Finite families of forbidden subgraphs for rainbow connection in graphs

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Abstract

A connected edge-colored graph G is rainbow-connected if any two distinct vertices of G are connected by a path whose edges have pairwise distinct colors; the rainbow connection number $\mathrm{rc}(G)$ of G is the minimum number of colors such that G is rainbow-connected. We consider families $\mathcal{F}$ of connected graphs for which there is a constant $k_{\mathcal{F}}$ such that, for every connected $\mathcal{F}$-free graph G, $\mathrm{rc}(G) \leq \mathrm{diam}(G) + k_{\mathcal{F}}$, where $\mathrm{diam}(G)$ is the diameter of G. In the paper, we finalize our previous considerations and give a complete answer for any finite family $\mathcal{F}$.

1 Introduction

We consider finite and simple graphs only, and for terminology and notation not defined here we refer to [3]. To avoid trivial cases, all graphs considered will be connected with at least one edge.

An edge-colored connected graph G is said to be *rainbow-connected* if each pair of distinct vertices of G is connected by a rainbow path, i.e., by a path whose edges have pairwise distinct colors. Note that the edge coloring need not be proper. The *rainbow connection number* of G, denoted by $\mathrm{rc}(G)$, is the minimum number of colors such that G is rainbow-connected.

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The concept of rainbow connection was introduced by Chartrand et al. [7]. It is easy to observe that, for any graph G, $\mathrm{rc}(G) \leq |V(G)| - 1$, since we can color the edges of a given spanning tree of G with different colors, and the remaining edges with one of the already used colors. In [7], the exact values of $\mathrm{rc}(G)$ were determined for several graph classes. The rainbow connection number has been studied for further graph classes in [4, 8, 10, 13] and for graphs with fixed minimum degree in [4, 11, 17]. The results are surveyed in [14] and in [15].

In [5, 12], it was shown that it is NP-hard to determine the exact value of $\mathrm{rc}(G)$. In fact, it is already NP-complete to decide whether $\mathrm{rc}(G) = 2$, and it is also NP-complete to decide whether a given edge-colored graph (with an unbounded number of colors) is rainbow-connected [5]. More generally, it has been shown in [12] that for any fixed $k \geq 2$, it is NP-complete to decide whether $\mathrm{rc}(G) = k$.

For the rainbow connection numbers of graphs the following results are known (and obvious).

Proposition A. *Let G be a connected graph of order n. Then*

(*i*) $1 \leq \mathrm{rc}(G) \leq n - 1$,

(*ii*) $\mathrm{rc}(G) \geq \mathrm{diam}(G)$,

(*iii*) $\mathrm{rc}(G) = 1$ *if and only if G is complete,*

(*iv*) $\mathrm{rc}(G) = n - 1$ *if and only if G is a tree.*

Note that the difference $\mathrm{rc}(G) - \mathrm{diam}(G)$ can be arbitrarily large. For $G \simeq K_{1,n-1}$ we have $\mathrm{rc}(K_{1,n-1}) - \mathrm{diam}(K_{1,n-1}) = (n-1) - 2 = n - 3$. Especially, each bridge requires a single color.

For connected bridgeless graphs, there is the following upper bound on $\mathrm{rc}(G)$, however, note that this bound is quadratic in terms of $\mathrm{rad}(G)$, and, since there is a constant c such that $c \cdot \mathrm{rad}(G) \geq \mathrm{diam}(G)$, also in $\mathrm{diam}(G)$.

Theorem B [1]. *For every connected bridgeless graph G with radius r,*

$$\mathrm{rc}(G) \leq r(r+2).$$

Moreover, for every integer $r \geq 1$, there exists a bridgeless graph G with radius r and $\mathrm{rc}(G) = r(r+2)$.

Let $\mathcal{F}$ be a family of connected graphs. We say that a graph G is $\mathcal{F}$*-free* if G does not contain an induced subgraph isomorphic to a graph from $\mathcal{F}$. Specifically, for $\mathcal{F} = \{X\}$, we say that G is X*-free*, and for $\mathcal{F} = \{X_1, \ldots, X_k\}$, we say that G is $(X_1, \ldots, X_k)$*-free.* The members of $\mathcal{F}$ will be referred to in this context as *forbidden induced subgraphs.* If $\mathcal{F} = \{X_1, \ldots, X_k\}$, we will also refer to the graphs $X_1, \ldots, X_k$ as a *forbidden k-tuple*, and for $|\mathcal{F}| = 2$, 3 and 4, we will also speak about *forbidden pair*, *forbidden triple* and *forbidden quadruple*, respectively.

Graphs characterized in terms of forbidden induced subgraphs are known to have many interesting properties. Although, as we know from Theorem B, $\mathrm{rc}(G)$ can be quadratic in terms of $\mathrm{diam}(G)$ even in bridgeless graphs, it turns out that the upper bound on $\mathrm{rc}(G)$ in terms of $\mathrm{diam}(G)$ can be remarkably lowered under forbidden subgraph conditions.

In [9], the authors considered the question for which families $\mathcal{F}$ of connected graphs, a connected $\mathcal{F}$-free graph satisfies $\mathrm{rc}(G) \leq \mathrm{diam}(G) + k_{\mathcal{F}}$, where $k_{\mathcal{F}}$ is a constant (depending on $\mathcal{F}$), and gave a complete answer for $1 \leq |\mathcal{F}| \leq 2$ by the following two results (where N denotes the *net*, i.e. the graph obtained by attaching a pendant edge to each vertex of a triangle).

Theorem C [9]. *Let X be a connected graph. Then there is a constant k_X such that every connected X-free graph G satisfies $\mathrm{rc}(G) \leq \mathrm{diam}(G) + k_X$, if and only if $X = P_3$.*

Theorem D [9]. *Let X, Y be connected graphs, $X, Y \neq P_3$. Then there is a constant k_{XY} such that every connected (X, Y)-free graph G satisfies $\mathrm{rc}(G) \leq \mathrm{diam}(G) + k_{XY}$, if and only if (up to a symmetry) either $X = K_{1,r}$, $r \geq 4$ and $Y = P_4$, or $X = K_{1,3}$ and Y is an induced subgraph of N.*

In this paper, we will consider an analogous question for a finite family $\mathcal{F}$ with $|\mathcal{F}| \geq 3$. Namely, we will consider the following question.

For which finite families $\mathcal{F} = \{X_1, X_2, \ldots, X_k\}$ (where $k \geq 3$ is an integer) of connected graphs, there is a constant $k_{\mathcal{F}}$ such that a connected graph G being $\mathcal{F}$-free implies $\mathrm{rc}(G) \leq \mathrm{diam}(G) + k_{\mathcal{F}}$?

We give a complete characterization for $|\mathcal{F}| = 3$ in Theorem 1, for $|\mathcal{F}| = 4$ in Theorem 9, and for an arbitrary finite family $\mathcal{F}$ in Theorem 10.

2 Preliminary results

In this section we summarize some further notations and facts that will be needed for the proofs of our results.

If G is a graph and $A \subset V(G)$, then $G[A]$ denotes the subgraph of G induced by the vertex set A, and $G - A$ the graph $G[V(G) \setminus A]$. Specifically, for $x \in V(G)$, $G - x$ is the graph $G[V(G) \setminus \{x\}]$, and for $e \in E(G)$, $G - e$ is the graph obtained from G by deleting the edge e. An edge $e \in E(G)$ such that $G - e$ is disconnected is called a *bridge*, and a graph with no bridges is called a *bridgeless graph*. An edge such that one of its vertices has degree one is called a *pendant edge*. For $x, y \in V(G)$, a path in G from x

to y will be referred to as an *(x, y)-path*, and, whenever necessary, it will be considered with orientation from x to y. For a subpath of a path P with origin u and terminus v (also referred to as a *$(u, v)-$arc of P*), we will use the notation uPv. Similarly, if C is a cycle with a fixed orientation, then uCv denotes the arc of C with origin u and terminus v, in the given orientation of C. If x is a vertex of a path or of a cycle (with a fixed orientation), then x^- and x^+ denote its predecessor and successor, respectively.

