On forbidden subgraphs and rainbow connection in graphs with minimum degree 2

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Abstract

A connected edge-colored graph G is said to be rainbow-connected if any two distinct vertices of G are connected by a path whose edges have pairwise distinct colors, and the rainbow connection number $\mathrm{rc}(G)$ of G is the minimum number of colors that can make G rainbow-connected. We consider families $\mathcal{F}$ of connected graphs for which there is a constant $k_{\mathcal{F}}$ such that every connected $\mathcal{F}$-free graph G with minimum degree at least 2 satisfies $\mathrm{rc}(G) \leq \mathrm{diam}(G) + k_{\mathcal{F}}$, where $\mathrm{diam}(G)$ is the diameter of G. In this paper, we give a complete answer for $|\mathcal{F}| = 1$, and a partial answer for $|\mathcal{F}| = 2$.

1 Introduction

We consider undirected finite simple graphs, and for terminology and notation not defined here we refer to [3]. To avoid trivial cases, all graphs considered here will be connected with at least one edge.

An edge-colored connected graph G is called *rainbow-connected* if each pair of distinct vertices of G is connected by a rainbow path, that is, by a path whose edges have pairwise distinct colors. Note that the edge coloring need not be proper. The *rainbow connection number* of G, denoted by $\mathrm{rc}(G)$, is the minimum number of colors that can make G rainbow-connected.

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The concept of rainbow connection was introduced by Chartrand et al. in [7]. It is easy to observe that if G has n vertices then $\mathrm{rc}(G) \leq n-1$, since we can color the edges of some spanning tree of G with different colors and then color the remaining edges with one of the already used colors. Chartrand et al. determined the precise value of the rainbow connection number for several graph classes including complete multipartite graphs [7]. The rainbow connection number has been studied for further graph classes in [4, 8, 11, 14] and for graphs with fixed minimum degree in [4, 6, 12, 16]. See [15] for a survey.

The computation of $\mathrm{rc}(G)$ is known to be NP-hard ([5, 13]). In fact, it is already NP-complete to decide whether $\mathrm{rc}(G) = 2$, and it is also NP-complete to decide whether a given edge-colored graph (with an unbounded number of colors) is rainbow-connected [5]. More generally, it has been shown in [13] that for any fixed $k \geq 2$ it is NP-complete to decide whether $\mathrm{rc}(G) = k$.

In the following proposition, we summarize some obvious facts and observations for the rainbow connection number of graphs.

Proposition A. *Let G be a connected graph of order n. Then*

(*i*) $1 \leq \mathrm{rc}(G) \leq n-1$,

(*ii*) $\mathrm{rc}(G) \geq \mathrm{diam}(G)$,

(*iii*) $\mathrm{rc}(G) = 1$ *if and only if G is complete,*

(*iv*) $\mathrm{rc}(G) = n-1$ *if and only if G is a tree.*

Note that the difference $\mathrm{rc}(G) - \mathrm{diam}(G)$ can be arbitrarily large, as can be seen by considering $G \simeq K_{1,n-1}$, for which $\mathrm{rc}(K_{1,n-1}) - \mathrm{diam}(K_{1,n-1}) = (n-1) - 2 = n-3$. Especially, each bridge of G requires a single color. Therefore, connected bridgeless graphs have been studied.

Theorem B [2]. *For every connected bridgeless graph G with radius r,*

$$\mathrm{rc}(G) \leq r(r+2).$$

Moreover, for every integer $r \geq 1$, there exists a bridgeless graph G with radius r and $\mathrm{rc}(G) = r(r+2)$.

Note that, since $\mathrm{rad}(G) \leq \mathrm{diam}(G)$, Theorem B gives in bridgeless graphs an upper bound on $\mathrm{rc}(G)$ which is quadratic in terms of the diameter of G. In this paper, we will be interested in finding conditions on a graph G that imply a linear upper bound on $\mathrm{rc}(G)$ in terms of $\mathrm{diam}(G)$.

Let $\mathcal{F}$ be a family of connected graphs. We say that a graph G is $\mathcal{F}$*-free* if G does not contain an induced subgraph isomorphic to a graph from $\mathcal{F}$. Specifically, for $\mathcal{F} = \{X\}$ we say that G is X*-free*, and for $\mathcal{F} = \{X, Y\}$ we say that G is (X, Y)*-free.* The members of $\mathcal{F}$ will be referred to in this context as *forbidden induced subgraphs*, and for $|\mathcal{F}| = 2$ we also say that $\mathcal{F}$ is a *forbidden pair.*

Graphs characterized in terms of forbidden induced subgraphs are known to have many interesting properties. Although, by virtue of Theorem B, $\mathrm{rc}(G)$ can be (even for bridgeless graphs) still quadratic in terms of $\mathrm{diam}(G)$, it turns out that forbidden subgraph conditions can remarkably lower the upper bound on $\mathrm{rc}(G)$.

In [10], the authors considered the question for which families $\mathcal{F}$ of connected graphs, a connected $\mathcal{F}$-free graph satisfies $\mathrm{rc}(G) \leq \mathrm{diam}(G)+k_{\mathcal{F}}$, where $k_{\mathcal{F}}$ is a constant (depending on $\mathcal{F}$), and gave a complete answer for $1 \leq |\mathcal{F}| \leq 2$ by the following two results (where N denotes the *net*, i.e. the graph obtained by attaching a pendant edge to each vertex of a triangle).

