Characterizing forbidden pairs for rainbow connection in graphs with minimum degree 2

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Abstract

A connected edge-colored graph G is rainbow-connected if any two distinct vertices of G are connected by a path whose edges have pairwise distinct colors; the rainbow connection number $\mathrm{rc}(G)$ of G is the minimum number of colors that are needed in order to make G rainbow connected. In this paper, we complete the discussion of pairs (X,Y) of connected graphs for which there is a constant k_{XY} such that, for every connected (X,Y)-free graph G with minimum degree at least 2, $\mathrm{rc}(G) \le \mathrm{diam}(G) + k_{XY}$ (where $\mathrm{diam}(G)$ is the diameter of G), by giving a complete characterization. In particular, we show that for every connected $(Z_3, S_{3,3,3})$-free graph G with $\delta(G) \ge 2$, $\mathrm{rc}(G) \le \mathrm{diam}(G) + 156$, and, for every connected $(S_{2,2,2}, N_{2,2,2})$-free graph G with $\delta(G) \ge 2$, $\mathrm{rc}(G) \le \mathrm{diam}(G) + 72$.

1 Introduction

We use [2] for terminology and notation not defined here and consider finite simple undirected graphs only. To avoid trivial cases, all graphs considered will be connected with at least one edge.

A subgraph of an edge-colored graph G is *rainbow* if all its edges have pairwise distinct colors, and G is *rainbow-connected* if, for any $x, y \in V(G)$, the graph G contains a rainbow path with x, y as endvertices. Note that the edge coloring need not be proper. The *rainbow connection number* of G, denoted by $\mathrm{rc}(G)$, is the minimum number of colors that are needed in order to make G rainbow connected.

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This concept of rainbow connection in graphs was introduced by Chartrand et al. in [5]. An easy observation is that if G has n vertices then $\mathrm{rc}(G) \leq n-1$, since one may color the edges of a given spanning tree of G with different colors and color the remaining edges with one of the already used colors. Chartrand et al. determined the precise value of the rainbow connection number for several graph classes including complete multipartite graphs [5]. The rainbow connection number has been studied for further graph classes in [3, 7, 11, 15] and for graphs with fixed minimum degree in [3, 12, 17]. See [16] for a survey.

There are various applications for such edge colorings of graphs. One interesting example is the secure transfer of classified information between agencies (see, e. g., [8]).

For the rainbow connection number of graphs, the following results are known (and obvious).

Proposition A. *Let G be a connected graph of order n. Then*

(i) $1 \leq \mathrm{rc}(G) \leq n-1$,

(ii) $\mathrm{rc}(G) \geq \mathrm{diam}(G)$,

(iii) $\mathrm{rc}(G) = 1$ *if and only if G is complete,*

(iv) $\mathrm{rc}(G) = n-1$ *if and only if G is a tree.*

Note that the difference $\mathrm{rc}(G) - \mathrm{diam}(G)$ can be arbitrarily large since e.g. for $G \simeq K_{1,n-1}$ we have $\mathrm{rc}(K_{1,n-1}) - \mathrm{diam}(K_{1,n-1}) = (n-1) - 2 = n-3$. Especially, each bridge requires a single color. For bridgeless graphs, the following upper bound is known.

Theorem B [1]. *For every connected bridgeless graph G with radius r,*

$$\mathrm{rc}(G) \leq r(r+2).$$

Moreover, for every integer $r \geq 1$, there exists a bridgeless graph G with radius r and $\mathrm{rc}(G) = r(r+2)$.

Note that this upper bound is still quadratic in terms of the diameter of G.

Let $\mathcal{F}$ be a family of connected graphs. We say that a graph G is *$\mathcal{F}$-free* if G does not contain an induced subgraph isomorphic to a graph from $\mathcal{F}$. Specifically, for $\mathcal{F} = \{X\}$ we say that G is *X-free*, and for $\mathcal{F} = \{X, Y\}$ we say that G is *(X, Y)-free*. The members of $\mathcal{F}$ will be referred to in this context as *forbidden induced subgraphs.*

If X_1, X_2 are graphs, we write $X_1 \overset{\text{IND}}{\subset} X_2$ if X_1 is an induced subgraph of X_2 (not excluding the possibility that $X_1 = X_2$), and if $\{X_1, Y_1\}$, $\{X_2, Y_2\}$ are pairs of graphs, we write $\{X_1, Y_1\} \overset{\text{IND}}{\subset} \{X_2, Y_2\}$ if either $X_1 \overset{\text{IND}}{\subset} Y_1$ and $X_2 \overset{\text{IND}}{\subset} Y_2$, or $X_1 \overset{\text{IND}}{\subset} Y_2$ and $X_2 \overset{\text{IND}}{\subset} Y_1$. It is straightforward to see that if $X_1 \overset{\text{IND}}{\subset} X_2$, then every X_1-free graph is X_2-free, and if $\{X_1, Y_1\} \overset{\text{IND}}{\subset} \{X_2, Y_2\}$, then every (X_1, Y_1)-free graph is (X_2, Y_2)-free.

Although, by Theorem B, $\mathrm{rc}(G)$ can be quadratic in terms of $\mathrm{diam}(G)$, it turns out that forbidden subgraph conditions can remarkably lower the upper bound on $\mathrm{rc}(G)$.

In [9], the authors considered the question for which families $\mathcal{F}$ of connected graphs, a connected $\mathcal{F}$-free graph satisfies $\mathrm{rc}(G) \leq \mathrm{diam}(G)+k_{\mathcal{F}}$, where $k_{\mathcal{F}}$ is a constant (depending on $\mathcal{F}$), and gave a complete answer for $1 \leq |\mathcal{F}| \leq 2$ by the following two results (where N denotes the *net*, i.e. the graph obtained by attaching a pendant edge to each vertex of a triangle).

Theorem C [9]. *Let X be a connected graph. Then there is a constant k_X such that every connected X-free graph G satisfies $\mathrm{rc}(G) \leq \mathrm{diam}(G) + k_X$, if and only if $X = P_3$.*

Theorem D [9]. *Let X, Y be connected graphs, $X, Y \neq P_3$. Then there is a constant k_{XY} such that every connected (X, Y)-free graph G satisfies $\mathrm{rc}(G) \leq \mathrm{diam}(G) + k_{XY}$, if and only if either $\{X, Y\} \overset{IND}{\subset} \{K_{1,r}, P_4\}$ for some $r \geq 4$, or $\{X, Y\} \overset{IND}{\subset} \{K_{1,3}, N\}$.*

Let $S_{i,j,k}$ denote the graph obtained by identifying one endvertex of three vertex disjoint paths of lengths i, j, k, Z_i the graph obtained by attaching a path of length i to a vertex of a triangle, and let $N_{i,j,k}$ denote the graph obtained by identifying each vertex of a triangle with an endvertex of one of three vertex disjoint paths of lengths i, j, k. We also use Z_1^t to denote the graph obtained by attaching a triangle to each vertex of degree 1 of a star $K_{1,t}$ (see Fig. 1).

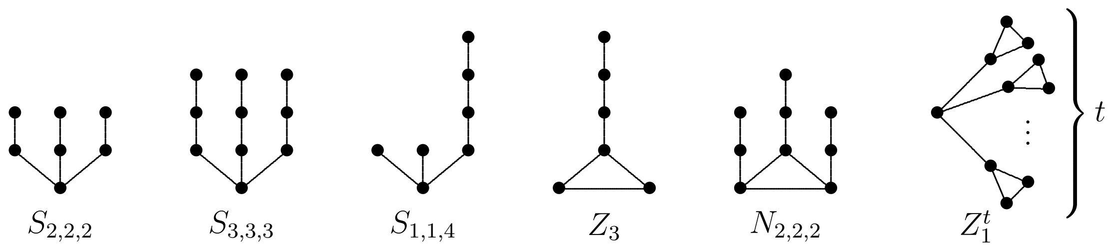

Figure 1: The graphs $S_{2,2,2}$, $S_{3,3,3}$, $S_{1,1,4}$, Z_3, $N_{2,2,2}$ and Z_1^t.

In [10], the authors considered an analogous question for graphs with minimum degree at least two and gave the following results.

Theorem E [10]. *Let X be a connected graph. Then there is a constant k_X such that every connected X-free graph G with $\delta(G) \geq 2$ satisfies $\mathrm{rc}(G) \leq \mathrm{diam}(G) + k_X$, if and only if $X \overset{IND}{\subset} P_5$.*

Theorem F [10]. *Let $X, Y \overset{IND}{\not\subset} P_5$ be a pair of connected graphs for which there is a constant k_{XY} such that every connected (X, Y)-free graph G with $\delta(G) \geq 2$ satisfies $\mathrm{rc}(G) \leq \mathrm{diam}(G)+k_{XY}$. Then $\{X, Y\} \overset{IND}{\subset} \{P_6, Z_1^r\}$ for some $r \in \mathbb{N}$, or $\{X, Y\} \overset{IND}{\subset} \{Z_3, P_7\}$, or $\{X, Y\} \overset{IND}{\subset} \{Z_3, S_{1,1,4}\}$, or $\{X, Y\} \overset{IND}{\subset} \{Z_3, S_{3,3,3}\}$, or $\{X, Y\} \overset{IND}{\subset} \{S_{2,2,2}, N_{2,2,2}\}$.*

In [10], it was also shown that for the first three of the forbidden pairs listed in Theorem F, the converse is also true.

Theorem G [10].

(i) *Let r be a positive integer and let G be a (P_6, Z_1^r)-free graph with $\delta(G) \geq 2$. Then* $\mathrm{rc}(G) \leq \mathrm{diam}(G) + 20 + r$.

(ii) *Let G be a connected (Z_3, P_7)-free graph with $\delta(G) \geq 2$. Then* $\mathrm{rc}(G) \leq \mathrm{diam}(G) + 30$.

(iii) *Let G be a connected $(Z_3, S_{1,1,4})$-free graph with $\delta(G) \geq 2$. Then* $\mathrm{rc}(G) \leq \mathrm{diam}(G) + 59$.

In this paper, we complete the characterization of forbidden pairs (X, Y) for which there is a constant k_{XY} such that every (X, Y)-free graph G with $\delta(G) \geq 2$ has $\mathrm{rc}(G) \leq \mathrm{diam}(G) + k_{XY}$, by proving sufficiency for the remaining pairs listed in Theorem F. In particular, we show the following:

- in Theorem 1, we show that every connected $(Z_3, S_{3,3,3})$-free graph G with $\delta(G) \geq 2$ has $\mathrm{rc}(G) \leq \mathrm{diam}(G) + 156$, and
- in Theorem 2, we show that every connected $(S_{2,2,2}, N_{2,2,2})$-free graph G with $\delta(G) \geq 2$ has $\mathrm{rc}(G) \leq \mathrm{diam}(G) + 72$.

Finally, in Theorem 3, we summarize these results and the results of the paper [10] and we give a complete characterization of all forbidden pairs $\{X, Y\}$ implying $\mathrm{rc}(G) \leq \mathrm{diam}(G) + k_{XY}$ in (X, Y)-free graphs G with $\delta(G) \geq 2$.

2 Definitions and notations

In this section, we summarize some further notations and known facts that will be needed for the proofs of our results.

