Rainbow connection and forbidden subgraphs

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Abstract

A connected edge-colored graph G is rainbow-connected if any two distinct vertices of G are connected by a path whose edges have pairwise distinct colors; the rainbow connection number $\mathrm{rc}(G)$ of G is the minimum number of colors such that G is rainbow-connected. We consider families $\mathcal{F}$ of connected graphs for which there is a constant $k_{\mathcal{F}}$ such that, for every connected $\mathcal{F}$-free graph G, $\mathrm{rc}(G) \leq \mathrm{diam}(G) + k_{\mathcal{F}}$, where $\mathrm{diam}(G)$ is the diameter of G. In this paper, we give a complete answer for $|\mathcal{F}| \in \{1, 2\}$.

1 Introduction

We use [2] for terminology and notation not defined here and consider finite and simple graphs only. To avoid trivial cases, all graphs considered here will be connected with at least one edge.

An edge-colored connected graph G is called *rainbow-connected* if each pair of distinct vertices of G is connected by a rainbow path, that is, by a path whose edges have pairwise distinct colors. Note that the edge coloring need not be proper. The *rainbow connection number* of G, denoted by $\mathrm{rc}(G)$, is the minimum number of colors such that G is rainbow-connected.

The concept of rainbow connection in graphs was introduced by Chartrand et al. in [7]. An easy observation is that if G has n vertices then $\mathrm{rc}(G) \leq n - 1$, since one may

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color the edges of a given spanning tree of G with different colors and color the remaining edges with one of the already used colors. Chartrand et al. determined the precise value of the rainbow connection number for several graph classes including complete multipartite graphs [7]. The rainbow connection number has been studied for further graph classes in [4, 10, 11, 15] and for graphs with fixed minimum degree in [4, 12, 17]. See [16] for a survey.

There are various applications for such edge colorings of graphs. One interesting example is the secure transfer of classified information between agencies (see, e. g., [9]).

The computational complexity of rainbow connectivity has been studied in [5, 13]. It is proved that the computation of $\mathrm{rc}(G)$ is NP-hard ([5, 13]). In fact, it is already NP-complete to decide whether $\mathrm{rc}(G) = 2$. It is also NP-complete to decide whether a given edge-colored graph (with an unbounded number of colors) is rainbow-connected [5]. More generally, it has been shown in [13] that for any fixed $k \geq 2$ it is NP-complete to decide whether $\mathrm{rc}(G) = k$.

For the rainbow connection numbers of graphs the following results are known (and obvious).

Proposition A. *Let G be a connected graph of order n. Then*

(i) $1 \leq \mathrm{rc}(G) \leq n - 1$,
(ii) $\mathrm{rc}(G) \geq \mathrm{diam}(G)$,
(iii) $\mathrm{rc}(G) = 1$ *if and only if G is complete,*
(iv) $\mathrm{rc}(G) = n - 1$ *if and only if G is a tree,*
(v) *if G is a cycle of length $n \geq 4$, then* $\mathrm{rc}(G) = \lceil \frac{n}{2} \rceil$.

Note that the difference $\mathrm{rc}(G) - \mathrm{diam}(G)$ can be arbitrarily large. For $G = K_{1,n-1}$ we have $\mathrm{rc}(K_{1,n-1}) - \mathrm{diam}(K_{1,n-1}) = (n - 1) - 2 = n - 3$. Especially, each bridge requires a single color.

Let $\mathcal{F}$ be a family of connected graphs. We say that a graph G is *$\mathcal{F}$-free* if G does not contain an induced subgraph isomorphic to a graph from $\mathcal{F}$. Specifically, for $\mathcal{F} = \{X\}$ we say that G is *X-free*, and for $\mathcal{F} = \{X, Y\}$ we say that G is *(X, Y)-free*. The members of $\mathcal{F}$ will be referred to in this context as *forbidden induced subgraphs*.

Graphs characterized in terms of forbidden induced subgraphs are known to have many interesting properties. Although, in general, there is no upper bound on $\mathrm{rc}(G)$ in terms of $\mathrm{diam}(G)$, and, in bridgeless graphs, by virtue of Theorem F, $\mathrm{rc}(G)$ can be quadratic in terms of $\mathrm{diam}(G)$, it turns out that forbidden subgraph conditions can remarkably lower the upper bound on $\mathrm{rc}(G)$.

Namely, we will consider the following question.

For which families $\mathcal{F}$ of connected graphs, there is a constant $k_{\mathcal{F}}$ such that a connected graph G being $\mathcal{F}$-free implies $\mathrm{rc}(G) \leq \mathrm{diam}(G) + k_{\mathcal{F}}$?

We give a complete answer for $|\mathcal{F}| = 1$ in Section 3, and for $|\mathcal{F}| = 2$ in Section 4.

2 Preliminary results

In this section we summarize some further notations and facts that will be needed for the proofs of our results.

An edge in a graph G is called a *bridge*, if its removal disconnects the graph. A graph with no bridges is called a *bridgeless graph*. An edge is called *pendant edge*, if one of its end vertices has degree one. For two vertices $x, y \in V(G)$, we denote by $\mathrm{dist}(x, y)$ the distance between x and y in G. The diameter and the radius of a graph G will be denoted by $\mathrm{diam}(G)$ and $\mathrm{rad}(G)$, respectively. For $M \subset V(G)$, we use $G[M]$ to denote the induced subgraph of G on M.