Theorem C [10]. *Let X be a connected graph. Then there is a constant k_X such that every connected X-free graph G satisfies $\mathrm{rc}(G) \leq \mathrm{diam}(G) + k_X$, if and only if $X = P_3$.*

Theorem D [10]. *Let X, Y be connected graphs, $X, Y \neq P_3$. Then there is a constant k_{XY} such that every connected (X, Y)-free graph G satisfies $\mathrm{rc}(G) \leq \mathrm{diam}(G) + k_{XY}$, if and only if (up to symmetry) either $X = K_{1,r}$, $r \geq 4$ and $Y = P_4$, or $X = K_{1,3}$ and Y is an induced subgraph of N.*

Moreover, it was also shown in [10] that the (seemingly more general) question of finding families $\mathcal{F}$, $1 \leq |\mathcal{F}| \leq 2$, implying a linear upper bound on $\mathrm{rc}(G)$, i.e., such that every connected $\mathcal{F}$-free graph G satisfies $\mathrm{rc}(G) \leq q_{XY} \cdot \mathrm{diam}(G) + k_{XY}$, where q_{XY}, k_{XY} are constants, has the same solution as in Theorems C, D.

In this paper, we will consider an analogous question under an additional assumption $\delta(G) \geq 2$. Under this assumption, such an upper bound on $\mathrm{rc}(G)$ is already known for graphs from some special classes of graphs, such as e.g. interval graphs, AT-free graphs, threshold graphs or circular arc graphs (see [6], or Theorem 5.2.2. in [15]). In this paper, we will consider the following question.

For which families $\mathcal{F}$ of connected graphs, there is a constant $k_{\mathcal{F}}$ such that a connected graph G with $\delta(G) \geq 2$ being $\mathcal{F}$-free implies $\mathrm{rc}(G) \leq \mathrm{diam}(G) + k_{\mathcal{F}}$?

We give a complete answer for $|\mathcal{F}| = 1$ in Section 3, and a partial answer for $|\mathcal{F}| = 2$ in Section 4. Finally, in Section 5 we show that there are no more families with $|\mathcal{F}| \leq 2$ that would imply a linear bound on $\mathrm{rc}(G)$ in terms of $\mathrm{diam}(G)$ for connected graphs G with $\delta(G) \geq 2$.

2 Preliminary results

In this section we summarize some further notations and facts that will be needed for the proofs of our results.

An edge $e \in E(G)$ such that $G - e$ is disconnected is called a *bridge*, and a graph with no bridges is called a *bridgeless graph*. An edge such that one of its vertices has degree one is called a *pendant edge*. The *subdivision* of a graph G is the graph obtained from G by adding a vertex of degree 2 to each edge of G. For graphs X, G, we write $X \subset G$ if X is a subgraph of G, $X \overset{\text{IND}}{\subset} G$ if X is an induced subgraph of G, and $X \simeq G$ if X and G are isomorphic. For two vertices $x, y \in V(G)$, we denote by $\operatorname{dist}(x, y)$ the distance between x and y in G. The diameter and the radius of a graph G will be denoted by $\operatorname{diam}(G)$ and $\operatorname{rad}(G)$, respectively. A shortest path joining two vertices at distance $\operatorname{diam}(G)$ will be referred to as a *diameter path*.

For a set $S \subset V(G)$ and an integer $k \geq 1$, the *neighborhood at distance* k of S is the set $N_G^k(S)$ of all vertices of G at distance k from S. In the special case when $k = 1$, we simply write $N_G(S)$ for $N_G^1(S)$, and if $|S| = 1$ with $x \in S$, we write $N_G(x)$ for $N_G(\{x\})$. For a set $M \subset V(G)$, we set $N_M(S) = N_G(S) \cap M$ and $N_M(x) = N_G(x) \cap M$, and for a subgraph $P \subset G$, we write $N_P(x)$ for $N_{V(P)}(x)$. We will also use the *closed neighborhood* of a vertex defined by $N_G[x] = N_G(x) \cup \{x\}$ and of a subgraph $P \subset G$ defined by $N_G[P] = N_G(V(P)) \cup V(P)$. Finally, we will use P_k to denote the path on k vertices.

A set $D \subset V(G)$ is *dominating* if every vertex in $V(G) \setminus D$ has a neighbor in D. A dominating set D in a graph G is called a *two-way dominating set* if D includes all vertices of G of degree 1. In addition, if $G[D]$ is connected, we call D a *connected two-way dominating set*. Note that if $\delta(G) \geq 2$, then every (connected) dominating set in G is a (connected) two-way dominating set.

Theorem E [6]. *If D is a connected two-way dominating set in a graph G, then* $\operatorname{rc}(G) \leq \operatorname{rc}(G[D]) + 3$.

In our proofs, we will also need the following result.

Theorem F [1]. *Let G be a connected P_5-free graph. Then G has a dominating clique or a dominating P_3.*

3 One forbidden subgraph

In this section, we characterize all connected graphs X such that every connected X-free graph G with $\delta(G) \geq 2$ satisfies $\operatorname{rc}(G) \leq \operatorname{diam}(G) + k_X$, where k_X is a constant.

In [10], we have shown that, without the assumption $\delta(G) \geq 2$, the only connected graph X for which there is a constant k_X such that $\operatorname{rc}(G) \leq \operatorname{diam}(G) + k_X$ for every connected X-free graph G, is the path $X = P_3$ (see Theorem C).

We show that, for graphs G with $\delta(G) \geq 2$, the only such graph X is the path P_5 (and its induced subgraphs).

Theorem 1. *Let X be a connected graph. Then there is a constant k_X such that every connected X-free graph G with minimum degree $\delta(G) \geq 2$ satisfies $\mathrm{rc}(G) \leq \mathrm{diam}(G)+k_X$, if and only if X is an induced subgraph of P_5.*
Furthermore, if G is connected P_5-free with $\delta(G) \geq 2$, then $\mathrm{rc}(G) \leq \mathrm{diam}(G) + 3$.

In the proof of Theorem 1, we will need the following fact.