A path with endvertices x, y will be referred to as an (x, y)*-path*, and for $F \subset G$, an (x, y)-path with $y \in V(F)$ is called an (x, F)*-path*. For a nontrivial (x, y)-path P, we set $\mathrm{int}(P) = V(P) \setminus \{x, y\}$, and two paths P, Q are said to be *internally vertex-disjoint* if $\mathrm{int}(P) \cap \mathrm{int}(Q) = \emptyset$. We use $\mathrm{rad}(G)$ for the radius of G and $\mathrm{diam}(G)$ for the diameter of G. If $x, y \in V(G)$ are at distance $\mathrm{diam}(G)$ and P is a shortest (x, y)-path, we say that P is a *diameter path*. If C is a cycle, then a subgraph of C which is a path is called an *arc of* C, and for $A, B \subset V(C)$, an arc of C with endvertices in A and B, respectively, is called an (A, B)*-arc of* C. For a path P and $x, y \in V(P)$, a subpath of P with origin at x and end at y is denoted by xPy, and for a cycle Q (with a fixed orientation), we use xQy to denote the (x, y)-arc of Q. The same arc, traversed in the opposite orientation, is denoted by $y\overleftarrow{Q}x$. For $X, Y \subset V(G)$, we use $E[X, Y]$ to denote the set of edges of G with one vertex in X and the other vertex in Y. We will also sometimes use $N_G[P]$ to denote the *closed neighborhood* of a subgraph $P \subset G$. Finally, a *bridge of* G is an edge $e \in E(G)$ such that $G - e$ has more components than G.

A dominating set D in a graph G is called a *two-way dominating set* if D includes all vertices of G of degree 1. In addition, if $G[D]$ is connected, we call D a *connected two-way dominating set*. Note that if $\delta(G) \geq 2$, then every (connected) dominating set in G is a (connected) two-way dominating set.

Theorem H [4]. *If D is a connected two-way dominating set in a graph G, then* $\mathrm{rc}(G) \leq \mathrm{rc}(G[D]) + 3$.

A set $D \subset V(G)$ is called a *k-step dominating set of* G, $k \geq 0$, if every vertex of G is at a distance at most k from D.

Theorem I [6]. *If G is a connected graph, and D^k is a connected k-step dominating set of G, then G has a connected $(k-1)$-step dominating set $D^{k-1} \supset D^k$ such that $\mathrm{rc}(G[D^{k-1}]) \leq \mathrm{rc}(G[D^k]) + \max\{2k+1, b_k\}$, where b_k is the number of bridges of G in $E(D^k, N(D^k))$.*

In our proofs, we will use several times the following easy consequence of Theorem I.

Corollary J. *Let G be a connected graph, and let D be a connected k-step dominating set of G such that $G[D]$ contains all bridges of G. Then* $\mathrm{rc}(G) \leq \mathrm{rc}(G[D]) + k\,(k+2)$.

Proof. Let D be a connected k-step dominating set of G such that $G[D]$ contains all bridges of G. By Theorem I, there is a connected $(k-1)$-step dominating set D^{k-1} in G such that $D^{k-1} \supset D$ and $\mathrm{rc}(G[D^{k-1}]) \leq \mathrm{rc}(G[D]) + (2k+1)$. By induction, we get a sequence of sets $D, D^{k-1}, D^{k-2}, \ldots, D^0$ such that $D^{i-1} \supset D^i$ and D^{i-1} is a connected $(i-1)$-step dominating set in G, $i = k, \ldots, 1$, (i.e., specifically, $D^0 = V(G)$), and such that $\mathrm{rc}(G) = \mathrm{rc}(G[D^0]) \leq \mathrm{rc}(G[D]) + (2k+1) + (2k-1) + \ldots + 3 = \mathrm{rc}(G[D]) + \frac{k\,(2k+4)}{2} = \mathrm{rc}(G[D]) + k\,(k+2)$. ■

We will also use the following two results on bridgeless graphs of small diameter.

Theorem K [13]. *If G is a connected bridgeless graph of diameter 2, then* $\mathrm{rc}(G) \leq 5$.

Theorem L [14]. *If G is a connected bridgeless graph of diameter 3, then* $\mathrm{rc}(G) \leq 9$.

3 Results

The following two theorems are the main results of this paper.

Theorem 1. *Let G be a connected $(Z_3, S_{3,3,3})$-free graph with $\delta(G) \geq 2$. Then* $\mathrm{rc}(G) \leq \mathrm{diam}(G) + 156$.

Theorem 2. *Let G be a connected $(S_{2,2,2}, N_{2,2,2})$-free graph with $\delta(G) \geq 2$. Then* $\mathrm{rc}(G) \leq \mathrm{diam}(G) + 72$.

Summarizing the statements of Theorems 1 and 2 with those of Theorems F and G, we obtain the following characterization.

Theorem 3. *Let $X, Y \overset{IND}{\not\subset} P_5$ be a pair of connected graphs. Then there is a constant k_{XY} such that every connected (X,Y)-free graph G with $\delta(G) \geq 2$ satisfies $\mathrm{rc}(G) \leq \mathrm{diam}(G) + k_{XY}$, if and only if either $\{X,Y\} \overset{IND}{\subset} \{P_6, Z_1^r\}$ for some $r \in \mathbb{N}$, or $\{X,Y\} \overset{IND}{\subset} \{Z_3, P_7\}$, or $\{X,Y\} \overset{IND}{\subset} \{Z_3, S_{1,1,4}\}$, or $\{X,Y\} \overset{IND}{\subset} \{Z_3, S_{3,3,3}\}$, or $\{X,Y\} \overset{IND}{\subset} \{S_{2,2,2}, N_{2,2,2}\}$.*

4 Proofs

Let $P = x_0, x_1, \ldots, x_\ell$ be a shortest (x_0, x_ℓ)-path in G and let $z \in V(G) \setminus V(P)$. If $|N_P(z)| \geq 2$ and $\{x_i, x_j\} \subset N_P(z)$, then $|i - j| \leq 2$ and $|N_P(z)| \leq 3$ since P is a shortest path. In the proofs of Theorems 1 and 2, we will use the following notation (for more details see Fig. 2):

- $M_i^1(P) := \{z \in V(G) \setminus V(P) |\, N_P(z) = \{x_i\}\}$ for $0 \leq i \leq \ell$,
- $N_i^1(P) := \{z \in V(G) \setminus V(P) |\, N_P(z) \supset \{x_{i-1}, x_{i+1}\}\}$ for $1 \leq i \leq \ell - 1$,
- $O_i^1(P) := \{z \in V(G) \setminus V(P) |\, N_P(z) = \{x_{i-1}, x_i\}\}$ for $1 \leq i \leq \ell$.

We set $M^1(P) = \bigcup_{i=0}^{\ell} M_i^1(P)$, $N^1(P) = \bigcup_{i=1}^{\ell-1} N_i^1(P)$, $O^1(P) = \bigcup_{i=1}^{\ell} O_i^1(P)$, and $R^1(P) = V(G) \setminus N_G[P]$. For $j = 1, 2, 3, \ldots, \ell - 1$ we further denote

- $M_i^{j+1}(P) = N_G(M_i^j(P)) \cap R^j(P)$ for $0 \leq i \leq \ell$, $N_i^{j+1}(P) = N_G(N_i^j(P)) \cap R^j(P)$ for $1 \leq i \leq \ell - 1$, $O_i^{j+1}(P) = N_G(O_i^j(P)) \cap R^j(P)$ for $1 \leq i \leq \ell$,
- $M^{j+1}(P) = \bigcup_{i=0}^{\ell} M_i^{j+1}(P)$, $N^{j+1}(P) = \bigcup_{i=1}^{\ell-1} N_i^{j+1}(P)$, $O^{j+1}(P) = \bigcup_{i=1}^{\ell} O_i^{j+1}(P)$,
- $R^{j+1}(P) = R^j(P) \setminus (M^{j+1}(P) \cup N^{j+1}(P) \cup O^{j+1}(P))$.

We also denote $M_i(P) = \bigcup_{j=1}^{\ell} M_i^j(P)$, $N_i(P) = \bigcup_{j=1}^{\ell} N_i^j(P)$ and $O_i(P) = \bigcup_{j=1}^{\ell} O_i^j(P)$.

If the path P is clear from the context, we will omit the letter P in all the above notations, i.e., we will shortly write M_i^j, N_i^j, O_i^j etc. for $M_i^j(P)$, $N_i^j(P)$, $O_i^j(P)$ etc., respectively.

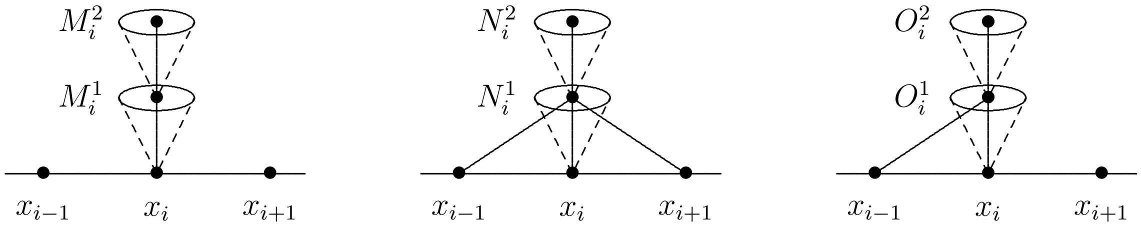

Figure 2: The sets M_i^j, N_i^j and O_i^j.

4.1 The pair $(Z_3, S_{3,3,3})$

Before proving sufficiency for the pair $X = Z_3$ and $Y = S_{3,3,3}$, we give some auxiliary statements.

Lemma 4. *Let G be a connected bridgeless Z_3-free graph with $\omega(G) \geq 3$ and $\delta(G) \geq 2$. Then* $\mathrm{rc}(G) \leq \min\{24, \mathrm{diam}(G) + 20\}$.

Proof. Let $S \subset V(G)$ denote the vertex set of a maximum clique in G. Suppose that there is a vertex $z \in V(G) \setminus S$ at distance $k \geq 4$ from $G[S]$ in G, and let P : $y_0, y_1, y_2, \ldots, y_k = z$, be a shortest path between z and some vertex $y_0 \in S$. Since P is shortest, y_0 is the only vertex of P belonging to S, P is induced and y_i has no neighbor in S for $2 \leq i \leq k$. Since S is maximum, $v_0y_1 \notin E(G)$ for some $v_0 \in S$. If there is another $v_1 \in S$ with $v_1 \neq v_0$ and $v_1y_1 \notin E(G)$, then $G[\{y_0, v_0, v_1, y_1, \ldots, y_k\}] \simeq Z_k$, otherwise, for some $v_1 \in S$, $v_1 \neq v_0$, we have $G[\{y_1, y_0, v_1, y_2, \ldots, y_k\}] \simeq Z_{k-1}$. Since $k \geq 4$, G contains an induced Z_3, a contradiction. Therefore, $\mathrm{dist}_G(x, y) \leq 3$ for every pair of vertices $x \in S$ and $y \in V(G) \setminus S$, implying $\mathrm{rad}(G) \leq 4$. By Theorem B, we have $\mathrm{rc}(G) \leq 24$. If $\mathrm{diam}(G) \geq 4$, then $\mathrm{rc}(G) \leq \mathrm{diam}(G) + 20$. Now, by Theorems K and L, we obtain $\mathrm{rc}(G) \leq \min\{24, \mathrm{diam}(G) + 20\}$. ∎

Lemma 5. *Let G be a connected Z_3-free graph with $\omega(G) \geq 3$ and $\delta(G) \geq 2$ such that G contains a bridge. Then* $\mathrm{rc}(G) \leq 4$.

Proof. Let xy be a bridge in G. Since G is connected, $G - xy$ has two components. Let G_1 denote a component containing a triangle and let G_2 denote the other component of $G - xy$. Up to a symmetry, suppose that $x \in V(G_1)$ and $y \in V(G_2)$. Every vertex of G_2 is adjacent to y, for otherwise we get an induced Z_3 with a triangle in G_1. Then $\omega(G_2) \geq 3$ since $\delta(G) \geq 2$. Now, every vertex of G_1 is adjacent to x, otherwise we get an induced Z_3 with a triangle in G_2. This implies that $D = \{x, y\}$ is a two-way dominating set in G and, by Theorem H, $\mathrm{rc}(G) \leq \mathrm{rc}(G[D]) + 3 = 4$. ∎

The following easy observation is a useful tool.