For $x \in V(G)$, we use $N_G(x)$ to denote the *neighborhood of x in G* and $N_G[x]$ to denote the *closed neighborhood of x in G* (i.e., $N_G(x) = \{y \in V(G)|\ xy \in E(G)\}$ and $N_G[x] = N_G(x) \cup \{x\}$). More generally, for sets $A, B \subset V(G)$, we denote $N_G(A) = \cup_{x \in A} N_G(x)$ and $N_B(A) = N_G(A) \cap B$, and for a subgraph $P \subset G$ we write $N_P(A)$ for $N_{V(P)}(A)$ and $N_G(P)$ for $N_G(V(P))$.

A dominating set D in a graph G is called a *two-way dominating set* if D includes all vertices of G of degree 1. In addition, if $G[D]$ is connected, we call D a *connected two-way dominating set*. Note that if $\delta(G) \geq 2$, then every (connected) dominating set in G is a (connected) two-way dominating set.

Theorem B [6]. *If D is a connected two-way dominating set in a graph G, then* $\mathrm{rc}(G) \leq \mathrm{rc}(G[D]) + 3$.

The following simple fact is implicit in the proof od Theorem B in [6]. However, since it is not stated explicitly, and since it will be used several times, we state it here, including its (easy) proof.

Proposition C [6]. *Let G be a graph and let $F \subset G$ be a connected subgraph of G such that every vertex in $V(G) \setminus V(F)$ has at least 2 neighbors in F. Then* $\mathrm{rc}(G) \leq \mathrm{rc}(F) + 2$.

Proof. Color the edges of G as follows:

- color the edges of F with colors $1, \ldots, k$, where $k = \mathrm{rc}(F)$,
- for each $x \in V(G) \setminus V(F)$, choose two edges from x to F and color them with colors $k + 1$ and $k + 2$,
- color the remaining edges arbitrarily (e.g., all of them with color $k + 2$).

Then G is rainbow-connected. ■

For the proofs of Theorem 4 and Theorem 6, we will also need the following two facts by Li et al. [14].

Theorem D [14]. *If G is a connected bridgeless graph of diameter 2, then* $\mathrm{rc}(G) \leq 5$.

Theorem E [14]. *If G is a connected graph of diameter 2 with $k \geq 1$ bridges, then* $\mathrm{rc}(G) \leq k+2$.

For connected bridgeless graphs, the following upper bound on $\mathrm{rc}(G)$ was proved by Basarajavu et al. [1].

Theorem F [1]. *For every connected bridgeless graph G with radius r,*

$$\mathrm{rc}(G) \leq r(r+2).$$

Moreover, for every integer $r \geq 1$, there exists a bridgeless graph G with radius r and $\mathrm{rc}(G) = r(r+2)$.

3 One forbidden subgraph

In this section, we characterize all connected graphs X such that every connected X-free graph G satisfies $\mathrm{rc}(G) \leq \mathrm{diam}(G) + k_X$, where k_X is a constant.

Theorem 1. *Let X be a connected graph. Then there is a constant k_X such that every connected X-free graph G satisfies* $\mathrm{rc}(G) \leq \mathrm{diam}(G) + k_X$, *if and only if $X = P_3$.*

Proof. If $X = P_3$, then G is a complete graph, implying $\mathrm{rc}(G) = \mathrm{diam}(G) = 1$.

Conversely, let $t_0 \geq 3$ and, for $t \geq t_0$, set $G_1^t = K_{1,t}$, and let G_2^t denote the graph obtained by attaching a pendant edge to each vertex of a complete graph K_t (see Fig. 1). Since $\mathrm{rc}(G_1^t) = t$ but $\mathrm{diam}(G_1^t) = 2$, G_1^t must contain an induced copy of X. Hence X is a star. Since $\mathrm{rc}(G_2^t) = t+1$ but $\mathrm{diam}(G_2^t) = 3$, G_2^t contains an induced copy of X. But X is a star and G_2^t is $K_{1,3}$-free, hence $X = K_{1,2} = P_3$. ■

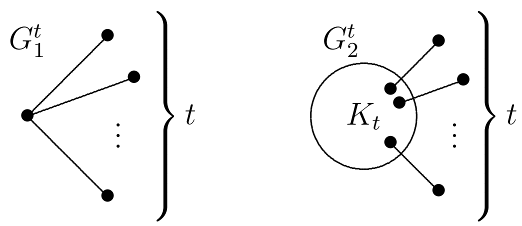

Figure 1: The graphs G_1^t and G_2^t

4 Pairs of forbidden subgraphs

The main result of this section, Theorem 2, characterizes all forbidden pairs X, Y for which there is a constant k_{XY} such that G being (X,Y)-free implies $\mathrm{rc}(G) \leq \mathrm{diam}(G) + k_{XY}$.

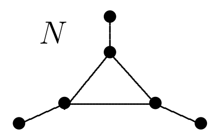

Figure 2: The net N

Here the *net* is the graph obtained by attaching a pendant edge to each vertex of a triangle (see Fig 2). By virtue of Theorem 1, we exclude the case that one of X, Y is P_3.

Theorem 2. *Let X, Y be connected graphs, $X, Y \neq P_3$. Then there is a constant k_{XY} such that every connected (X, Y)-free graph G satisfies $\mathrm{rc}(G) \leq \mathrm{diam}(G) + k_{XY}$, if and only if (up to symmetry) either $X = K_{1,r}$, $r \geq 4$ and $Y = P_4$, or $X = K_{1,3}$ and Y is an induced subgraph of N.*

The proof of Theorem 2 will be subdivided into three separate results: in Proposition 3, we prove necessity, and Theorems 4 and 6 will establish sufficiency of the forbidden pairs given in Theorem 2.