Proposition 2. *Let G be a connected P_5-free graph with n_1 vertices of degree 1. Then $\mathrm{rc}(G) \leq 5 + n_1$.*

Proof. By Theorem F, the graph G has a dominating set D which induces a clique or a P_3. Hence $\mathrm{rc}(G[D]) \leq 2$. Now let D_1 be the set of all vertices of degree 1 in G, and set $D^+ = D \cup D_1$. Then $|D^+| = |D|+n_1$ and $\mathrm{rc}(G[D^+]) \leq 2+n_1$. Moreover, D^+ is connected since D is connected and dominating. Therefore, D^+ is a connected two-way dominating set and hence $\mathrm{rc}(G) \leq \mathrm{rc}(G[D^+]) + 3 \leq 5 + n_1$. ■

Proof of Theorem 1. Let G be connected P_5-free. If $\mathrm{diam}(G) = 1$, then G is a clique, and then $\mathrm{rc}(G) = 1 = \mathrm{diam}(G)$. If $\mathrm{diam}(G) \geq 2$, then, immediately by Proposition 2, $\mathrm{rc}(G) \leq 5 \leq \mathrm{diam}(G) + 3$.

Now we show that there is no other such graph X. Let $t_0 \geq 3$ and, for $t \geq t_0$, let (see Fig. 1):

- G_1^t be the graph obtained by attaching a pendant edge to each vertex of a complete graph K_t,
- G_2^t be the graph obtained by attaching a triangle to each vertex of degree 1 of a star $K_{1,t}$,
- G_3^t be the graph obtained by attaching a cycle of length 4 to each vertex of degree 1 of a star $K_{1,t}$,
- G_4^t be the graph obtained by attaching a triangle to each vertex of degree 1 of the graph G_1^t.

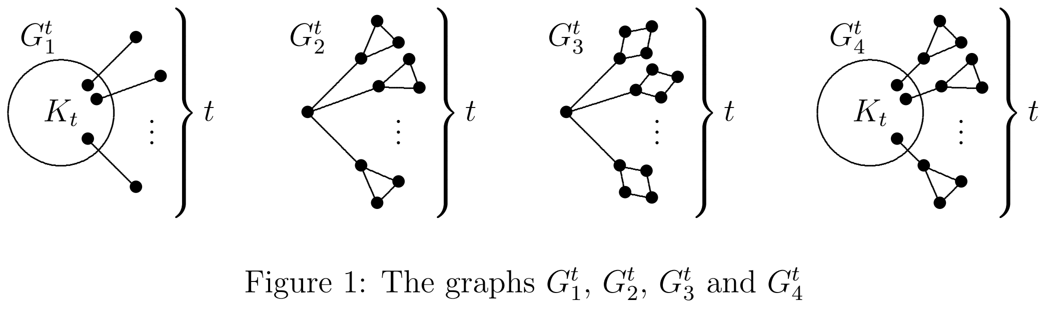

Figure 1: The graphs G_1^t, G_2^t, G_3^t and G_4^t

Clearly $\mathrm{rc}(G_2^t) \geq t$ but $\mathrm{diam}(G_2^t) = 4$, hence X is an induced subgraph of a subdivision of a star or X contains a triangle. Since $\mathrm{rc}(G_3^t) \geq t$ but $\mathrm{diam}(G_3^t) = 6$, and G_3^t is triangle free, X is an induced subgraph of a subdivision of a star. Finally, $\mathrm{rc}(G_4^t) \geq t$ but $\mathrm{diam}(G_4^t) = 5$, and G_4^t is $K_{1,3}$-free, hence X is a subdivision of $K_{1,2}$, i.e. the path P_5 (or its induced subgraph). ■

4 Pairs of forbidden subgraphs

Let $S_{i,j,k}$ denote the graph obtained by identifying one endvertex of three vertex disjoint paths of lengths i, j, k; Z_i the graph obtained by attaching a path of length i to a vertex of a triangle, and let $N_{i,j,k}$ denote the graph obtained by identifying each vertex of a triangle with an endvertex of one of three vertex disjoint paths of lengths i, j, k. In this context, we will also write Z_1^t for the graph G_2^t introduced in the proof of Theorem 1 (see Fig. 2).

It is easy to observe that if $X \overset{\text{IND}}{\subset} X'$, then every (X, Y)-free graph is also (X', Y)-free. Thus, when considering forbidden pairs implying some graph property, we will be always interested in finding *maximal pairs* for the property, i.e., pairs X, Y such that, if replacing one of X, Y, say, X, with a graph $X' \neq X$ such that $X \overset{\text{IND}}{\subset} X'$, then the statement under consideration is not true for (X', Y)-free graphs.

The following statement gives a list of all possible maximal pairs of forbidden subgraphs X, Y for which there can be a constant k_{XY} such that $\mathrm{rc}(G) \leq \mathrm{diam}(G) + k_{XY}$ for any connected (X, Y)-free graph G with $\delta(G) \geq 2$. By virtue of Theorem 1, we exclude the case when one of X, Y is P_5.

Theorem 3. *Let $X, Y \neq P_5$ be a maximal pair of connected graphs for which there is a constant k_{XY} such that every connected (X, Y)-free graph G with $\delta(G) \geq 2$ satisfies $\mathrm{rc}(G) \leq \mathrm{diam}(G) + k_{XY}$. Then (up to symmetry) either $X = S_{2,2,2}$ and $Y = N_{2,2,2}$, $X = P_6$ and $Y = Z_1^r$ ($r \in \mathbb{N}$), or $Y = Z_3$ and $X \in \{P_7, S_{3,3,3}, S_{1,1,4}\}$.*

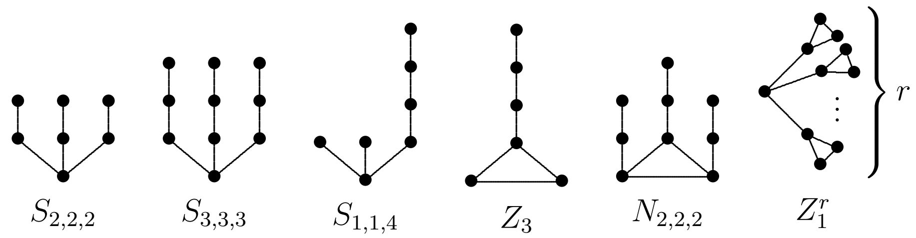

Figure 2: The graphs $S_{2,2,2}$, $S_{3,3,3}$, $S_{1,1,4}$, Z_3, $N_{2,2,2}$ and Z_1^r.

