Coloring the cliques of line graphs

Gábor Bacsó [1] Zdeněk Ryjáček [2*] Zsolt Tuza [3,4,†]

[1] Computer and Automation Institute, Hungarian Academy of Sciences
H–1111 Budapest, Kende u. 13–17, Hungary

[2] Department of Mathematics, University of West Bohemia
306 14 Pilsen, Czech Republic

[3] Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences
H–1053 Budapest, Reáltanoda u. 13–15, Hungary

[4] Department of Computer Science and Systems Technology, University of Pannonia
H–8200 Veszprém, Egyetem u. 10, Hungary

November 2, 2016

Abstract

The *weak chromatic number*, or *clique chromatic number* (CCHN) of a graph is the minimum number of colors in a vertex coloring, such that every maximal clique gets at least two colors. The *weak chromatic index*, or *clique chromatic index* (CCHI) of a graph is the CCHN of its line graph.

Most of the results here are upper bounds for the CCHI, as functions of some other graph parameters, and contrasting with lower bounds in some cases. Algorithmic aspects are also discussed; the main result within this scope (and in the paper) shows that testing whether the CCHI of a graph equals 2 is NP-complete. We deal with the CCHN of the graph itself as well.

Keywords:
Chromatic number, weak coloring, clique chromatic number, clique chromatic index, line graph, Ramsey theory

2000 Mathematics Subject Classification: 05C15 (05C85, 05C76)

*Research supported by project P202/12/G061 of the Czech Science Foundation
† Research supported in part by the Hungarian State and the European Union under the grant TAMOP-4.2.2.A-11/1/KONV-2012-0072.

1 Introduction

Covering and coloring (partitioning) are very general fundamental problems in combinatorics. In the present paper we investigate them for inclusionwise maximal complete subgraphs of graphs, mostly in line graphs. Quantitative and also algorithmic results will be presented. For the detailed history of covering and coloring the cliques of graphs, which started in the mid-1980's, we refer to [3] where numerous references are given. Before stating results explicitly, let us introduce the terminology that will be used throughout the paper.

1.1 Definitions and notation

In this paper, by a *graph* we mean a finite graph $G = (V, E)$ without loops or multiple edges; except if we explicitly emphasize the contrary. For a vertex x, $N_G(x)$ denotes the set of neighbors of x in G, and $\Gamma_G(x)$ stands for the induced subgraph on $N_G(x)$. We use $\alpha(G)$ to denote the *independence number* of G, i.e., the maximum size of an independent (stable) set in G, and throughout the paper, 'ln' means the natural logarithm. Throughout the paper, by a *clique* in a graph G we always mean an inclusionwise nontrivial maximal complete subgraph of G.

For a graph G, $L(G)$ is the *line graph of* G, i.e., the vertex set of $L(G)$ is $E(G)$, and two edges e and f are adjacent if and only if they have some common endpoint(s). We emphasize here that in the case of defining line graphs, we allow multiple edges in the original graph G (or, more exactly, we allow G to be a multigraph). Our definition also means that $L(G)$ is defined to be a simple graph (unlike some also frequently used definitions).

A hypergraph $\mathcal{H} = (V(\mathcal{H}), \mathcal{E}(\mathcal{H}))$ is *k-colorable* if the elements of $V(\mathcal{H})$ can be colored with k colors so that no monochromatic hyperedge occurs, except the singletons (if $\mathcal{H}$ contains 1-element edges). The minimum possible k for which $\mathcal{H}$ is k-colorable is called the *chromatic number* of $\mathcal{H}$. In the context of the present work, a *transversal* of a hypergraph is a set of vertices meeting every *non-singleton hyperedge*; and the minimum size of a transversal in $\mathcal{H}$ is denoted by $\tau(\mathcal{H})$ and called the *transversal number* of $\mathcal{H}$. It is important to emphasize that here we deviate from the standard definition, which would require that a transversal contains all 1-element hyperedges.

In the context of this paper, the following concept is of crucial importance: given a graph G, the *clique hypergraph of* G is the hypergraph $\mathcal{C}(G)$ having $V(G)$ as vertex set, and the (maximal) cliques of G as hyperedges (recall that we have defined cliques in G to be *maximal* complete subgraphs of G). A graph G is *weakly k-colorable* (or *k-clique-colorable*), if its clique hypergraph is k-colorable (or, equivalently, if there is a k-vertex-coloring of G such that there is no monochromatic clique). The *weak chromatic number of* G, denoted $\chi_C(G)$, is the chromatic number of the clique hypergraph of G (or, equivalently, $\chi_C(G)$ is the minimum number k, such that G is weakly k-colorable). Similarly, the *clique-transversal number* $\tau_C(G)$ *of* G is the transversal number of the clique hypergraph of G.

