Rainbow cycles in edge-colored graphs

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Abstract

Let G be a graph of order n with an edge coloring c, and let $\delta^c(G)$ denote the minimum color degree of G, i.e., the largest integer such that each vertex of G is incident with at least $\delta^c(G)$ edges having pairwise distinct colors. A subgraph $F \subset G$ is rainbow if all edges of F have pairwise distinct colors. In this paper, we prove that (i) if G is triangle-free and $\delta^c(G) > \frac{n}{3} + 1$, then G contains a rainbow C_4, and (ii) if $\delta^c(G) > \frac{n}{2} + 2$, then G contains a rainbow cycle of length at least 4.

Keywords: edge coloring, rainbow coloring, rainbow cycle, color degree condition
2000 MSC: 05C15

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1. Introduction

We consider finite simple undirected graphs, and for notations and terminology not defined here we refer e.g. to [1]. By an *edge-colored graph* we mean a triple $G = (V(G), E(G), c)$, where $(V(G), E(G))$ is a (simple finite undirected) graph and $c : E(G) \to \mathbb{Z}^+$. The function c is called an *edge coloring* of G. If the edge coloring is clear from the context, we will simply speak of an edge-colored graph G. If $G = (V(G), E(G), c)$ and $H \subset G$ is a subgraph of G, then we automatically consider H to be edge-colored by the restriction of the function c

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[1] Research partly supported by project P202/12/G061 of the Czech Science Foundation and by the European Regional Development Fund (ERDF), project NTIS - New Technologies for Information Society, European Centre of Excellence, CZ.1.05/1.1.00/02.0090.

[2] Research supported by JSPS KAKENHI Grant Number 26400190.

to $E(H)$. An edge set $F \subset E(G)$ is called *rainbow* if no two distinct edges in F receive the same color, and a graph is called *rainbow* if its edge set is rainbow.

For an edge $e \in E(G)$, $c(e)$ denotes the color of e, and for a vertex $u \in V(G)$, $E(u)$ denotes the set of all edges incident to u. The cardinality of the set $c(E(u)) = \{c(e) : e \in E(u)\}$ is called the *color degree* of u and denoted by $d^c_G(u)$. The *minimum color degree* of G is denoted by $\delta^c(G)$ (or simply δ^c).

For $S_1, S_2 \subset V(G)$, $S_1 \cap S_2 = \emptyset$, we set $E(S_1, S_2) = \{xy \in E(G) : x \in S_1, y \in S_2\}$ and, in the special case when $S_1 = \{u\}$, we write $E(u, S)$ for $E(\{u\}, S)$. For a subgraph $F \subset G$ and a vertex $u \in V(G) \setminus V(F)$, we simply write $c(u, F)$ for $c(E(u, V(F)))$ and F^C for $G - F$.

For a subgraph $F \subset G$, we use $N_F(x)$ to denote the neighborhood of a vertex $x \in V(G)$ in F, i.e., the set of all vertices that are adjacent to x in F, and we write $d_F(x)$ for the degree of x in F. A path with endvertices $x, y \in V(G)$ is sometimes referred to as an (x, y)-path, and, for a path P and $u, v \in V(P)$, we use uPv to denote the subpath of P with endvertices u, v.

The existence of rainbow substructures in edge-colored graphs has been widely studied in the literature. We mention here those of the known results that are related to our paper; for more information we refer the reader to the survey paper by Kano and Li [5].

It turns out that the problem of existence of rainbow cycles is closely related to the problem of existence of cycles in (uncolored) directed graphs. Let D be a directed graph with vertex set $V(D) = \{u_1, u_2, \ldots, u_p\}$ and arc set $A(D)$, and let G be the underlying (undirected) graph of D. Li [7] constructed an edge-coloring $c : E(G) \to V(D)$ of G by defining $c(u_i u_j) = u_j$ for each arc $u_i u_j \in A(D)$. It is easy to see that D has a directed cycle of length ℓ if and only if $(V(G), E(G), c)$ has a rainbow cycle of length ℓ. Hence the problem of the existence of rainbow cycles in edge-colored graphs is a generalization of the corresponding problem on directed cycles in directed graphs. Indeed, the problems on rainbow cycles seems to be more difficult than the directed problem. For instance, it is known that if the minimum out-degree $\delta^+(D)$ is at least one, then D has a directed cycle; however, for rainbow cycles, the corresponding problem to determine the minimum color degree which guarantees the existence of a rainbow cycle is not solved yet.