Proof. Let $t \geq 1$ and let (see Fig. 3):

- $G_5^{s,t}$ be the graph obtained by attaching a cycle of length $s \geq 4$ to each pendant edge of $S_{t,t,t}$ for any $t \geq 1$,
- G_6^t be the graph obtained by attaching a triangle to each pendant edge of the graph $N_{t,t,t}$ for any $t \geq 1$,
- G_7^t be the graph obtained by attaching a C_4 to each pendant edge of the graph G_1^t for any $t \geq 1$.

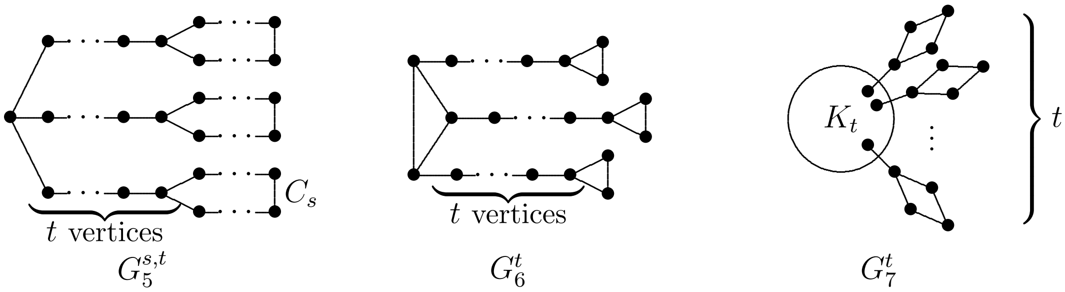

Figure 3: The graphs $G_5^{s,t}$, G_6^t and G_7^t.

We will also use the graphs $G_2^t(=Z_1^t)$, G_3^t and G_4^t shown in Fig. 1.

Consider the graph $G_5^{s,t}$. Clearly $\operatorname{diam}(G_5^{s,t}) = 2\left(\lfloor \frac{s}{2} \rfloor + t\right)$. Since all bridges of $G_5^{s,t}$ must have mutually distinct colors, $\operatorname{rc}(G_5^{s,t}) \geq 3t$. Specifically, for the graph $G_5^{2t,3t}$, $t \geq 2$, we obtain $\operatorname{diam}(G_5^{2t,3t}) = 8t$ and $\operatorname{rc}(G_5^{2t,3t}) \geq 9t = \frac{9}{8}\operatorname{diam}(G_5^{2t,3t})$. Similarly, for the graph G_6^t we have $\operatorname{diam}(G_6^t) = 2t+3$ and $\operatorname{rc}(G_6^t) \geq 3t$, implying $\operatorname{rc}(G_6^t) \geq \frac{3}{2}\operatorname{diam}(G_6^t) - \frac{9}{2}$, and for the graph G_7^t we have $\operatorname{diam}(G_7^t) = 7$ while $\operatorname{rc}(G_7^t) \geq t$. Thus, for sufficiently large t, each of the graphs $G_5^{2t,3t}$, G_6^t, G_7^t must contain an induced subgraph isomorphic to some of the graphs X, Y.

We can suppose that, up to symmetry, $X \overset{\text{IND}}{\subset} G_5^{s,t}$, implying $X = S_{i,j,k}$, $i,j,k \geq 1$, or $X = P_i$, $i \geq 6$ (note that $X \overset{\text{IND}}{\subset} G_5^{s,t}$ must be true for any sufficiently large integers s and t).

Now consider the graph G_2^t. There are two possibilities:

(*i*) $Y \overset{\text{IND}}{\subset} G_2^t$. Then $Y \overset{\text{IND}}{\subset} Z_1^r$ for some $r \geq 1$. Consider the graph G_4^t. First, if $X \overset{\text{IND}}{\subset} G_4^t$, we have $X = P_6$ and $Y \overset{\text{IND}}{\subset} Z_1^r$ since G_4^t is $S_{i,j,k}$-free for any $i,j,k \geq 1$. Secondly, if $Y \overset{\text{IND}}{\subset} G_4^t$, observing that the only common induced subgraph of G_2^t and G_4^t is Z_3, we have $Y = Z_3$. Now, $Y \overset{\text{IND}}{\subset} G_3^t$ implies $Y = P_5$, which is excluded by the assumptions, hence we have $X \overset{\text{IND}}{\subset} G_3^t$, from which $X = P_7$, $X = S_{3,3,3}$ or $X = S_{1,1,4}$.

(*ii*) $X \overset{\text{IND}}{\subset} G_2^t$. Then the only common induced subgraphs of both $G_5^{s,t}$ and G_2^t are $S_{2,2,2}$ and P_5 (or their induced subgraphs). But since P_5 is excluded by the assumptions, we have $X = S_{2,2,2}$. Now consider the graphs G_4^t, G_6^t and G_7^t. Since all of them are $S_{i,j,k}$-free for any $i,j,k \geq 1$, we get $Y \overset{\text{IND}}{\subset} G_4^t$, $Y \overset{\text{IND}}{\subset} G_6^t$ and $Y \overset{\text{IND}}{\subset} G_7^t$, implying that $Y = N_{2,2,2}$, $Y = Z_3$ or $Y = P_6$. The case $X = S_{2,2,2}$ and $Y = P_6$ is covered by case (*i*) since $S_{2,2,2} \overset{\text{IND}}{\subset} Z_1^r$ for any $r \geq 3$, and the case $X = S_{2,2,2}, Y = Z_3$ is covered by the pair $X = S_{3,3,3}$, $Y = Z_3$ in case (*i*).