Proposition 3. *Let $X, Y \neq P_3$ be connected graphs for which there is a constant k_{XY} such that every connected (X, Y)-free graph G satisfies $\mathrm{rc}(G) \leq \mathrm{diam}(G) + k_{XY}$. Then (up to symmetry) either $X = K_{1,r}$, $r \geq 4$ and $Y = P_4$, or $X = K_{1,3}$ and Y is an induced subgraph of N.*

Proof. Let $t_0 \geq 3$ and, for $t \geq t_0$, let (see Fig. 3):

- G_3^t be the graph obtained by attaching an endvertex of a path P_t to every vertex of a triangle,
- G_4^t be the graph obtained by attaching a pendant edge to every internal vertex of a path P_t.

We will also use the graphs G_1^t and G_2^t introduced in the proof of Theorem 1.

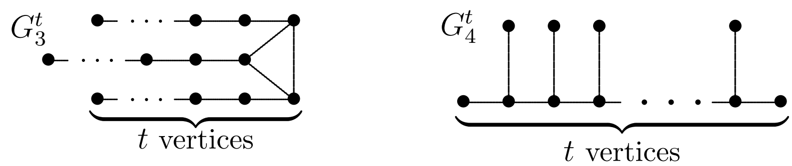

Figure 3: The graphs G_3^t and G_4^t

Consider the graph $G_1^t = K_{1,t}$. Since $\mathrm{rc}(G_1^t) = t$ while $\mathrm{diam}(G_1^t) = 2$, we have, up to symmetry, $X = K_{1,r}$ for some $r \geq 3$. Now we consider the graphs G_2^t and G_3^t. Clearly $\mathrm{rc}(G_2^t) = t + 1$ while $\mathrm{diam}(G_2^t) = 3$, and for G_3^t we observe that $\mathrm{diam}(G_3^t) = 2t - 1$ while $\mathrm{rc}(G_3^t) \geq 3(t - 1)$ (since all edges of the three paths must have mutually distinct colors), from which $\mathrm{rc}(G_3^t) \geq \frac{3}{2}(\mathrm{diam}(G_3^t) - 1)$. Since both G_2^t and G_3^t are claw-free, neither of them contains X, implying that both G_2^t and G_3^t contain Y. Since the maximum common induced subgraph of G_2^t and G_3^t is the net, we have that $Y = N$, or Y is an induced subgraph of N.

Now consider the graph G_4^t. Obviously, $\mathrm{diam}(G_4^t) = t-1$ and $\mathrm{rc}(G_4^t) = |V(G_4^t)| - 1 = 2t-3$, from which $\mathrm{rc}(G_4^t) = 2\,\mathrm{diam}(G_4^t) - 1$. We have two possibilities:

(i) G_4^t contains X. Then we obtain that $X = K_{1,3}$ and $Y = N$ (or an induced subgraph of N).

(ii) G_4^t contains Y. As the only induced subgraph of the net N contained in G_4^t and different from P_3 (or an induced subgraph) is the path P_4, and the case $X = K_{1,3}$ is already covered by case (i), we have that $X = K_{1,r}$, $r \geq 4$, and $Y = P_4$.

■

Theorem 4. *Let G be a connected $(K_{1,r}, P_4)$-free graph for some $r \geq 4$. Then $\mathrm{rc}(G) \leq r+1$.*

Proof. We have $\mathrm{diam}(G) \leq 2$ since G is P_4-free. If $\mathrm{diam}(G) = 1$, then G is a complete graph and $\mathrm{rc}(G) = 1$. Hence we may assume that $\mathrm{diam}(G) = 2$.

If G is bridgeless, then, by Theorem D, $\mathrm{rc}(G) \leq 5$, implying $\mathrm{rc}(G) \leq 5 \leq r+1$ and we are done. Thus, let $e = uv$ be a bridge in G. Since $\mathrm{diam}(G) = 2$, one of u, v, say, u, is of degree 1, and v is adjacent to all the other vertices of G. Since G is $K_{1,r}$-free, G has at most $r-1$ bridges. By Theorem E, we then have $\mathrm{rc}(G) \leq (r-1)+2 = r+1$. ■

Corollary 5. *Let G be a connected $(K_{1,r}, P_4)$-free graph for some $r \geq 4$. Then $\mathrm{rc}(G) \leq \mathrm{diam}(G) + r - 1$.*

Proof. If $\mathrm{diam}(G) = 1$, then $\mathrm{rc}(G) = 1 \leq \mathrm{diam}(G) + r - 1$, and if $\mathrm{diam}(G) = 2$, then, by Theorem 4, $\mathrm{rc}(G) \leq r+1 = \mathrm{diam}(G) + r - 1$. ■

Note that, for any $r \geq 3$, the graph G_1^{r-1} in Fig. 1 is $(K_{1,r}, P_4)$-free and has $\mathrm{rc}(G_1^{r-1}) = r-1 = \mathrm{diam}(G_1^{r-1}) + r - 3$. This shows that the constant in Theorem 4 and Corollary 5 has to depend on r.

Theorem 6. *Let G be a connected $(K_{1,3}, N)$-free graph. Then $\mathrm{rc}(G) \leq \mathrm{diam}(G) + 3$.*

For the proof of Theorem 6, we will need some observations on cycles and paths in $(K_{1,3}, N)$-free graphs. The first of them deals with induced cycles.