Finally, the *weak chromatic index* $\chi'_C(G)$ *of a graph* G is defined as the weak chromatic number of its line graph, i.e., $\chi'_C(G) := \chi_C(L(G))$. (It is a well-known fact that a clique in $L(G)$ corresponds to a triangle or to a star in G, hence $\chi'_C(G)$ can be equivalently defined as the minimum number k for which there is a k-coloring of edges of G such that there is neither a monochromatic nontrivial star nor a monochromatic triangle.)

In this paper, we show the following:

- in Section 2, we deal with the weak chromatic index $\chi'_C(G)$, and we show
  – NP-completeness of deciding whether $\chi'_C(G) \leq 2$,
  – some upper bounds on $\chi'_C(G)$ in terms of properties of vertex neighborhoods and in terms of the chromatic number $\chi(G)$;

- in Section 3, we consider the weak chromatic number $\chi_C(G)$, and we give an upper bound on χ_C based on the clique transversal number τ_C;

- in Section 4, we consider algorithmic aspects of the above problems, and we show polynomial algorithms giving a weak coloring for some special types of inputs.

1.2 Former results on sufficient conditions for $\chi_C(G) = 2$

It is immediate to observe that $\chi_C(G) \leq \chi(G)$ with $\chi_C(G) = \chi(G)$ if G is triangle-free, and that $\chi(G) - \chi_C(G)$ can be arbitrarily large (an example is a complete graph of order $n \geq 2$, for which $\chi(G) = n$ while $\chi_C(G) = 2$). In this paragraph, we mention some cases when a graph is known to be weakly 2-colorable.

Strongly perfect graphs. For these graphs, in every induced subgraph H, there exists a stable set, meeting all the maximal cliques of H, by definition. It can be easily seen that such graphs are perfect. Moreover, they are obviously 2-clique-colorable. Strongly perfect graphs were introduced by Berge and Duchet in 1984 [4]. It is known that the following graphs are strongly perfect:

– perfectly orderable graphs [5], [6];
– graphs with Dilworth number at most 3 [12];
– (quasi-) Meyniel graphs [10], [7].

Small independence number. From Corollary 2 in [2] (which states that if $G \neq C_5$ and $\alpha(G) \geq 2$, then $\chi_C(G) \leq \alpha(G)$), we immediately have the following result.

Theorem 1 *If* $\alpha(G) = 2$ *and* $G \notin \{C_5, 2K_1\}$, *then* $\chi_C(G) = 2$.

Remark 2 *The condition on α cannot be replaced by the local assumption that G is claw-free (neither that G is a line graph) because if $n > ck!$ for a suitably chosen constant c, then $\chi_C(L(K_n)) > k$, from old known results in Ramsey theory. Later on (in Section 2.2.2) we will treat this question in more detail.*

Later on, in Theorem 16 and Corollary 17, we will see that $\chi_C(G) = 2$ also if G is the line graph of a 4-colorable graph and not an odd cycle of length at least five.

2 The weak chromatic index

Most of the questions discussed in this paper, are strongly related to the following characterization problem:

WEAKLINE-k
Given a natural number k, for which graphs H does $\chi'_C(H)) \leq k$ hold?

In [1], we can find a method, which reduces the weak edge coloring problem to the coloring problem of the triangle hypergraph of the graph, where triangles are viewed as edge triplets. Thus, we do not have to care of the monochromatic stars, only of the monochromatic triangles. This method is reconsidered in Section 4.1 from algorithmic point of view and a quick algorithm is extracted (Theorem 21 and Algorithm 1).

2.1 A complexity result

Now we consider the following algorithmic problem, corresponding to a special case of the (theoretical) problem above:

WEAKLINE-2
Input: Graph $G = (V, E)$.
Question: Can the edges of G be 2-colored without monochromatic triangles?

Remark 3 *By the fact mentioned at the beginning of the present chapter, this is equivalent to ask – "Is $L(G)$ weakly 2-colorable?"*

Theorem 4 WEAKLINE-2 *is NP-complete.*

Before proving Theorem 4, we state a claim. Here, in a 2-edge-coloring of a graph, the *majority color* of a triangle T is the color which is used (at least) twice on the edges of T.

Claim 5 *Take any 2-coloring of the edges of K_5 on the vertices a, b, c, d, e without monochromatic triangles, and let the majority color in the triangle abc be red. Then the edge de will be colored necessarily red.*

Proof The assertion follows easily from the fact that such a coloring yields always two edge-disjoint C_5's in both colors. □

Proof of Theorem 4 The membership of WeakLine-2 in NP is clear. We will prove its NP-completeness by a polynomial reduction from the following algorithmic problem, which is well-known to be NP-complete [11].

