Closure and Hamilton-connected claw-free hourglass-free graphs

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September 18, 2014

Abstract

The closure cl(G) of a claw-free graph G is the graph obtained from G by a series of local completions at eligible vertices, as long as this is possible. The construction of an SM-closure of G follows the same operations, but if G is not Hamilton-connected, then the construction terminates once every local completion at an eligible vertex leads to a Hamilton-connected graph.

Although (see e.g. [7]) cl(G) may be Hamilton-connected even if G is not, we show that if G is a 2-connected claw-free graph with minimum degree at least 3 such that its SM-closure is hourglass-free, then G is Hamilton-connected if and only if the closure cl(G) of G is Hamilton-connected.

Keywords: closure; SM-closure; claw-free; Hamilton-connected; hourglass

1 Introduction

All graphs considered here are finite undirected graphs and for terminology and notation not defined here we refer to [1].

Let $G = (V, E)$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The *claw* is the graph $K_{1,3}$ and the *hourglass* is the only graph with degree sequence $4, 2, 2, 2, 2$ (i.e. two triangles with exactly one common vertex). The *square* of a graph G is the graph G^2 whose vertex set is $V(G)$, two distinct vertices being adjacent in G^2 if and only if their distance in G is at most 2. Specifically, the square of a path P_6 on six vertices will be denoted $(P_6)^2$ (see Fig. 1). A graph is called *S-free* if it has no induced subgraph isomorphic to S. Specifically, a graph is called *claw-free* if $S = K_{1,3}$ and *hourglass-free*

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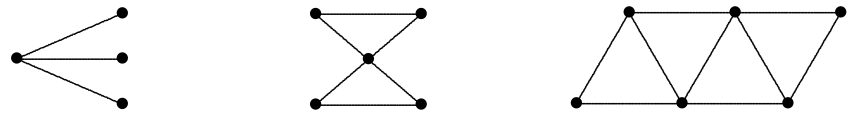

Figure 1: The graphs $K_{1,3}, \Gamma, (P_6)^2$

if $S = \Gamma$, respectively. A graph G is *hamiltonian* if G contains a *hamiltonian cycle*, i.e., a cycle passing through all its vertices, and G is *Hamilton-connected* if, for any pair of distinct vertices $x, y \in V(G)$, G contains a *hamiltonian* (x, y)*-path*, i.e., an (x, y)-path passing through all vertices of G.

For a vertex $x \in V(G)$, the set $N_G(x) = \{y \in V(G) : xy \in E(G)\}$ is called the *neighborhood* of x in G. A vertex $x \in V(G)$ is *locally connected* if the subgraph induced by $N_G(x)$ is connected, x is *eligible* if the subgraph induced by $N_G(x)$ is connected and noncomplete, and x is *simplicial* if the subgraph induced by $N_G(x)$ is complete, respectively. The set of all eligible vertices in G will be denoted $V_{EL}(G)$.

Let x be a vertex of a claw-free graph G. If the subgraph induced by $N_G(x)$ is connected, we add edges joining all pairs of nonadjacent vertices in $N_G(x)$ and obtain the graph G^*_x. This operation is called the *local completion of* G *at* x. Note that if a graph G is $\{K_{1,3}, \Gamma\}$-free and $x \in V_{EL}(G)$, then G^*_x is not necessarily Γ-free (an easy example is the graph $(P_6)^2$, in which we choose x as one of its vertices of degree 3).

The *closure* $\text{cl}(G)$ of a graph G is the graph obtained from G by recursively repeating the local completion operation, as long as this is possible. Note that the closure $\text{cl}(G)$ of a given graph G is uniquely determined and $\text{cl}(G)$ is the line graph of a simple triangle-free graph [6].

As in [5], for a given claw-free graph G, we construct a graph G^M by the following construction.

(i) If G is Hamilton-connected, we set $G^M = \text{cl}(G)$.

(ii) If G is not Hamilton-connected, we recursively perform the local completion operation at such eligible vertices for which the resulting graph is still not Hamilton-connected, as long as this is possible. We obtain a sequence of graphs $G_1, \cdots G_k$ such that

(1) $G_1 = G$,

(2) $G_{i+1} = (G_i)^*_{x_i}$ for some $x_i \in V_{EL}(G_i), i = 1, \cdots, k$,

(3) G_k has no hamiltonian (a, b)-path for some $a, b \in V(G_k)$,

(4) for any $x \in V_{EL}(G_k)$, $(G_k)^*_x$ is Hamilton-connected,

and we set $G^M = G_k$.