Lemma 6. *Let G be a triangle-free graph with $\delta(G) \geq 2$, let $b = uv$ be a bridge in G, and let G_u, G_v denote the components of $G - b$ such that $u \in V(G_u)$ and $v \in V(G_v)$. Then there is a vertex at distance two from u in G_u and a vertex at distance two from v in G_v.*

Proof. Consider a component G_u of the graph $G - b$. If every vertex x of G_u is a neighbor of u, then $d_G(x) = 1$ since G is triangle-free, a contradiction. The proof for G_v is symmetric. ■

Proof of Theorem 1. Let G be a $(Z_3, S_{3,3,3})$-free graph. If G contains a triangle, then the statement follows from Lemma 4 and 5. Thus, we assume that G is triangle-free.

First suppose that $\mathrm{diam}(G) \leq 5$. If G is bridgeless, then $\mathrm{rc}(G) \leq 35 \leq \mathrm{diam}(G) + 35$ by Theorem B. Thus, we assume that G contains a bridge b, and let G_1, G_2 denote the components of $G - b$. By Lemma 6, $\mathrm{diam}(G) = 5$ and b is the central edge of a diameter path in G. Since $\mathrm{diam}(G) = 5$, $\mathrm{rad}(G_1) = \mathrm{rad}(G_2) = 2$ and G_1, G_2 are both bridgeless. Then, by Theorem B, $\mathrm{rc}(G_1) \leq 8$ and $\mathrm{rc}(G_2) \leq 8$. Thus we get $\mathrm{rc}(G) \leq \mathrm{rc}(G_1) + \mathrm{rc}(G_2) + 1$ since we need an extra color for the bridge b, implying that $\mathrm{rc}(G) \leq 17 = \mathrm{diam}(G) + 12$.

For the sets M_i^j and N_i^j, we have the following statement.

<u>Claim 1.1.</u> *Let $a, b \in V(G)$ be vertices at distance $\mathrm{dist}_G(a, b) \geq 6$, and let $P : a = x_0, x_1, \ldots, x_k = b$ ($k \geq 6$) be a shortest (a, b)-path in G. Then*

(i) $M_i^j = \emptyset$ *for* $3 \leq i \leq k - 3$ *and* $j \geq 3$,

(ii) $N_i^j = \emptyset$ *for* $3 \leq i \leq k - 3$ *and* $j \geq 4$,

(iii) $N_G(y) \subset (M^1 \cup N^1)$ *for every* $y \in M_i^2$, $3 \leq i \leq k - 3$,

(iv) $N_G(y) \subset N_i^2$ *for every* $y \in N_i^3$, $3 \leq i \leq k - 3$.

<u>Proof.</u> We prove the statement (i). Let, to the contrary, $z \in M_i^3$ for some $3 \leq i \leq k - 3$. Let Q be a shortest (z, x_i)-path in G. Then the set of vertices $\{x_{i-3}, x_{i-2}, \ldots x_{i+3}\} \cup V(Q)$ induces an $S_{3,3,3}$, a contradiction. The proof of (ii) and (iii) is analogous. To prove (iv), let $y \in N_i^3$ (for some i, $3 \leq i \leq k - 3$), let $y^2 \in N(y) \cap N_i^2$, and let $y^1 \in N(y^2) \cap N_i^1$. If $z \in N(y) \setminus \{y^2\}$, then the set $\{y^1, y^2, y, z, x_{i-1}, x_{i-2}, x_{i-3}, x_{i+1}, x_{i+2}, x_{i+3}\}$ induces an $S_{3,3,3}$, unless $z \in N(y^1)$, implying $z \in N_i^2$. □

For the rest of the proof, we suppose that $d = \mathrm{diam}(G) \geq 6$. We choose a diameter path $P : x_0, x_1, \ldots, x_d$ in G. Unless otherwise stated, the sets M_i^j, N_i^j and R^j, as introduced above, will be always understood with respect to this fixed diameter path P. Since G is triangle-free, $N_P(z) = \{x_{i-1}, x_{i+1}\}$ for any $z \in N_i^1$, $1 \leq i \leq d - 1$, and $O_i^1 = \emptyset$ for $1 \leq i \leq d$.

<u>Claim 1.2.</u> *The path P contains all bridges of G.*

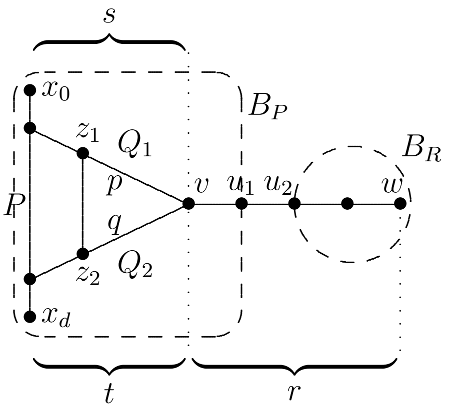

Figure 3: The paths P, Q_1 and Q_2 in the proof of Claim 1.2

Proof. Let, to the contrary, $b = u_1u_2$ be a bridge with $b \notin E(P)$, and choose the notation such that u_1 is in the component B_P of $G-b$ containing P, and u_2 is in the other component B_R of $G - b$ (see Fig. 3). By Lemma 6, B_R contains a vertex w with $\mathrm{dist}(u_2, w) = 2$ (i.e., $\mathrm{dist}(u_1, w) = 3$). Let Q_1 denote a shortest (w, x_0)-path in G and Q_2 a shortest (w, x_d)-path in G. Let v be the last common vertex of Q_1 and Q_2, i.e., a vertex such that the paths vQ_1x_0 and vQ_2x_d are internally vertex-disjoint. Denote $s = \mathrm{dist}(v, x_0) = |E(vQ_1x_0)|$, $t = \mathrm{dist}(v, x_d) = |E(vQ_2x_d)|$, and $r = \mathrm{dist}(v, w) = |E(vQ_1w)| = |E(vQ_2w)|$. Obviously, $r \geq 3$.

Now, choose the paths Q_1, Q_2 such that

(i) $|E(Q_1)|$ is minimum and $|E(Q_2)|$ is minimum (as already mentioned), and

(ii) subject to (i), $s + t = |E(vQ_1x_0)| + |E(vQ_2x_d)|$ is minimum.

Since $d = \mathrm{diam}(G)$, we have

$$\mathrm{dist}(x_0, w) = s + r \leq d, \tag{1}$$

$$\mathrm{dist}(x_d, w) = t + r \leq d. \tag{2}$$

Since $x_0\overleftarrow{Q_1}vQ_2x_d$ is an (x_0, x_d)-path and P is a diameter path, we have

$$\mathrm{dist}(x_0, v) + \mathrm{dist}(x_d, v) = s + t \geq d. \tag{3}$$

We show that $t \geq 3$: if $t \leq 2$, then from (3) we have $s + 2 \geq s + t \geq d$, from which $s \geq d-2$, and then (1) implies $d \geq s+r \geq d-2+r \geq d-2+3 = d+1$, a contradiction. Hence $t \geq 3$, and, symmetrically, (using (2) instead of (1)), $s \geq 3$. Hence the graph F, consisting of the paths vQ_1x_0, vQ_2x_d and $v\overleftarrow{Q_1}w$, contains a subgraph isomorphic to the graph $S_{3,3,3}$ (with center at v). Since G is $S_{3,3,3}$-free, F is not an induced subgraph of G.

Thus, let $h = z_1z_2$ be an arbitrary edge with $z_1, z_2 \in V(F)$ but $h \in E(G) \setminus E(F)$. Since both Q_1 and Q_2 are shortest (hence chordless), up to a symmetry, $z_1 \in V(v^{+Q_1}Q_1x_0)$ and $z_2 \in V(v^{+Q_2}Q_2x_d)$, where v^{+Q_1} and v^{+Q_2} denotes the successor of v on Q_1 and Q_2, respectively. Set $p = \mathrm{dist}(v, z_1) = |E(vQ_1z_1)|$ and $q = \mathrm{dist}(v, z_2) = |E(vQ_2z_2)|$. Obviously, $p \geq 1$ and $q \geq 1$.

We show that $p = q$. First suppose that $p \geq q+1$. Then, considering the paths Q_1 and $Q_2' = wQ_1z_1z_2Q_2x_d$, we have a contradiction with the choice of Q_1 and Q_2: if $p > q+1$, then Q_2' is shorter than Q_2, contradicting (i), and if $p = q+1$, then $|E(Q_2)| = |E(Q_2')|$, but $|E(z_1Q_1x_0)| + |E(z_1Q_2'x_d)| < |E(vQ_1x_0)| + |E(vQ_2x_d)|$, contradicting (ii) (where z_1 plays the role of v). Hence $p < q+1$. Symmetrically, $q < p+1$, implying $p = q$. Moreover, since G is triangle-free, we have

$$p = q \geq 2. \tag{4}$$

Now, we choose the edge $h = z_1z_2$ such that

(iii) subject to (i) and (ii), $p = q$ is maximum.

By (4), and since $r \geq 3$, $q+r \geq 5$, i.e., $r \geq 5-q$. From (2) we then have $d \geq t+r \geq t+5-q$, from which $t - q \leq d - 5$. Since the path $x_0\overleftarrow{Q_1}z_1z_2Q_2x_d$ is an (x_0, x_d)-path of length $s - p + 1 + t - q$, and P is a diameter path, $d \leq s - p + 1 + t - q \leq s - p + 1 + d - 5$, from which we conclude that $s - p \geq 4$. Thus, $\mathrm{dist}(z_1, x_0) = |E(z_1Q_1x_0)| = s - p \geq 4$. Symmetrically, $\mathrm{dist}(z_2, x_d) = |E(z_2Q_2x_d)| = t - q \geq 4$, and hence $|E(z_1z_2Q_2x_d)| \geq 5$.

Thus, the graph consisting of the paths $z_1Q_1x_0$, $z_1z_2Q_2x_d$ and $z_1\overleftarrow{Q_1}w$ contains a subgraph F' isomorphic to $S_{3,3,3}$ (with center at z_1). By the choice (iii), F' is an induced subgraph of G, a contradiction. $\square$

We set
$$J_C = \bigcup_{i=3}^{d-3} (M_i \cup N_i \cup \{x_i\}).$$
By Claim 1.1, $\mathrm{dist}_G(x, P) \leq 3$ for each $x \in J_C$. Moreover, the vertices in J_C at distance 3 from P have no neighbors in $V(G) \setminus J_C$, as shown in the following statement.

<u>Claim 1.3.</u> *Let $x \in J_C$. If $\mathrm{dist}_G(x, P) = 3$, then x has no neighbor in $V(G) \setminus J_C$.*

<u>Proof.</u> If $x \in J_C$ is at distance three from P, then $x \in N_i^3$ by Claim 1.1(i) for some i, $3 \leq i \leq d-3$, and by Claim 1.1(iv), every neighbor of x belongs to N_i^2, and hence to J_C. $\square$

We also show the following observation.