Li and Wang [8] constructed an edge-colored bipartite graph and an edge-colored complete graph, both having minimum color degree $\log_2 n$ and no rainbow cycles. Erdös and Gallai [4] showed that every graph G with $m(\geq n)$ edges has a cycle of length at least $\frac{2m}{n-1}$. For the number of colors $c(G) = |c(E(G))|$, Broersma et at. [2] pointed out that the Erdös and Gallai's theorem immediately implies that if $c(G) \geq n$, then G contains a rainbow cycle of length at least $\frac{2}{n-1} c(G)$. On the other hand, Li et al. [6] constructed an edge-coloring of the complete graph K_n as follows: let $V(K_n) = \{u_1, \ldots, u_n\}$ and let $c : E(K_n) \to V(K_n)$ be the edge-coloring defined by $c(u_i u_j) = u_j$ for all $i < j$. Then, obviously, K_n with this edge-coloring contains no rainbow cycle, and $c(K_n) = n - 1$. Thus, the lower bound of $c(G)$ obtained by Broersma et al. is best possible. Li et al. [6] showed that if $e(G) + c(G) \geq n(n+1)/2$, then G contains a rainbow triangle, and the above example implies that this lower bound of $e(G) + c(G)$ is also best possible.

For directed graphs, the following conjecture by Caccetta and Häggkvist [3] is well-known: a directed graph with $\delta^+ \geq d$ has a directed cycle of length at most $\lceil \frac{n}{d} \rceil$. For rainbow cycles, the following example can be considered as an extremal graph for the analogue of the Caccetta-Häggkvist conjecture. Let $V_1, \dots, V_{d+1}$ be disjoint sets of vertices such that

$$\sum_{1 \leq i \leq d+1} |V_i| = n \text{ and } \lfloor n/(d+1) \rfloor \leq |V_i| \leq \lceil n/(d+1) \rceil \text{ for all } 1 \leq i \leq d+1,$$

and let $V_i = \{u_1^i, \dots, u_{|V_i|}^i\}$ for each $1 \leq i \leq d+1$. Let G be the graph with the vertex set $\bigcup_{1 \leq i \leq d+1} V_i$ and the edge set $\bigcup_{1 \leq i \leq d-1} \{u_j^i u_k^{i+1} : u_j^i \in V_i, u_k^{i+1} \in V_{i+1}\}$. An edge-coloring $c : E(G) \to V(G)$ is defined by $c(u_j^i u_k^{i+1}) = u_k^{i+1}$ for all edges $u_j^i u_k^{i+1} \in E(G)$. Then obviously G contains no rainbow cycle of length at most d and $\delta^c = \lfloor \frac{n}{d+1} \rfloor + 1$. Thus, as in the case of directed graphs, we need a minimum color degree at least $\lfloor \frac{n}{d} \rfloor$ for the existence of a rainbow cycle of length at most d.

Our research is motivated by the following recent results. For short cycles, Broersma et al. [2] gave a neighborhood union-type condition by showing that if $|c(E(u) \cup E(v))| \geq n - 1$ for all pairs $u, v \in V(G)$, then G has a rainbow cycle of length at most four. A minimum degree condition for the existence of a rainbow triangle was given by Li [7].

Theorem A [7]. *Let $(V(G), E(G), c)$ be an edge-colored graph of order $n \geq 3$. If*

$$\delta^c(G) > \frac{n}{2},$$

then G contains a rainbow triangle.

Li et al. [6] improved Theorem A as follows.

Theorem B [6]. *Let $(V(G), E(G), c)$ be an edge-colored graph of order $n \geq 3$ satisfying one of the following conditons:*

(*i*) $\displaystyle\sum_{u \in V(G)} d^c(u) \geq \frac{n(n+1)}{2}$,

(*ii*) $\delta^c(G) \geq \dfrac{n}{2}$ and $G \notin \{K_{n/2,n/2}, K_4, K_4 - e\}$.

Then G contains a rainbow triangle.