■

Now we will consider sufficiency of some of the forbidden pairs given in Theorem 3. Namely, in this paper, we prove sufficiency for the pair $X = P_6$, $Y = Z_1^r$ ($r \in \mathbb{N}$) in

Theorem 4, for the pair $X = P_7$, $Y = Z_3$ in Theorem 6, and for the pair $X = S_{1,1,4}$, $Y = Z_3$ in Theorem 8.

The sufficiency proofs for the remaining two pairs $X = S_{3,3,3}$, $Y = Z_3$ and $X = S_{2,2,2}$, $Y = N_{2,2,2}$ are much more complicated and require different techniques, and these will be therefore published (and the characterization will be completed) in a separate paper [9].

Theorem 4. *Let r be a positive integer and let G be a connected (P_6, Z_1^r)-free graph with $\delta(G) \geq 2$. Then $\mathrm{rc}(G) \leq \mathrm{diam}(G) + 20 + r$.*

Proof. Since G is P_6-free, $\mathrm{diam}(G) = d \leq 4$. If $d = 1$, then G is complete and we are done. So we assume that $d \geq 2$.

If G is bridgeless, then, by Theorem B, $\mathrm{rc}(G) \leq \mathrm{rad}(G)(\mathrm{rad}(G) + 2) \leq d(d + 2) = d + d(d+1) \leq d + 20$. Hence suppose that G contains a bridge $e = xy$. Since $\delta(G) \geq 2$, we have $d \geq 3$. If $d = 3$, then $V(G) = N_G[x] \cup N_G[y]$, the bridge xy is a two-way dominating set in G and, by Theorem E, $\mathrm{rc}(G) \leq 1 + 3 = 4 = \mathrm{diam}(G) + 1$.

Thus, it remains to consider the case $d = 4$. Let X, Y be the components of $G - e$ (X containing x). Since $\delta(G) \geq 2$, both X and Y are nontrivial. Let $u \in V(X)$ be at maximum distance from x and, similarly, let $v \in V(Y)$ be at maximum distance from y. Since G is P_6-free, we get, up to symmetry, $\mathrm{dist}(x, u) = 1$ and $\mathrm{dist}(y, v) = 2$. Thus, $V(X) \subset N_G[x]$ and, since $\delta(G) \geq 2$, X is bridgeless. Similarly, since $\delta(G) \geq 2$ and every vertex of Y is at distance at most 2 from y, every bridge in Y is incident with the vertex y. Thus, every bridge in G is incident with y.

Let $B \subset G$ denote the subgraph determined (i.e., edge-induced) by the set of all bridges in G (note that B is a star with center at y). Since G is $Z_{1,r}$-free, every vertex of G is at distance at most 2 from y, and since $\delta(G) \geq 2$, we have $|E(B)| < r$. Clearly, each component of $G - E(B)$ is bridgeless. Let A denote the only (possibly trivial) component of $G - E(B)$ containing y. Then A is bridgeless and of radius at most 2, implying that $\mathrm{rc}(A) \leq 2{\cdot}4 = 8$ by Theorem B. In the rest of G, i.e., in the graph $G_1 = G[V(G - A) \cup \{y\}]$, $V(B)$ is a two-way dominating set, implying that $\mathrm{rc}(G_1) \leq \mathrm{rc}(B) + 3 = r - 1 + 3 = r + 2$ since each bridge must have a distinct color. Therefore $\mathrm{rc}(G) \leq r + 2 + 8 = r + 10 = \mathrm{diam}(G) + r + 6$. ■

Now we turn our attention to the forbidden pairs (Z_3, P_7) and $(Z_3, S_{1,1,4})$. Since all such graphs are Z_3-free, the following lemma will be useful in our proofs.

Lemma 5. *Let G be a connected Z_3-free graph with $\omega(G) \geq 3$ and $\delta(G) \geq 2$ such that G contains a bridge. Then $\mathrm{rc}(G) \leq 4$.*

Proof. Let xy be a bridge in G. Then there are two components of $G - xy$. Let G_1 denote a component containing a triangle and let G_2 denote the other component of $G - xy$. Up to symmetry, suppose that $x \in V(G_1)$ and $y \in V(G_2)$. Then every vertex of G_2 is adjacent to y, for otherwise we get an induced Z_3 with a triangle in G_1. This implies that $\omega(G_2) \geq 3$ since $\delta(G) \geq 2$. Now, every vertex of G_1 is adjacent to x since otherwise we get an induced Z_3 with a triangle in G_2. This implies that $D = \{x, y\}$ is a two-way dominating set in G and, by Theorem E, $\mathrm{rc}(G) \leq \mathrm{rc}(G[D]) + 3 = 4$. ■

Theorem 6. *Let G be a connected (Z_3, P_7)-free graph with $\delta(G) \geq 2$. Then $\mathrm{rc}(G) \leq \mathrm{diam}(G) + 30$.*

Proof. Since G is P_7-free, $\mathrm{diam}(G) \leq 5$. If G is bridgeless, we have $\mathrm{rc}(G) \leq \mathrm{rad}(G)(\mathrm{rad}(G) + 2) \leq \mathrm{diam}(G)(\mathrm{diam}(G) + 2) \leq \mathrm{diam}(G) + 30$ by Theorem B. Hence we assume that G has a bridge $e = xy$. By Lemma 5, we can suppose that G is triangle-free.

Let A, B denote the components of $G - e$. Since $\delta(G) \geq 2$, both A and B are nontrivial, and since G is triangle-free, each of them contains a vertex at distance 2 from $e = xy$. Let $u \in V(A)$ be at maximum distance from x, and $v \in V(B)$ be at maximum distance from y.