3-Uniform Hypergraph 2-Colorability (3HG-2COL)
Input: A hypergraph $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$.
Question: Is $\mathcal{H}$ 2-colorable?

We construct a graph $G = G(\mathcal{H})$ from any 3-uniform hypergraph $\mathcal{H}$ in polynomial time such that the answers to the two algorithmic questions are YES (or NO) at the same time.

In order to avoid ambiguity, exceptionally, in the context of hypergraphs below, the elements of $V(\mathcal{H})$ will be called *points*, while the elements of $V(G)$ will be called vertices, as usual.

In the construction, a triangle $T(p)$ will correspond to each point $p \in V(\mathcal{H})$ and a triangle $U(H)$ will correspond to each hyperedge $H \in E(\mathcal{H})$. These triangles yield a partition of $V(G)$. Let us consider a point p and a hyperedge H containing p, say $H = \{p, q, r\}$. Now we make the correspondence more precise. The triangle $U(H)$ will have the vertices $z(H, p), z(H, q), z(H, r)$. We build a K_5 on the vertex set $\{z(H, r), z(H, q)\} \cup T(p)$, by adding the six edges between $T(p)$ and $\{z(H, r), z(H, q)\}$.

At the end, the edge set of G will consist of three parts:

– the edges of the triangles $T(p)$ (the 'point-edges');

– the edges of the triangles $U(H)$ (the 'set-edges');

– the edges added to obtain the K_5 subgraphs (the 'additional edges').

We shall prove that the points of $\mathcal{H}$ can be properly colored with two colors if and only if G has a 2-edge-coloring without monochromatic triangles. We abbreviate the two directions of this statement as $\mathcal{H} \Rightarrow G$ and $G \Rightarrow \mathcal{H}$.

First, we prove $\mathcal{H} \Rightarrow G$.

We are given a proper 2-coloring ϕ of the points of $\mathcal{H}$. We shall define a 2-edge-coloring of G, using ϕ.

Assume that $p \in V(\mathcal{H})$ is colored green, say. Let us consider the three point-edges of the triangle $T(p)$. We color them by exactly two colors in such a way that the majority color of $T(p)$ be green (the position of the red edge is arbitrary).

Assume that $H = \{p, q, r\} \in \mathcal{H}$. Take a set-edge $e = Z(H, q)Z(H, r)$ and try to define the color of e in G. It will be the majority color of $T(p)$; that is, green.

Finally we have to color the additional edges. We are relatively free in doing this. The only requirement is that the edges of the K_5 being built has to be colored without monochromatic triangles, given the colors of point-edges and set-edges already. Clearly, this requirement can be fulfilled.

We have to show that in the coloring of G, obtained from ϕ, no triangle M is monochromatic. For $M = T(p)$, $p \in V(\mathcal{H})$, this is obvious. Let $M = U(H)$, $H \in \mathcal{H}$. Any edge $e = xy \subseteq M$ of the three, will get the color of $H - \{q, r\}$ in ϕ where $x = z(H, q), y = z(H, r)$. So, we can directly use the fact that H is not monochromatic in ϕ. The remaining triangles of G are contained in one of the K_5's and they are properly colored because so is the whole K_5.

Second, we prove $G \Rightarrow \mathcal{H}$.

Now we are given a 2-edge-coloring ψ of G without monochromatic triangles.

The color of vertex $p_0 \in V(\mathcal{H})$ will be defined as the color of the edge of the form $e = z(H, q)z(H, r)$ in ψ where $H = \{p_0, q, r\}$ is any hyperedge containing p_0. We observe that the color is uniquely determined. The reason is that, by Claim 5, taking the majority color of $T(p_0)$ in ψ, say red, it determines the color of all the edges of the form above; that is, they will be red.

We state that the hyperedge $H_0 = \{p_0, q_0, r_0\}$ will get two different colors. Let us consider the triangle $U(H_0)$, corresponding to H_0 in G. As we have seen, the color of p_0, q_0 and r_0 will coincide with the majority color (under ψ) in $T(p_0)$, $T(q_0)$ and $T(r_0)$, respectively. But these colors coincide with the colors of the edges $z(H_0, q_0)z(H_0, r_0)$, $z(H_0, r_0)z(H_0, p_0)$ and $z(H_0, p_0)z(H_0, q_0)$. These three edges yield a triangle in G, proving for H_0 what we needed. □

2.2 Upper bounds for the weak chromatic index

In the series of results below we will investigate the connection between the traditional chromatic number and the weak chromatic index.