<u>Claim 1.4.</u> *Let $u \in V(G) \setminus J_C$ and $v \in J_C$ be such that $uv \in E(G)$. Then $v \in M_i \cup N_i \cup \{x_3\}$ for some $i \leq 6$, or $v \in M_i \cup N_i \cup \{x_{d-3}\}$ for some $i \geq d-6$.*

<u>Proof.</u> Up to a symmetry, suppose that $u \in (M_0 \cup M_1 \cup N_1 \cup M_2 \cup N_2 \cup \{x_0, x_1, x_2\}) \setminus J_C$. If $v \in M_i$ for some $i \geq 7$, then $\mathrm{dist}_G(u, P) + \mathrm{dist}_G(v, P) \leq 3$ by Claim 1.1 (iii), implying that $i \leq 6$, for otherwise a path consisting of a shortest (x_0, u)-path, the edge uv and a shortest (v, x_d)-path is an (x_0, x_d)-path shorter than P, a contradiction. Analogously, if $v \in N_i$ for some $i \geq 7$, then $\mathrm{dist}_G(u, P) + \mathrm{dist}_G(v, P) \leq 4$ by Claim 1.1(iv), implying that $i \leq 6$, for otherwise a path consisting of a shortest (x_0, u)-path, the edge uv and a shortest (v, x_d)-path is shorter than P, a contradiction. Similarly, if $v = x_i$ for some $i \geq 5$, then there is a shorter (x_0, x_d)-path containing the edge uv, a contradiction again. Finally, if $v = x_4$, then $u \in M_4 \cup N_3 \cup N_5$, contradicting the choice of u. $\square$

We now distinguish two cases.

Case 1: *The set J_C is a cutset of G.*

We show that there is no vertex at distance greater than 5 from P in G.

Claim 1.5. *For every $z \in V(G) \setminus J_C$, $\mathrm{dist}_G(z, P) \leq 5$.*

Proof. Let, to the contrary, $\ell = \mathrm{dist}_G(z, P) \geq 6$ for some $z \in V(G) \setminus J_C$. Up to a symmetry, suppose that $z \in M_0 \cup M_1 \cup N_1 \cup M_2 \cup N_2$. Let Q_1 denote a shortest (z, x_d)-path in G, and let y' denote the last vertex of Q_1 in J_C and y the successor of y' on Q_1, both in an orientation of Q_1 from x_d (note that y' exists since J_C is a cutset). From Claim 1.3 and from the fact that $y \notin J_C$, $\mathrm{dist}_G(y, P) \leq 2$. Then clearly $\mathrm{dist}_G(x_0, y) \leq 4$ and $\mathrm{dist}_G(y, z) \geq \ell - 2 \geq 4$. We have $d \geq \mathrm{dist}_G(z, x_d) \geq \mathrm{dist}_G(z, y) + \mathrm{dist}_G(y, x_d) \geq 4 + \mathrm{dist}_G(y, x_d)$, implying that $\mathrm{dist}_G(y, x_d) \leq d - 4$. But $d \leq \mathrm{dist}_G(x_0, y) + \mathrm{dist}_G(y, x_d)$, implying that $\mathrm{dist}_G(y, x_d) \geq d - 4$. Hence $\mathrm{dist}_G(y, x_d) = d - 4$, $\mathrm{dist}_G(y, x_0) = 4$, $\mathrm{dist}_G(z, y) = 4$, $y \in M_2^2$ (by Claim 1.3) and $z \in M_2^6$. We denote Q_2 a shortest (y, x_2)-path. Then the path $x_1 x_2 Q_2 y Q_1 z$ is induced, and the path $x_0 x_1 x_2 Q_2 y Q_1 x_d$ is a diameter path. Recall that Q_1 is a shortest path. Now, if $d \geq 7$, then the subgraph consisting of the paths $y Q_2 x_2 x_1$, $y Q_1 x_d$ and $y \overleftarrow{Q_1} z$ contains an induced $S_{3,3,3}$ (with center at y). Hence $d = 6$, and then $\mathrm{dist}_G(y, x_d) = 2$, implying $y' x_6 \in E(G)$. But then, by the definition of J_C, $y' \in N_3^1 \cup M_3^1$, contradicting the fact that P is a shortest path. $\square$

Now, by Claim 1.5, the set $V(P)$ is 5-step dominating in G, hence by Corollary J and by Claim 1.2, we have $\mathrm{rc}(G) \leq \mathrm{diam}(G) + 5 \cdot 7 \leq \mathrm{diam}(G) + 35$.

Case 2: *The set J_C is not a cutset of G.*

If G is not bridgeless, then all bridges of G are on P by Claim 1.2, and at least one vertex of each bridge is in J_C by Lemma 6. But then J_C is a cutset of G, contradicting the assumption of Case 2. Thus, G is bridgeless.

First suppose that $d = \mathrm{diam}(G) \leq 12$. Since G is bridgeless, by Theorem B, we have $\mathrm{rc}(G) \leq \mathrm{rad}(G)(\mathrm{rad}(G) + 2) \leq d(d + 2)$. It is easy to verify that, for $d \leq 12$, $d(d + 2) = d + d(d + 1) \leq d + 156$, we have $\mathrm{rc}(G) \leq \mathrm{diam}(G) + 156$, and we are done.

Thus, for the rest of the proof, we suppose that $d = \mathrm{diam}(G) \geq 13$. We introduce the following notation:

$$J_1 = \bigcup_{i=3}^{6} (M_i \cup N_i \cup \{x_i\}),$$
$$J_2 = \bigcup_{i=d-6}^{d-3} (M_i \cup N_i \cup \{x_i\}),$$
$$J_V = J_C \setminus (J_1 \cup J_2).$$

We further denote P' a shortest (x_0, x_d)-path in $G - J_C$. Note that $J_1 \cap J_2 = \emptyset$ since $d \geq 13$.

Note that, by Claim 1.4, there is no edge between J_V and $V(G) \setminus J_C$.

If $F \subset G$ is a cycle or a path, and $A^I = v_1Fv_2$ is an arc of F, we say that A^I is J_C*-internal* if $V(A^I) \subset J_C$, A^I is maximal (in terms of the number of vertices) with this property and $v_1 \in J_j$ and $v_2 \in J_{3-j}$ for some $j \in \{1,2\}$. We also say that an arc $A^E = w_1Fw_2$ is J_C*-external* if no internal vertex of A^E belongs to J_C, and $w_1 \in J_j$ and $w_2 \in J_{3-j}$ for some $j \in \{1,2\}$. We will use $j_C^I(F)$ to denote the number of internally vertex-disjoint J_C-internal arcs of F and $j_C^E(F)$ for the number of internally vertex-disjoint J_C-external arcs of F. Finally, we say that an arc A of F is a J_C-arc if A is J_C-internal or J_C-external.

Let now C be a shortest cycle in G such that $j_C^I(C)$ is odd (note that C exists since the subgraph $G[V(P) \cup V(P')]$ certainly contains such a cycle.) We observe that $j_C^E(C)$ is also odd. Clearly, the total number of arcs of C between some vertex of J_1 and some vertex of J_2 is even. Since $j_C^I(C)$ is odd, there must be an arc of C between J_1 and J_2 which is not J_C-internal. Let A' be such an arc and choose A' shortest possible. Since A' is not J_C-internal, A' contains some vertex $z \in V(G) \setminus J_C$, and since A' is shortest, $\mathrm{int}(A') \cap (J_1 \cup J_2) = \emptyset$. By Claim 1.4, we also have $\mathrm{int}(A') \cap J_V = \emptyset$, since there is no edge between z and J_V. Thus, A' is J_C-external. This means that every arc of C between J_1 and J_2 is either J_C-internal or J_C-external, hence a J_C-arc. Thus $j_C^I(C) + j_C^E(C)$ is even and, since $j_C^I(C)$ is odd, $j_C^E(C)$ must be also odd.

<u>Claim 1.6.</u> *Let A be a J_C-internal (v_1, v_2)-arc of C, let v_1', v_2' denote the neighbor of v_1 or v_2 in $V(C) \setminus \mathrm{int}(A)$, respectively. Then $\mathrm{dist}_G(v_1', v_2') \geq d - 8$.*

<u>Proof.</u> By the definition of a J_C-internal arc, $v_1', v_2' \notin J_C$. By symmetry, we can suppose that $v_1' \in (M_0 \cup M_1 \cup N_1 \cup M_2 \cup N_2 \cup \{x_0, x_1, x_2\}) \setminus J_C$. Then $\mathrm{dist}_G(v_1', P) \leq 2$ and $\mathrm{dist}_G(v_2', P) \leq 2$ by Claim 1.3 and since $v_1', v_2' \notin J_C$. Hence $\mathrm{dist}_G(x_0, v_1') \leq 4$ and $\mathrm{dist}_G(v_2', x_d) \leq 4$. Since $\mathrm{dist}_G(x_0, v_1') + \mathrm{dist}_G(v_1', v_2') + \mathrm{dist}_G(v_2', x_d) \geq d$, we get $\mathrm{dist}_G(v_1', v_2') \geq d - \mathrm{dist}_G(x_0, v_1') - \mathrm{dist}_G(v_2', x_d) \geq d - 8$. □

Note that, using Claim 1.6, we immediately observe that $|V(C)| \geq 10$.

<u>Claim 1.7.</u> *The cycle C can be chosen such that $j_C^I(C) = j_C^E(C) = 1$.*

<u>Proof.</u> Since both $j_C^I(C)$ and $j_C^E(C)$ are odd, hence nonzero, there is a pair A^I, A^E of J_C-arcs of C such that A^I is J_C-internal, A^E is J_C-external, and A^I, A^E are consecutive on C, i.e., at least one of the components of $C - (\mathrm{int}(A^I) \cup \mathrm{int}(A^E))$ contains no J_C-arc. Let v_i^I, v_i^E denote the endvertex of A^I, A^E in J_i $(i = 1, 2)$, respectively, let $(v_i^I)'$ denote the neighbor of v_i^I on $C - A^I$, and let $(v_i^E)'$ denote the neighbor of v_i^E on A^E. Then $(v_1^I)', (v_1^E)' \in (M_0 \cup M_1 \cup N_1 \cup M_2 \cup N_2 \cup \{x_0, x_1, x_2\}) \setminus J_C$ and $(v_2^I)', (v_2^E)' \in (M_{d-2} \cup N_{d-2} \cup M_{d-1} \cup N_{d-1} \cup M_d \cup \{x_d, x_{d-1}, x_{d-2}\}) \setminus J_C$. By Claim 1.3, $\mathrm{dist}_G((v_i^I)', P) \leq 2$ and $\mathrm{dist}_G((v_i^E)', P) \leq 2$, implying that $\mathrm{dist}_G((v_i^I)', (v_i^E)') \leq 6$ for $i = 1, 2$. Since A^I and A^E are consecutive on C, we may assume (up to a symmetry) that there is no J_C-arc between v_1^I and v_1^E.

Now, if, say, $j_C^I(C) > 1$, then the cycle C' consisting of A^I, A^E, the arc $v_1^I C v_1^E$, and a shortest $((v_2^I)', (v_2^E)')$-path, has length at most $|V(C)| - 2 \cdot 5 + 6 = |V(C)| - 4$ since we delete from C at least two J_C-internal arcs and we add a shortest $((v_2^I)', (v_2^E)')$-path. But this contradicts the fact that C is shortest possible. Therefore $j_C^I(C) = 1$, and analogously $j_C^E(C) = 1$. □

By Claim 1.7, for the rest of the proof we suppose that the cycle C is chosen such that $j_C^I(C) = j_C^E(C) = 1$.

Claim 1.8. *Every (y, y')-arc of C of length at most $\frac{|V(C)|}{2}$ is a shortest (y, y')-path in G.*

Proof. Suppose, to the contrary, that there is an arc yCy' of length at most $\frac{|V(C)|}{2}$ that is not a shortest path in G, let Q be a shortest (y, y')-path in G, and, among all such arcs in C, choose the arc $A_1 : yCy'$ such that the path Q is shortest possible. By the same argument as in the proof of Claim 1.7, $j_C^I(Q) \leq 1$ and $j_C^E(Q) \leq 1$.