Li [7] also gives the following condition for the existence of a rainbow 4-cycle in a balanced bipartite graph.

Theorem C [7]. *Let $(V(G), E(G), c)$ be an edge-colored balanced bipartite graph of order $2n$. If*

$$\delta^c(G) > \frac{3n}{5} + 1,$$

then G contains a rainbow C_4.

The analogous question in edge-colored triangle-free (not necessarily bipartite) graphs was considered by Wang et al. [9].

Theorem D [9]. *Let $(V(G), E(G), c)$ be an edge-colored triangle-free graph of order $n \geq 9$. If*

$$\delta^c(G) \geq \frac{3 - \sqrt{5}}{2}\, n + 1,$$

then G contains a rainbow C_4.

Note that, in Theorem D, $\frac{3-\sqrt{5}}{2} \doteq 0.382$.

Finally, for long rainbow cycles, we know that there is a rainbow cycle of length at least $\frac{2}{n-1}\, c(G)$ if $c(G) \geq n$, as mentioned before. Li and Wang [8] showed the following.

Theorem E [8]. *Let $(V(G), E(G), c)$ be an edge-colored triangle-free graph of order $n \geq 8$. If*

$$\delta^c(G) \geq \frac{3}{4}\, n + 1,$$

then G contains a rainbow cycle of length at least $\delta^c(G) - \frac{3}{4}\, n + 2$.

Note that, for the existence of a rainbow cycle of length at least four, Theorem E gives a sufficient condition $\delta^c(G) \geq \frac{3}{4}\, n + 2$.

In the present paper, we prove the following two results.

- In Section 2, we give our first main result, Theorem 1, which strengthens Theorem D by showing the following.

  If $(V(G), E(G), c)$ is an edge-colored triangle-free graph of order n such that $\delta^c(G) > \frac{n}{3} + 1$, then G contains a rainbow 4-cycle.

- Our second main result, Theorem 3 in Section 3, strengthens Theorem E in the special case of a cycle of length at least 4 by showing that even assumptions similar to those of Theorem A are sufficient for the existence of a rainbow cycle of length at least 4. Namely, we show the following.

  If $(V(G), E(G), c)$ is an edge-colored graph of order n such that $\delta^c(G) > \frac{n}{2} + 2$, then G contains a rainbow cycle of length at least four.

Finally, in Section 4, we give a conjecture concerning a possible generalization of Theorem 3.

2. Rainbow C_4's in triangle-free graphs

The main result of this section, Theorem 1, is a strengthening of Theorem D (note that, in Theorem D, $\frac{3-\sqrt{5}}{2} \doteq 0,382$).

Theorem 1. *Let $(V(G), E(G), c)$ be an edge-colored graph of order n. If G is triangle-free and*

$$\delta^c(G) > \frac{n}{3} + 1,$$

then G contains a rainbow 4-cycle.

For the proof of Theorem 1, we will need the following lemma.

Lemma 2. *Let $(V(G), E(G), c)$ be an edge-colored graph containing no rainbow 4-cycle and let $\{xy_iz\}_{i=1}^p$ be a set of rainbow (x,z)-paths of length two in G. If $\{xy_i\}_{i=1}^p$ is rainbow, then $|\{c(y_iz)\}_{i=1}^p| \leq 3$.*

Proof. Suppose, to the contrary, that $|\{c(y_iz)\}_{i=1}^p| \geq 4$, choose the notation such that $|\{c(y_iz)\}_{i=1}^4| = 4$, and set $c(y_iz) = a_i$, $i = 1, 2, 3, 4$.

Since the paths xy_1z and xy_2z are rainbow, $c(xy_1) \neq a_1$ and $c(xy_2) \neq a_2$. Since the cycle $C = xy_1zy_2x$ cannot be a rainbow C_4, we have $c(xy_1) = a_2$ or $c(xy_2) = a_1$. Symmetrically, $c(xy_3) = a_4$ or $c(xy_4) = a_3$. Thus, we have the following 4 possibilities.