2.2.1 Neighborhood sets

Notation 6 *Here, for any graph G and for any set P of edges, $G - P$ means the graph obtained from G by omitting the edges in P.*

Theorem 7 *Let H be a connected graph that is not an odd cycle of length at least five and let $k \geq 2$ be an integer such that, for any $x \in V(H)$, $\chi(\Gamma_H(x) - P) \leq k$ for some $P \subset E(\Gamma_H(x))$ with $0 \leq |P| \leq k - 1$. Then $\chi'_C(H) \leq k$.*

Proof The result can be easily verified for small graphs on 3 or 4 vertices. Thus, let H be a graph and $k \geq 2$ an integer satisfying the assumptions of Theorem 7, set

$|V(H)| = n$ and suppose the theorem is true for all graphs on at most $n-1$ vertices. Choose a vertex $x \in V(H)$ and set $H' = H - x$.

If H' is an odd cycle of length at least five, then H is a subgraph of wheel and it is straightforward to verify that $\chi_C(L(H)) \leq 2$. Thus, we may suppose H' is not an odd cycle of length at least five.

First assume that H' is connected. Since $N_{H'}(x) \subset N_H(x)$ for any $x \in V(H')$, the graph H' satisfies the assumptions of Theorem 7. By the induction hypothesis, H' has a k-edge-coloring ϕ without monochromatic stars and triangles.

Let $F := \Gamma_H(x)$, let χ_F be a (proper) k-vertex-coloring of $F - P$ (using the colors as above), for some $P \subset E(F)$ with $0 \leq |P| \leq k-1$ and suppose such a set P is chosen smallest possible. Let $A_1, \ldots, A_k$ be the color classes of χ_F. By the minimality of P, every $e \in P$ has both vertices in the same color class. Denote $P_i = \{e \in P \mid e \text{ has both ends in } A_i\}$, $B_i = \{\phi(e) \mid e \in P_i\}$ and $C_i = \{1, \ldots, k\} \setminus B_i$, $i = 1, \ldots, k$.

Suppose that for some $I_0 \subset \{1, \ldots, k\}$, $I_0 \neq \emptyset$, we have $|C_i| \leq |I_0| - 1$ for all $i \in I_0$. Then for any $i \in I_0$ we have $k - |B_i| \leq |I_0| - 1$ or, equivalently, $|B_i| \geq k - |I_0| + 1$, implying $|P| \geq \sum_{i \in I_0} |P_i| \geq \sum_{i \in I_0} |B_i| \geq |I_0|(k - |I_0| + 1) = |I_0|(k - |I_0| + 1) - k + k = (k - |I_0|)(|I_0| - 1) + k \geq k$, a contradiction. Hence for every $I \subset \{1, \ldots, k\}$, $I \neq \emptyset$, we have $|C_i| \geq |I|$ for some $i \in I$. This implies that $|\bigcup_{i \in I} C_i| \geq |I|$ for all $I \subset \{1, \ldots, k\}$. By Hall's theorem, the system $\{C_i\}_{i=1}^k$ has a transversal, i.e., there are $c_1, \ldots, c_k$ such that $c_i \in C_i$ and $c_i \neq c_j$ for $i \neq j$, $1 \leq i, j \leq k$.

We extend the k-edge-coloring ϕ of H' to a k-edge-coloring ψ of H by setting $\psi(xu) = c_i$ for all edges xu with $u \in A_i$, $i = 1, \ldots, k$. Then ψ is a k-edge-coloring of H with no monochromatic triangle. Possible monochromatic stars (that may occur at vertices of F of degree 1 in H') are avoided by the methods of [1] (see Theorem 21 and Algorithm 1).

Second, if x is a cutpoint, we apply the induction hypothesis for the blocks, that is, for each subgraph induced by x and one of the components.

Obviously, this can be done also in general:

Claim 8 *If all the blocks (with respect to x) are k-clique-colorable then so is the whole graph.* □

Example 9 Let $H = K_6$. Then, for any $x \in V(H)$, $\Gamma_H(x) \simeq K_5$, and removing from a K_5 any two independent edges we get a 3-colorable graph. Hence Theorem 7 gives $\chi'_C(H) \leq 3$. This bound is sharp since obviously $\chi'_C(H) \geq 3$ by a very special case of Ramsey's theorem.

For a graph H, let $\chi_{loc}(H) = \max\{\chi(\Gamma_H(x)) \mid x \in V(H)\}$. Thus, from Theorem 7 we have:

Corollary 10 *Let H be a graph that is not an odd cycle of length at least five. Then $\chi'_C(H) \leq \chi_{loc}(H)$.*

We present another application of Theorem 7. A connected graph is *unicyclic* if $|E(H)| \leq |V(H)|$ (that is, H has at most one cycle). A graph is *locally unicyclic* if all the components of $\Gamma_H(x)$ are unicyclic for every $x \in V(H)$.