Lemma 7. *Let G be a $(K_{1,3}, N)$-free graph and let $C \subset G$ be a chordless cycle of length at least 5 in G. Then $V(C) \cup N_G(C) = V(G)$ and every vertex in $V(G) \setminus V(C)$ has at least 2 consecutive neighbors on C.*

Proof. Let first $x \in V(G) \setminus V(C)$ be at distance 1 from C, let $y \in N_C(x)$, and let y_1, y_2 be the neighbors of y on C. If neither y_1 nor y_2 is adjacent to x, then $G[\{y, y_1, y_2, x\}] \simeq K_{1,3}$, a contradiction.

Secondly, let $x \in V(G) \setminus V(C)$ be at distance 2 from C, and let y be a neighbor of x at distance 1 from C. By the above, y has 2 consecutive neighbors y_1, y_2 on C. Let y_1' be the neighbor of y_1 on C distinct from y_2, and, symmetrically, let y_2' be the neighbor of y_2 on C distinct from y_1. If $y_1'y \in E(G)$, then $G[\{y, x, y_1', y_2\}] \simeq K_{1,3}$, if $y_2'y \in E(G)$, then $G[\{y, x, y_1, y_2'\}] \simeq K_{1,3}$, and if neither y_1' nor y_2' is adjacent to y, then $G[\{y_1, y_2, y, y_1', y_2', x\}] \simeq N$. ■

We will also need the following simple observations on shortest paths and their neighborhoods in $(K_{1,3}, N)$-free graphs. Their main idea can be found in [3] (and, in fact, already in [8]), however, for the sake of completeness, we include them here as well.

Let G be a claw-free graph, let $x, y \in V(G)$ and let $P: \ x = v_0v_1v_2 \dots v_k = y$, $k \geq 3$, be a shortest xy-path in G. Let $z \in V(G) \setminus V(P)$.

1. If $|N_P(z)| = 1$, then, since G is claw-free, z is adjacent to x or to y.
2. If $|N_P(z)| \geq 2$ and $\{v_i, v_j\} \subset N_P(z)$, then, since P is a shortest path, $|i - j| \leq 2$.
3. By (1) and (2), since G is claw-free and since P is a shortest path, $|N_P(z)| \leq 3$ for every vertex $z \in V(G) \setminus V(P)$, and the vertices of $N_P(z)$ are consecutive on P.

This motivates the following notation:

$$N_i := \{z \in V(G) \setminus V(P) |\ N_P(z) = \{v_{i-1}, v_i, v_{i+1}\}\} \text{ for } 1 \leq i \leq k-1,$$

$$M_i := \{z \in V(G) \setminus V(P) |\ N_P(z) = \{v_{i-1}, v_i\}\} \text{ for } 1 \leq i \leq k,$$

$$M_0 := \{z \in V(G) \setminus V(P) |\ N_P(z) = \{v_0\}\},$$

$$M_{k+1} := \{z \in V(G) \setminus V(P) |\ N_P(z) = \{v_k\}\}.$$

Then, by (1), (2) and (3), we have $N(P) \cup V(P) = (\bigcup_{i=1}^{k-1} N_i) \cup (\bigcup_{i=0}^{k+1} M_i) \cup V(P)$. We further denote $S = V(P) \cup N(P)$ and $R = V(G) \setminus S$. The sets M_i and N_i have the following properties.

Lemma 8. *Let G be a $(K_{1,3}, N)$-free graph, let $x, y \in V(G)$ be vertices at distance $\mathrm{dist}_G(x, y) \geq 3$, and let $P: \ x = v_0v_1v_2 \dots v_k = y$ be a shortest xy-path in G. Then*
(i) $N_G(M_i) \subset V(P) \cup N_G(P)$, $i = 2, \dots, k-1$,
(ii) $N_G(N_i) \subset V(P) \cup N_G(P)$, $i = 1, \dots, k-1$,
(iii) $N_P(R) = \emptyset$,
(iv) $N_S(R) \subseteq M_0 \cup M_1 \cup M_k \cup M_{k+1}$.

Proof. If $zy \in E(G)$ for some $z \in R$ and $y \in M_i$, $2 \leq i \leq k-1$, then we have $G[\{v_{i-2}, v_{i-1}, v_i, v_{i+1}, y, z\}] \simeq N$, a contradiction. Hence $N_G(M_i) \subset S$, implying (i). Similarly, if $zy \in E(G)$ for $z \in R$ and $y \in N_i$, $1 \leq i \leq k-1$, then $G[\{y, z, v_{i-1}, v_{i+1}\}] \simeq K_{1,3}$, a contradiction. Hence $N_G(N_i) \subset S$, implying (ii). Part (iii) follows immediately by the definition of R, and (iv) follows immediately by (i) and (ii). ■

Proof of Theorem 6. Let G be a $(K_{1,3}, N)$-free graph. If $\mathrm{diam}(G) = 1$, G is a complete graph and there is nothing to do.

Let now $\mathrm{diam}(G) = 2$. If G is bridgeless, we have $\mathrm{rc}(G) \leq 5 = \mathrm{diam}(G) + 3$ by Theorem D; if G has $k \geq 1$ bridges, then, by $\mathrm{diam}(G) = 2$, all bridges in G have a vertex in common, implying $k \leq 2$ (since G is $K_{1,3}$-free), and we have $\mathrm{rc}(G) \leq k + 2 \leq 4 = \mathrm{diam}(G) + 2$ by Theorem E.