Case	Rainbow C_4
$c(xy_1) = a_2$, $c(xy_3) = a_4$	xy_1zy_3x
$c(xy_1) = a_2$, $c(xy_4) = a_3$	xy_1zy_4x
$c(xy_2) = a_1$, $c(xy_3) = a_4$	xy_2zy_3x
$c(xy_2) = a_1$, $c(xy_4) = a_3$	xy_2zy_4x

In each of the cases, we have obtained a contradiction. Hence $|\{c(y_iz)\}_{i=1}^p| \leq 3$. ■

Proof of Theorem 1. Let G be a graph satisfying the assumptions of Theorem 1 and suppose, to the contrary, that G contains no rainbow C_4. Let $x^0 \in V(G)$, and let $N_1 \subset N_G(x^0)$ be such that $|N_1| = |c(x^0, N_1)| = \delta^c$. Set $N_1 = \{x_1^1, \ldots, x_{\delta^c}^1\}$. Similarly, let $N_2 \subset N_G(x_{\delta^c}^1) \setminus \{x^0\}$ be such that $|N_2 \cup \{x^0\}| = |c(x_{\delta^c}^1, N_2 \cup \{x^0\})| = \delta^c$ (hence $|N_2| = \delta^c - 1$), and set $N_2 = \{x_1^2, \ldots, x_{\delta^c-1}^2\}$. Since G is simple and triangle-free, $N_2 \cap (N_1 \cup \{x^0\}) = \emptyset$.

Now, let $H \subset G$ be the graph with $V(H) = \{x^0\} \cup N_1 \cup N_2$ and with $E(H) = E(x^0, N_1) \cup \{x_i^1x_j^2 \in E(N_1, N_2) : c(x_i^1x_j^2) \neq c(x^0x_i^1)\}$. Note that, by the definition of H, $E(x_{\delta^c}^1, N_2) \subset E(H)$, and, for every $x_j^2 \in N_2$, any (x^0, x_j^2)-path of length 2 in H is rainbow. Also note that each of the sets N_1, N_2 is independent since G is triangle-free. Finally, let $M \subset G$ be the graph with $V(M) = N_1 \cup N_2$ and with $E(M) = \{x_i^1x_j^2 \in E(N_1, N_2) : c(x_i^1x_j^2) = c(x^0x_i^1)\}$.

We distinguish two cases.

Case 1: *there is a vertex* $x^2_{j_0} \in N_2$ *such that* $d_M(x^2_{j_0}) = 0$. Then $E(x^2_{j_0}, N_1) \subset E(H)$ and hence, by Lemma 2, we have $|c(x^2_{j_0}, N_1)| \le 3$. Obviously $x^2_{j_0}x^0 \notin E(G)$ since G is triangle-free. Hence there is a set $N_3 \subset N_G(x^2_{j_0}) \setminus (\{x^0\} \cup N_1)$ such that $|c(x^2_{j_0}, N_3)| \ge \delta^c - 3$, implying $|N_3| \ge \delta^c - 3$. Since N_2 is independent, $N_3 \cap N_2 = \emptyset$. Hence we have

$$n \ge 1 + |N_1| + |N_2| + |N_3| = 1 + \delta^c + (\delta^c - 1) + (\delta^c - 3) = 3\delta^c - 3,$$

from which

$$\delta^c \le \frac{n}{3} + 1,$$

a contradiction.

Case 2: *for every* $x^2_j \in N_2$, $d_M(x^2_j) \ge 1$. Set $N_1^M = \{x^1_i \in N_1 : d_M(x^1_i) \ge 1\}$. Since $|N_1| = \delta^c$ and $d_M(x^1_{\delta^c}) = 0$, we have $|N_1^M| \le \delta^c - 1$. Recall that $|N_2| = \delta^c - 1$. Now, if $d_M(x^1_i) < d_M(x^2_j)$ for every edge $x^1_i x^2_j \in E(M)$, then $|N_1^M| > |N_2|$, a contradiction (recall that M is a bipartite graph). Hence there is an edge $x^1_{i_0}x^2_{j_0} \in E(M)$ such that $d_M(x^1_{i_0}) \ge d_M(x^2_{j_0})$.