Corollary 11 *If H is locally unicyclic and not an odd cycle of length at least five, then $\chi'_C(H) \leq 2$.*

Proof Pick a vertex $x \in V(H)$ and let $\hat{H} := H - x$. We consider the blocks, namely the subgraphs made from x and the components of $\hat{H}$. Every block B satisfies the conditions of Theorem 7, thus $\chi'_C(B) \leq 2$, and we can accomplish the proof by using the trivial claim above. □

2.2.2 Direct relations between chromatic number and weak chromatic index

The fact mentioned in *Remark 2* will be proved here, in a wider frame. We use the classical notation $R_k(3)$ for

$$\min\{n \mid \text{in every } k\text{-edge-coloring of } K_n, \text{ there exists some monochromatic triangle}\}.$$

We shall find a direct connection between the two quantities in the title.

Proposition 12 *Let $\chi(G) = M$. Then $M < R_k(3)$ implies $\chi'_C(G) \leq k$, and for $G = K_M$ the converse is also true.*

Proof Take an arbitrary graph G having a minimum coloring of the vertices by M colors. Let us identify the vertices in each color class with one vertex. Thus, we obtain a K_M which does have a k-edge-coloring without monochromatic triangles, by $M < R_k(3)$ and by Ramsey theory. If we transform this edge-coloring back to G in the natural way, no monochromatic triangle appears. Moreover, by [1], G has some k-edge-coloring which yields a weak coloring of $L(G)$.

For the second part of the assertion, suppose $\chi'_C(K_M) \leq k$. Then we can color the edges of K_M in such a way that (among others) no monochromatic triangle occurs. By the definition of $R_k(3)$, $M < R_k(3)$ follows. □

We remark that the second part makes sense by the fact that the latter quantity is finite.

Now we express the connection in a more concise form. For this aim, let $\lambda(M)$ be the maximum of $\chi'_C(H)$, for graphs with $\chi(H) = M$.

Proposition 13

$$\lambda(M) = \min\{k \mid R_k(3) > M\}.$$

Proof This is implied by Proposition 12. □

The following assertion is a numerical consequence of the former proposition.

Corollary 14 *For large enough* M *we have*

$$\ln M/\ln\ln M < \lambda(M) \le \log_\delta M,$$

where $\delta > 3.199$.

Before proving the corollary, we will need some preliminaries. First we prove a folklore statement which will be useful later too.

Lemma 15 *For* J *large enough, "*$J < c(k+1)!$ *for some constant* c*" implies* $k > L := \ln J/\ln\ln J$.

Proof First note that $c(k+1)! < k^k$ for k large enough. Also, clearly

$$(k+1)! = 2(k+1)\prod_{i=3}^{k} i \le 2(k+1)\cdot k^{k-2}$$

Obviously,

$$(\ln J)^{\ln J/\ln\ln J} = J.$$

Thus $L^L < J$ with L above (for J large enough). We obtain

$$L^L < k^k$$

and

$$k > L.$$

□

Proof of Corollary 14
Lower bound

A well-known upper bound is

$$R_k(3) \le ck! \tag{1}$$

with $c = 3$ (and smaller c for larger k) [9]. Now we apply Lemma 15, with $J = M$.

Upper bound

The up-to-date lower bound seems to be $R_k(3) > \delta^k$ for large k (see [13], [8]). □

An alternative approach

The method below yields a coloring that is not much worse than the bound in Corollary 14 and can be found directly.

Theorem 16 *For a connected graph H with $\chi(H) \geq 3$ that is not an odd cycle of length at least five, a weak K-edge-coloring can be constructed by a quick algorithm where*

$$K = \lfloor \log_2(\chi(H) - 1) \rfloor + 1.$$

Proof Set $\chi(H) = M$, and let χ be an M-vertex-coloring of the vertices of H, say, by $0, 1, \ldots, M-1$. For $uv \in E(H)$, set

$$\phi(uv) := \lfloor \log_2(|\chi(v) - \chi(u)|) \rfloor + 1,$$

or, equivalently,

$$\phi(uv) = i \quad \text{if and only if} \quad 2^{i-1} \leq |\chi(v) - \chi(u)| < 2^i.$$

Since obviously $1 \leq |\chi(v) - \chi(u)| \leq M - 1$ for every $uv \in E(H)$, the coloring ϕ uses $\lfloor \log_2(M-1) \rfloor + 1$ colors. Although ϕ itself is not necessarily a weak edge coloring of H, by [1] (see Theorem 21 and Algorithm 1), for the proof of Theorem 16, it is enough to verify that there is no monochromatic triangle. (Note that ϕ uses at least two colors by the assumption that $\chi(H) \geq 3$.)