A graph G^M obtained by the above construction will be called *a strong M-closure* (or briefly an *SM-closure*) of the graph G, and a graph G equal to its SM-closure will be said to be *SM-closed*. If G^M is an SM-closure of a claw-free graph G, then clearly G^M is Hamilton-connected if and only if so is G. Note that, for a given graph G, its SM-closure is not necessarily uniquely determined.

As shown in [6], a claw-free graph G is hamiltonian if and only if $\mathrm{cl}(G)$ is hamiltonian, however, it is known (see [2], [7]) that there are infinitely many claw-free graphs G such that G is not Hamilton-connected while $\mathrm{cl}(G)$ is Hamilton-connected. However, it could be still possible that G is Hamilton-connected if and only if $\mathrm{cl}(G)$ is Hamilton-connected, if the graph G satisfies some additional assumptions. This question was considered in [4]. The first observation in [4] shows that the local completion operation preserves the property of being $\{K_{1,3}, (P_6)^2, \Gamma\}$-free.

Proposition A [4]. *Let G be a connected $\{K_{1,3}, (P_6)^2, \Gamma\}$-free graph and let x be a locally connected vertex of G. Then G_x^* is also $\{K_{1,3}, (P_6)^2, \Gamma\}$-free.*

Using Proposition A, the following was proved in [4].

Theorem B [4]. *Let G be a 3-connected $\{K_{1,3}, (P_6)^2, \Gamma\}$-free graph with minimum degree at least 4. Then G is Hamilton-connected if and only if $\mathrm{cl}(G)$ is Hamilton-connected.*

In this paper, we prove the following results strengthening Theorem B.

Theorem 1. *Let G be a 2-connected claw-free graph with minimum degree $\delta(G) \geq 3$ and let G^M be an SM-closure of G. If G^M is hourglass-free, then G^M is the only SM-closure of G and $G^M = \mathrm{cl}(G)$.*

Proof of Theorem 1 is postponed to Section 3.

Note that, in Theorem 1, it is possible that G is of connectivity 2 while $\mathrm{cl}(G)$ is 3-connected. Then clearly G is not Hamilton-connected, and Theorem 1 guarantees that neither is $\mathrm{cl}(G)$ (regardless of its connectivity).

From Theorem 1 we easily obtain the following fact.

Theorem 2. *Let G be a 2-connected claw-free graph with minimum degree $\delta(G) \geq 3$ such that G has an hourglass-free SM-closure G^M. Then G is Hamilton-connected if and only if $\mathrm{cl}(G)$ is Hamilton-connected.*

Proof. If G is Hamilton-connected, then clearly so is $\mathrm{cl}(G)$. Conversely, suppose that G is not Hamilton-connected, and let G^M be an hourglass-free SM-closure of G. Then G^M is not Hamilton-connected either. By Theorem 1, $G^M = \mathrm{cl}(G)$ and the result follows. ■

Theorem 2 implies the following result, which is a direct strengthening of Theorem B.

Corollary 3. *Let G be a 2-connected $\{K_{1,3}, (P_6)^2, \Gamma\}$-free graph with minimum degree $\delta(G) \geq 3$. Then G is Hamilton-connected if and only if $\mathrm{cl}(G)$ is Hamilton-connected.*

Proof. Let G^M be an SM-closure of G. If G is $\{K_{1,3}, (P_6)^2, \Gamma\}$-free, then so is G^M by Proposition A. The statement now follows by Theorem 2. ■

2 Notations and preliminary results

Before presenting the proof of Theorem 1, we first introduce some additional terminology and notation. The *line graph* $L(G)$ of a graph G is the graph with $V(L(G)) = E(G)$ and $E(L(G)) = \{e_ie_j : e_i \text{ and } e_j \text{ have a common vertex in } G\}$. If $G = L(H)$ and $e \in E(H)$, we will also use the notation $L(e)$ to denote the vertex of G corresponding to e. For simple graphs, it is well-known that if G is a line graph (of some graph), then the graph H such that $G = L(H)$ is uniquely determined (with one exception of the graphs C_3 and $K_{1,3}$, for which both $L(C_3)$ and $L(K_{1,3})$ are isomorphic to C_3). However, this is not true for multigraphs, as it is easy to construct infinitely many examples of nonisomorphic multigraphs with isomorphic line graphs. This drawback can be overcome by imposing an additional requirement that if $G = L(H)$, then simplicial vertices in G correspond to pendant edges in H (where a *pendant edge* is an edge one vertex of which is of degree 1). It can be shown [8] that for any line graph G there is a uniquely determined multigraph H such that $G = L(H)$ and simplicial vertices in G correspond to pendant edges in H. This graph H will be called the *preimage* of G and denoted $H = L^{-1}(G)$.