For graphs X and G, we write $X \subset G$ if X is a subgraph of G, $X \overset{\text{IND}}{\subset} G$ if X is an induced subgraph of G, and $X \simeq G$ if X and G are isomorphic. We use $d_G(x)$ for the degree of a vertex x, and, for two vertices $x, y \in V(G)$, we denote by $\text{dist}(x, y)$ the distance between x and y in G. The diameter and the radius of a graph G will be denoted by $\text{diam}(G)$ and $\text{rad}(G)$, respectively. A shortest path joining two vertices at distance $\text{diam}(G)$ will be referred to as a *diameter path*. We use $\alpha(G)$ for the independence number of G, $\overline{G}$ for the complement of a graph G, $\delta(G)$ for the minimum degree of G, and $\overline{\delta}(G)$ for the average degree of G (i.e., $\overline{\delta}(G) = \frac{1}{|V(G)|}\sum_{x \in V(G)} d_G(x)$). Throughout the paper, $\mathbb{N}$ denotes the set of all positive integers.

For a set $S \subset V(G)$ and $k \in \mathbb{N}$, the *neighborhood at distance* k of S is the set $N_G^k(S)$ of all vertices of G at distance k from S. In the special case when $k = 1$, we simply write $N_G(S)$ for $N_G^1(S)$, and if $|S| = 1$ with $x \in S$, we write $N_G(x)$ for $N_G(\{x\})$. For a set $M \subset V(G)$, we set $N_M(S) = N_G(S) \cap M$ and $N_M(x) = N_G(x) \cap M$, and for a subgraph $P \subset G$, we write $N_P(x)$ for $N_{V(P)}(x)$. Finally, we will also use the *closed neighborhood* of a vertex $x \in V(G)$ defined by $N_G^k[x] = (\cup_{i=1}^k N_G^i(x)) \cup \{x\}$ and of a subgraph $P \subset G$ defined by $N_G^k[P] = \cup_{i=1}^k N_G^i(V(P)) \cup V(P)$.

A set $D \subset V(G)$ is *dominating* if every vertex in $V(G) \setminus D$ has a neighbor in D. A dominating set D in a graph G is called a *two-way dominating set* if D includes all vertices of G of degree 1. In addition, if $G[D]$ is connected, we call D a *connected two-way dominating set*. Note that if $\delta(G) \geq 2$, then every (connected) dominating set in G is a (connected) two-way dominating set.

Theorem E [6]. *If D is a connected two-way dominating set in a graph G, then* $\text{rc}(G) \leq \text{rc}(G[D]) + 3$.

We also recall the famous theorem by Ramsey [16].

Theorem F [16]. *For every $a, b \in \mathbb{N}$ there exists a positive integer n such that every graph of order at least n contains either K_a or $\overline{K_b}$ as an induced subgraph.*

The smallest integer n associated with a, b as in Theorem F is called the *Ramsey number $R(a, b)$ of a, b*.

The following fact is an easy consequence of the Turán's theorem [18, 19]. For its proof see e.g. [2] (Chapter 13 "Stability number", page 280, Corollary 2 of Theorem 5).

Theorem G [18, 19, 2]. *Let G be a graph of order n and average degree $\overline{\delta}(G)$. Then $\alpha(G) \geq \frac{n}{\overline{\delta}(G)+1}$.*

3 Forbidden triples

For $i, j, k \in \mathbb{N}$, let $S_{i,j,k}$ denote the graph obtained by identifying one endvertex of three vertex disjoint paths of lengths i, j, k, $N_{i,j,k}$ the graph obtained by identifying each vertex of a triangle with an endvertex of one of three vertex disjoint paths of lengths i, j, k, and let K_t^h denote the graph obtained by attaching a pendant edge to every vertex of a complete graph K_t (see Fig. 1).

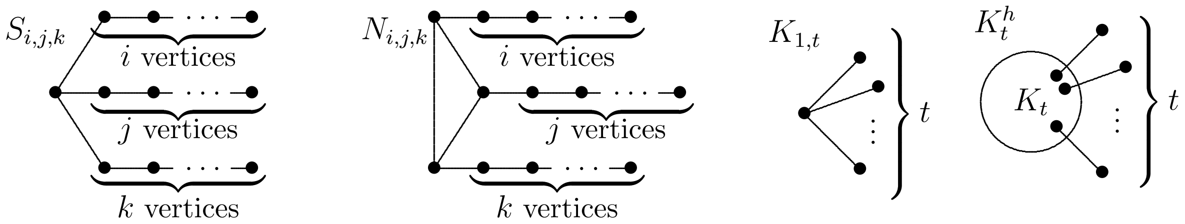

Figure 1: The graphs $S_{i,j,k}$, $N_{i,j,k}$, $K_{1,t}$ and K_t^h

For $k \in \mathbb{N}$, it is easy to see that if $\mathcal{F} = \{X_1, X_2, \ldots, X_k\}$ and $\mathcal{F}' = \{X_1', X_2, \ldots, X_k\}$, where $X_1 \overset{\text{IND}}{\subset} X_1'$, then every $\mathcal{F}$-free graph is also $\mathcal{F}'$-free. More generally, if $\mathcal{F}$, $\mathcal{F}'$ are finite families of connected graphs, we write $\mathcal{F} \overset{\text{IND}}{\subset} \mathcal{F}'$ if there is a bijection $\varphi : \mathcal{F} \to \mathcal{F}'$ such that $X \overset{\text{IND}}{\subset} \varphi(X)$ for any $X \in \mathcal{F}$. Clearly, if $\mathcal{F} \overset{\text{IND}}{\subset} \mathcal{F}'$, then every $\mathcal{F}$-free graph is also $\mathcal{F}'$-free. We set:

$\mathcal{F}_1 = \{\{P_3\}\}$,
$\mathcal{F}_2 = \{\{X, Y\} |\ \{X, Y\} \overset{\text{IND}}{\subset} \{K_{1,3}, N\}\}$,
$\mathcal{F}_3 = \{\{K_{1,r}, P_4\} |\ r \geq 4\}$.

In this notation, Theorems C and D can be equivalently reformulated as follows.

Let $\mathcal{F}$ be a finite family of connected graphs with $1 \leq |\mathcal{F}| \leq 2$. Then there is a constant $k_{\mathcal{F}}$ such that every connected $\mathcal{F}$-free graph satisfies $\mathrm{rc}(G) \leq \mathrm{diam}(G) + k_{\mathcal{F}}$, if and only if $\mathcal{F} \in \mathcal{F}_1$, $\mathcal{F} \in \mathcal{F}_2$ or $\mathcal{F} \in \mathcal{F}_3$.

Now we set

$\overline{\mathcal{F}_4} = \{\{K_{1,3}, K_s^h, N_{1,j,k}\} |\ s > 3, 1 \leq j \leq k, j + k > 2\}$,
$\overline{\mathcal{F}_5} = \{\{K_{1,r}, K_s^h, P_\ell\} |\ r > 3, s > 3, \ell > 4\}$,
$\overline{\mathcal{F}_6} = \{\{K_{1,r}, S_{1,j,k}, N\} |\ r > 3, 1 \leq j \leq k, j + k > 2\}$,

and

$\mathcal{F}_i = \{\{X, Y, Z\} |\ \{X, Y, Z\} \overset{\text{IND}}{\subset} \mathcal{F}$ for some $\mathcal{F} \in \overline{\mathcal{F}_i}\}$, $i = 4, 5, 6$.

The following statement, which is the main result of this section, gives a complete answer to the same question for $|\mathcal{F}| = 3$. We exclude the cases which are covered by Theorems C and D.