Thus, for the rest of the proof we suppose that $\mathrm{diam}(G) = d \geq 3$. Let $v_0, v_d \in V(G)$ be at distance d, let $P: \ v_0 v_1 v_2 \dots v_d$ be a diameter path in G, and let M_i, N_i, S, R be as above. Set $B_c = (\cup_{i=1}^{d-1} N_i) \cup (\cup_{i=2}^{d-1} M_i) \cup \{v_1, \dots, v_{d-1}\}$. By virtue of Lemma 8, we have $N_G(B_c) \subset V(P) \cup N_G(P)$.

We distinguish two cases.

<u>Case 1: *B_c is a cutset of G.*</u>

We claim that $R = \emptyset$. Let, to the contrary, $z \in R$ be at distance 2 from P. Then, by Lemma 8, by the assumption of Case 1 and by symmetry, we can suppose that $N_S(z) \subset M_0 \cup M_1$. Let Q be a shortest (z, v_d)-path, let w be the first vertex of Q in B_c (it exists by the assumption of Case 1), and let w^- be the predecessor of w on Q. By Lemma 8, $\mathrm{dist}(w^-, P) = 1$, implying $w^- \in M_0 \cup M_1$. Then $\mathrm{dist}_G(w^-, v_d) \geq d - 1$ (otherwise the path $v_0 w^- Q v_d$ is a (v_0, v_d)-path shorter than d), implying $\mathrm{dist}_G(w^-, v_d) = d - 1$ and $w^- z \in E(G)$. But then $G[\{w^-, z, v_0, w\}] \simeq K_{1,3}$, a contradiction. Thus, $V(P)$ is a connected dominating set in G. Moreover, if $M_0 \neq \emptyset$, then $M_0 \subset N_G(M_2)$, for otherwise again M_0 contains a vertex at distance $d+1$ from v_d (note that an edge from M_0 to M_3 is not possible since it would create an induced net). This specifically implies that every vertex in M_0 is of degree at least 2. Thus, the only vertices of G that can possibly be of degree 1, are the vertices v_0 and v_d. Consequently, $V(P)$ is a connected two-way dominating set in G, and, by Theorem B, we have $\mathrm{rc}(G) \leq \mathrm{rc}(P) + 3 = \mathrm{diam}(G) + 3$.

<u>Case 2: *B_c is not a cutset of G.*</u>

In this case, our strategy is to construct in G an induced cycle of length at least 5 and to use Lemma 7 and Proposition C. However, for $d = 3$, it is possible that G contains an edge xy with $x \in M_1$ and $y \in M_3$, in which case the general construction does not work. Thus, we consider the possibility when $d = 3$ separately.

Set $H = G - B_c$.

Subcase 2.1: $d = 3$.

First suppose that H contains a (v_0, v_3)-path which neither contains an edge from M_1 to M_3 nor has such an edge as a chord, and, among all such paths, let $P' : v_3v_4 \ldots v_{3+\ell} = v_0$ be a shortest one. Clearly, $\ell \geq 3$. Set $P^3 : v_2v_3v_4$ if $v_2v_4 \notin E(G)$ or $P^3 : v_2v_4$ if $v_2v_4 \in E(G)$, and, symmetrically, set $P^0 : v_{3+\ell-1}v_0v_1$ if $v_{3+\ell-1}v_1 \notin E(G)$ or $P^0 : v_{3+\ell-1}v_1$ if $v_{3+\ell-1}v_1 \in E(G)$, respectively. Set $C : v_1v_2P^3v_4 \ldots v_{3+\ell-1}P_0v_1$. By the choice of P', at least one of the paths P^0, P^3, $v_4P'v_{3+\ell-1}$ has length at least 2, hence C is a cycle of length at least 5 and it is straightforward to verify that C is chordless.

Claim 1. $\ell \leq 5$.

Proof. Suppose that $\ell \geq 6$, and let Q be a shortest (v_0, v_5)-path in G. Then $|E(Q)| \leq 3$ (since $\mathrm{diam}(G) = 3$), and, since $\ell \geq 6$ and P' is shortest in $H = G - B_c$, we have $\mathrm{dist}_H(v_0, v_5) \geq 4$. Hence either Q contains an edge between M_1 and M_3, or Q contains a vertex from B_c. However, in the first case, if $x \in V(Q) \cap M_3$ and x^-, x^+ are the predecessor and successor of x on Q, then $G[\{x, x^-, x^+, v_3\}] \simeq K_{1,3}$, a contradiction. Hence Q contains a vertex from B_c.

Let w^- be the last vertex of Q in B_c, and let w be its successor on Q (it exists since $v_5 \notin B_c$ by the definition of P'). By Lemma 8, w is at distance at most 1 from P. Since clearly $w \notin \{v_0, v_3\}$, either $wv_0 \in E(G)$ or $wv_3 \in E(G)$. If $wv_0 \in E(G)$, then, replacing in Q the subpath v_0Qw by the edge v_0w, we get a (v_0, v_5)-path in G shorter than Q, a contradiction. Hence $wv_3 \in E(G)$. Now, $w \neq v_5$ since C is chordless, therefore $\mathrm{dist}_G(v_0, w) = 2$, implying that $v_0, w^- \in E(G)$ and $wv_5 \in E(Q)$. But then $G[\{w, w^-, v_5, v_3\}] \simeq K_{1,3}$, a contradiction. Hence $\ell \leq 5$. $\square$

Now, C is a chordless cycle of length at least 5 and at most $3 + \ell \leq 8$. Thus, by Lemma 7, Proposition C and Proposition A(v), we have $\mathrm{rc}(G) \leq \mathrm{rc}(C) + 2 \leq 6 = \mathrm{diam}(G) + 3$.