Let $A_2 : y'Cy$ denote the complementary arc to A_1 (i.e., $V(A_1) \cup V(A_2) = V(C)$ and $V(A_1) \cap V(A_2) = \{y, y'\}$). Then clearly A_1, A_2 and Q are pairwise internally vertex-disjoint paths with common endvertices, hence both $C_1 : yA_1y'\overleftarrow{Q}y$ and $C_2 : y'A_2yQy'$ are cycles in G. By the definition of Q and by the assumption that A_1 is of length at most $\frac{|V(C)|}{2}$, we have $|E(Q)| < |E(A_1)| \leq |E(A_2)|$, hence both C_1 and C_2 are shorter than C. Let $A : v_1Cv_2$ be the (only) J_C-internal arc of C, and choose the notation such that $v_1 \in J_1$ and $v_2 \in J_2$. According to the position of y and y' with respect to A, we have the following three possibilities.

(α) $y, y' \notin J_C$. Then either $A \subset A_1$, or $A \subset A_2$, thus, for each value of $j_C^I(Q)$, either $j_C^I(C_1) = 1$ or $j_C^I(C_2) = 1$.

(β) $y, y' \in J_C$. Then both y and y' are vertices of A (possibly $A = A_1$, or $\{y, y'\} \cap \mathrm{int}(A) \neq \emptyset$). If $j_C^E(Q) = 0$, then $j_C^I(C_2) = 1$, and if $j_C^E(Q) = 1$, then $j_C^I(C_1) = 1$.

(γ) $y \in J_C$ and $y' \notin J_C$. Let z be the vertex in $Q \cap (J_1 \cup J_2)$ such that $\mathrm{dist}_Q(z, y)$ is maximal (i.e., z is the last vertex of Q in J_C, in the orientation from y to y'). Now, if $z \in J_1$, then $j_C^I(C_1) = 1$, and if $z \in J_2$, then $j_C^I(C_2) = 1$.

In each of the possible cases, we have obtained a contradiction with the choice of C. □

Recall that, by Claim 1.6, $|V(C)| \geq 10$. We show that the set $V(C)$ is 3-step dominating in G. Let, to the contrary, y_4 be a vertex at distance 4 from C, and let $Q : y_4, y_3, y_2, y_1, y_0$ be a shortest (y_4, C)-path in G (i.e., $\{y_0\} = V(C) \cap V(Q)$). Let y_0^{+i} (y_0^{-i}) denote the i-h successor (predecessor) of y_0 on C, respectively, and set $A = y^{-3}Cy^{+3}$. Since G is $S_{3,3,3}$-free, the subgraph $G[V(A) \cup \mathrm{int}(Q)]$ is not isomorphic to $S_{3,3,3}$, and since both Q and C are induced, there is an edge $uv \in E(G)$ with $u \in V(Q) \setminus \{y_0\}$ and $v \in V(A) \setminus \{y_0\}$. Then $u = y_1$ since Q is shortest, and by Claim 1.8 and since G is triangle-free, $v \in \{y_0^{-2}, y_0^{+2}\}$. By symmetry, let $v = y_0^{+2}$. But then $G[(V(Q) \cup V(A) \cup \{y_0^{+4}\}) \setminus \{y_0^{+1}, y_0^{-3}\}]$ is an induced $S_{3,3,3}$ with center at y_1, a contradiction. Thus, the set $V(C)$ is 3-step dominating in G.

Recall that G is bridgeless since J_C is not a cutset. Then, by Corollary J, we have $\mathrm{rc}(G) \leq \mathrm{diam}(C) + 1 + 3 \cdot 5 \leq \mathrm{diam}(G) + 16$. ■

4.2 The pair $(S_{2,2,2}, N_{2,2,2})$

The proof of Theorem 2 basically follows the same strategy as the proof of Theorem 1. We first handle the cases with small diameter, show that all bridges are on a diameter path, and then we again distinguish two cases according to whether the set J_C is a cutset of G or not: in the first case, we obtain a 3-step domination by a diameter path, while in the second case we obtain a 2-step domination by a certain chordless cycle. However, there are 2 major differences:

- the graph G does not have to be triangle-free (implying that the sets $O_i^j(P)$ can be nonempty and vertices in the sets $N_i^1(P)$ can have three neighbours),
- all distances are smaller since we work with an $S_{2,2,2}$ instead of an $S_{3,3,3}$.

Consequently, some parts of the proof are identical with the corresponding parts of the proof of Theorem 1, some parts are almost identical with only different constants, and some parts are substantially different. In order to avoid unnecessary (and tedious) repetitions, for the identical parts, we refer to the corresponding parts of the proof of Theorem 1.

Proof of Theorem 2. Let G be an $(S_{2,2,2}, N_{2,2,2})$-free graph. First suppose that $d = \mathrm{diam}(G) \leq 4$. If G is bridgeless, then $\mathrm{rc}(G) \leq 24 \leq \mathrm{diam}(G)+20$ by Theorem B. Thus we assume that G contains a bridge $b = uv$. If $d = 3$, then uv is a two-way dominating set in G since $\delta(G) \geq 2$, implying that $\mathrm{rc}(G) \leq 4$ by Theorem H. Hence we suppose that $d = 4$. Let G_u, G_v denote the components of $G - b$ such that $u \in V(G_u)$ and $v \in V(G_v)$. Up to a symmetry, suppose that every vertex of G_u is adjacent to u. Then $\mathrm{rad}(G_u) = 1$ and G_u is bridgeless, and $\mathrm{rad}(G_v) = 2$. If G_v is also bridgeless, then $\mathrm{rc}(G_v) \leq 8$ by Theorem B, implying that $\mathrm{rc}(G) = \mathrm{rc}(G_u) + 1 + \mathrm{rc}(G_v) \leq 3 + 1 + 8 = 12$. Thus, we assume that G_v contains a bridge. Since G is $S_{2,2,2}$-free and $\delta(G) \geq 2$, G_v contains only one bridge b', for otherwise v would be a center of an induced $S_{2,2,2}$. Moreover, b' is incident with v. Let G_{v_1}, G_{v_2} denote the components of $G_v - b'$ such that G_{v_1} contains v. Then G_{v_1}, G_{v_2} are both bridgeless, $\mathrm{rad}(G_{v_2}) = 1$, and $\mathrm{rad}(G_{v_1}) = 1$ since otherwise v would be a center of an induced $S_{2,2,2}$. Thus $\mathrm{rc}(G) = \mathrm{rc}(G_u) + 2 + \mathrm{rc}(G_{v_1}) + \mathrm{rc}(G_{v_2}) \leq 3 + 2 + 3 + 3 = 11$.

For the rest of the proof, we suppose that $d = \mathrm{diam}(G) \geq 5$. We choose a diameter path $P : x_0, x_1, \ldots, x_d$ in G. Unless otherwise stated, the sets M_i^j, N_i^j, O_i^j and R^j, as introduced above, will be always understood with respect to this fixed diameter path P. For these sets M_i^j, N_i^j and O_i^j, we can prove the following statement.

<u>Claim 2.1.</u>

(*i*) $M_i^j = \emptyset$ *for* $2 \leq i \leq d-2$ *and* $j \geq 2$,

(*ii*) $O_i^j = \emptyset$ *for* $3 \leq i \leq d-2$ *and* $j \geq 3$,

(*iii*) $N_i^j = \emptyset$ *for* $2 \leq i \leq d-2$ *and* $j \geq 3$,

(*iv*) *if* $x \in O_i^2$, $3 \leq i \leq d-2$, *and* $y \in R^1$ *is such that* $xy \in E(G)$, *then* $y \in O_i^2$,

(*v*) *if* $x \in N_i^2$, $2 \leq i \leq d-2$, *and* $y \in R^1$ *is such that* $xy \in E(G)$, *then* $y \in N_i^2$.

<u>Proof.</u> The statements (*i*), (*ii*) and (*iii*) follow from the fact that G is $S_{2,2,2}$-free or $N_{2,2,2}$-free. Now we show (*iv*). Let $x \in O_i^2$ for some i, $3 \leq i \leq d-2$, let $y \in R^1$ be such that $xy \in E(G)$, and let x_i' denote a neighbor of x in O_i^1. Then $x_i'y \in E(G)$, for

otherwise the set $\{x_i', x_{i-1}, x_i, x, y, x_{i-2}, x_{i-3}, x_{i+1}, x_{i+2}\}$ induces an $N_{2,2,2}$, a contradiction. The statement (v) can be proved analogously. $\square$

Claim 2.2. *The path P contains all bridges of G.*

Proof. For the proof of Claim 2.2, we can basically follow the proof of Claim 1.2, and we refer to Figure 3. Differently from Claim 1.2, for the vertex w in the component B_R, we have only $\mathrm{dist}(u_2, w) = 1$, i.e., $\mathrm{dist}(u_1, w) = 2$, and for $r = \mathrm{dist}(v, w) = |E(vQ_1w)| = |E(vQ_2w)|$, we have $r \geq 2$.

We choose the paths Q_1, Q_2 satisfying (i) and (ii), and we have the inequalities (1), (2) and (3) as in the proof of Claim 1.2.

We show that $t \geq 2$: if $t \leq 1$, then from (3) we have $s + 1 \geq s + t \geq d$, from which $s \geq d-1$, and then (1) implies $d \geq s+r \geq d-1+r \geq d-1+2 = d+1$, a contradiction. Hence $t \geq 2$, and, symmetrically, (using (2) instead of (1)), $s \geq 2$. Hence the graph F, consisting of the paths vQ_1x_0, vQ_2x_d and $v\overleftarrow{Q_1}w$, contains a subgraph isomorphic to the graph $S_{2,2,2}$ (with center at v). Since G is $S_{2,2,2}$-free, F is not an induced subgraph of G.

Now, as in the proof of Claim 1.2, we have an edge $h = z_1z_2$ with the same properties, and, in the same way, we show that $p = q$. However, since G need not be triangle-free, inequality (4) now reads

$$p = q \geq 1. \tag{4}$$

As in Claim 1.2, we choose the edge $h = z_1z_2$ such that

(iii) subject to (i) and (ii), $p = q$ is maximum.

By (4), and since $r \geq 2$, we have $q + r \geq 3$, i.e., $r \geq 3 - q$. From (2) we then have $d \geq t+r \geq t+3-q$, from which $t-q \leq d-3$. Since the path $x_0\overleftarrow{Q_1}z_1z_2Q_2x_d$ is an (x_0, x_d)-path of length $s-p+1+t-q$, and P is a diameter path, $d \leq s-p+1+t-q \leq s-p+1+d-3$, from which we conclude that $s - p \geq 2$. Thus, $\mathrm{dist}(z_1, x_0) = |E(z_1Q_1x_0)| = s - p \geq 2$. Symmetrically, $\mathrm{dist}(z_2, x_d) = |E(z_2Q_2x_d)| = t - q \geq 2$, and hence $|E(z_1z_2Q_2x_d)| \geq 3$.

If $p = q = 1$ the subgraph $[\{z_1z_2v, z_1Q_1x_0, z_2Q_2x_d, vQ_1w\}]_G$ contains an induced $N_{2,2,2}$, a contradiction

For $p = q \geq 2$, the graph consisting of the paths $z_1Q_1x_0$, $z_1z_2Q_2x_d$ and $z_1\overleftarrow{Q_1}w$ contains a subgraph F' isomorphic to $S_{2,2,2}$ (with center at z_1). By the choice (iii), F' is an induced subgraph of G, a contradiction. $\square$

We define the set $J_C = \left(\bigcup_{i=2}^{d-2}(M_i \cup N_i \cup \{x_i\})\right) \cup \left(\bigcup_{i=3}^{d-2} O_i\right)$.