Theorem 1. *Let $\mathcal{F}$ be a finite family of connected graphs with $|\mathcal{F}| = 3$ such that $\mathcal{F} \not\supset \mathcal{F}'$ for any $\mathcal{F}' \in \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$. Then there is a constant $k_{\mathcal{F}}$ such that every connected $\mathcal{F}$-free graph satisfies $\mathrm{rc}(G) \leq \mathrm{diam}(G) + k_{\mathcal{F}}$, if and only if $\mathcal{F} \in \mathcal{F}_4 \cup \mathcal{F}_5 \cup \mathcal{F}_6$.*

Proof of Theorem 1 will be subdivided into several parts. In Proposition 2, we prove the necessity of the triples given by the conditions (i), (ii) and (iii) of Theorem 1. We then prove several auxiliary statements, and, using them, we establish sufficiency in Proposition 8.

Proposition 2. *Let $X, Y, Z \neq P_3$ be connected graphs, $\{X, Y, Z\} \not\supset \mathcal{F}'$ for any $\mathcal{F}' \in \mathcal{F}_2 \cup \mathcal{F}_3$, for which there is a constant k_{XYZ} such that every connected (X, Y, Z)-free graph G satisfies $\mathrm{rc}(G) \leq \mathrm{diam}(G) + k_{XYZ}$. Then the graphs X, Y, Z satisfy, up to a permutation, one of the following conditions:*

(i) $X = K_{1,3}$, $Y = K_s^h$ $(s > 3)$, $Z = N_{1,p,q}$ $(p + q > 2,\ 1 \leq p \leq q)$,

(ii) $X = K_{1,r}$ $(r > 3)$, $Y = K_s^h$ $(s > 3)$, $Z = P_\ell$ $(\ell > 4)$,

(iii) $X = K_{1,r}$ $(r > 3)$, $Y = S_{1,p,q}$ $(p + q > 2,\ 1 \leq p \leq q)$, $Z = N$.

Proof. Let $t_0 \geq 3$ and, for $t \geq t_0$, let (see Fig. 1 and Fig. 2):

- G_1^t be the star $K_{1,t}$;
- G_2^t be the graph K_t^h;
- $G_3^{i,t}$ $(i \geq 1)$ be the graph obtained from the path $P = x_0, x_1, \ldots, x_{(i+2)(t-1)+3}$ by adding t new vertices $y_0, \ldots, y_{t-1}$ with $N_P(y_j) = \{x_{(i+2)j+1}, x_{(i+2)j+2}\}$, and attaching a pendant edge to every y_j, $j = 0, \ldots, t-1$;
- $G_4^{i,t}$ $(i \geq 1)$ be the graph obtained from the path $P = x_0, x_1, x_2, \ldots x_{(t-1)i+2}$ by attaching a pendant edge to each of the vertices $\{x_1, x_{i+1}, x_{2i+1}, \ldots, x_{(t-1)i+1}\}$.

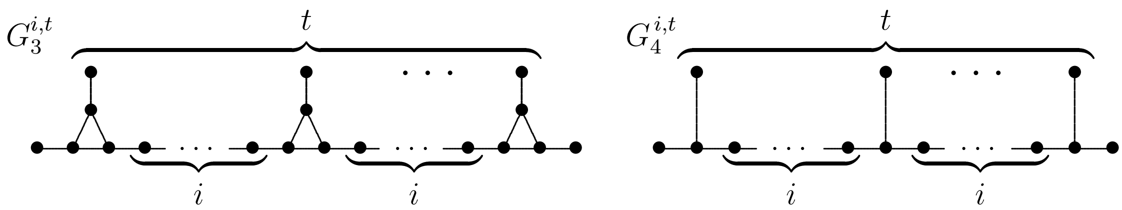

Figure 2: The graphs $G_3^{i,t}$ and $G_4^{i,t}$

For the graphs G_1^t and G_2^t, we have $\mathrm{diam}(G_1^t) = 2$, but $\mathrm{rc}(G_1^t) = t$, and $\mathrm{diam}(G_2^t) = 3$, but $\mathrm{rc}(G_2^t) = t+1$, respectively. For the graph $G_3^{i,t}$, we have $\mathrm{diam}(G_3^{i,t}) = (i+2)(t-1)+3$ but $\mathrm{rc}(G_3^{i,t}) \geq (i+2)(t-1)+3+t \geq \left(1 + \frac{t}{(i+2)(t-1)+3}\right) \mathrm{diam}(G_3^{i,t})$, since all edges of the path P must have distinct colors and none of the pendant edges can be colored with a color used on P. Analogously, for the graph $G_4^{i,t}$, we have $\mathrm{diam}(G_4^{i,t}) = (t-1)i+1$, while $\mathrm{rc}(G_4^{i,t}) = (t-1)i+1+t = \left(1+\frac{t}{(t-1)i+1}\right)\mathrm{diam}(G_4^{i,t})$. Thus, each of the graphs G_1^t, G_2^t, $G_3^{i,t}$, $G_4^{i,t}$ must contain an induced subgraph isomorphic to some of X, Y, Z.

Consider the graph $G_1^t = K_{1,t}$. Up to a symmetry, we have $X = K_{1,r}$ for some $r \geq 3$ (for $r \leq 2$, we get $X \overset{\text{IND}}{\subset} P_3$, which is excluded by the assumptions). Now we consider the graph G_2^t. Obviously, G_2^t is $K_{1,3}$-free, hence G_2^t does not contain X. Thus, up to a symmetry, G_2^t contains Y, implying $Y = K_s^h$ for some $s \geq 3$ (for $s \leq 2$ we get $Y \overset{\text{IND}}{\subset} P_4$, but the pair $X = K_{1,r}$, $Y = P_4$ is excluded by the assumptions).

Now consider the graph $G_3^{i,t}$. Clearly $X \overset{\text{IND}}{\not\subset} G_3^{i,t}$ since $G_3^{i,t}$ is $K_{1,3}$-free, hence $Y \overset{\text{IND}}{\subset} G_3^{i,t}$ or $Z \overset{\text{IND}}{\subset} G_3^{i,t}$.

<u>Case 1: $G_3^{i,t}$ contains Y.</u> Then $Y = N$. Since the pair $X = K_{1,3}$, $Y = N$ is excluded by the assumptions, $X = K_{1,r}$ with $r > 3$. Now we consider the graph $G_4^{i,t}$. Clearly $X \overset{\text{IND}}{\not\subset} G_4^{i,t}$ since $G_4^{i,t}$ is $K_{1,4}$-free, and if $Y \overset{\text{IND}}{\subset} G_4^{i,t}$, then we get $X = K_{1,r}$, $r > 3$, and $Y = P_4$, which is excluded by the assumptions. Thus, $G_4^{i,t}$ contains Z and we have $X = K_{1,r}$, $r > 3$, $Y = N$, and $Z = S_{1,p,q}$ $(1 \leq p \leq q)$. Since the pair $Y = N$, $Z = K_{1,3}$ is excluded by the assumptions, $q \geq 2$.

<u>Case 2: $G_3^{i,t}$ contains Z.</u> Then $Z = N_{1,p,q}$ for some $1 \leq p \leq q$. Consider the graph $G_4^{i,t}$. If $X \overset{\text{IND}}{\subset} G_4^{i,t}$, then we obtain $X = K_{1,3}$, $Y = K_s^h$ and $Z = N_{1,p,q}$ $(q \geq 2)$, since for $q = 1$, $(X, Z) = (K_{1,3}, N)$, which is excluded by the assumptions. If $Y \overset{\text{IND}}{\subset} G_4^{i,t}$, then $Y = P_4$ and we get $X = K_{1,r}$, $Y = P_4$ which is also excluded by the assumptions. Finally, if $G_4^{i,t}$ contains Z, then $Z = P_\ell$ $(\ell > 4)$, since for $\ell \leq 4$ we get $(X, Z) = (K_{1,r}, P_4)$, which is excluded by the assumptions. Hence we obtain $X = K_{1,r}$ $(r \geq 3)$, $Y = K_s^h$ $(s \geq 3)$ and $Z = P_\ell$ $(\ell > 4)$. But for $r \leq 3$, the triple $(K_{1,3}, K_s^h, P_\ell)$ is covered by the triple $(K_{1,3}, K_s^h, N_{1,p,q})$, hence $r > 3$ and, analogously, for $s \leq 3$, the triple $(K_{1,r}, N, P_\ell)$ is covered by the triple $(K_{1,r}, N, S_{1,p,q})$. Therefore we obtain $X = K_{1,r}$ $(r > 3)$, $Y = K_s^h$ $(s > 3)$ and $Z = P_\ell$ $(\ell > 4)$. ■

Now we prove several lemmas and propositions which will be needed for the proof of Proposition 8.