Thus, we finally suppose that every (v_0, v_d)-path in H either contains an edge from M_1 to M_3, or has such an edge as a chord.

Claim 2. *The set $V(P) \cup M_1 \cup B_c \cup M_3 \subset V(G)$ can be covered by 4 complete graphs K_1, K_2, K_3, K_4 such that $V(K_1) = \{v_0, v_1\} \cup M_1$, $V(K_2) = \{v_1, v_2\} \cup N_1 \cup M_2$, $V(K_3) = \{v_1, v_2\} \cup N_2$, and $V(K_4) = \{v_2, v_3\} \cup M_3$.*

Proof. If there are $x_1, x_2 \in M_1$ with $x_1x_2 \notin E(G)$, then $G[\{v_1, x_1, x_2, v_2\}] \simeq K_{1,3}$, a contradiction. Hence K_1 is complete. Similarly, if some $x_1, x_2 \in (N_1 \cup M_1)$ are nonadjacent, then $G[\{v_2, x_1, x_2, v_3\}] \simeq K_{1,3}$, hence K_2 is also complete. The proof for K_3 and K_4 is symmetric. $\square$

Set $F = G[V(P) \cup M_1 \cup B_c \cup M_3]$.

Claim 3. $\mathrm{rc}(F) \leq 4$.

Proof. Color $E(K_1)$ with color 1, $E(K_i)\setminus E(K_{i-1})$ with color i, $i = 2, 3, 4$, and remaining edges of F arbitrarily (e.g., all of them with color 4). Then F is rainbow-connected. $\square$

Claim 4. *$V(F) \cup N_G(F) = V(G)$ and every vertex in $V(G) \setminus V(F)$ has at least 2 neighbors in F.*

Proof. Suppose that a vertex $x \in V(G) \setminus V(F)$ at distance 1 from F has exactly one neighbor in F, and set $N_F(x) = \{y\}$. Then, by Lemma 8, up to symmetry, either $x \in M_0$ or $y \in M_1$. Let Q be a shortest (x, v_3)-path in H. By the assumption, Q contains an edge from M_1 to M_3, implying that, in both cases, the successor of x on Q is in M_1. Thus, if $x \in M_0$, x has 2 neighbors in F and we are done, and, if $y \in M_1$, the successor y^+ of y on Q is in M_3 and we have $G[\{y, x, v_0, y^+\}] \simeq K_{1,3}$, a contradiction. Hence every vertex at distance 1 from F has at least 2 neighbors in F.

It remains to show that $V(F) \cup N_G(F) = V(G)$. Let, to the contrary, $z \in V(G)$ be at distance 2 from F, let y be a neighbor of z at distance 1 from F, and, by the previous part, let y_1, y_2 be neighbors of y in $V(F)$. Then $y_1y_2 \in E(G)$, for otherwise $G[\{y, z, y_1, y_2\}] \simeq K_{1,3}$. Since $\mathrm{dist}(z, v_0) \leq 3$ and $\mathrm{dist}(z, v_3) \leq 3$, we have, up to symmetry, $y_1 \in M_1 \cup \{v_0\}$ and $y_2 \in M_3 \cup \{v_3\}$. If e.g. $y_2 = v_3$, then $v_0y_1y_2$ is a (v_0, v_3)-path of length 2, a contradiction. Hence $y_2 \in M_3$, and, symmetrically, $y_1 \in M_1$. But then $G[\{y, y_1, y_2, z, v_0, v_3\}] \simeq N$, a contradiction. $\square$

Now, by Claim 4, by Claim 3 and by Proposition C, we have $\mathrm{rc}(G) \leq \mathrm{rc}(F) + 2 \leq 6 = \mathrm{diam}(G) + 3$.

Subcase 2.2: $d \geq 4$.

Let $P' : v_dv_{d+1}v_{d+2} \ldots v_{d+\ell-1}v_{d+\ell} = v_0$ be a shortest v_dv_0-path in H. Since P is a diameter path, $\ell \geq d$. Since H is $(K_{1,3}, N)$-free and P' is a shortest path in H, we can define analogously the sets M_i, N_i for $i = d+1, \ldots d+\ell$, and we set $B'_c = (\cup_{i=d+1}^{d+\ell-1} N_i) \cup (\cup_{i=d+2}^{d+\ell-1} M_i) \cup \{v_{d+1}, \ldots, v_{d+\ell-1}\}$. By Lemma 8, we have $N_G(B'_c) \subset V(P') \cup N_G(P')$.

Let P^d be the path $P^d : v_{d-1}v_dv_{d+1}$ if $v_{d-1}v_{d+1} \notin E(G)$, or the edge $P^d : v_{d-1}v_{d+1}$ if $v_{d-1}v_{d+1} \in E(G)$, respectively, and, symmetrically, set $P^0 : v_{d+\ell-1}v_0v_1$ if $v_{d+\ell-1}v_1 \notin E(G)$, or $P^0 : v_{d+\ell-1}v_1$ if $v_{d+\ell-1}v_1 \in E(G)$, respectively. Finally, let C be the cycle $C : v_1 \ldots v_{d-1}P^dv_{d+1} \ldots v_{d+\ell-1}P^0v_1$. Then C is a cycle of length at least $2d-2$.