Claim 1. *All vertices of G are at distance at most 2 from e.*

Proof. If, say, $\mathrm{dist}(y, v) \geq 3$, then $\mathrm{dist}(u, v) \geq 6$, a contradiction. □

Claim 2. *A, B are bridgeless.*

Proof. Let, say, f be a bridge in B. If $y \notin f$, then, by Claim 1, f is a pendant edge, contradicting the assumption $\delta(G) \geq 2$. Hence $y \in f$. Set $f = yz$. By Claim 1, all vertices of B are adjacent to z. Since $\delta(G) \geq 2$, B contains a triangle, a contradiction. □

Now, by Claim 1, A and B have radius 2, and, by Claim 2, A and B are bridgeless. Thus, by Theorem B, $\mathrm{rc}(A) \leq 8$, $\mathrm{rc}(B) \leq 8$, and, with one extra color for e, we have $\mathrm{rc}(G) \leq 8 + 1 + 8 = 17 \leq \mathrm{diam}(G) + 14$, since $\mathrm{diam}(G) \geq 3$. ■

For the pair $(Z_3, S_{1,1,4})$, we will need the following lemma.

Lemma 7. *Let G be a $(Z_3, S_{1,1,4})$-free graph, let $x_0, x_d \in V(G)$ be vertices at distance $d \geq 8$, let $P = x_0 x_1 \ldots x_d$ be a shortest (x_0, x_d)-path and let $y \in V(G) \setminus V(P)$ be at distance 1 from P. Then y satisfies one of the following:*

(*i*) $N_P(y) = \{x_0\}$,
(*ii*) $N_P(y) = \{x_d\}$,
(*iii*) $d = 8$ *and* $N_P(y) = N_G(y) = \{x_3, x_5\}$.

Proof. If $|N_P(y)| = 1$, i.e., $N_P(y) = \{x_i\}$ for some i, $0 \leq i \leq d$, then, for $1 \leq i \leq d-4$ we have $G[\{x_i, y, x_{i-1}, x_{i+1}, x_{i+2}, x_{i+3}, x_{i+4}\}] \simeq S_{1,1,4}$ and for $4 \leq i \leq d-1$ we have $G[\{x_i, y, x_{i+1}, x_{i-1}, x_{i-2}, x_{i-3}, x_{i-4}\}] \simeq S_{1,1,4}$. Thus, we have a contradiction in all cases except (i) and (ii).

If $|N_P(y)| \geq 3$, then $N_P(y) = \{x_{i-1}, x_i, x_{i+1}\}$ for some i, $1 \leq i \leq d-1$, since P is shortest. But then either $G[\{x_{i+1}, y, x_i, x_{i+2}, x_{i+3}, x_{i+4}\}]$ (for $i \leq d-4$), or $G[\{x_{i-1}, y, x_i, x_{i-2}, x_{i-3}, x_{i-4}\}]$ (for $i \geq 4$) is a Z_3, a contradiction.

Thus, $|N_P(y)| = 2$. If the neighbors of y on P are consecutive, say, $N_P(y) = \{x_i, x_{i+1}\}$, then either $G[\{x_{i+1}, y, x_i, x_{i+2}, x_{i+3}, x_{i+4}\}]$ or $G[\{x_i, y, x_{i+1}, x_{i-1}, x_{i-2}, x_{i-3}\}]$ is a Z_3, a contradiction. Hence $N_P(y) = \{x_{i-1}, x_{i+1}\}$ for some i, $1 \leq i \leq d-1$ (recall that the neighbors of y on P are at distance at most 2 since P is shortest). But then either $G[\{x_{i+1}, y, x_i, x_{i+2}, x_{i+3}, x_{i+4}, x_{i+5}\}]$ or $G[\{x_{i-1}, y, x_i, x_{i-2}, x_{i-3}, x_{i-4}, x_{i-5}\}]$ is an $S_{1,1,4}$, unless $d = 8$ and $N_P(y) = \{x_3, x_5\}$.

Thus, to finish the proof, it remains to show that in this case also $N_G(y) = \{x_3, x_5\}$. Let, to the contrary, $z \in V(G) \setminus V(P)$ be a neighbor of y. Suppose that z has a neighbor on P. Obviously $zx_0 \notin E(G)$ and $zx_8 \notin E(G)$ since P is shortest; by what we have already proved, we have $N_P(z) = \{x_3, x_5\}$. But then $G[\{x_3, z, y, x_2, x_1, x_0\}] \simeq Z_3$, a contradiction. Thus, z has no neighbor on P, implying $G[\{y, z, x_3, x_5, x_6, x_7, x_8\}] \simeq S_{1,1,4}$, a contradiction again. ■

Theorem 8. *Let G be a connected $(Z_3, S_{1,1,4})$-free graph with $\delta(G) \geq 2$. Then* $\mathrm{rc}(G) \leq \mathrm{diam}(G) + 56$.

Proof. Suppose first that $\mathrm{diam}(G) \geq 8$, and let $P = x_0 x_1 \dots x_d$, $d \geq 8$, be a diameter path in G. Since $\delta(G) \geq 2$, x_0 has a neighbor, say, y, outside P. Since P is a diameter path, there is a shortest (y, x_d)-path Q' of length $d-1$ or d. The paths P and Q' are internally vertex-disjoint, for otherwise, if x_j, $1 \leq j \leq d$, is the first internal vertex of P that is on Q' and w is the predecessor of x_j on Q', then x_j is an internal vertex of P having a neighbor outside P, contradicting Lemma 7, unless $d = 8$ and $j = 5$, in which case we have a similar contradiction on the predecessor of w on Q'.

Set $Q = x_0 y Q' x_d$. The graph $G_1 = G - x_0$ is $(Z_3, S_{1,1,4})$-free and, by Lemma 7, $P_1 = y Q x_d P x_1$ is a shortest (y, x_1)-path in G_1 of length greater than 8. By Lemma 7, no internal vertex of P_1 has a neighbor outside P_1 in G_1, and, by the distance, also in G. Using a symmetric argument in $G_2 = G - x_d$, we conclude that G is a cycle, implying $\mathrm{rc}(G) \leq \mathrm{diam}(G) + 1$.