For $c \in V(G)$ and $\ell \in \mathbb{N}$, we set $\alpha_\ell(G, c) = \max\{|M| \mid M \subset N_G^\ell[c], M \text{ is independent}\}$.

Lemma 3. *Let $r, s, \ell \in \mathbb{N}$. Then there is a constant $\alpha(r, s, \ell)$ such that, for every $(K_{1,r}, K_s^h)$-free connected graph G and for every $c \in V(G)$, $\alpha_\ell(G, c) < \alpha(r, s, \ell)$.*

Proof. Let $r, s \in \mathbb{N}$. We prove the lemma by induction on ℓ.

For $\ell = 1$, we have $\alpha(r, s, 1) = r - 1$, for otherwise G contains $K_{1,r}$ as an induced subgraph.

Let, to the contrary, ℓ be the smallest integer for which $\alpha(r,s,\ell)$ does not exist (i.e., $\alpha_\ell(G,c)$ can be arbitrarily large), choose a graph G and a vertex $c \in V(G)$ such that $\alpha_\ell(G,c) \geq (r-2)R\big(s\,(2r-3)\,,\alpha(r,s,\ell-1)\big) + \alpha(r,s,\ell-1)$, and let $M \subseteq N_G^\ell[c]$ be an independent set in G of size $\alpha_\ell(G,c)$. Let $M^0 = \{x_1^0,...,x_k^0\}$ be a subset of M such that, for every $x \in M^0$, $\operatorname{dist}(c,x) = \ell$. Obviously, $|M^0| \geq (r-2)R\big(s\,(2r-3)\,,\alpha(r,s,\ell-1)\big)$. Let Q_i be a shortest (x_i^0,c)-path in G, $i = 1,\ldots,k$. Clearly, the length of each Q_i is ℓ, $i = 1,...,k$. We denote M^1 the set of all successors of the vertices from M^0 on Q_i, $i = 1,...,k$, and x_i^1 the successor of x_i^0 on Q_i (note that some distinct vertices in M^0 can have a common successor in M^1). Every vertex in M^1 has at most $r-2$ neighbors in M^0 since G is $K_{1,r}$-free. Thus, $|M^1| \geq \frac{|M^0|}{r-2} \geq R\big(s\,(2r-3)\,,\alpha(r,s,\ell-1)\big)$. By the induction assumption and by Theorem F, $\langle M^1\rangle_G$ contains a complete subgraph $K_{s(2r-3)}$ with $s(2r-3)$ vertices. Choose the notation such that $V(K_{s(2r-3)}) = \{x_1^1,...,x_{s(2r-3)}^1\}$, and set $\widetilde{M^0} = N_{M^0}(K_{s(2r-3)})$.

Now, using a matching between $K_{s(2r-3)}$ and $\widetilde{M^0}$, we can easily find in G a $K_{s(2r-3)}^h$ with vertices of degree 1 in $\widetilde{M^0}$; however, such a $K_{s(2r-3)}^h$ does not have to be induced. To reach a contradiction, we need to find an induced matching of size at least s between $K_{s(2r-3)}^h$ and $\widetilde{M^0}$. We define a digraph $\vec{H}$ with $V(\vec{H}) = \widetilde{M^0}$ and $E(\vec{H}) = \{x_i^0x_j^0 |\ x_j^1 \in N_{K_{s(2r-3)}}(x_i^0)\}$. Since G is $K_{1,r}$-free, $d_{\vec{H}}^-(x_i^0) \leq r-2$ (one neighbor is on Q_i) for every $x_i^0 \in \widetilde{M^0}$. Hence $\sum_{x_i^0 \in \widetilde{M^0}} d_{\vec{H}}^-(x_i^0) \leq |\widetilde{M^0}|(r-2)$. By the directed version of the "handshaking lemma", we have also $\sum_{x_i^0 \in \widetilde{M^0}} d_{\vec{H}}^+(x_i^0) \leq |\widetilde{M^0}|(r-2)$.

Let H be the symmetrization of $\vec{H}$. Then $\sum_{x_i^0 \in \widetilde{M^0}} d_H(x_i^0) \leq 2|\widetilde{M^0}|(r-2)$. Hence $\overline{\delta}(H) \leq 2(r-2)$. By Theorem G, $\alpha(H) \geq \frac{|V(H)|}{\overline{\delta}(H)+1} \geq \frac{|\widetilde{M^0}|}{1+2(r-2)} = \frac{|\widetilde{M^0}|}{2r-3}$, and since $|\widetilde{M^0}| \geq s(2r-3)$, we have $\alpha(H) \geq \frac{s(2r-3)}{2r-3} = s$. Let A denote an independent set in H with $|A| \geq s$. Then G contains an induced K_s^h with vertex set $V(K_{s(2r-3)}) \cup A$, a contradiction. ■

Lemma 4. *Let $r,s,\ell \in \mathbb{N}$. Then there is a constant $B(r,s,\ell)$ such that every connected $(K_{1,r},K_s^h)$-free graph G with $\operatorname{diam}(G) \leq \ell$ has at most $B(r,s,\ell)$ bridges.*

Proof. Let, to the contrary, G be a connected $(K_{1,r},K_s^h)$-free graph containing more than $\alpha(r,s,\operatorname{diam}(G)) \cdot \operatorname{diam}(G)$ bridges. Let B be the set of all bridges in $G, b = |B|$, and let $c \in V(G)$. For every bridge $e \in B$, take a shortest path P_e connecting e with c. Note that some path P_e can be a subpath of some other P_f, $e,f \in B$. Let $\mathcal{P}$ be the set of all maximal paths P_e, $e \in B$, under the ordering by inclusion. On each $P \in \mathcal{P}$, there are at most $\operatorname{diam}(G)$ bridges, implying that $|\mathcal{P}| \geq \frac{b}{\operatorname{diam}(G)}$. Clearly, the endvertices of all $P \in \mathcal{P}$ are independent, therefore $\alpha \geq \alpha(r,s,\operatorname{diam}(G))$, contradicting Lemma 3. ■

Proposition 5. *Let $r,s,\ell \in \mathbb{N}$. Then there is a constant $K(r,s,\ell)$ such that every connected $(K_{1,r},K_s^h)$-free graph G with $\operatorname{diam}(G) \leq \ell$ satisfies $\operatorname{rc}(G) \leq \operatorname{diam}(G) + K(r,s,\ell)$.*

Proof. If G is bridgeless, then $\mathrm{rc}(G) \leq \ell(\ell+2)$ by Theorem B. Thus, suppose that G contains a bridge, and let B be the set of all bridges in G. By Lemma 4, $|B| \leq B(r,s,\ell)$. By Theorem B, we can color every component of $G - B$ with at most $\ell(\ell+2)$ colors, and for the bridges we use other at most $B(r,s,\ell)$ colors. Therefore, we obtain $\mathrm{rc}(G) \leq (B(r,s,\ell)+1)\ell(\ell+2) + B(r,s,\ell) \leq \mathrm{diam}(G) + (B(r,s,\ell)+1)\ell(\ell+2) + B(r,s,\ell)$. ■

Let now G be an $(S_{1,p,q}, N_{1,p,q})$-free graph ($p \leq q$), let x, y be vertices at distance $\ell \geq p+q+1$, let $P : x = x_0, x_1, \ldots, x_\ell = y$ be a shortest (x,y)-path in G, and let $z \in V(G) \setminus V(P)$. We easily observe the following.