Let, to the contrary, $T = uvw$ be a ϕ-monochromatic triangle, i.e., $\phi(uv) = \phi(v,w) = \phi(u,w) = i$ for some i, $1 \leq i \leq \lfloor \log_2(M-1) \rfloor + 1$. Since χ is proper, we can choose the notation such that $\chi(u) < \chi(v) < \chi(w)$. Then from $\phi(uw) = i$ we have $\chi(w) - \chi(u) < 2^i$, however, from $\phi(uv) = i$ and $\phi(vw) = i$ we obtain $\chi(w) - \chi(u) = \chi(w) - \chi(v) + \chi(v) - \chi(u) \geq 2^{i-1} + 2^{i-1} = 2^i$, a contradiction. □

From Theorem 16, we immediately have the following corollary, showing that $\chi'_C(H) \leq 2$, among others, in planar graphs (with the obvious exception of an odd cycle of length at least five).

Corollary 17 *If $\chi(H) \leq 4$ and H is not an odd cycle of length at least five, then $\chi'_C(H) \leq 2$.*

3 Relations between τ_C and χ_C

Theorem 18 *For every constant δ with $0 < \delta < 1$ there exists a constant $\tilde{\delta} > 0$ with the following property. If G is a graph of order n such that all induced subgraphs G' of G satisfy $\tau_C(G') \leq \delta n'$ (where n' denotes the number of vertices in G'), then*

$$\chi_C(G) \leq \tilde{\delta} \ln n \tag{2}$$

for n large enough.

Proof We construct a weak coloring of $G_1 := G$, by giving the list of its color classes.The first class C_1 will be the complement of a minimum clique transversal T_2 of G. By the definition of a clique transversal, C_1 satisfies the property, required for being a weak color class. The procedure will go on within the subgraph G_2 induced by the set T_2. We pick a minimum clique transversal T_3 of G_2, its complement, and so on. When we have a graph on one vertex only (or any graph without edges), we obtain a weak coloring of G, thus, repeating the steps of the procedure above $(N-1)$ times, with N satisfying

$$\delta^{N-1} n < 2 \tag{3}$$

then $\chi_C(G) \le N$.

Inequality (3) is equivalent to

$$N - 1 > \delta_0 \ln(n/2) \tag{4}$$

where $\delta_0 := 1/\ln(1/\delta) = -1/\ln\delta$.

This is implied by $N \ge \tilde{\delta} \ln n$ for $\tilde{\delta} > \delta_0$ and n large enough, consequently, any $\tilde{\delta} > \delta_0$ can play the role of the constant in the theorem. □

Remark 19 *We cannot guarantee any absolute upper bound for $\tilde{\delta}$, only an upper bound which is the function of δ, easily obtainable from the proof above.*

Considering the other direction, we can prove the following result.

Theorem 20 *Given any $\epsilon > 0$, for infinitely many n, there exists a graph G on n vertices and satisfying the conditions in Theorem 18 for the induced subgraphs of G but having*

$$\chi_C(G) > (\ln n/\ln\ln n)/(2-\epsilon) \tag{5}$$

Proof Let $G := L(K_r)$, with some r large enough. First we show that the induced subgraph conditions are satisfied. We prove that $\delta = 3/5$ is good.

Let us pick any induced subgraph G' of G. By *Theorem 1* in [1], $\tau_C(G') \le 1/2\, n'$ except if some component of G' is an odd cycle of length $n' = 2\ell + 1 \ge 5$. But for such a cycle, $\tau = \ell + 1 \le 3/5\, n'$.

By applying Lemma 15 again, for $J = r$ in this case, we obtain that, for r large enough, $r < c(k+1)!$ implies $k > L := \ln r/\ln\ln r$. Using $n := |V(G)| = \binom{r}{2}$, and, as a consequence, $n^{1/2} < r < n^{1/(2-\epsilon)}$, inequality (5) easily follows. □

4 Efficient algorithms

As we have seen, weak k-colorings are hard to find in general. In this section we describe some algorithms which run in polynomial time and whose output is a weak coloring if the input satisfies some conditions.

4.1 Elimination of monochromatic stars

From a (not necessarily proper) edge k-coloring of a graph G without monochromatic triangles, we can construct a weak k-coloring of the line graph. The method is an algorithmic transcription of the 'proof by contradiction' given in [1]. We use the following notation:

$$A_G := \{x \in V(G) \mid d(x) = 1\} \cup \{x \in \cup_{C \in \mathcal{C}} V(C) \mid d(x) \geq 3\}$$

where $\mathcal{C}$ is the set of cycles of G if G is triangle-free, and it is the set of triangles otherwise. Note that A_G is empty if and only if each component of G is a cycle.