Secondly, suppose that $\mathrm{diam}(G) = d \leq 7$. If G is bridgeless, then, by Theorem B, we have $\mathrm{rc}(G) \leq d(d+2) = d + d(d+1) \leq d + 56$. Thus, let $e = xy$ be a bridge in G, and let A, B be the components of $G - e$ (A containing x). Since $\delta(G) \geq 2$, both A and B are nontrivial. Let $u \in V(A)$ be at maximum distance from x and, similarly, let $v \in V(B)$ be at maximum distance from y. By Lemma 5, we can suppose that G is triangle-free. Thus, since $\delta(G) \geq 2$, we have $\mathrm{dist}(u,x) \geq 2$ and $\mathrm{dist}(v,y) \geq 2$, implying $\mathrm{diam}(G) = \mathrm{dist}(u,x) + \mathrm{dist}(v,y) + 1 \geq 5$.

Claim 1. *Both* $\mathrm{dist}(u,x) = 2$ *and* $\mathrm{dist}(v,y) = 2$.

Proof. Let, say, $\mathrm{dist}(u,x) \geq 3$. Let z be the first vertex on a shortest (y,v)-path Q (in the orientation from y to v) that is of degree at least 3 (such a vertex must exist since $\delta(G) \geq 2$), and let w be a neighbor of z outside Q. Clearly $z \neq v$ (since v is at maximum distance from y), hence let z^+ denote the successor of z on Q. Now $wz^+ \notin E(G)$ since G is triangle-free, but then z, w, z^+ together with the first four vertices of a shortest (z,u)-path induce an $S_{1,1,4}$, a contradiction. □

Claim 2. *Both* A *and* B *are bridgeless.*

Proof. Let f be a bridge in, say, A. By Claim 1 and $\delta(G) \geq 2$, we have $x \in f$, but then $\delta(G) \geq 2$ implies that A contains a triangle, a contradiction. □

Now, both A and B are of radius at most 2 by Claim 1, and are bridgeless by Claim 2. By Theorem B, we have $\mathrm{rc}(A) \leq \mathrm{rad}(A)(\mathrm{rad}(A)+2) \leq 8$, and similarly $\mathrm{rc}(B) \leq 8$. Using one extra color for the edge e, we obtain $\mathrm{rc}(G) \leq 8 + 1 + 8 = 17 \leq \mathrm{diam}(G) + 12$, since $\mathrm{diam}(G) \geq 5$. ■

5 Concluding remarks

In Sections 3 and 4, we have studied forbidden families $\mathcal{F}$ with $|\mathcal{F}| \leq 2$ implying that $\mathrm{rc}(G) \leq \mathrm{diam}(G) + k_{\mathcal{F}}$. As a next step, it is natural to ask for forbidden families $\mathcal{F}$ implying that $\mathrm{rc}(G)$ is bounded by a linear function of $\mathrm{diam}(G)$. Thus, we can address the following question.

For which families $\mathcal{F}$ of connected graphs, there are constants $q_{\mathcal{F}}$, $k_{\mathcal{F}}$ such that a connected graph G with $\delta(G) \geq 2$ being $\mathcal{F}$-free implies $\mathrm{rc}(G) \leq q_{\mathcal{F}} \cdot \mathrm{diam}(G) + k_{\mathcal{F}}$?

In [10], we have shown that, without the assumption $\delta(G) \geq 2$, the answer is the same as in the case $q_{\mathcal{F}} = 1$. By a slight modification of the argument from [10], we will show an analogous result for $\delta(G) \geq 2$.

For $|\mathcal{F}| = 1$, it is easy to observe that all the graphs G_2^t, G_3^t, G_4^t, used in the necessity part of the proof of Theorem 1, have bounded diameter but unbounded rainbow connection number for $t \to \infty$. Thus, for $|\mathcal{F}| = 1$, the answer is the same as in Theorem 1, i.e., the only such graph X is the path $X = P_5$.

Our last result, which is a counterpart to Theorem 3, shows that the situation is the same also for $|\mathcal{F}| = 2$, i.e., for pairs of forbidden subgraphs.

Theorem 9. *Let $X, Y \neq P_5$ be a maximal pair of connected graphs for which there are constants q_{XY}, k_{XY} such that every connected (X,Y)-free graph G with $\delta(G) \geq 2$ satisfies $\mathrm{rc}(G) \leq q_{XY} \cdot \mathrm{diam}(G) + k_{XY}$. Then (up to symmetry) either $X = S_{2,2,2}$ and $Y = N_{2,2,2}$, $X = P_6$ and $Y = Z_1^r$ ($r \in \mathbb{N}$), or $Y = Z_3$ and $X \in \{P_7, S_{3,3,3}, S_{1,1,4}\}$.*

Proof. Let q, k be arbitrary constants and let s be a positive integer such that $3 \cdot 2^{s-3} > q+1$. Let

- T_s be a balanced cubic tree of depth $s+1$, i.e., with $3 \cdot 2^s$ leaves (vertices of degree 1; for $s = 2$, see Fig. 4 left),
- T_s' be the subdivision of T_s (for $s = 2$, see Fig. 4 middle),
- $T_{s,r}$ be the tree obtained by identifying each leaf of a tree T_s with an endvertex of a path P_{r+1},
- $T_{s,r}'$ be the tree obtained by identifying each leaf of a tree T_s' with an endvertex of a path P_{r+1} (for $s = 2$, see Fig. 4 right).

