Characterisation of Graphs with Exclusive Sum Labelling

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Abstract

A *sum graph* G is a graph with a mapping of the vertex set of G onto a set of positive integers S in such a way that two vertices of G are adjacent if and only if the sum of their labels is an element of S. In an *exclusive sum graph* the integers of S that are the sum of two other integers of S form a set of integers that label a collection of isolated vertices associated with the graph G. A graph bears a *k-exclusive sum labelling* (abbreviated k-ESL), if the set of isolated vertices is of cardinality k.

In this paper, observing that the property of having a k-ESL is hereditary, we provide a characterisation of graphs that have a k-exclusive sum labelling, for any $k \geq 1$, in terms of describing a universal graph for the property.

Keywords: Graph labelling, exclusive sum graph labelling, hyperdiamond, hereditary property, induced subgraph, universal graph

1 Introduction

All graphs considered here are simple and undirected unless otherwise stated. All graphs are also connected except for the isolated vertices necessary to maintain the labelling. We will define terms specific to this article, for all other terms used the reader is referred to [2].

A *sum graph* G is a graph with a mapping of the vertex set of G onto a set of positive integers S in such a way that two vertices of G are adjacent if and only if the sum of their labels is an element of S. More formally, for a *sum labelling* $L : V(G) \to S$, we have $u, v \in V(G), uv \in E(G)$, if and only if there is a $w \in V(G)$ such that $L(u) + L(v) = L(w)$. In this case the vertex w is said to be a *working vertex* whose work is to *witness* the edge uv.

Sum graphs were introduced by Harary in [4] as a terse way of storing and communicating graphs. An easy observation is that they must be disconnected; the vertex with the largest label must be an isolate. Any graph can be sum labelled by adding sufficiently many isolated vertices. The *sum number* of a graph G, $\sigma(G)$ is the smallest cardinality of a set of isolates that must be included with G in order for it to have a sum labelling.

A sum graph with all working vertices being confined to the set of isolates was postulated in [7] and given the name *exclusive sum graph*. More precisely, for a given positive integer k, a *k-exclusive sum labelling* (abbreviated k-ESL) of a graph G is a sum labelling L of the graph $G \cup \overline{K_k}$ such that, for $u, v \in V(G \cup \overline{K_k})$, we have $uv \in E(G \cup \overline{K_k})$ if and only if $L(u) + L(v) = L(w)$ for some $w \in \overline{K_k}$ (and, similarly as above, we say that the isolate w *witnesses* the edge uv). We will use $\mathcal{E}_k$ to represent the class of graphs having a k-ESL.

Thus, a (given) k-ESL assigns to every edge of G an isolate by which it is witnessed. This assignment determines an edge colouring of G, in which the colour of an edge equals the label of the isolate by which it is witnessed, and since all labels of vertices have to be distinct, this edge colouring is proper. Moreover, also conversely, once the assignment of labels to the edges of G (i.e., the edge colouring of G) is given, then the labelling L of the vertices of G is uniquely determined, up to an additive constant (provided G is connected; otherwise this is true in each component of G). However, note that not every proper k-edge-colouring of G determines a k-ESL of G: for example, the graph $K_{2,2,2}$ is 4-edge-colourable while any its k-ESL requires $k \geq 7$.

Obviously, if G has a k-ESL, then G has a k'-ESL for every $k' \geq k$. The *exclusive sum number* of a graph G, $\epsilon(G)$ is the smallest k for which G has a k-ESL. Clearly $\sigma(G) \leq \epsilon(G)$ and, by the above observations on edge-colourings,

$\chi'(G) \leq \epsilon(G)$, where $\chi'(G)$ is the edge chromatic number (also called the chromatic index) of G. By Vizing's theorem then $\epsilon(G) \geq \chi'(G) \geq \Delta(G)$. Exclusive sum numbers are known for various special families of graphs. For details, we refer to the survey [8] by Ryan and to the dynamic survey [3] by Gallian.

We say that a graph property $\mathcal{P}$ is *hereditary* if, whenever a graph G has $\mathcal{P}$, so does every its induced subgraph. Similarly, a class $\mathcal{C}$ of graphs is *hereditary* if, when $G \in \mathcal{C}$, all induced subgraphs of G are also in $\mathcal{C}$. Note that if $\mathcal{F}$ is a given (finite or infinite) family of graphs, then the class of all *$\mathcal{F}$-free graphs* (i.e., graphs that do not contain an induced subgraph isomorphic to any graph from $\mathcal{F}$), is a hereditary class.