- If $|N_P(z)| = 1$, then z has no neighbor in $\{x_p, \ldots, x_{\ell-p}\}$ since G is $S_{1,p,q}$-free.
- If $|N_P(z)| \geq 2$ and $\{x_i, x_j\} \subset N_P(z)$, then $|i-j| \leq 2$, for otherwise there is an (x,y)-path in G shorter than P.

We will use the following notation:

- $M_i = \{z \in V(G) \setminus V(P) | N_P(z) = \{x_{i-1}, x_i\}\}$ for $1 \leq i \leq \ell$,
- $N_i = \{z \in V(G) \setminus V(P) | N_P(z) \supset \{x_{i-1}, x_{i+1}\}\}$ for $1 \leq i \leq \ell - 1$,
- $D_i = M_i \cup N_i \cup M_{i+1} \cup \{x_i\}$ for $i = 1, \ldots, \ell-1$.

We further set $S = V(P) \cup N(P)$, $R = V(G) \setminus S$, $\widehat{J} = \bigcup_{i=p+1}^{\ell-p-1} D_i$, and $R_{\widehat{J}} = G[V(G) \setminus \widehat{J}]$.

Lemma 6. *Let G be an $(S_{1,p,q}, N_{1,p,q})$-free graph, let x, y be vertices at distance $\ell \geq p+q+1$, and let $P : x = x_0, x_1, \ldots, x_\ell = y$ be a shortest (x,y)-path in G. Then*
(*a*) $N_R(M_i) = \emptyset$ *for* $i = p+1, \ldots, \ell-p$,
(*b*) $N_R(N_i) = \emptyset$ *for* $i = p, p+1, \ldots, \ell-p$,
(*c*) $N_{D_i}(D_j) = \emptyset$ *for* $|i-j| > 3$ *and* $N_{N_i}(N_j) = \emptyset$ *for* $|i-j| > 1$,
(*d*) $N_{R_{\widehat{J}}}(D_i) = \emptyset$ *for* $i = p+4, \ldots, \ell-p-4$.

Proof. Statement (a) follows from the fact that G is $N_{1,p,q}$-free, and (b) follows from the fact that G is $S_{1,p,q}$-free. To show (c), we observe that if there is a vertex $z \in N_{D_i}(D_j)$ for $|i-j| > 3$, or $z \in N_{N_i}(N_j)$ for $|i-j| > 1$, then there is an (x_0, x_ℓ)-path through z shorter than P, a contradiction. For proving (d), let, to the contrary, $z \in N_{R_{\widehat{J}}}(D_i)$. Then $\mathrm{dist}_G(z, P) = 1$, for otherwise we have a contradiction with (a), (b) or (c). But then there is an (x_0, x_ℓ)-path in G shorter than P, a contradiction again. ■

Proposition 7. *Let $r, s, p, q \in \mathbb{N}$. Then there is a constant $K(r,s,p,q)$ such that every connected $(K_{1,r}, K_s^h, S_{1,p,q}, N_{1,p,q})$-free graph G with* $\mathrm{diam}(G) \geq 3p+q+19$ *satisfies* $\mathrm{rc}(G) \leq \mathrm{diam}(G) + K(r,s,p,q)$.

Proof. Let $d = \text{diam}(G)$ and let $P : x_0, x_1, \ldots, x_d$ be a diameter path in G. Let $M_i, N_i, D_i, \widehat{J}$ be defined as above. We distinguish two possibilities.

Case 1: $\widehat{J}$ is a cutset of G. Then $G - \widehat{J}$ contains two components (since G is $S_{1,p,q}$-free). Denote A_0 the component containing x_0 and A_d the component containing x_d.

Obviously, $\text{rad}(A_0) \leq p+1$ and also $\text{rad}(A_d) \leq p+1$. By Proposition 5, there is a constant $K(r,s,\ell)$ such that $\text{rc}(A_0) \leq \text{diam}(A_0) + K(r,s,2(p+1)) \leq 2(p+1) + K(r,s,2(p+1))$ since $\text{diam}(A_0) \leq 2\ \text{rad}(A_0) \leq 2(p+1)$. Analogously, $\text{rc}(A_d) \leq 2(p+1) + K(r,s,2(p+1))$. Now we consider the graph $G[\widehat{J}] = G - (V(A_0) \cup V(A_d))$. By Lemma 6 (a), (b), the subpath $x_p P x_{d-p}$ of P is a two-way dominating set in $G[\widehat{J}]$ and hence $\text{rc}(G[\widehat{J}]) \leq \text{rc}(x_p P x_{d-p}) + 3 = d - 2p + 3$ by Theorem E. For the graph G we then have $\text{rc}(G) \leq \text{rc}(A_0) + \text{rc}(A_d) + \text{rc}(G[\widehat{J}]) \leq 2\big(2(p+1) + K(r,s,2(p+1))\big) + d - 2p + 3 = d + 2(p+1) + 2K(r,s,2(p+1)) + 5$, which completes the proof.

Case 2: $\widehat{J}$ is not a cutset of G. Let $J = \bigcup_{i=p+6}^{d-p-6} D_i$. Since $J \subset \widehat{J}$ and $\widehat{J}$ is not a cutset of G, J is not a cutset of G as well. Let $P' : x'_0 = x_0, x'_1, \ldots, x'_{d'} = x_d$ be a shortest (x_0, x_d)-path in $G - J$ (of length $d' \geq d$). For the path P' we define the sets M'_i, N'_i, D'_i, J' analogously as the sets M_i, N_i, D_i, J for P. Note that $J' \subset N_G(P')$. We further set $B_1 = D_{p+6} \cup D_{p+7} \cup D_{p+8}$, $B_2 = D_{d-p-6} \cup D_{d-p-7} \cup D_{d-p-8}$, and we define B'_1 and B'_2 analogously. Finally, we also denote $J_V = J \setminus \{B_1 \cup B_2\}$ and $J'_V = J' \setminus \{B'_1 \cup B'_2\}$.

Claim 1. *For every $x \in N_G(P')$ and every $y \in J$, $xy \notin E(G)$.*

Proof. Let, to the contrary, $x \in N_G(P')$ and $y \in J$ be vertices such that $xy \in E(G)$. Let $j < \frac{d}{2}$ be the maximal index of a vertex x_j on P such that $N_G(x_j)$ contains a vertex of P', and $j' < \frac{d'}{2}$ the maximal index of a vertex $x'_{j'}$ on P' such that $x'_{j'}$ has a neighbor on P. Let x'_a denote the neighbor of x_j on P' with maximal index, $x_{a'}$ the neighbor of $x'_{j'}$ on P with maximal index, and x_i the vertex of $P \cap P'$ with maximal index ($i \leq \frac{d}{2}$). By Lemma 6 (a) and (b), $a' \leq p$. Let P'' denote the path consisting of the subpath $x_i P' x'_a$ and the edge $x'_a x_j$.

Now we show that $j \leq p+1$. If $x'_a = x'_{j'}$, then, by Lemma 6, $N_R(x_j) = \emptyset$, implying $j \leq p$. Hence we suppose that $x'_a \neq x'_{j'}$. Since P is a shortest (x_0, x_d)-path in G, $\text{dist}_P(x_i, x_j) \leq \text{dist}_{P''}(x_i, x'_a) + 1$. Analogously, since P' is shortest in $G - J$, $\text{dist}_P(x_i, x_{a'}) + 1 \geq \text{dist}_{P'}(x_i, x'_a) + \text{dist}_{P'}(x'_a, x'_{j'}) = \text{dist}_{P''}(x_i, x'_a) + \text{dist}_{P'}(x'_a, x'_{j'})$. Comparing these two inequalities, we get

$$\text{dist}_P(x_i, x_{a'}) + 1 - \text{dist}_{P'}(x'_a, x'_{j'}) \geq \text{dist}_{P''}(x_i, x'_a) \geq \text{dist}_P(x_i, x_j) - 1,$$

from which $\text{dist}_P(x_i, x_{a'}) + 1 \geq \text{dist}_P(x_i, x_j)$ since $\text{dist}_{P'}(x'_a, x'_{j'}) \geq 1$. This implies that $a' + 1 \geq j$, and hence $j \leq p+1$.