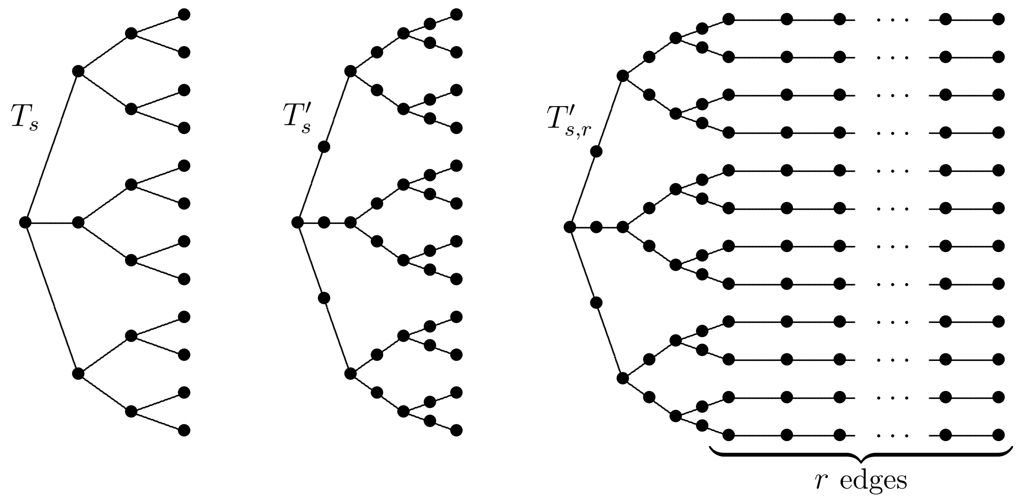

Figure 4: The trees T_2, T_2' and $T_{2,r}'$

Now, for $t \geq s+1$, let:

- $G_8^{s,t}$ be the graph obtained by identifying each leaf of a tree $T_{s,2t}'$ with one vertex of a cycle C_{2t},
- $G_9^{s,t}$ be the line graph of the graph obtained by attaching two pendant edges to each leaf of a tree $T_{s,2t}$

(for $s = 1$, see Fig. 5).

For the graph $G_8^{s,t}$, we have $\mathrm{diam}(G_8^{s,t}) = 2(2s+2+3t)$ and $\mathrm{rc}(G_8^{s,t}) \geq |E(T_{s,2t})| > 3 \cdot 2^s 2t \geq 3 \cdot 2^{s-2}(2t+3t) \geq 3 \cdot 2^{s-2}(2s+2+3t) = 3 \cdot 2^{s-3} \cdot \mathrm{diam}(G_8^{s,t}) > (q+1) \cdot \mathrm{diam}(G_8^{s,t})$ since every bridge has to be colored with a different color. Hence there is a t_1 such that, for $t \geq t_1$, $\mathrm{rc}(G_8^{s,t}) > q \cdot \mathrm{diam}(G_8^{s,t}) + k$.

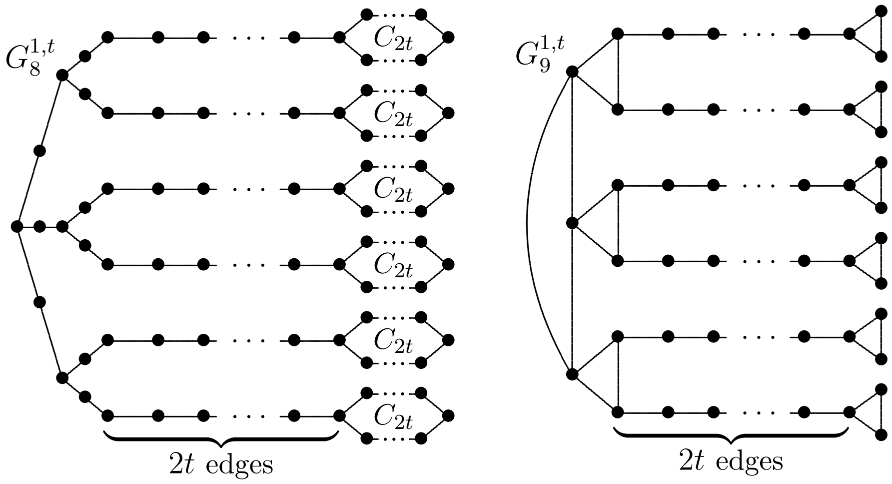

Figure 5: The graphs $G_8^{1,t}$ and $G_9^{1,t}$

For the graph $G_9^{s,t}$, we analogously have $\mathrm{diam}(G_9^{s,t}) = 2s+1+4t+2 = 2s+4t+3$ and, since $G_9^{s,t}$ has $3 \cdot 2^s 2t = 3 \cdot 2^{s+1}t$ bridges, we have $\mathrm{rc}(G_9^{s,t}) \geq 3 \cdot 2^{s+1}t = 3 \cdot 2^{s-2}(4t+4t) > 3 \cdot 2^{s-2}(4t+2s+3) = 3 \cdot 2^{s-2} \cdot \mathrm{diam}(G_9^{s,t}) > (q+1) \cdot \mathrm{diam}(G_9^{s,t})$. Hence there is a t_2 such that, for $t \geq t_2$, $\mathrm{rc}(G_9^{s,t}) > q \cdot \mathrm{diam}(G_9^{s,t}) + k$.

We will also use the graphs G_2^t, G_3^t, G_4^t and G_7^t introduced in the proofs of Theorems 1 and 3, which, as already noted, have bounded diameter but their rainbow connection number is unbounded for $t \to \infty$; hence there is a t_3 such that $\mathrm{rc}(G_i^t) > q \cdot \mathrm{diam}(G_i^t) + k$ for $t \geq t_3$ and $i = 2, 3, 4, 7$.

Now, let X, Y be connected graphs implying that every connected (X, Y)-free graph G satisfies $\mathrm{rc}(G) \leq q \cdot \mathrm{diam}(G) + k$, and set $t_0 = \max\{t_1, t_2, t_3\}$. Then, by the above discussion, for $t \geq t_0$, each of the graphs G_2^t, G_3^t, G_4^t, G_7^t, $G_8^{s,t}$ and $G_9^{s,t}$ contains an induced X or Y.

We can suppose that, up to symmetry, $X \overset{\text{IND}}{\subset} G_8^{s,t}$, implying that X is a tree of maximum degree 3, in which no two vertices of degree 3 are adjacent. Now the remaining part of the proof proceeds by exactly the same argument as the final part of the proof of Theorem 3, with the only difference that, instead of the graphs $G_5^{s,t}$ and G_6^t, we use the graphs $G_8^{s,t}$ and $G_9^{s,t}$. ■

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