Set $d_M(x^1_{i_0}) = k$. Then $|N_H(x^1_{i_0}) \cap N_2| \le (\delta^c - 1) - k$, i.e., there are at most $\delta^c - k - 1$ edges from $x^1_{i_0}$ to N_2 in the subgraph H. Since all edges in M incident to $x^1_{i_0}$ have the same color $c(x^0 x^1_{i_0})$ and since N_1 is independent, we have $|c(x^1_{i_0}, (\{x^0\} \cup N_1 \cup N_2))| \le \delta^c - k$. Hence there is a set $N_2' \subset N_G(x^1_{i_0}) \setminus (\{x^0\} \cup N_1 \cup N_2)$ such that $|c(x^1_{i_0}, N_2')| \ge k$, implying $|N_2'| \ge k$.

Now we consider colors at the vertex $x^2_{j_0}$. Since $d_M(x^2_{j_0}) \le d_M(x^1_{i_0}) = k$, there are at most k edges in M incident to $x^2_{j_0}$. These edges in M incident to $x^2_{j_0}$ can have at most k distinct colors, and, by Lemma 2, the edges in H incident to $x^2_{j_0}$ can have at most 3 distinct colors. Thus, we have $|c(x^2_{j_0}, N_1)| \le k + 3$. Since $x^0 x^2_{j_0} \notin E(G)$, and since N_2 is independent, there is a set $N_3 \subset N_G(x^2_{j_0}) \setminus (\{x^0\} \cup N_1 \cup N_2)$ such that $|c(x^2_{j_0}, N_3)| \ge \delta^c - k - 3$, implying $|N_3| \ge \delta^c - k - 3$.

Finally, we observe that $N_2' \cap N_3 = \emptyset$ since $x^1_{i_0}x^2_{j_0} \in E(G)$ and G is triangle-free. Hence we have

$$n \ge 1 + |N_1| + |N_2| + |N_3| + |N_2'| \ge 1 + \delta^c + (\delta^c - 1) + (\delta^c - k - 3) + k = 3\delta^c - 3,$$

from which

$$\delta^c \le \frac{n}{3} + 1,$$

a contradiction. ■

3. Rainbow cycles of length at least 4

The following theorem is our second main result.

Theorem 3. *Let $(V(G), E(G), c)$ be an edge-colored graph of order n. If*

$$\delta^c(G) > \frac{n}{2} + 2,$$

then G contains a rainbow cycle of length at least four.

For the proof of Theorem 3, we will need the following lemma.

Lemma 4. *Let $(V(G), E(G), c)$ be an edge-colored graph and let $P = u_1 \dots u_p$ be a longest rainbow path in G. If G has no rainbow cycle of length at least k ($k \le |P|$), then for any positive integers s, t such that $s + t = k$,*

$$|c(u_1, u_k P u_{p-(t-1)}) \cap c(u_p, u_s P u_{p-(k-1)})| \le 1.$$

Proof. Suppose that G contains no rainbow cycle of length at least k.

Claim 1. *For any $a \in c(u_1, u_k P u_p)$ and $u_i \in V(u_k P u_p)$ such that $c(u_1 u_i) = a$, there is an edge $e \in E(u_1 P u_i)$ such that $c(e) = a$.*

Proof. If there is no edge of color a in $u_1 P u_i$, then $u_1 P u_i u_1$ is a rainbow cycle of length at least k, a contradiction. □

Note that also, by symmetry, for any $a \in c(u_p, u_1 P u_{p-(k-1)})$ and $u_i \in V(u_1 P u_{p-(k-1)})$ such that $c(u_p u_i) = a$, there is an edge $e \in E(u_i P u_p)$ such that $c(e) = a$.

Suppose now that $c(u_1, u_k P u_{p-(t-1)}) \cap c(u_p, u_s P u_{p-(k-1)})$ contains two colors a_1, a_2. Let $u_\ell \in V(u_k P u_{p-(t-1)})$ and $u_{\ell'} \in V(u_s P u_{p-(k-1)})$ be such that $c(u_1 u_\ell) = c(u_p u_{\ell'}) = a_1$. Similarly, let $u_m \in V(u_k P u_{p-(k-1)})$ and $u_{m'} \in V(u_s P u_{p-(k-1)})$ be such that $c(u_1 u_m) = c(u_p u_{m'}) = a_2$. By Claim 1, obviously $\ell' < \ell$ and $m' < m$. Note that $\ell \ne m$ and $\ell' \ne m'$ since the edges $u_1 u_\ell$ and $u_1 u_m$, or $u_p u_{\ell'}$ and $u_p u_{m'}$, respectively, must have different colors by definition.