Claim 5. *The cycle C is chordless.*

Proof. Let, to the contrary, $v_iv_j \in E(G)$ be a chord in C. Since both P and P' are chordless, we can choose the notation such that $1 \leq i \leq d-1$ and $d+1 \leq j \leq d+\ell-1$. Since $v_j \in V(P')$, we have $v_j \notin B_c$ by the definition of P', implying that $i = d-1$ and $v_j \in M_d$, or, symmetrically, $i = 1$ and $v_j \in M_1$. This implies that in the first case $v_j = v_{d+1}$ and in the second case $v_j = v_{d+\ell-1}$, in both cases, $v_iv_j \in V(C)$ by the definition of C. Thus, C is chordless. $\square$

Claim 6. $\ell \le d+2$.

Proof. Suppose that $\ell \ge d+3$, and let Q be a shortest (v_0, v_{d+2})-path in G. Then $|E(Q)| \le d$ (since $\mathrm{diam}(G) = d$), and, since $\ell \ge d+3$ and P' is shortest in $H = G - B_c$, we have $\mathrm{dist}_H(v_0, v_{d+2}) \ge d+1$. Hence Q contains a vertex from B_c. Let w^- be the last vertex of Q in B_c, and let w be its successor on Q (it exists since $v_{d+2} \notin B_c$ by the definition of P'). By Lemma 8, w is at distance at most 1 from P. Since clearly $w \notin \{v_0, v_d\}$, either $wv_0 \in E(G)$ or $wv_d \in E(G)$. If $wv_0 \in E(G)$, then, replacing in Q the subpath v_0Qw by the edge v_0w, we get a (v_0, v_{d+2})-path in G shorter than Q, a contradiction. Hence $wv_d \in E(G)$. Now, $w \ne v_{d+2}$ since C is chordless, implying $\mathrm{dist}_G(v_0, w) \le d-1$. However, if $\mathrm{dist}_G(v_0, w) \le d-2$, then v_0Qwv_d is a (v_0, v_d)-path of length at most $d-1$, contradicting the fact that $\mathrm{diam}(G) = d$. Hence $\mathrm{dist}_G(v_0, w) = d-1$, implying that $\mathrm{dist}_G(v_0, w^-) = d-2$ and $wv_{d+2} \in E(Q)$. But then $G[\{w, w^-, v_{d+2}, v_d\}] \simeq K_{1,3}$, a contradiction. Hence $\ell \le d+2$. □

By Claim 5, C is a chordless cycle of length at least $2d-2 \ge 6$, thus, by Lemma 7 and by Proposition C, $\mathrm{rc}(G) \le \mathrm{rc}(C) + 2$. By Claim 6, the length of C is at most $d + \ell \le 2d+2$, hence, by Proposition A(v), $\mathrm{rc}(C) \le \lceil \frac{2d+2}{2} \rceil = d+1$. Summarizing, we have $\mathrm{rc}(G) \le \mathrm{rc}(C) + 2 \le d+3$. ■

5 Concluding remarks

In Sections 3 and 4, we have characterized forbidden families $\mathcal{F}$ with $|\mathcal{F}| \le 2$ implying that $\mathrm{rc}(G) \le \mathrm{diam}(G) + k_{\mathcal{F}}$. As a next step, it is natural to ask for forbidden families $\mathcal{F}$ implying that $\mathrm{rc}(G)$ is bounded by a linear function of $\mathrm{diam}(G)$. Thus, we can address the following question.

For which families $\mathcal{F}$ of connected graphs, there are constants $q_{\mathcal{F}}$, $k_{\mathcal{F}}$ such that a connected graph G being $\mathcal{F}$-free implies $\mathrm{rc}(G) \le q_{\mathcal{F}} \cdot \mathrm{diam}(G) + k_{\mathcal{F}}$?

For $|\mathcal{F}| = 1$, it is easy to observe that both graphs G_1^t, G_2^t, used in the proof of the "only if" part of Theorem 1, have bounded diameter but their rainbow connection number is unbounded for $t \to \infty$. Thus, for $|\mathcal{F}| = 1$, the answer to the above question is the same as in Theorem 1, i.e., the only such graph X is the path $X = P_3$.

Our last result shows that the situation is the same also for $|\mathcal{F}| = 2$.

Theorem 9. *Let $X, Y \ne P_3$ be connected graphs. Then there are constants q_{XY}, k_{XY} such that every connected (X,Y)-free graph G satisfies $\mathrm{rc}(G) \le q_{XY} \cdot \mathrm{diam}(G) + k_{XY}$, if and only if (up to symmetry) either $X = K_{1,r}$, $r \ge 4$ and $Y = P_4$, or $X = K_{1,3}$ and Y is an induced subgraph of N.*

Proof. Sufficiency follows from Theorems 4 and 6; it remains to show necessity.

Let q, k be arbitrary constants, let s be a positive integer such that $3 \cdot 2^{s-2} > q+1$, and let T_s be a balanced cubic tree of depth $s+1$, i.e., with $3 \cdot 2^s$ leaves (vertices of degree 1) and $3 \cdot 2^s - 2$ non-leaves of degree 3, thus with $|V(T_s)| = 3 \cdot 2^{s+1} - 2$ vertices and $|E(T_s)| = 3 \cdot 2^{s+1} - 3$ edges (for $s = 2$, see Fig. 4).