By Claim 2.1, $\mathrm{dist}_G(x, P) \leq 2$ for each $x \in J_C$.

We also show the following observation.

Claim 2.3. *Let $u \in V(G) \setminus J_C$ and $v \in J_C$ be such that $uv \in E(G)$. Then $v \in M_i \cup N_i \cup O_{i+1} \cup \{x_2\}$ for some $i \leq 4$, or $v \in M_i \cup N_i \cup O_i \cup \{x_{d-2}\}$ for some $i \geq d-4$.*

Proof. By Claim 2.1(i), (ii) and (iii), we have $\mathrm{dist}_G(v, P) \leq 2$, and by Claim 2.1($iv$), ($v$), $\mathrm{dist}_G(u, P) \leq 2$. Up to a symmetry, suppose that $u \in (M_0 \cup M_1 \cup N_0 \cup N_1 \cup O_1 \cup O_2 \cup \{x_0, x_1\}) \setminus J_C$ (the case $u \in (M_d \cup M_{d-1} \cup N_d \cup N_{d-1} \cup O_d \cup O_{d-1} \cup \{x_d, x_{d-1}\}) \setminus J_C$ is symmetric).

If $v \in M_i$ for some $i \geq 5$, then $\mathrm{dist}_G(u, P) + \mathrm{dist}_G(v, P) \leq 2$ by Claim 2.1(i), implying that $i \leq 4$, for otherwise a path consisting of a shortest (x_0, u)-path, the edge uv and a shortest (v, x_d)-path is an (x_0, x_d)-path shorter than P, a contradiction. If $v \in N_i$ for some $i \geq 5$, then $\mathrm{dist}_G(u, P) + \mathrm{dist}_G(v, P) \leq 3$ by Claim 2.1(v), implying that $i \leq 4$, for otherwise a path consisting of a shortest (x_0, u)-path, the edge uv and a shortest (v, x_d)-path is an (x_0, x_d)-path shorter than P, a contradiction. Analogously, if $v \in O_i$ for some $i \geq 6$, then $\mathrm{dist}_G(u, P) + \mathrm{dist}_G(v, P) \leq 3$ by Claim 2.1(iv), implying that $i \leq 5$, since otherwise a path consisting of a shortest (x_0, u)-path, the edge uv and a shortest (v, x_d)-path is an (x_0, x_d)-path shorter than P, a contradiction. Similarly, if $v = x_i$ for some $i \geq 4$, then there is a shorter (x_0, x_d)-path containing the edge uv, a contradiction again. Finally, if $v = x_3$ then $u \in M_3 \cup N_2 \cup N_3 \cup N_4 \cup O_3 \cup O_4$, contradicting the choice of u. $\square$

We now distinguish two cases.

Case 1: *The set J_C is a cutset of G.*

We show that there is no vertex at distance greater than 3 from P in G.

Claim 2.4. *For every $x \in V(G) \setminus J_C$, $\mathrm{dist}_G(x, P) \leq 3$.*

Proof. Let, to the contrary, $x \in V(G) \setminus J_C$ be at distance 4 from P in G. Up to a symmetry, suppose that $\mathrm{dist}_G(x, x_0) < \mathrm{dist}_G(x, x_d)$. Let Q denote a shortest x, x_d-path, z_i' the first vertex of Q in J_C (in an orientation of Q from x) and z_i the predecessor of z_i' on Q in the same orientation. By Claim 2.1 and by the definition of J_C, $\mathrm{dist}_G(z_i, P) = 1$ and $\mathrm{dist}_G(z_i, x_0) \leq 2$, implying that $\mathrm{dist}_G(z_i, x_d) \geq d - 2$. Then $\mathrm{dist}_G(x, z_i) \geq 3$, implying that $\mathrm{dist}_G(x, x_d) \geq 3 + d - 2 > d$, a contradiction. $\square$

By Claim 2.4, the set $V(P)$ is 3-step dominating in G, hence by Corollary J and by Claim 2.2, we have $\mathrm{rc}(G) \leq \mathrm{diam}(G) + 3 \cdot 5 \leq \mathrm{diam}(G) + 15$.

Case 2: *The set J_C is not a cutset of G.*

First suppose that $d = \mathrm{diam}(G) \leq 8$. If G is not bridgeless, then all bridges of G are on P by Claim 2.2, and at least one vertex of each bridge is in J_C. But then J_C is a cutset of G, contradicting the assumption of Case 2. Hence G is bridgeless, and then, by Theorem B, we have $\mathrm{rc}(G) \leq \mathrm{rad}(G)(\mathrm{rad}(G) + 2) \leq d(d+2)$. It is easy to verify that, for $d \leq 8$, $d(d+2) = d + d(d+1) \leq d + 72$, we have $\mathrm{rc}(G) \leq \mathrm{diam}(G) + 72$, and we are done.

Thus, for the rest of the proof, we suppose that $d = \mathrm{diam}(G) \geq 9$. We introduce the following notation:

$$J_1 = \bigcup_{i=2}^{4} (M_i \cup N_i \cup O_{i+1} \cup \{x_i\}),$$
$$J_2 = \bigcup_{i=d-4}^{d-2} (M_i \cup N_i \cup O_i \cup \{x_i\}),$$
$$J_V = J_C \setminus (J_1 \cup J_2).$$
We further denote P' a shortest (x_0, x_d)-path in $G - J_C$. Note that $J_1 \cap J_2 = \emptyset$ since $d \geq 9$.

Note that, by Claim 2.3, there is no edge between J_V and $V(G) \setminus J_C$.

As in the proof of Theorem 1, we introduce the concepts of a J_C*-internal arc* and a J_C*-external arc* and the numbers $j^I_C(C)$ and j^E_C, and we define the cycle C to be a shortest cycle in G such that $j^I_C(C)$ is odd. Using Claim 2.3 (which is a counterpart of Claim 1.4), we show in the same way that j^E_C is also odd.

Claim 2.5. *Let A be a J_C-internal (v_1, v_2)-arc of C, let v_1', v_2' denote the neighbor of v_1 or v_2 in $V(C) \setminus \mathrm{int}(A)$, respectively. Then $\mathrm{dist}_G(v_1', v_2') \geq d - 4$.*

Proof. By the definition of a J_C-internal arc, $v_1', v_2' \notin J_C$. By symmetry, we can suppose that $v_1' \in (M_0 \cup M_1 \cup N_1 \cup M_2 \cup N_2 \cup \{x_0, x_1\}) \setminus J_C$. Then $\mathrm{dist}_G(v_1', P) \leq 1$ and $\mathrm{dist}_G(v_2', P) \leq 1$ by Claim 2.1 and since $v_1', v_2' \notin J_C$. Hence $\mathrm{dist}_G(x_0, v_1') \leq 2$ and $\mathrm{dist}_G(v_2', x_d) \leq 2$. Since $\mathrm{dist}_G(x_0, v_1') + \mathrm{dist}_G(v_1', v_2') + \mathrm{dist}_G(v_2', x_d) \geq d$, we get $\mathrm{dist}_G(v_1', v_2') \geq d - \mathrm{dist}_G(x_0, v_1') - \mathrm{dist}_G(v_2', x_d) \geq d - 4$. $\square$

Note that, using Claim 2.5, we immediately observe that $|V(C)| \geq 10$.

Claim 2.6. *The cycle C can be chosen such that $j^I_C(C) = j^E_C(C) = 1$.*

Proof. Since both $j^I_C(C)$ and $j^E_C(C)$ are odd, hence nonzero, there is a pair A^I, A^E of J_C-arcs of C such that A^I is J_C-internal, A^E is J_C-external, and A^I, A^E are consecutive on C, i.e., at least one of the components of $C - (\mathrm{int}(A^I) \cup \mathrm{int}(A^E))$ contains no J_C-arc. Let v_i^I, v_i^E denote the endvertex of A^I, A^E in J_i $(i = 1, 2)$, respectively, let $(v_i^I)'$ denote the neighbor of v_i^I on $C - A^I$, and let $(v_i^E)'$ denote the neighbor of v_i^E on A^E. Then $(v_1^I)', (v_1^E)' \in (M_0 \cup M_1 \cup N_1 \cup O_1 \cup O_2 \cup \{x_0, x_1\}) \setminus J_C$ and $(v_2^I)', (v_2^E)' \in (O_{d-2} \cup O_d \cup M_{d-1} \cup N_{d-1} \cup M_d \cup \{x_d, x_{d-1}\}) \setminus J_C$. By Claim 2.1, $\mathrm{dist}_G((v_i^I)', P) \leq 1$ and $\mathrm{dist}_G((v_i^E)', P) \leq 1$, implying that $\mathrm{dist}_G((v_i^I)', (v_i^E)') \leq 4$ for $i = 1, 2$. Since A^I and A^E are consecutive on C, we may assume (up to a symmetry) that there is no J_C-arc between v_1^I and v_1^E.

Now, if, say, $j^I_C(C) > 1$, then the cycle C' consisting of A^I, A^E, the arc $v_1^I C v_1^E$, and a shortest $((v_2^I)', (v_2^E)')$-path, has length at most $|V(C)| - 2 \cdot 5 + 3 = |V(C)| - 7$ since we delete from C at least two J_C-internal arcs and we add a shortest $((v_2^I)', (v_2^E)')$-path. But this contradicts the fact that C is shortest possible. Therefore $j^I_C(C) = 1$, and analogously $j^E_C(C) = 1$. $\square$

By Claim 2.6, for the rest of the proof we suppose that the cycle C is chosen such that $j^I_C(C) = j^E_C(C) = 1$.

The next claim is identical with Claim 1.8, and since the proof of Claim 1.8 uses only metric arguments on cycles and arcs, the proof of Claim 2.7 is also identical with that of Claim 1.8 (using Claim 2.6, which is a counterpart to Claim 1.7). We therefore include here only the statement of Claim 2.7, and for its proof, we refer to the proof of Claim 1.8.

Claim 2.7. *Every (y, y')-arc of C of length at most $\frac{|V(C)|}{2}$ is a shortest (y, y')-path in G.* □

Recall that, by Claim 2.5, $|V(C)| \geq 10$. Now, every vertex of G is at distance at most 2 from C by the same arguments as in the proof of Claim 2.1. But then the set $V(C)$ is 2-step dominating in G, hence by Corollary J and by Claim 2.2, we have $\mathrm{rc}(G) \leq \mathrm{diam}(C) + 1 + 2 \cdot 4 \leq \mathrm{diam}(G) + 9$. ■

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5 Appendix (for referees only, not for publication)

We include here a full proof of Theorem 2, containing also the omitted parts.

Proof of Theorem 2. First suppose that $d = \mathrm{diam}(G) \leq 4$. If G is bridgeless, then $\mathrm{rc}(G) \leq 24 \leq \mathrm{diam}(G) + 20$ by Theorem B. Thus we assume that G contains a bridge $b = uv$. If $d = 3$, then uv is a two-way dominating set in G since $\delta(G) \geq 2$, implying that $\mathrm{rc}(G) \leq 4$ by Theorem H. Hence we suppose that $d = 4$. Let G_u, G_v denote the components of $G - b$ such that $u \in V(G_u)$ and $v \in V(G_v)$. Up to a symmetry, suppose that every vertex of G_u is adjacent to u. Then $\mathrm{rad}(G_u) = 1$ and G_u is bridgeless, and $\mathrm{rad}(G_v) = 2$. If G_v is also bridgeless, then $\mathrm{rc}(G_v) \leq 8$ by Theorem B, implying that $\mathrm{rc}(G) = \mathrm{rc}(G_u) + 1 + \mathrm{rc}(G_v) \leq 3 + 1 + 8 = 12$. Thus, we assume that G_v contains a bridge. Since G is $S_{2,2,2}$-free and $\delta(G) \geq 2$, G_v contains only one bridge b', for otherwise v would be a center of induced $S_{2,2,2}$. Moreover, b' is incident with v. Let G_{v_1}, G_{v_2} denote the components of $G_v - b'$ such that G_{v_1} contains v. Then G_{v_1}, G_{v_2} are both bridgeless, $\mathrm{rad}(G_{v_2}) = 1$, and $\mathrm{rad}(G_{v_1}) = 1$ since otherwise v would be a center of an induced $S_{2,2,2}$. Thus $\mathrm{rc}(G) = \mathrm{rc}(G_u) + 2 + \mathrm{rc}(G_{v_1}) + \mathrm{rc}(G_{v_2}) \leq 3 + 2 + 3 + 3 = 11$.