Now, for $y \in J$, there is a vertex x_t on P such that y is a neighbor of x_t. Among all such vertices on P, we choose the vertex x_t with maximal index t. Since $y \in J$, we have $t \geq p+6$. By Lemma 6 (a) and (b), $\mathrm{dist}_G(x, P) \leq 1$, hence there is a vertex x_u on P such that $xx_u \in E(G)$. Choose x_u such that u is minimal. If $x_u \notin \widehat{J}$, then the path consisting of the arc x_0Px_u, the edges x_ux, xy, yx_t and the arc x_tPx_d is shorter than P, a contradiction.

Thus $x_u \in \widehat{J}$, implying that also $x \in \widehat{J}$. Since $x \in N_G(P')$, there is a vertex z on P' such that x is a neighbor of z. Since $x \in \widehat{J}$, $\mathrm{dist}_G(z, P) \leq 1$ by Lemma 6 (a) and (b). Let x_z denote a neighbor of z on P closest to x_0. We already know that $z \leq p+1$. But then the path consisting of the subpath x_0Px_z, the edges x_zz, zx, xy, yx_t and the subpath x_tPx_d is shorter than P, a contradiction. $\square$

<u>Claim 2.</u> *If $x \in J_V$ and $y \in V(G) \setminus J$, then $xy \notin E(G)$.*

<u>Proof.</u> Let, to the contrary, $xy \in E(G)$. By Lemma 6, we have $\mathrm{dist}_G(x, P) \leq 1$. Again by Lemma 6, $N_R(x) = \emptyset$, implying that $\mathrm{dist}_G(y, P) \leq 1$. Thus $\mathrm{dist}_G(y, P) = 1$, but then the path x_0PxyPx_d is an (x_0, x_d)-path in G which is shorter than P, a contradiction. $\square$

<u>Claim 3.</u> *The set $J \cup J'$ is a cutset of G.*

<u>Proof.</u> Suppose, to the contrary, that $J \cup J'$ is not a cutset of G, and let $P'' : x_0 = x''_0, x''_1, \ldots, x''_{d''} = x_d$ be a shortest (x_0, x_d)-path in $G - (J \cup J')$. For the path P'' we define the sets M''_i, N''_i, D''_i, J'' analogously as the sets M_i, N_i, D_i, J for P. Consider the subgraph S_{x_0} of G consisting of the paths x_0Px_{d-p-9} (a subpath of P), $x_0P'x'_{d'-p-9}$ (a subpath of P') and $x_0P''x''_{d''-p-9}$ (a subpath of P''). Note that the paths P, P' and P'' are not necessarily vertex-disjoint, hence the subgraph S_{x_0} is not necessarily induced in G, and can have several vertices of degree more than 2.

We take a minimal (with respect to the number of vertices) subgraph $S_{\min}$ of $G[S_{x_0}]$ such that $S_{\min}$ is isomorphic to S_{i_1,j_1,k_1} (for some $i_1, j_1, k_1 \in \mathbb{N}$), and contains the vertices x_{d-p-9}, $x'_{d'-p-9}$ and $x''_{d''-p-9}$. We show that $S_{\min}$ is induced. Suppose, to the contrary, that there is an edge uv in $G[V(S_{\min})]$ which is not an edge of $S_{\min}$. By the minimality of $S_{\min}$, both u and v are neighbors of the (only) vertex $z \in V(S_{\min})$ of degree 3 in $S_{\min}$. If there are two such edges (satisfying the conditions for uv), then we have a contradiction with the minimality of $S_{\min}$. Thus, there is exactly one such edge uv, but then there is an induced copy of N_{i_1-1,j_1-1,k_1}. By Claim 2, there are no edges between $P \cap J$ and $(P' \cap J') \cup (P'' \cap J'')$. Analogously, there are no edges between $P' \cap J'$ and $P'' \cap J''$ by Claim 2 used for $G - J$. Thus, since $d \geq 3p + q + 19$, G contains an induced copy of $N_{1,p,q}$, a contradiction. This contradiction shows that $S_{\min}$ is induced in G, but then, since $d \geq 3p + q + 19$, $S_{\min}$ contains an induced $S_{1,p,q}$, our final contradiction. $\square$

If $F \subset G$ is a cycle or a path, and $A : v_1Fv_2$ is an arc of F, then A is said to be J_V-*crossing* if $v_1 \in B_j$ and $v_2 \in B_{3-j}$ for some $j \in \{1, 2\}$, and $V(v_1^+Fv_2^-) \subset J_V$. We will use $j_V(F)$ to denote the number of internally vertex-disjoint J_V-crossing arcs of F. For the set J'_V, the concept of a J'_V-*crossing* arc and the number $j'_V(F)$ are defined analogously.

Let now C_0 be a shortest cycle in G such that $j_V(C_0)$ is odd (note that C_0 exists since the subgraph $G[V(P)\cup V(P')]$ certainly contains such a cycle.) We show that $j'_V(C_0) > 0$ (i.e., C_0 contains a J'_V-crossing arc). By Claim 3, $J \cup J'$ is a cut set, therefore some arc of C_0 passes through J'_V. Let $Q \subset C_0$ denote such an arc. By Claim 1, $N_G^2[P'] = N_{G-J}^2[P']$ and therefore Lemma 6 and Claim 2 are applicable also to P'. Thus, by Claim 2 used for $G - J$, there are no edges between any vertex of J'_V and any vertex of $G - (J \cup J')$, implying that Q contains some vertex of B'_1 and some vertex of B'_2. The desired arc is then obtained as a shortest subarc of Q with all internal vertices in J'_V and with one endvertex in B'_1 and the other in B'_2. Hence $j'_V(C_0) > 0$. Moreover, also by Claim 3, we observe that $j'_V(C_0)$ is odd.

Claim 4. $\quad j_V(C_0) = j'_V(C_0) = 1$.

Proof. Suppose, to the contrary, that $j_V(C_0) \geq 3$, and choose the orientation of C_0 and two J_V-crossing arcs $A^1 : v_1^1C_0v_2^1$ and $A^2 : v_1^2C_0v_2^2$ in C_0 such that $v_1^i \in B_1$ and $v_2^i \in B_2$, $i = 1, 2$, and such that the arc $A' : v_2^2C_0v_1^1$ satisfies $j_V(A') = 0$ and $j'_V(A') \geq 1$ (by Claim 3, it is straightforward to verify that this is always possible). Set $t = \lceil \frac{d}{2} \rceil$. By Lemma 6 (c), there are vertices $w_1 \in V(A^1)\cap D_t$ and $w_2 \in V(A^2)\cap D_t$. Then, for the cycle $C'_0 : w_1^-w_1x_tw_2w_2^+C_0w_1^-$ we have $j_V(C'_0) = 1$, and C'_0 is shorter than C_0, a contradiction. The proof for $j'_V(C_0)$ is symmetric. $\square$

Claim 5. *Every (y, y')-arc of C_0 of length at most $\frac{|V(C_0)|}{2}$ is a shortest (y, y')-path in G.*

Proof. Suppose, to the contrary, that there is an arc yC_0y' that is not a shortest path in G, let Q be a shortest (y, y')-path in G, and, among all such arcs in C_0, choose the arc $A_1 : yC_0y'$ such that the path Q is shortest possible. By the same argument as in the proof of Claim 4, $j_V(Q) \leq 1$ and $j'_V(Q) \leq 1$.