By symmetry, we can suppose that $\ell' < m'$. Note that, by Claim 1, there is an edge e_1 in $u_{\ell'} P u_\ell$ with $c(e_1) = a_1$, and there is an edge e_2 in $u_{m'} P u_m$ with $c(e_2) = a_2$.

If $\ell < m$, then the cycle $u_1 P u_{\ell'} u_p P u_m u_1$ is a rainbow cycle of length at least k (see Fig. 1), a contradiction. Hence $m < \ell$, i.e., $\ell' < m' < m < \ell$. But now, if $e_1 \in E(u_{\ell'} P u_m)$, then $u_1 P u_{\ell'} u_p P u_m u_1$ is a rainbow cycle of length at least k (see Fig. 2a), a contradiction, and if $e_1 \in E(u_m P u_\ell)$, then $u_1 u_\ell P u_p u_{m'} P u_1$ is a rainbow cycle of length at least k (see Fig. 2b), a contradiction again. Thus, $|c(u_1, u_k P u_{p-(t-1)}) \cap c(u_p, u_s P u_{p-(k-1)})| \le 1$. ■

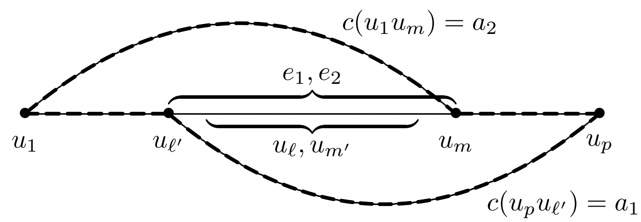

Figure 1: The case $\ell' < m'$ and $\ell < m$

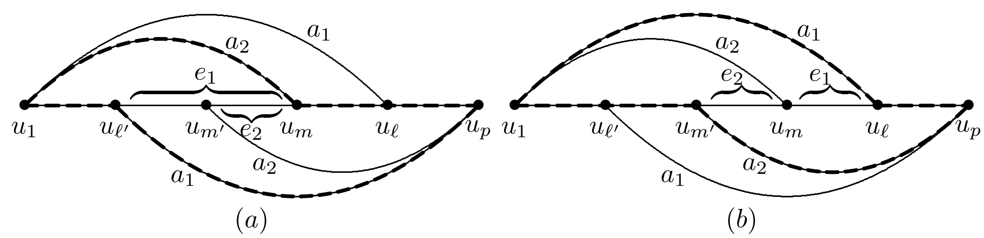

Figure 2: The case $\ell' < m'$ and $m < \ell$

Proof of Theorem 3. Let G be a graph satisfying the assumptions of Theorem 3. Suppose to the contrary, that G contains no rainbow cycle of length at least 4.

Set

$$A = c(u_1, u_4Pu_{p-1}),\ B = c(u_p, u_2Pu_{p-3})$$

and

$$C_0 = (c(u_1, P^C) \setminus c(u_1, P)) \cap (c(u_p, P^C) \setminus c(u_p, P)).$$

Note that $C_0 \cap (A \cup B) = \emptyset$ and for any $a \in C_0$, there is an edge in P whose color is a (since otherwise P is not a longest rainbow path).

It is easy to observe that $c(u_1u_2) \notin B$ and $c(u_pu_{p-1}) \notin A$ (see Claim 1 in the proof of Lemma 4). However, possibly $c(u_1u_2) \in A$ or $c(u_pu_{p-1}) \in B$. Set

$$\varepsilon_1 = \begin{cases} 1 & \text{if } c(u_1u_2) \notin A, \\ 0 & \text{if } c(u_1u_2) \in A; \end{cases}$$

$$\varepsilon_2 = \begin{cases} 1 & \text{if } c(u_pu_{p-1}) \notin B, \\ 0 & \text{if } c(u_pu_{p-1}) \in B; \end{cases}$$

$$\varepsilon_1' = \begin{cases} 1 & \text{if } c(u_1u_p) \notin A \cup \{c(u_1u_2)\}, \\ 0 & \text{otherwise;} \end{cases}$$

$$\varepsilon_2' = \begin{cases} 1 & \text{if } c(u_1u_p) \notin B \cup \{c(u_pu_{p-1})\}, \\ 0 & \text{otherwise.} \end{cases}$$