Now, it is immediate to observe that, for a given k, if L is a k-ESL of a graph G and G' is an induced subgraph of G, then the restriction of L to $V(G') \cup \overline{K_k}$ is a k-ESL of G'. Thus, the property of "having a k-ESL" is a hereditary property, and the class $\mathcal{E}_k$ of all graphs having a k-ESL is a hereditary class.

There are two ways of characterising hereditary classes of graphs: in terms of *forbidden induced subgraphs*, i.e., a family $\mathcal{F}$ of graphs such that $G \in \mathcal{C}$ if and only if G is $\mathcal{F}$-free, and (sometimes also) in terms of a *universal graph*, i.e., a graph $\mathcal{G}$ such that $G \in \mathcal{C}$ if and only if G is an induced subgraph of $\mathcal{G}$. While a forbidden subgraph characterisation seems to be complicated (note that even for $k = 2$, the family $\mathcal{F}$ for $\mathcal{E}_2$ consists of all cycles and the claw $K_{1,3}$, hence is infinite), we will succeed in finding a universal graph for a generalised version of the problem. The following structure will play a crucial role in the characterisation.

Definition 1.1 *A* hyperdiamond *is a generalisation of the honeycomb grid and is defined as follows.*

(i) *H_1 is one edge (i.e., K_2).*

(ii) *Take an infinite sequence of copies of H_i: $\ldots, H_i^{-2}, H_i^{-1}, H_i^0, H_i^1, H_i^2, \ldots$*

(iii) *Colour the vertices with 2 colours (black and white) so that corresponding vertices in H_i^j and H_i^{j+1} have different colours.*

(iv) *For every j, join every black vertex in H_i^j to its corresponding (white) vertex in H_i^{j+1} with a copy of H_1.*

So H_1 is a single edge, H_2 is an infinite path, H_3 is the infinite honeycomb grid, H_4 is the infinite diamond (sometimes also called the "diamond structure"). Figure 1 shows H_3 being constructed from copies of the path H_2, and H_4 being constructed from copies of the H_3.
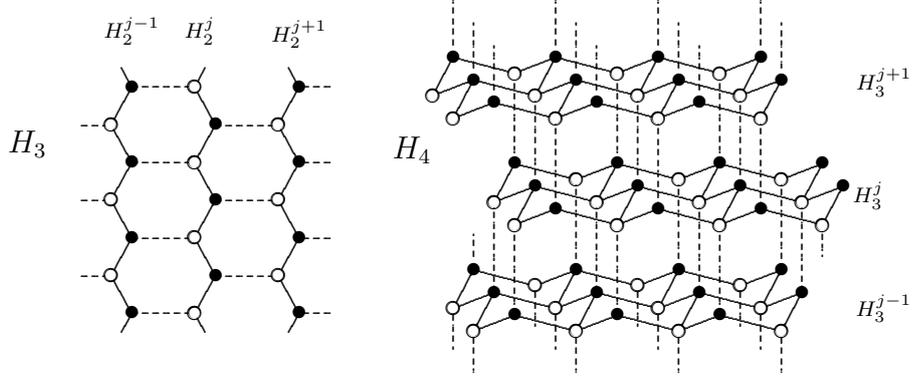
Fig. 1. The honeycomb grid H_3 and the diamond structure H_4

By the definition of H_k, we easily observe that, for $i = 1$, the sequence given in Step 2 is an infinite matching, denoted M_1, and for each $i = 2, \ldots, k$, in Step 4, a perfect matching, denoted M_i, is added to join the copies of H_{i-1}. Thus, for any $k \geq 2$, the matchings $M_1, \ldots, M_k$ define a decomposition of $E(H_k)$ into k perfect matchings. This decomposition will be called the *canonical decomposition* of H_k. Obviously, removing any one of the matchings $M_1, \ldots, M_k$ from the H_k will leave an infinite number of copies of H_{k-1}.

Note that, from purely geometrical point of view, H_3 is 2-dimensional (being in the plane), and H_4 is 3-dimensional (being a crystallographic structure). However, for our purposes, we will consider k (i.e., the number of perfect matchings in a canonical decomposition) to be the *dimension of H_k*.

All graphs considered herein, except the hyperdiamonds, will be finite.

In this article we provide a characterisation of graphs having a k-ESL for any $k \geq 1$. A surprising feature of the result is the central role of a universal graph played by the hyperdiamond structure.

2   Graphs having a k-ESL

First we consider the (easy) cases of graphs having 1-ESL and 2-ESL. Since $\Delta(G) \leq \epsilon(G)$, the only graph with 1-ESL is K_2, and a graph has a 2-ESL if and only if it is a path [7]. Thus, the first nontrivial case is that of a 3-ESL.