For the rest of the proof, we suppose that $d = \mathrm{diam}(G) \geq 5$. We choose a diameter path $P : x_0, x_1, \ldots, x_d$ in G. Unless otherwise stated, the sets M_i^j, N_i^j, O_i^j and R^j, as introduced above, will be always understood with respect to this fixed diameter path P. For these sets M_i^j, N_i^j and O_i^j, we can prove the following statement.

<u>Claim 2.1.</u>
(*i*) $M_i^j = \emptyset$ *for* $2 \leq i \leq d-2$ *and* $j \geq 2$,
(*ii*) $O_i^j = \emptyset$ *for* $3 \leq i \leq d-2$ *and* $j \geq 3$,
(*iii*) $N_i^j = \emptyset$ *for* $2 \leq i \leq d-2$ *and* $j \geq 3$,
(*iv*) *if* $x \in O_i^2$, $3 \leq i \leq d-2$, *and* $y \in R^1$ *is such that* $xy \in E(G)$, *then* $y \in O_i^2$,
(*v*) *if* $x \in N_i^2$, $2 \leq i \leq d-2$, *and* $y \in R^1$ *is such that* $xy \in E(G)$, *then* $y \in N_i^2$.

<u>Proof.</u> The statements (i), (ii) and (iii) follow from the fact that G is $S_{2,2,2}$-free or $N_{2,2,2}$-free, respectively. Now we show (iv). Let $x \in O_i^2$ for some i, $3 \leq i \leq d-2$, let $y \in R^1$ be such that $xy \in E(G)$, and let x_i' denote a neighbor of x in O_i^1. Then $x_i'y \in E(G)$, for otherwise the set $\{x_i', x_{i-1}, x_i, x, y, x_{i-2}, x_{i-3}, x_{i+1}, x_{i+2}\}$ induces an $N_{2,2,2}$, a contradiction. The statement (v) can be proved analogously. $\square$

<u>Claim 2.2.</u> *The path P contains all bridges of G.*

<u>Proof.</u> Let, to the contrary, $b = u_1u_2$ be a bridge with $b \notin E(P)$, and choose the notation such that u_1 is in the component B_P of $G-b$ containing P, and u_2 is in the other component B_R of $G-b$ (see Fig. 4). The component B_R contains a vertex w with $\mathrm{dist}(u_2, w) = 1$ (i.e., $\mathrm{dist}(u_1, w) = 2$). Let Q_1 denote a shortest (w, x_0)-path in G and Q_2 a shortest (w, x_d)-path in G. Let v be the last common vertex of Q_1 and Q_2, i.e., a vertex such that the paths vQ_1x_0 and vQ_2x_d are internally vertex-disjoint. Denote $s = \mathrm{dist}(v, x_0) = |E(vQ_1x_0)|$,

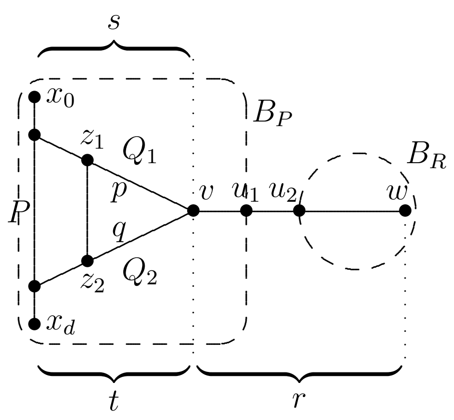

Figure 4: The paths P, Q_1 and Q_2 in the proof of Claim 2.2

$t = \mathrm{dist}(v, x_d) = |E(vQ_2x_d)|$, and $r = \mathrm{dist}(v, w) = |E(vQ_1w)| = |E(vQ_2w)|$. Obviously, $r \geq 2$.

Now, choose the paths Q_1, Q_2 such that
(i) $|E(Q_1)|$ is minimum and $|E(Q_2)|$ is minimum (as already mentioned), and
(ii) subject to (i), $s + t = |E(vQ_1x_0)| + |E(vQ_2x_d)|$ is minimum.
Since $d = \mathrm{diam}(G)$, we have

$$\mathrm{dist}(x_0, w) = s + r \leq d, \tag{1}$$

$$\mathrm{dist}(x_d, w) = t + r \leq d. \tag{2}$$

Since $x_0\overleftarrow{Q_1}vQ_2x_d$ is an (x_0, x_d)-path and P is a diameter path, we have

$$\mathrm{dist}(x_0, v) + \mathrm{dist}(x_d, v) = s + t \geq d. \tag{3}$$

We show that $t \geq 2$: if $t \leq 1$, then from (3) we have $s + 1 \geq s + t \geq d$, from which $s \geq d - 1$, and then (1) implies $d \geq s + r \geq d - 1 + r \geq d - 1 + 2 = d + 1$, a contradiction. Hence $t \geq 2$, and, symmetrically, (using (2) instead of (1)), $s \geq 2$. Hence the graph F, consisting of the paths vQ_1x_0, vQ_2x_d and $v\overleftarrow{Q_1}w$, contains a subgraph isomorphic to the graph $S_{2,2,2}$ (with center at v). Since G is $S_{2,2,2}$-free, F is not an induced subgraph of G.

Thus, let $h = z_1z_2$ be an arbitrary edge with $z_1, z_2 \in V(F)$ but $h \in E(G) \setminus E(F)$. Since both Q_1 and Q_2 are shortest (hence chordless), up to a symmetry, $z_1 \in V(v^{+Q_1}Q_1x_0)$ and $z_2 \in V(v^{+Q_2}Q_2x_d)$, where v^{+Q_1} and v^{+Q_2} denotes the successor of v on Q_1 and Q_2, respectively. Set $p = \mathrm{dist}(v, z_1) = |E(vQ_1z_1)|$ and $q = \mathrm{dist}(v, z_2) = |E(vQ_2z_2)|$. Obviously, $p \geq 1$ and $q \geq 1$.

We show that $p = q$. First suppose that $p \geq q+1$. Then, considering the paths Q_1 and $Q_2' = wQ_1z_1z_2Q_2x_d$, we have a contradiction with the choice of Q_1 and Q_2: if $p > q + 1$, then Q_2' is shorter than Q_2, contradicting (i), and if $p = q + 1$, then $|E(Q_2)| = |E(Q_2')|$, but $|E(z_1Q_1x_0)| + |E(z_1Q_2'x_d)| < |E(vQ_1x_0)| + |E(vQ_2x_d)|$, contradicting ($ii$) (where z_1

plays the role of v). Hence $p < q+1$. Symmetrically, $q < p+1$, implying $p = q$. Moreover we have

$$p = q \geq 1. \tag{4}$$

Now, we choose the edge $h = z_1 z_2$ such that

(iii) subject to (i) and (ii), $p = q$ is maximum.

By (4), and since $r \geq 2$, $q+r \geq 3$, i.e., $r \geq 3-q$. From (2) we then have $d \geq t+r \geq t+3-q$, from which $t - q \leq d - 3$. Since the path $x_0 \overleftarrow{Q_1} z_1 z_2 Q_2 x_d$ is an (x_0, x_d)-path of length $s - p + 1 + t - q$, and P is a diameter path, $d \leq s - p + 1 + t - q \leq s - p + 1 + d - 3$, from which we conclude that $s - p \geq 2$. Thus, $\mathrm{dist}(z_1, x_0) = |E(z_1 Q_1 x_0)| = s - p \geq 2$. Symmetrically, $\mathrm{dist}(z_2, x_d) = |E(z_2 Q_2 x_d)| = t - q \geq 2$, and hence $|E(z_1 z_2 Q_2 x_d)| \geq 3$.

If $p = q = 1$ the subgraph $[\{z_1 z_2 v, z_1 Q_1 x_0, z_2 Q_2 x_d, v Q_1 w\}]_G$ contains an induced $N_{2,2,2}$, a contradiction

For $p = q \geq 2$, the graph consisting of the paths $z_1 Q_1 x_0$, $z_1 z_2 Q_2 x_d$ and $z_1 \overleftarrow{Q_1} w$ contains a subgraph F' isomorphic to $S_{2,2,2}$ (with center at z_1). By the choice (iii), F' is an induced subgraph of G, a contradiction. $\square$

We define the set $J_C = \left(\bigcup_{i=2}^{d-2} (M_i \cup N_i \cup \{x_i\}) \right) \cup \left(\bigcup_{i=3}^{d-2} O_i \right)$.

By Claim 2.1, $\mathrm{dist}_G(x, P) \leq 2$ for each $x \in J_C$.

We also show the following observation.

<u>Claim 2.3.</u> *Let $u \in V(G) \setminus J_C$ and $v \in J_C$ be such that $uv \in E(G)$. Then $v \in M_i \cup N_i \cup O_{i+1} \cup \{x_2\}$ for some $i \leq 4$, or $v \in M_i \cup N_i \cup O_i \cup \{x_{d-2}\}$ for some $i \geq d-4$.*

<u>Proof.</u> By Claim 2.1(i), (ii) and (iii), we have $\mathrm{dist}_G(v, P) \leq 2$, and by Claim 2.1($iv$), ($v$), $\mathrm{dist}_G(u, P) \leq 2$. Up to a symmetry, suppose that $u \in (M_0 \cup M_1 \cup N_0 \cup N_1 \cup O_1 \cup O_2 \cup \{x_0, x_1\}) \setminus J_C$ (the case $u \in (M_d \cup M_{d-1} \cup N_d \cup N_{d-1} \cup O_d \cup O_{d-1} \cup \{x_d, x_{d-1}\}) \setminus J_C$ is symmetric).

If $v \in M_i$ for some $i \geq 5$, then $\mathrm{dist}_G(u, P) + \mathrm{dist}_G(v, P) \leq 2$ by Claim 2.1(i), implying that $i \leq 4$, for otherwise a path consisting of a shortest (x_0, u)-path, the edge uv and a shortest (v, x_d)-path is an (x_0, x_d)-path shorter than P, a contradiction. If $v \in N_i$ for some $i \geq 5$, then $\mathrm{dist}_G(u, P) + \mathrm{dist}_G(v, P) \leq 3$ by Claim 2.1(v), implying that $i \leq 4$, for otherwise a path consisting of a shortest (x_0, u)-path, the edge uv and a shortest (v, x_d)-path is an (x_0, x_d)-path shorter than P, a contradiction. Analogously, if $v \in O_i$ for some $i \geq 6$, then $\mathrm{dist}_G(u, P) + \mathrm{dist}_G(v, P) \leq 3$ by Claim 2.1(iv), implying that $i \leq 5$, since otherwise a path consisting of a shortest (x_0, u)-path, the edge uv and a shortest (v, x_d)-path is an (x_0, x_d)-path shorter than P, a contradiction. Similarly, if $v = x_i$ for some $i \geq 4$, then there is a shorter (x_0, x_d)-path containing the edge uv, a contradiction again. Finally, if $v = x_3$ then $u \in M_3 \cup N_2 \cup N_3 \cup N_4 \cup O_3 \cup O_4$, contradicting the choice of u. $\square$

We now distinguish two cases.