If $u_1u_p \notin E_G$, then we define $\varepsilon_1' = \varepsilon_2' = 0$. By Lemma 1, $|A \cap B| \leq 1$. Hence

$$\begin{aligned} |E(P)| &\geq |\{e \in E_P : c(e) \in A \cup B\}| + \varepsilon_1 + \varepsilon_2 + \varepsilon_1'\varepsilon_2' \\ &\quad + |\{e \in E_P : c(e) \in C_0\}| \\ &= |A| + |B| - 1 + |C_0| + \varepsilon_1 + \varepsilon_2 + \varepsilon_1'\varepsilon_2'. \end{aligned}$$

Set

$$\begin{aligned} C_1 &= \{c_1, c_2, \ldots, c_{|C_1|}\} = c(u_1, P^C) \setminus (C_0 \cup c(u_1, P)), \\ C_2 &= \{c_1', c_2', \ldots, c_{|C_2|}'\} = c(u_p, P^C) \setminus (C_0 \cup c(u_p, P)). \end{aligned}$$

Let $x_i \in N_{P^C}(u_1)$ be such that $c(u_1x_i) = c_i$ and $y_i \in N_{P^C}(u_p)$ such that $c(u_py_i) = c_i'$. Obviously,

$$|\{x_1, \ldots, x_{|C_1|}\} \cap \{y_1, \ldots, y_{|C_2|}\}| \leq 1,$$

for otherwise there is a rainbow 4-cycle. Therefore,

$$|V(P^C)| \geq |C_0| + |C_1| + |C_2| - 1.$$

Since $|c(u_1, P) \setminus (A \cup \{c(u_1u_2)\})| \leq 2$ and $|c(u_p, P) \setminus (B \cup \{c(u_pu_{p-1})\})| \leq 2$, we have

$$\begin{aligned} |A| + |C_0| + |C_1| + \varepsilon_1 + \varepsilon_1' + 1 &\geq d^c(u_1) \geq \delta^c, \\ |B| + |C_0| + |C_2| + \varepsilon_2 + \varepsilon_2' + 1 &\geq d^c(u_p) \geq \delta^c. \end{aligned}$$

Summarizing, we have

$$|V(P)| = |E(P)| + 1 \geq |A| + |B| + |C_0| + \varepsilon_1 + \varepsilon_2 + \varepsilon_1'\varepsilon_2'$$

and

$$|V(P^C)| \geq |C_0| + |C_1| + |C_2| - 1,$$

from which

$$\begin{aligned} n &\geq |V(P)| + |V(P^C)| \geq |A| + |C_0| + |C_1| + \varepsilon_1 + \varepsilon_1'\varepsilon_2' + |B| + |C_0| + |C_2| + \varepsilon_2 - 1 \\ &\geq 2\delta^c - 4. \end{aligned}$$

This implies $\delta^c(G) \leq \frac{n}{2} + 2$, a contradiction. ■

4. Concluding remarks

1. Our results, although improving known results, are still far from the potential sharpness examples mentioned in the introduction. Therefore, we do not know whether our results are sharp.

2. We believe that Theorem 3 could be possibly generalized. We propose the following conjecture.

Conjecture 5. *Let $(V(G), E(G), c)$ be an edge-colored graph of order n and let k be a positive integer. If $\delta^c(G) \geq \frac{n+k}{2}$, then G contains a rainbow cycle of length at least k.*

We include here two examples of colorings of small complete graphs supporting the conjecture. Let $V(K_n) = \{u_1, \ldots, u_n\}$.

For K_4 consider the following edge-coloring: decompose K_4 into three perfect matchings and color them with different colors. Thus $\delta^c(K_4) = 3$.

For K_6 color edges of two 3-cycles $u_1u_3u_5u_1$ and $u_2u_4u_6u_2$ with the same color. Decompose all the remaining edges into three perfect matchings (arbitrarily) and color these matchings with different colors. Thus $\delta^c(K_6) = 4$.

Note that these graphs have then $\delta^c(G) = \frac{n+k-2}{2}$ for $k = 4$. On the other hand, in each of them every cycle of length at least 4 is not rainbow.

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