Let $A_2 : y'C_0y$ denote the complementary arc to A_1 (i.e., $V(A_1) \cup V(A_2) = V(C_0)$ and $V(A_1) \cap V(A_2) = \{y, y'\}$). Then clearly A_1, A_2 and Q are pairwise internally vertex-disjoint paths with common endvertices, hence both $C_1 : yA_1y'\overleftarrow{Q}y$ and $C_2 : y'A_2yQy'$ (where $\overleftarrow{Q}$ denotes Q traversed in the opposite orientation) are cycles in G. By the definition of Q and by the assumption of Claim 5, we have $|E(Q)| < |E(A_1)| \leq |E(A_2)|$, hence both C_1 and C_2 are shorter than C_0. Let $A : v_1C_0v_2$ be the (only) J_V-crossing arc of C_0. According to the position of y and y' with respect to A, we have, up to symmetry, the following three possibilities.

(α) $y, y' \notin J_V$. Then either $A \subset A_1$, or $A \subset A_2$, thus, for each value of $j_V(Q)$, either $j_V(C_1) = 1$ or $j_V(C_2) = 1$.

(β) $y, y' \in J_V$. Then both y and y' are internal vertices of A, hence $j_V(A_1) = 0$. If $j'_V(Q) = 0$, then $j_V(C_2) = 1$, and if $j'_V(Q) = 1$, then $j_V(C_1) = 1$.

(γ) $y \notin J_V$ and $y' \in J_V$. In this case, we first observe that we can suppose that $V(Q) \cap B_1 = \emptyset$ or $V(Q) \cap B_2 = \emptyset$. Let, to the contrary, A_Q be a J_V-crossing arc of Q, let t, $p+10 \le t \le d-p-10$, be such that either $x_t = y'$ or $x_t y' \in E(G)$ (this is always possible since, by Lemma 6, $V(x_{p+10}Px_{d-p-10})$ is a dominating set in J_V), and let $w \in V(A_Q) \cap D_t$. Then (similarly as in the proof of Claim 4), the (y, y')-path $Q' : yQw(x_t)y'$ is not longer than Q and contains vertices of only one of B_1, B_2. Now, if Q passes through B_1, then $j_V(C_2) = 1$, and if Q passes through B_2, then $j_V(C_1) = 1$.

In each of the possible cases, we have obtained a contradiction with the choice of C_0. $\square$

<u>Claim 6.</u> *If $v \in V(C_0) \cap B_1$ and $w \in V(C_0) \cap B_2$, then $\mathrm{dist}_G(v, w) \ge p + q + 1$.*

<u>Proof.</u> By Lemma 6, $\mathrm{dist}_G(v, P) \le 1$ and $\mathrm{dist}_G(w, P) \le 1$. Let $v \in D_{i_0}$ and $w \in D_{i_1}$ (for some $i_0 \in \{p+6, p+7, p+8\}$ and $i_1 \in \{d-p-6, d-p-7, d-p-8\}$). Since $d \ge 3p+q+19$, $\mathrm{dist}_G(x_{i_0-1}, x_{i_1+1}) \ge d - 2(p+7) \ge p+q+5$. Clearly $\mathrm{dist}_G(x_{i_0-1}, v) \le 2$ and $\mathrm{dist}_G(w, x_{i_1+1}) \le 2$. Then $\mathrm{dist}_G(x_{i_0-1}, v) + \mathrm{dist}_G(v, w) + \mathrm{dist}_G(w, x_{i_1+1}) \ge p+q+5$, implying that $\mathrm{dist}_G(v, w) \ge p+q+1$. $\square$

By a symmetric argument to that of Claim 6, we also have $\mathrm{dist}_G(v', w') \ge p+q+1$ for any $v' \in V(C_0) \cap B'_1$ and $w' \in V(C_0) \cap B'_2$. Consequently, the length of C_0 is at least $2(p+q+1)$.

Summarizing, we have the cycle C_0 of length at least $|V(C_0)| \ge 2(p+q+1)$ such that every path $Q \subset C_0$ of length at most $\frac{|V(C_0)|}{2}$ is shortest possible. Clearly C_0 has length at most $2d+1$. By Lemma 6, C_0 is a two-way dominating set in G. Therefore $\mathrm{rc}(G) \le \mathrm{rc}(C_0) + 3 \le \mathrm{diam}(C_0) + 4 \le \mathrm{diam}(G) + 4$. $\blacksquare$

The following result completes the proof of Theorem 1 by establishing sufficiency.

Proposition 8. *Let (X, Y, Z) be one of the following triples of graphs:*

(*i*) $X = K_{1,3}$, $Y = K_s^h$ $(s > 3)$, $Z = N_{1,j,k}$ $(j+k > 2,\ 1 \le j \le k)$,

(*ii*) $X = K_{1,r}$ $(r > 3)$, $Y = K_s^h$ $(s > 3)$, $Z = P_\ell$ $(\ell > 4)$,

(*iii*) $X = K_{1,r}$ $(r > 3)$, $Y = S_{1,j,k}$ $(j+k > 2,\ 1 \le j \le k)$, $Z = N$.

Then there is a constant k_{XYZ} such that every connected (X, Y, Z)-free graph G satisfies $\mathrm{rc}(G) \le \mathrm{diam}(G) + k_{XYZ}$.

Proof. It is straightforward to verify that in each of the cases (i), (ii) and (iii), every (X, Y, Z)-free graph is $(K_{1,r_0}, K^h_{s_0})$-free, for suitable values of $r_0, s_0 \in \mathbb{N}$. Thus, for graphs with small diameter, the proof follows immediately from Proposition 5. Similarly, for large diameter, every (X, Y, Z)-free graph is also $(K_{1,r_0}, K^h_{s_0}, S_{1,j_0,k_0}, N_{1,j_0,k_0})$-free in each of the cases (i), (ii) and (iii) for suitable values of $r_0, s_0, j_0, k_0 \in \mathbb{N}$. Hence, for large diameter, the proof follows from Proposition 7. ■

4 Forbidden quadruples

In this section, we prove an analogous result which characterizes all forbidden quadruples $\mathcal{F}$ for which there is a constant $k_{\mathcal{F}}$ such that $\mathrm{rc}(G) \leq \mathrm{diam}(G) + k_{\mathcal{F}}$ for any connected $\mathcal{F}$-free graph G. We exclude the cases which are covered by Theorems C, D and 1. We set

$\overline{\mathcal{F}_7} = \{\{K_{1,r}, K^h_s, N_{1,j,k}, S_{1,\bar{j},\bar{k}}\} |\ r > 3, s > 3, 1 \leq j \leq k, j+k > 2, 1 \leq \bar{j} \leq \bar{k}, \bar{j}+\bar{k} > 2\}$,

and

$\mathcal{F}_7 = \{\{X, Y, Z, W\} |\ \{X, Y, Z, W\} \overset{\text{IND}}{\subset} \mathcal{F}$ for some $\mathcal{F} \in \overline{\mathcal{F}_7}\}$.

Theorem 9. *Let $\mathcal{F}$ be a finite family of connected graphs with $|\mathcal{F}| = 4$ such that $\mathcal{F} \not\supset \mathcal{F}'$ for any $\mathcal{F}' \in \mathcal{F}_1 \cup \mathcal{F}_2 \cup \ldots \cup \mathcal{F}_6$. Then there is a constant $k_{\mathcal{F}}$ such that every connected $\mathcal{F}$-free graph satisfies $\mathrm{rc}(G) \leq \mathrm{diam}(G) + k_{\mathcal{F}}$, if and only if $\mathcal{F} \in \mathcal{F}_7$.*

Proof. Sufficiency follows immediately from Proposition 5 (for bounded diameter) and Proposition 7 (for large diameter), respectively. Thus, it remains to prove necessity. The proof basically follows the proof of Proposition 2, with only some parts different. For the sake of completeness, we include a complete proof here.

Let $\mathcal{F} = \{X, Y, Z, W\}$. Let $t_0 \geq 3$ and, for $t \geq t_0$, let G^t_1, G^t_2, $G^{i,t}_3$, $G^{i,t}_4$ be the graphs used in the proof of Proposition 2 (see Fig. 1 and Fig. 2).