Moreover, given an edge-coloring, adopting the terminology of [1] let us say that a vertex x is called an MCS-vertex ('monochromatic star vertex') if all edges incident to x have the same color and either x has degree ≥ 3 or $d(x) = 2$ but x is not contained in a triangle. For an edge-coloring ϕ, $M(\phi)$ is the set of MCS vertices with respect to ϕ.

Theorem 21 *Let G be a connected graph other than an odd cycle, and let $k \geq 2$ be an integer. If ϕ is a given (not necessarily proper) k-edge-coloring of G in which no monochromatic triangles occur, then a weak k-coloring of the line graph of G can be constructed from ϕ in polynomial time.*

Proof (Sketch.) It was proved in [1] that $L(G)$ is weakly k-colorable if and only if G admits an edge k-coloring without monochromatic triangles. The argument in [1] considers a coloring ϕ of $E(G)$ with the extremal properties that $|M(\phi)|$ is minimum and if $M(\phi) \neq \emptyset$ still holds then also $\mathrm{dist}(M(\phi), A_G)$ is as small as possible. It is then shown that in such a minimal coloring neither $M(\phi) \cap A_G \neq \emptyset$ nor $\mathrm{dist}(M(\phi), A_G) > 0$ can occur, consequently $M(\phi) = \emptyset$ must be valid. We have extracted the steps of the proof into Algorithm 1. Its polynomial running time is obvious, and its soundness follows by the arguments given in [1]. □

4.2 Weak 2-colorings of chordal graphs

As we mentioned in the introduction, the definitions immediately imply that strongly perfect graphs are weakly 2-colorable. In chordal graphs, a weak 2-coloring can also be determined efficiently.

Proposition 22 (*i*) *Chordal graphs are weakly 2-colorable in $O(S(n))$ time, where n denotes the number of vertices and $S(n)$ is the number of steps needed to determine a simplicial order.*

(*ii*) *If a simplicial order is given, then a weak 2-coloring can be found in $O(m)$ time, where m denotes the number of edges.*

Algorithm 1 BISTAR(G) — Elimination of monochromatic stars in edge colorings

Require: Connected graph G (not a cycle), edge coloring $\phi : E(G) \to \{1, \dots, k\}$ ($k \geq 2$) without monochromatic triangles

Ensure: A weak edge k-coloring of G

```
1: A := A_G, M := M(φ) {initialization}
2: if M = ∅ then
3:    go to Step 33
4: Find x ∈ M nearest to A {assume: color at x is not '1'}
5: if x ∈ A then
6:    if G is triangle-free then
7:       Select a cycle C = x_1 ... x_m with x_1 = x
8:       Recolor C with two colors 1, 2 s.t. the color of x_i x_{i+1} is 1 for i odd and 2
         for i even {x_{m+1} = x_1}
9:       return to Step 2
10:   else
11:      Select triangle xyz
12:      if some further edge beside xy has color ≠ 1 at y then
13:         Recolor xy to '1'
14:         return to Step 2
15:      if some further edge beside xz has color ≠ 1 at z then
16:         Recolor xz to '1'
17:         return to Step 2
18:      if all edges but xy, xz are colored '1' at y and z then
19:         Recolor xy, xz to '1' and yz to '2'
20:         return to Step 2
21: else
22:    Select neighbor z of x with dist(A, z) = dist(A, x) − 1 {now x ∉ A}
23:    Recolor xz to '1'
24:    M := M \ {x}, x := z
25:    if x is MCS then
26:       M := M ∪ {x}
27:       if x ∈ A then
28:          return to Step 6
29:       else
30:          return to Step 22
31:    else
32:       return to Step 2
33: print the current coloring and STOP
```

Proof A simplicial vertex is contained in a unique maximal clique. Hence, if x is a non-isolated simplicial vertex, then a weak 2-coloring of $G - x$ can be extended to that of G by just ensuring that the color of x is different from the color of at least one of its neighbors. A formal description of this idea is shown in Algorithm 2.

To meet the given time bounds, assume that the input graph is connected and $x_1, \ldots, x_n$ is a simplicial order. It will suffice to specify a spanning tree T in which each x_i with $1 \le i < n$ has precisely one neighbor x_j with $i < j \le n$, because then the unique proper vertex 2-coloring of T is a weak 2-coloring of G. Without additional attention to structure, this may be time consuming if many high-degree vertices are preceded in the simplicial order by many of their neighbors. One explicit tree is obtained, however, if we join each x_i to its neighbor x_j of largest subscript $j \in \{i+1, \ldots, n\}$. If G is given with its adjacency lists, then largest j can be determined in $d(x_i)$ steps, hence the total running time is $O(\sum_{i=1}^{n} d(x_i)) = O(m)$ steps.