It is an easy observation that in the special case when G is a line graph and $H = L^{-1}(G)$, a vertex $x \in V(G)$ is locally connected if and only if the edge $e = L^{-1}(x)$ is in a triangle or in a multiedge in H, and $G_x^* = L(H|_e)$, where the graph $H|_e$ is obtained from H by contraction of e into a vertex and replacing the created loop(s) by pendant edge(s).

A *walk* in G is an alternating sequence $v_0, e_0, v_1, e_1, \cdots, e_{k-1}, v_k$ of vertices and edges of G such that $e_i = v_iv_{i+1}$ for all $i = 0, 1, \cdots, k-1$. A *trail* in G is a walk with no repeated edges. For a trail $T = v_0, e_0, v_1, e_1, \cdots, e_{k-1}, v_k$ and two elements a, b of the sequence T, a preceding b in T (a, b can be vertices or edges), the subtrail T' determined by the subsequence between a and b will be denoted $T' = aTb$; the trail T'', determined by the same sequence in the reverse order is denoted $T'' = b\overleftarrow{T}a$. For $h, f \in E(G)$, an (h, f)-*trail* in G is a trail such that $e_0 = h, e_{k-1} = f$.

Given a trail T and an edge e in a multigraph G, we say that e is *dominated (internally dominated)* by T if e is incident to a vertex (to an interior vertex) of T, respectively. A trail T in G is called an *internally dominating trail*, shortly *IDT*, if T

internally dominates all the edges in G. The following result shows that Hamilton paths in a line graph correspond to internally dominating trails in its preimage.

Theorem C [3]. *Let H be a multigraph with $|E(H)| \geq 3$. Then $G = L(H)$ is Hamilton-connected if and only if, for any pair of edges $e_1, e_2 \in E(H)$, H has an internally dominating (e_1, e_2)-trail.*

It can be shown (see [5]) that if G is SM-closed, then G is a line graph, and if $H = L^{-1}(G)$, then H contains no multiple edge of multiplicity more than 2, no multitriangle (a triangle with a multiple edge) and no diamond (a pair of triangles with a common edge). The following two theorems summarize further basic properties of the SM-closure operation which will be of importance for our proof.

Theorem D [5]. *Let G be a claw-free graph and let G^M be its SM-closure. Then:*

1. *G is Hamilton-connected if and only if G^M is Hamilton-connected;*
2. *G^M is a line graph, and $H = L^{-1}(G^M)$ satisfies one of the following conditions:*
   - *(1) H is a triangle-free simple graph;*
   - *(2) There are $e, f \in E(H)$ such that there is no (e, f)-IDT and either*
     - *(α) H is triangle-free and $\{e, f\}$ is the only multiedge in H, or*
     - *(β) H is a simple graph containing at most 2 triangles, each triangle in H contains at least one of e, f, and if H contains 2 triangles, then the triangles have no common edge.*

Theorem E [9]. *Let G be an* SM-closed *graph and let $H = L^{-1}(G)$. Then H does not contain a triangle with a vertex of degree 2.*

3 Proof of Theorem 1

In this section we provide the proof of Theorem 1.

Proof of Theorem 1. Let G be a 2-connected claw-free graph with minimum degree $\delta(G) \geq 3$, let G^M be an hourglass-free SM-closure of G and let $H = L^{-1}(G^M)$. Then H satisfies the condition (1), (2)(α) or (2)(β) of Theorem D, and we need to prove that $G^M = \mathrm{cl}(G)$. If G^M satisfies condition (1) of Theorem D, then H is a triangle-free simple graph, hence $G^M = L(H)$ is a closed graph, i.e., $\mathrm{cl}(G^M) = G^M$, implying $G^M = \mathrm{cl}(G)$. For the proof of Theorem 1, it is sufficient to show that neither one of the cases (2)(α) and (2)(β) of Theorem D is possible.