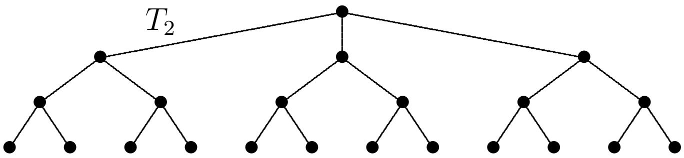

Figure 4: The tree T_2

For $t \geq s+1$, let:

- $G_5^{s,t}$ be the graph obtained by identifying each leaf of a tree T_s with an endvertex of a path P_{t+1},
- $G_6^{s,t}$ be the line graph of the graph $G_5^{s,t}$

(for $s = 1$, see Fig. 5).

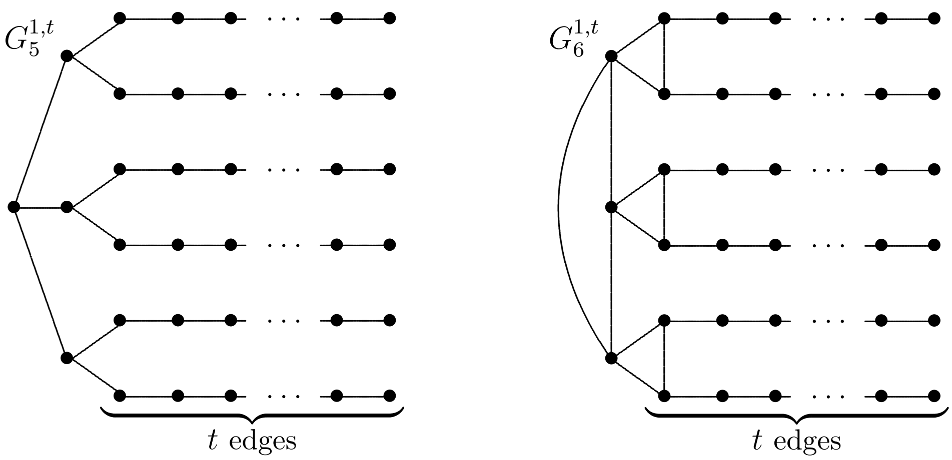

Figure 5: The graphs $G_5^{1,t}$ and $G_6^{1,t}$

For the graph $G_5^{s,t}$, we have $\mathrm{diam}(G_5^{s,t}) = 2(s+t+1)$ and $\mathrm{rc}(G_5^{s,t}) = |E(G_5^{s,t})| > 3 \cdot 2^s t \geq 3 \cdot 2^{s-1}(t+s+1) = 3 \cdot 2^{s-2} \cdot \mathrm{diam}(G_5^{s,t}) > (q+1) \cdot \mathrm{diam}(G_5^{s,t})$ since $G_5^{s,t}$ is a tree. Hence there is a t_1 such that, for $t \geq t_1$, $\mathrm{rc}(G_5^{s,t}) > q \cdot \mathrm{diam}(G_5^{s,t}) + k$.

For the graph $G_6^{s,t}$, we analogously have $\mathrm{diam}(G_6^{s,t}) = 2s + 1 + 2t = 2s + 2t + 1$ and, since $G_6^{s,t}$ has $3 \cdot 2^s t$ bridges, we have $\mathrm{rc}(G_6^{s,t}) \geq 3 \cdot 2^s t \geq 3 \cdot 2^{s-1}(t+s+1) = 3 \cdot 2^{s-2}(2t+2s+2) > 3 \cdot 2^{s-2} \cdot \mathrm{diam}(G_6^{s,t}) > (q+1) \cdot \mathrm{diam}(G_6^{s,t})$. Hence there is a t_2 such that, for $t \geq t_2$, $\mathrm{rc}(G_6^{s,t}) > q \cdot \mathrm{diam}(G_6^{s,t}) + k$.

We will also use the graphs G_1^t and G_2^t introduced in the proof of Theorem 1, which, as already noted, have bounded diameter but their rainbow connection number is unbounded for $t \to \infty$; hence there are t_3 and t_4 such that $\mathrm{rc}(G_1^t) > q \cdot \mathrm{diam}(G_1^t) + k$ for $t \geq t_3$ and $\mathrm{rc}(G_2^t) > q \cdot \mathrm{diam}(G_2^t) + k$ for $t \geq t_4$.

Now, let X, Y be connected graphs implying that every connected (X, Y)-free graph G satisfies $\mathrm{rc}(G) \leq q \cdot \mathrm{diam}(G) + k$, and set $t_0 = \max\{t_1, t_2, t_3, t_4\}$. Then, by the above discussion, for $t \geq t_0$, each of the graphs G_1^t, G_2^t, $G_5^{s,t}$, $G_6^{s,t}$ contains an induced X or Y. By symmetry, we can suppose that G_1^t contains X, implying $X = K_{1,r}$ for some $r \geq 3$. Since both G_2^t and $G_6^{s,t}$ are claw-free, Y is an induced subgraph of both G_2^t and $G_6^{s,t}$, implying that $Y = N$ (or an induced subgraph).

Considering $G_5^{s,t}$, we have two possibilities:

(*i*) $G_5^{s,t}$ contains X, and then $X = K_{1,3}$ and $Y = N$,

(*ii*) $G_5^{s,t}$ contains Y, and then, since the only induced subgraph of N contained in $G_5^{s,t}$ and different from P_3 is P_4, and the case $X = K_{1,3}$ is already covered in (*i*), we conclude that $X = K_{1,r}$, $r \geq 4$, and $Y = P_4$.

■

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