We first consider the graph $G^t_1 = K_{1,t}$. Then, up to a symmetry, we have $X = K_{1,r}$ for some $r \geq 3$ (for $r \leq 2$, we get $X \overset{\text{IND}}{\subset} P_3$, which is excluded by the assumptions).

Secondly, we consider the graph G^t_2. Obviously, G^t_2 is $K_{1,3}$-free, implying $X \overset{\text{IND}}{\not\subset} G^t_2$. Thus, up to a symmetry, $Y \overset{\text{IND}}{\subset} G^t_2$, implying $Y = K^h_s$ for some $s \geq 3$ (for $s \leq 2$, we get $Y \overset{\text{IND}}{\subset} P_4$, but the pair $X = K_{1,4}$, $Y = P_4$ is excluded by the assumptions).

Now we consider the graph $G^{i,t}_3$. Clearly $X \overset{\text{IND}}{\not\subset} G^{i,t}_3$ since $G^{i,t}_3$ is $K_{1,4}$-free, hence, up to a symmetry, $Y \overset{\text{IND}}{\subset} G^{i,t}_3$ or $Z \overset{\text{IND}}{\subset} G^{i,t}_3$.

Case 1: $G^{i,t}_3$ contains Y. Then $Y = N$. Since the pair $X = K_{1,3}$, $Y = N$ is excluded by the assumptions, $X = K_{1,r}$ with $r > 3$. Now we consider the graph $G^{i,t}_4$. Clearly $X \overset{\text{IND}}{\not\subset} G^{i,t}_4$ since $G^{i,t}_4$ is $K_{1,4}$-free, and if $Y \overset{\text{IND}}{\subset} G^{i,t}_4$, then we get $X = K_{1,r}$, $r > 3$ and

$Y = P_4$, which is excluded by the assumptions. Hence, up to a symmetry, $Z \overset{\text{IND}}{\subset} G_4^{i,t}$, but then $X = K_{1,r}$, $r > 3$, $Y = N$ and $Z = S_{1,j,k}$ $(1 \leq j \leq k)$, which is also excluded by the assumptions.

Case 2: $G_3^{i,t}$ contains Z. Then $Z = N_{1,j,k}$ for some $1 \leq j \leq k$. Consider the graph $G_4^{i,t}$. If $X \overset{\text{IND}}{\subset} G_4^{i,t}$, then $X = K_{1,3}$, $Y = K_s^h$ and $Z = N_{1,j,k}$ $(k \geq 2)$, which is excluded by the assumptions, if $Y \overset{\text{IND}}{\subset} G_4^{i,t}$, then $Y = P_4$ and we get $X = K_{1,r}$, $Y = P_4$ which is excluded by the assumptions, and if $Z \overset{\text{IND}}{\subset} G_4^{i,t}$, then we get $X = K_{1,r}$ $(r \geq 3)$, $Y = K_s^h$ $(s \geq 3)$ and $Z = P_\ell$ $(\ell > 3)$, which is also excluded by the assumptions. Thus, $G_4^{i,t}$ contains W. Then we get $W = S_{1,\bar{j},\bar{k}}$ for some $\bar{j}, \bar{k} \in \mathbb{N}$, $1 \leq \bar{j} \leq \bar{k}$, which gives the quadruple $(X, Y, Z, W) = (K_{1,r}, K_s^h, N_{1,j,k}, S_{1,\bar{j},\bar{k}})$ with $r \geq 3$, $s \geq 3$, $1 \leq j \leq k$, $j + k > 2$ and $1 \leq \bar{j} \leq \bar{k}$.

However, if $r = 3$, then $K_{1,3} \overset{\text{IND}}{\subset} S_{1,\bar{j},\bar{k}}$ and we get the triple $(K_{1,3}, K_t^h, N_{1,j,k})$, which is excluded by the assumptions, hence $r > 3$. Analogously, for $s = 3$, $K_3^h \overset{\text{IND}}{\subset} N_{1,j,k}$ and we get the triple $(K_{1,r}, N, S_{1,\bar{j},\bar{k}})$ which is also excluded by the assumptions, thus $s > 3$. Finally, for $\bar{j} = \bar{k} = 1$, $S_{1,1,1} \overset{\text{IND}}{\subset} K_{1,r}$ and we get the triple $(S_{1,1,1} = K_{1,3}, K_s^h, N_{1,j,k})$, which is also excluded by the assumptions, hence $\bar{k} \geq 2$. Thus, we obtain the quadruple $(X, Y, Z, W) = (K_{1,r}, K_s^h, N_{1,j,k}, S_{1,\bar{j},\bar{k}})$ with $r > 3$, $s > 3$, $1 \leq j \leq k$, $j + k > 2$, $1 \leq \bar{j} \leq \bar{k}$ and $\bar{j} + \bar{k} > 2$. ∎

5 Forbidden k-tuples for any $k \in \mathbb{N}$

In Theorems C, D, 1 and 9, we have given a characterization of all families $\mathcal{F}$ with $|\mathcal{F}| \leq 4$. Now, let $\mathcal{F} = \{X_1, \ldots, X_k\}$ with $k \geq 5$. Then, repeating the proof of Theorem 9, we easily observe that some four of the graphs $X_1, \ldots, X_k$, say, X_1, X_2, X_3, X_4, must satisfy the conditions given by Theorem 9. But now, for any graphs $X_5, \ldots, X_k$, every $(X_1, \ldots, X_k)$-free graph is trivially also (X_1, X_2, X_3, X_4)-free. Thus, considering k-tuples for $k \geq 5$ does not give anything new. A similar observation can be also applied to the cases of a single forbidden subgraph (Theorem C), to forbidden pairs (Theorem D), and to forbidden triples (Theorem 1).

Before stating our final result, we recall the notation of the families of forbidden subgraphs under consideration:

$\mathcal{F}_1 = \{\{P_3\}\}$,
$\mathcal{F}_2 = \{\{X, Y\} |\ \{X, Y\} \overset{\text{IND}}{\subset} \{K_{1,3}, N\}\}$,
$\mathcal{F}_3 = \{\{K_{1,r}, P_4\} |\ r \geq 4\}$.
$\overline{\mathcal{F}_4} = \{\{K_{1,3}, K_s^h, N_{1,j,k}\} |\ s > 3, 1 \leq j \leq k, j + k > 2\}$,
$\overline{\mathcal{F}_5} = \{\{K_{1,r}, K_s^h, P_\ell\} |\ r > 3, s > 3, \ell > 4\}$,
$\overline{\mathcal{F}_6} = \{\{K_{1,r}, S_{1,j,k}, N\} |\ r > 3, 1 \leq j \leq k, j + k > 2\}$,

$\mathcal{F}_i = \{\{X, Y, Z\}|\ \{X, Y, Z\} \overset{\text{IND}}{\subset} \mathcal{F}$ for some $\mathcal{F} \in \overline{\mathcal{F}_i}\}$, $i = 4, 5, 6$,

$\overline{\mathcal{F}_7} = \{\{K_{1,r}, K_s^h, N_{1,j,k}, S_{1,\bar{j},\bar{k}}\}|\ r > 3, s > 3, 1 \le j \le k, j+k > 2, 1 \le \bar{j} \le \bar{k}, \bar{j}+\bar{k} > 2\}$,

and

$\mathcal{F}_7 = \{\{X, Y, Z, W\}|\ \{X, Y, Z, W\} \overset{\text{IND}}{\subset} \mathcal{F}$ for some $\mathcal{F} \in \overline{\mathcal{F}_7}\}$.

Now we can summarize our observations in the following theorem.

Theorem 10. *Let $\mathcal{F}$ be a finite family of connected graphs. Then there is a constant $k_{\mathcal{F}}$ such that every connected $\mathcal{F}$-free graph satisfies $\mathrm{rc}(G) \le \mathrm{diam}(G) + k_{\mathcal{F}}$, if and only if $\mathcal{F}$ contains a subfamily $\mathcal{F}' \in \mathcal{F}_1 \cup \mathcal{F}_2 \cup \ldots \cup \mathcal{F}_7$.*

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