Case 1: *The set J_C is a cutset of G.*

We show that there is no vertex at distance greater than 3 from P in G.

Claim 2.4. *For every $x \in V(G) \setminus J_C$, $\mathrm{dist}_G(x, P) \leq 3$.*

Proof. Let, to the contrary, $x \in V(G) \setminus J_C$ be at distance 4 from P in G. Up to a symmetry, suppose that $\mathrm{dist}_G(x, x_0) < \mathrm{dist}_G(x, x_d)$. Let Q denote a shortest x, x_d-path, z_i' the first vertex of Q in J_C (in an orientation of Q from x) and z_i the predecessor of z_i' on Q in the same orientation. By Claim 2.1 and by the definition of J_C, $\mathrm{dist}_G(z_i, P) = 1$ and $\mathrm{dist}_G(z_i, x_0) \leq 2$, implying that $\mathrm{dist}_G(z_i, x_d) \geq d-2$. Then $\mathrm{dist}_G(x, z_i) \geq 3$, implying that $\mathrm{dist}_G(x, x_d) \geq 3 + d - 2 > d$, a contradiction. $\square$

By Claim 2.4, the set $V(P)$ is 3-step dominating in G, hence by Corollary J and by Claim 2.2, we have $\mathrm{rc}(G) \leq \mathrm{diam}(G) + 3 \cdot 5 \leq \mathrm{diam}(G) + 15$.

Case 2: *The set J_C is not a cutset of G.*

First suppose that $d = \mathrm{diam}(G) \leq 8$. If G is not bridgeless, then all bridges of G are on P by Claim 2.2, and at least one vertex of each bridge is in J_C. But then J_C is a cutset of G, contradicting the assumption of Case 2. Hence G is bridgeless, and then, by Theorem B, we have $\mathrm{rc}(G) \leq \mathrm{rad}(G)(\mathrm{rad}(G)+2) \leq d(d+2)$. It is easy to verify that, for $d \leq 8$, $d(d+2) = d + d(d+1) \leq d + 72$, we have $\mathrm{rc}(G) \leq \mathrm{diam}(G) + 72$, and we are done.

Thus, for the rest of the proof, we suppose that $d = \mathrm{diam}(G) \geq 9$. We introduce the following notation:

$$J_1 = \bigcup_{i=2}^{4} (M_i \cup N_i \cup O_{i+1} \cup \{x_i\}),$$
$$J_2 = \bigcup_{i=d-4}^{d-2} (M_i \cup N_i \cup O_i \cup \{x_i\}),$$
$$J_V = J_C \setminus (J_1 \cup J_2).$$

We further denote P' a shortest (x_0, x_d)-path in $G - J_C$. Note that $J_1 \cap J_2 = \emptyset$ since $d \geq 9$.

Note that, by Claim 2.3, there is no edge between J_V and $V(G) \setminus J_C$.

If $F \subset G$ is a cycle or a path, and $A^I = v_1 F v_2$ is an arc of F, we say that A^I is *J_C-internal* if $V(A^I) \subset J_C$, A^I is maximal (in terms of the number of vertices) with this property and $v_1 \in J_j$ and $v_2 \in J_{3-j}$ for some $j \in \{1, 2\}$. We also say that an arc $A^E = w_1 F w_2$ is *J_C-external* if no internal vertex of A^E belongs to J_C, and $w_1 \in J_j$ and $w_2 \in J_{3-j}$ for some $j \in \{1, 2\}$. We will use $j_C^I(F)$ to denote the number of internally vertex-disjoint J_C-internal arcs of F and $j_C^E(F)$ for the number of internally vertex-disjoint J_C-external arcs of F. Finally, we say that an arc A of F is a J_C-arc if A is J_C-internal or J_C-external.

Let now C be a shortest cycle in G such that $j_C^I(C)$ is odd (note that C exists since the subgraph $G[V(P) \cup V(P')]$ certainly contains such a cycle.) We observe that $j_C^E(C)$

is also odd. Clearly, the total number of arcs of C between some vertex of J_1 and some vertex of J_2 is even. Since $j_C^I(C)$ is odd, there must be an arc of C between J_1 and J_2 which is not J_C-internal. Let A' be such an arc and choose A' shortest possible. Since A' is not J_C-internal, A' contains some vertex $z \in V(G) \setminus J_C$, and since A' is shortest, $\mathrm{int}(A') \cap (J_1 \cup J_2) = \emptyset$. By Claim 2.3, we also have $\mathrm{int}(A') \cap J_V = \emptyset$, since there is no edge between z and J_V. Thus, A' is J_C-external. This means that every arc of C between J_1 and J_2 is either J_C-internal or J_C-external, hence a J_C-arc. Thus $j_C^I(C) + j_C^E(C)$ is even and, since $j_C^I(C)$ is odd, $j_C^E(C)$ must be also odd.

Claim 2.5. *Let A be a J_C-internal (v_1, v_2)-arc of C, let v_1', v_2' denote the neighbor of v_1 or v_2 in $V(C) \setminus \mathrm{int}(A)$, respectively. Then* $\mathrm{dist}_G(v_1', v_2') \geq d - 4$.

Proof. By the definition of a J_C-internal arc, $v_1', v_2' \notin J_C$. By symmetry, we can suppose that $v_1' \in (M_0 \cup M_1 \cup N_1 \cup M_2 \cup N_2 \cup \{x_0, x_1\}) \setminus J_C$. Then $\mathrm{dist}_G(v_1', P) \leq 1$ and $\mathrm{dist}_G(v_2', P) \leq 1$ by Claim 2.1 and since $v_1', v_2' \notin J_C$. Hence $\mathrm{dist}_G(x_0, v_1') \leq 2$ and $\mathrm{dist}_G(v_2', x_d) \leq 2$. Since $\mathrm{dist}_G(x_0, v_1') + \mathrm{dist}_G(v_1', v_2') + \mathrm{dist}_G(v_2', x_d) \geq d$, we get $\mathrm{dist}_G(v_1', v_2') \geq d - \mathrm{dist}_G(x_0, v_1') - \mathrm{dist}_G(v_2', x_d) \geq d - 4$. □

Note that, using Claim 2.5, we immediately observe that $|V(C)| \geq 10$.

Claim 2.6. *The cycle C can be chosen such that $j_C^I(C) = j_C^E(C) = 1$.*

Proof. Since both $j_C^I(C)$ and $j_C^E(C)$ are odd, hence nonzero, there is a pair A^I, A^E of J_C-arcs of C such that A^I is J_C-internal, A^E is J_C-external, and A^I, A^E are consecutive on C, i.e., at least one of the components of $C - (\mathrm{int}(A^I) \cup \mathrm{int}(A^E))$ contains no J_C-arc. Let v_i^I, v_i^E denote the endvertex of A^I, A^E in J_i ($i = 1, 2$), respectively, let $(v_i^I)'$ denote the neighbor of v_i^I on $C - A^I$, and let $(v_i^E)'$ denote the neighbor of v_i^E on A^E. Then $(v_1^I)', (v_1^E)' \in (M_0 \cup M_1 \cup N_1 \cup O_1 \cup O_2 \cup \{x_0, x_1\}) \setminus J_C$ and $(v_2^I)', (v_2^E)' \in (O_{d-2} \cup O_d \cup M_{d-1} \cup N_{d-1} \cup M_d \cup \{x_d, x_{d-1}\}) \setminus J_C$. By Claim 2.1, $\mathrm{dist}_G((v_i^I)', P) \leq 1$ and $\mathrm{dist}_G((v_i^E)', P) \leq 1$, implying that $\mathrm{dist}_G((v_i^I)', (v_i^E)') \leq 4$ for $i = 1, 2$. Since A^I and A^E are consecutive on C, we may assume (up to a symmetry) that there is no J_C-arc between v_1^I and v_1^E.

Now, if, say, $j_C^I(C) > 1$, then the cycle C' consisting of A^I, A^E, the arc $v_1^I C v_1^E$, and a shortest $((v_2^I)', (v_2^E)')$-path, has length at most $|V(C)| - 2 \cdot 5 + 3 = |V(C)| - 7$ since we delete from C at least two J_C-internal arcs and we add a shortest $((v_2^I)', (v_2^E)')$-path. But this contradicts the fact that C is shortest possible. Therefore $j_C^I(C) = 1$, and analogously $j_C^E(C) = 1$. □

By Claim 2.6, for the rest of the proof we suppose that the cycle C is chosen such that $j_C^I(C) = j_C^E(C) = 1$.

Claim 2.7. *Every (y, y')-arc of C of length at most $\frac{|V(C)|}{2}$ is a shortest (y, y')-path in G.*

Proof. Suppose, to the contrary, that there is an arc yCy' of length at most $\frac{|V(C)|}{2}$ that is not a shortest path in G, let Q be a shortest (y, y')-path in G, and, among all such arcs in C, choose the arc $A_1 : yCy'$ such that the path Q is shortest possible. By the same argument as in the proof of Claim 2.6, $j_C^I(Q) \leq 1$ and $j_C^E(Q) \leq 1$.

Let $A_2 : y'Cy$ denote the complementary arc to A_1 (i.e., $V(A_1) \cup V(A_2) = V(C)$ and $V(A_1) \cap V(A_2) = \{y, y'\}$). Then clearly A_1, A_2 and Q are pairwise internally vertex-disjoint paths with common endvertices, hence both $C_1 : yA_1y'\overleftarrow{Q}y$ and $C_2 : y'A_2yQy'$ are cycles in G. By the definition of Q and by the assumption that A_1 is of length at most $\frac{|V(C)|}{2}$, we have $|E(Q)| < |E(A_1)| \leq |E(A_2)|$, hence both C_1 and C_2 are shorter than C. Let $A : v_1Cv_2$ be the (only) J_C-internal arc of C, and choose the notation such that $v_1 \in J_1$ and $v_2 \in J_2$. According to the position of y and y' with respect to A, we have the following three possibilities.

(α) $y, y' \notin J_C$. Then either $A \subset A_1$, or $A \subset A_2$, thus, for each value of $j_C^I(Q)$, either $j_C^I(C_1) = 1$ or $j_C^I(C_2) = 1$.

(β) $y, y' \in J_C$. Then both y and y' are vertices of A (possibly $A = A_1$, or $\{y, y'\} \cap \mathrm{int}(A) \neq \emptyset$). If $j_C^E(Q) = 0$, then $j_C^I(C_2) = 1$, and if $j_C^E(Q) = 1$, then $j_C^I(C_1) = 1$.

(γ) $y \in J_C$ and $y' \notin J_C$. Let z be the vertex in $Q \cap (J_1 \cup J_2)$ such that $\mathrm{dist}_Q(z, y)$ is maximal (i.e., z is the last vertex of Q in J_C, in the orientation from y to y'). Now, if $z \in J_1$, then $j_C^I(C_1) = 1$, and if $z \in J_2$, then $j_C^I(C_2) = 1$.

In each of the possible cases, we have obtained a contradiction with the choice of C. $\square$

Recall that, by Claim 2.5, $|V(C)| \geq 10$. Now, every vertex of G is at distance at most 2 from C by the same arguments as in the proof of Claim 2.1. But then the set $V(C)$ is 2-step dominating in G, hence by Corollary J and by Claim 2.2, we have $\mathrm{rc}(G) \leq \mathrm{diam}(C) + 1 + 2 \cdot 4 \leq \mathrm{diam}(G) + 9$. $\blacksquare$