Case 1: *H satisfies condition* $(2)(\alpha)$ *of Theorem D.*
Let $e, f \in E(H)$ be the (only) pair of parallel edges in H, and let a, b be their vertices. If one of a, b is of degree 2, then e, f are non-pendant edges in H corresponding to simplicial vertices in G^M, contradicting the fact that simplicial vertices in G^M correspond to pendant edges in $H = L^{-1}(G^M)$. Hence both a and b are of degree at least 3. If both $d_H(a) \geq 4$ and $d_H(b) \geq 4$, then, except for e and f, a is incident to some edges e_1, e_2 and b is incident to some edges f_1, f_2, but then $G^M[\{L(e), L(e_1), L(e_2), L(f_1), L(f_2)\}]$ is an induced hourglass because H is triangle-free, a contradiction. Hence, we can assume $d_H(a) = 3$. Let a' be the vertex of $N_H(a) \setminus \{b\}$. We distinguish three possibilities according to the degree of a'.

If $d_H(a') = 1$, then $\{e, f\}$ is an edge-cut of H, separating the edge aa' from the rest of H. Hence the vertex $L(aa')$ is of degree 2 in G^M, contradicting the fact that $\delta(G^M) \geq \delta(G) \geq 3$.

If $d_H(a') \geq 3$, we choose arbitrary vertices $a_1, a_2 \in N_H(a') \setminus \{a\}$, and then clearly $G^M[\{L(aa'), L(a'a_1), L(a'a_2),\ L(e), L(f)\}]$ is an induced hourglass in G^M, a contradiction.

Thus, $d_H(a') = 2$. Let a'' denote the second neighbor of a' and consider the graph $H|_e$. Since e is in a multiedge, $L(e) \in V_{EL}(G^M)$ and, by the definition of the SM-closure, $(G^M)^*_{L(e)} = L(H|_e)$ is Hamilton-connected. According to Theorem C, for any pair of edges $e_1, e_2 \in E(H|_e)$, $H|_e$ has an internally dominating (e_1, e_2)-trail. Let T be an $(a'a, a'a'')$-IDT in $H|_e$. Since $d_H(a') = 2$, we have $T = a', a'a, aTa'', a''a', a'$ (i.e., a' is not an interior vertex of T), and since in $H|_e$ we have $a = b$ while in H, a and b are the vertices of both e and f, the trail $T' = e, aTa', a'a, a, f$ is an (e, f)-IDT in H, a contradiction.

Case 2: *H satisfies condition* $(2)(\beta)$ *of Theorem D.*
Let $C_3 = abca$ be a triangle in H containing the edge e, and assume $e = ac$. Since G^M is SM-closed, by Theorem E, each of the vertices a, b, c has degree at least 3. If, say, $d_H(a) > 3$, we consider a pair of vertices $a_1, a_2 \in N_H(a) \setminus \{b, c\}$ and a vertex $b_1 \in N_H(b) \setminus \{a, c\}$ and then, since H contains neither a multiedge nor a diamond, the graph $G^M[\{L(ab), L(aa_1), L(aa_2), L(bc), L(bb_1)\}]$ is an induced hourglass in G^M, a contradiction. Hence $d(a) = d(b) = d(c) = 3$.

Let a', b' and c' denote the (only) neighbors of a, b and c outside C_3, respectively. If one of a', b', c' has degree 1, say $d_H(a') = 1$, then $\{ab, ac\}$ is an edge-cut of H, separating the edge aa' from the rest of H, but then $d_{G^M}(L(aa')) = 2$, a contradiction. If one of a', b', c' has degree at least 3, say, $d_H(a') \geq 3$, we choose vertices $a_1, a_2 \in N_H(a') \setminus \{a\}$ and then $G^M[\{L(aa'), L(a'a_1), L(a'a_2), L(ac), L(ab)\}]$ is an induced hourglass in G^M, a contradiction again. Hence $d_H(a') = d_H(b') = d_H(c') = 2$.

Denote $N_H(a') \setminus \{a\} = \{a''\}$, $N_H(b') \setminus \{b\} = \{b''\}$ and $N_H(c') \setminus \{c\} = \{c''\}$, and set $V_1 = \{a, b, c, a', b', c', a'', b'', c''\}$ and $H_1 = H[V_1]$ (see Fig. 2). Now we consider possible positions of the edge f in $E(H_1)$ or in $E(H) \setminus E(H_1)$. Since $e = ac$, there are, up to

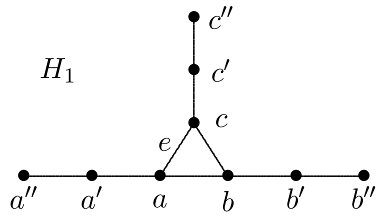

Figure 2: The graph H_1

symmetry, 5 possible positions of f in H_1, namely, $f \in \{bc, aa', bb', a'a'', b'b''\} \subseteq E(H_1)$, and one possibility when $f \in E(H) \setminus E(H_1)$. We consider these cases separately. In each of the cases, we use the fact that, for each edge $h \in E(H)$, the corresponding vertex $L(h)$ of $G^M = L(H)$ is eligible and hence, by the definition of the SM-closure, the graph $H|_h$ has an (h_1, h_2)-IDT for any pair of edges $h_1, h_2 \in E(H)$.