An alternative way to organize the algorithm is to traverse G in a breadth-first-like manner, as described in Algorithm 3. Each time we select all non-visited neighbors of the current vertex, but proceeding in decreasing order of subscript rather than in increasing distance from the root. Then every vertex is visited from its neighbor of largest subscript. □

Algorithm 2 CHORDAL(G) — Weak 2-coloring in chordal graphs

Require: Chordal graph $G = (V, E)$
Ensure: Weak 2-coloring $\phi : V \to \{r, g\}$
if G has isolated vertices **then**
  Find isolated vertex x; let $y :=$ NIL, $\phi(y) := \{r\}$ {dummy vertex y}
else
  Find simplicial vertex x and neighbor y of x
CHORDAL($G - x$)
$\phi(x) := \{r, g\} \setminus \{\phi(y)\}$

Algorithm 3 SIMPL_CHORDAL(G) — Weak 2-coloring from simplicial order

Require: Simplicial order $x_1, \ldots, x_n$ of connected input graph $G = (V, E)$
Ensure: Weak 2-coloring $\phi : V \to \{1, 2\}$
$\phi(x_n) = 1$, $\phi(x_1) = \ldots = \phi(x_{n-1}) = 0$
for $j = n$ **downto** 2 **do**
  for all $x_i \in N_G(x_j)$ **do**
    if $\phi(x_i) = 0$ **then**
      $\phi(x_i) := 3 - \phi(x_j)$

5 Conclusion

Many questions remain open, some of them we mention here. The first one is a natural extension of Theorem 4.

Problem 23 *Is* WeakLine-k NP*-complete for every fixed* $k \geq 2$*?*

The comparison of Theorems 18 and 20 leads to another interesting question. Let us say that a set system $\mathcal{F}$ over an underlying set X is *δ-sparse-coverable* if, for every $X' \subseteq X$, there exists a set T' such that $|T'| \leq \delta\,|X'|$ and T' meets all of those $F \in \mathcal{F}$ which are contained in X'. Applying this notion to the clique hypergraphs of graphs, the following problem arises.

Problem 24 *Determine the asymptotic growth of the maximum of* χ_C *as a function of the number of vertices, in graph classes where the clique hypergraphs are δ-sparse-coverable for some* $0 < \delta < 1$ *fixed.*

Beyond the context of the present paper, an analogous problem is of interest for many important classes of hypergraphs as well.

References

[1] T. Andreae, M. Schughart, Zs. Tuza: Clique-transversal sets of line graphs and complements of line graphs, *Discrete Math.* 88 (1991), 11–20.

[2] G. Bacsó, S. Gravier, A. Gyárfás, M. Presissmann, and A. Sebő: Coloring the maximal cliques of graphs, *SIAM J. Discrete Math.* 17 (2004), 361–376.

[3] G. Bacsó and Zs. Tuza: Clique-transversal sets and weak 2-colorings in graphs of small maximum degree, *Discrete Math. Theoret. Computer Sci.* 11 (2009), 15–24.

[4] C. Berge, P. Duchet: Strongly perfect graphs, *Ann. Discrete Math.* 21 (1984), 57–61.

[5] V. Chvátal: Perfectly ordered graphs, *Ann. Discrete Math.* 21 (1984), 63–65.

[6] V. Chvátal: Star-cutsets and perfect graphs, *J. Combin. Theory B* 39 (1985), 189–199.

[7] H. Everett, C.M.H. de Figueiredo, C. Linhares-Sales, F. Maffray, O. Porto and B.A. Reed: Path parity and perfection, *Discrete Math.* 165/166 (1997), 233–252.

[8] H. Fredricksen, M.M. Sweet: Symmetric Sum-free Partitions and Lower Bound for Schur Numbers, *Electron. J. Combin.* 7 (2000), #R32.

[9] R.L. Graham, B.L. Rothschild and J.H. Spencer: Ramsey Theory, John Wiley and Sons, 1990.

[10] A. Hertz: A fast algorithm for colouring Meyniel graphs, *J. Combin. Theory B* 50 (1990), 231–240.

[11] L. Lovász: Coverings and colorings of hypergraphs, *Congr. Numer.* 8 (1973), 3–12.

[12] M. Preissmann: A class of strongly perfect graphs, *Discrete Math.* 54 (1985), 117–120.

[13] Xu Xiaodong, Xie Zheng, G. Exoo and S.P. Radziszowski: Constructive Lower Bounds on Classical Multicolor Ramsey Numbers, *Electron. J. Combin.* Vol 11 Issue 1(2004), #R35,24 pages.