Let u_e and v_e represent end vertices of an edge e of G, witnessed by an isolate w_e. Define the function f on the edges of G as the sum of labels of end points of an edge minus the edge color (i.e., the label of the witnessing isolate), formally $f(e) = L(u_e) + L(v_e) - L(w_e) = L(u_e) + L(v_e) - \chi(e)$. In an

exclusively labelled sum graph,

(1)
$$f(e) = 0, \ \forall e \in E(G).$$

Sum labellings and exclusive sum labellings are not necessarily unique. For clarity we will employ the following definition.

Definition 2.1 *A particular labelling is an exclusive sum labelling in which all labels are distinct positive integers.*

A general labelling is an exclusive sum labelling in which the vertices are labelled with parameters indicating a relationship between labels such that equation (1) holds for all edges.

Figure 2a) shows an example of a graph with a particular labelling, and Figure 2b) gives an example of a general labelling for the same graph. Setting $x = 1, a = 6, b = 10, c = 14$ gives the particular labelling in Figure 2a). Another particular labelling can result from setting $x = 3, a = 11, b = 22, c = 30$. Other particular labellings can be obtained from appropriate settings of any three of x, a, b, c and solving $(a - x) + (b - x) - c = 0$.
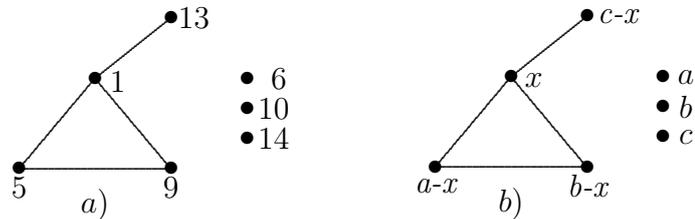
Fig. 2. Particular and general labellings

Definition 2.2 *A generic labelling is a general labelling such that equation (1) is satisfied for all choices of parameters $x, a, b, ...$ In the case where the labelling requires k isolates, we may use the term generic k-labelling.*

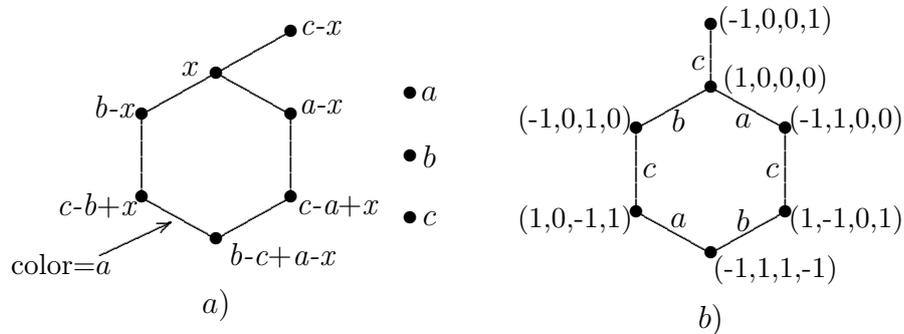Figure 3a) gives an example of a graph with a generic 3-labelling.

Fig. 3. A generic labelling as a restriction of a canonical labelling

Now we introduce a special type of a generic labelling on H_k.

Definition 2.3 *Let $k \geq 1$, let $M_1, M_2, \ldots, M_k$ be the canonical decomposition of H_k, and let ϕ be a labelling on $V(H_k)$ defined by the following construction:*

(i) *select an origin and label it $L(u) := x$,*

(ii) *label the isolates with $a_1, a_2, \ldots, a_k$, respectively,*

(iii) *color every edge $e \in E(M_i)$ with color $\chi(e) = a_i$, $i = 1, 2, \ldots, k$, (or, equivalently, assign to the edges of M_i the isolate labelled a_i as a witness, $i = 1, 2, \ldots, k$),*

(iv) *for every edge $uv \in E(H_k)$ such that $L(u)$ is already defined while $L(v)$ is not, set $L(v) := \chi(uv) - L(u)$.*

Then each vertex $u \in V(H_k)$ is labelled with an expression $\psi * x + \alpha_1 * a_1 + \alpha_2 * a_2 + \ldots + \alpha_k * a_k$, where $\psi \in \{-1, 1\}$, and we set $\phi(u) = (\psi, \alpha_1, \alpha_2, \ldots, \alpha_k)$. The labelling ϕ is called the canonical labelling of H_k. When we need to specify the dimension, we speak of a canonical k-labelling.