If $f = bc$, we consider the graph $H|_{ab}$. Since $L(ab) \in V_{EL}(G^M)$, the graph $(G^M)^*_{L(ab)} = L(H|_{ab})$ is Hamilton-connected. Let T be an (e, f)-IDT in $H|_{ab}$. Then T contains the edge cc' and, by symmetry, we can suppose that T passes through $bb', b'b''$ and dominates (but does not pass through) $aa', a'a''$. Then $T' = e, a, ab, bTc, f$ is an (e, f)-IDT in H, a contradiction.

If $f = aa'$, consider $H|_{ab}$, and let T be an $(a'a, a'a'')$-IDT in $H|_{ab}$. There are two possibilities: either T passes through the edges $bb', b'b''$ (and does not contain the edges $cc', c'c''$, which are dominated by T but not contained in it), or T passes through $cc', c'c''$ and dominates but does not contain $bb', b'b''$. Then, in the first case the trail $T' = e, c, cb, bTa', f$, and in the second case the trail $T' = e, a, ab, b, bc, cTa', f$ is an (e, f)-IDT in H, a contradiction.

If $f = bb'$, we consider the graph $H|_{ab}$, and let T be a $(b'b, b'b'')$-IDT in $H|_{ab}$. Then either T passes through $aa', a'a''$ and dominates $cc', c'c''$, in which case $T' = e, c, cb, b, ba, aTb', f$ is an (e, f)-IDT in H, or T passes through $cc', c'c''$ and dominates $aa', a'a''$, and then $T' = e, a, ab, b, bc, cTb', f$ is an (e, f)-IDT in H. In both cases, we have a contradiction.

If $f = a'a''$, we consider $H|_{bc}$, and choose T as an $(a'a'', bb')$-IDT in $H|_{bc}$. Then either T passes through $a'a$ (and necessarily also through $cc', c'c''$ and $b''b'$), in which case $T' = e, cTb, ba, a, aa', a', f$ is an (e, f)-IDT in H, or T does not pass through aa' (and necessarily passes through $cc', c'c''$ but not through $b'b''$), and then $T' = e, a, ab, b, bc, c\overleftarrow{T}a'', f$ is an (e, f)-IDT in H. In both cases, we have a contradiction.

If $f = b'b''$, we consider $H|_{ac}$, and T is a $(bb', b'b'')$-IDT in $H|_{ac}$. Up to symmetry, we can consider only one possibility, namely, that T passes through $aa', a'a''$ and dominates $cc', c'c''$. Then $T' = e, c, cb, b, ba, aTb'', f$ is an (e, f)-IDT in H, a contradiction.

Finally, let $f \in E(H) \setminus E(H_1)$. Then we consider the graph $H|_{ab}$, and T is a (bb', f)-IDT in $H|_{ab}$.

First suppose that T does not pass through the edge $b'b''$. Then either T passes through $aa', a'a''$ (and $cc', c'c''$ are dominated), or T passes through $cc', c'c''$ (and $aa', a'a''$ are dominated); then in the first case $T' = e, c, cb, b, ba, a, aa'Tf$ and in the second case $T' = e, a, ab, b, bc, c, cc'Tf$ is an (e, f)-IDT in H, a contradiction.

Secondly, suppose that T passes through $b'b''$. Then necessarily T passes through both $a''a', a'a$ and $cc', c'c''$, either in this or in the opposite orientation; in the first case $T' = e, a, aa'\overleftarrow{T}b'b, b, bc, c, cc'Tf$ and in the second case $T' = e, c, cc'\overleftarrow{T}b'b, b, ba, a, aa'Tf$ is an (e, f)-IDT in H, a contradiction.

This final contradiction completes the proof of Theorem 1. ∎

Acknowledgements. This research is supported by project No. P202/12/G061 of the Czech Science Foundation, by Nature Science Funds of China (No. 11171129, No. 11471037), and by Specialized Research Fund for the Doctoral Program of Higher Education (No.20131101110048).

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