof of the main result of the paper [6] as we applied to the proof of Theorem 5 to obtain our Theorem 7. We leave details to the reader.

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