For example, in terms of the canonical labelling ϕ on H_3, the origin and its three neighbours are labelled $(1, 0, 0, 0), (-1, 1, 0, 0), (-1, 0, 1, 0), (-1, 0, 0, 1)$. Of course, the canonical labelling of H_k can be restricted to any finite induced subgraph G of H_k, and it immediately gives a generic k-labelling of G (see Fig. 3b) for $k = 3$).

The main result of this paper is given in the following theorem.

Theorem 2.4 *A graph G has a generic k-labelling if and only if G is an induced subgraph of H_k.*

Recall that for a graph to have an exclusive sum labelling, the vertices must be labelled with distinct positive integers. Therefore any generic (or even general) labelling must have a solution in the positive integers. The following result shows that this is always possible.

Theorem 2.5 *Any graph bearing a generic k-labelling has also a k-ESL, i.e., is an exclusive sum graph with k isolates.*

3 Non-embeddable exclusive sum graphs

In Section 2, we have described, for $k \geq 1$, all graphs having a generic k-labelling, and we have shown that

- these graphs are exactly all induced subgraphs of the H_k, and
- each of these graphs also has a (particular) k-ESL.

However, there still remain many graphs that have a (particular) k-ESL but not a generic k-labelling. The next result shows that there are no such graphs among trees.

Proposition 3.1 *Let T be a tree and $k \in \mathbb{N}$. Then T has a k-ESL if and only if T has a generic k-labelling.*

Since all connected graphs have a spanning tree, Proposition 3.1 implies that any graph G with a particular k-ESL has a spanning subgraph F with a generic k-labelling. This spanning subgraph may not necessarily be a tree and may not require all k isolates for labelling. However, as the next theorem demonstrates, the k-ESL of G can always be constructed by embedding F in H_k and solving the restriction of the canonical labelling for the remaining edges in G, which appear as additional edges, joining vertices of H_k, but not in H_k (sometimes called "false edges"). See Figure 4 for an example.
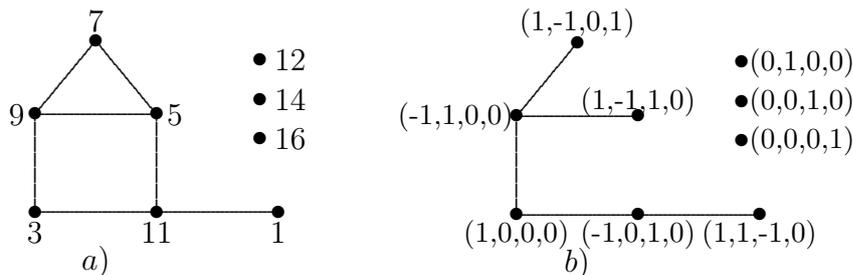

Fig. 4. Non-embeddable graph and its spanning subgraph that lies in H_3

Thus, answering the original question, we observe that graphs with k-ESL are exactly those which can be obtained by taking an induced subgraph of H_k with the corresponding restriction of the canonical labelling, and substituting appropriate values for the parameters. This characterisation is stated more precisely in the following theorem.

Theorem 3.2 *Let G be a graph and L a k-ESL of G. Then there is a spanning subgraph $F \subset G$ having a generic k-labelling and such that F can be embedded in H_k in such a way that $L = \phi(x, a_1, a_2, \ldots, a_k)$ for some values of $x, a_1, a_2, \ldots, a_k$, where ϕ is the canonical labelling of H_k.*

4   Conclusion

However, even having a maximal spanning subgraph embeddable in H_k is not enough to ensure that the graph has a particular k-labelling. For example, each of the graphs K_5, W_7 and $K_{2,2,2}$ has $\epsilon(G) \geq 7$, while they all possess spanning subgraphs that can be embedded in H_3. The fact that the embedding

dimension of a maximal spanning subgraph gives no information about the value of k necessary to support a k-ESL then begs the following open question.

Open Question 1. *How difficult is it to determine the exclusive sum number of a graph without actually providing an exclusive sum labelling?*

As mentioned in Section 1, a forbidden subgraph characterisation for $\mathcal{E}_k$ appears to be difficult. We noted that, even for $k = 2$ the family of forbidden subgraphs for $\mathcal{E}_2$ is infinite containing, as it does, all cycles as well as the claw $K_{1,3}$. While we suspect the family of forbidden subgraphs for $\mathcal{E}_k$ is infinite for all k, we pose a perhaps more approachable problem.

Open Question 2. *Describe the family of forbidden subgraphs for $\mathcal{E}_3$.*

Acknowledgement

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