Hamiltonian properties of 3-connected {claw,hourglass}-free graphs

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November 10, 2017

Abstract

We show that some sufficient conditions for hamiltonian properties of claw-free graphs can be substantially strengthened under an additional assumption that G is hourglass-free (where hourglass is the graph with degree sequence $4, 2, 2, 2, 2$).

Let G be a 3-connected claw-free and hourglass-free graph of order n. We show that

(i) if G is P_{20}-free, Z_{18}-free, or $N_{2i,2j,2k}$-free with $i + j + k \leq 9$, then G is hamiltonian,

(ii) if G is P_{12}-free, then G is Hamilton-connected,

(iii) G contains a cycle of length at least $\min\{\sigma_{12}(G), n\}$, unless $L^{-1}(\mathrm{cl}(G))$ has a nontrivial contraction to the Petersen graph,

(iv) if $\sigma_{13}(G) \geq n + 1$, then G is hamiltonian, unless $L^{-1}(\mathrm{cl}(G))$ has a nontrivial contraction to the Petersen graph.

Here P_i denotes the path on i vertices, Z_i ($N_{i,j,k}$) denotes the graph obtained by attaching a path of length $i \geq 1$ (three vertex-disjoint paths of lengths $i, j, k \geq 1$) to a triangle, $\sigma_k(G)$ denotes the minimum degree sum over all independent sets of size k, and $L^{-1}(\mathrm{cl}(G))$ is the line graph preimage of the closure of G.

Keywords: claw-free; hourglass-free; hamiltonian; Hamilton-connected; circumference; forbidden subgraph; degree condition.

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1 Introduction.

We basically follow the most common graph-theoretical terminology and notation and for concepts not defined here we refer the reader to [2].

Specifically, by a *graph* we always mean a simple finite graph $G = (V(G), E(G))$; in some situations, where we admit multiple edges (specifically, in Subsection 3.3), we speak about a *multigraph.*

We use $N_G(x)$ to denote the neighborhood and $d_G(x)$ to denote the degree of a vertex $x \in V(G)$. A *pendant vertex* is a vertex of degree 1, and a *pendant edge* is an edge having a pendant vertex. We denote $V_i(G) = \{x \in V(G)| \ d_G(x) = i\}$, $V_{\leq i}(G) = \{x \in V(G)| \ d_G(x) \leq i\}$, and $V_{\geq i}(G) = \{x \in V(G)| \ d_G(x) \geq i\}$. We use $\delta(G)$ to denote the minimum degree of G, and, for a positive integer k, we set $\sigma_k(G) = \min\{\sum_{x \in I} d_G(x)| \ I \subset V(G) \text{ independent}, |I| = k\}$ if G contains an independent set of size k, and $\sigma_k(G) = \infty$ otherwise. For $M \subset V(G)$, $\langle M \rangle_G$ denotes the induced subgraph on M. If F, G are graphs, we write $F \subset G$ if F is a subgraph of G, $F \overset{\text{IND}}{\subset} G$ if F is an induced subgraph of G, and $F \simeq G$ if F and G are isomorphic. By a *clique* we mean a complete subgraph of G, not necessarily maximal, and we say that a vertex $x \in V(G)$ is *simplicial* if $\langle N_G(x) \rangle_G$ is a clique.

For a set $X \subset E(G)$, an *X-contraction of G* is the graph $G|_X$ obtained from G by identifying the vertices of each edge in X and removing the resulting loops. For a connected subgraph $F \subset G$, we set $G|_F = G|_{E(F)}$, we use $\mathrm{con}(F)$ to denote the vertex in $G|_F$ to which F is contracted, and we also say that F is the *contraction preimage* of the vertex $v = \mathrm{con}(F)$, denoted $F = \mathrm{con}^{-1}(v)$. Finally, if F and G are graphs, we say that G *has a nontrivial contraction to F* if there is $X \subset E(G)$ such that $G|_X \simeq F$ and for every $v \in V(F)$, $\mathrm{con}^{-1}(v)$ is nontrivial.

Throughout the paper, $c(G)$ denotes the *circumference of G*, i.e., the length of a longest cycle in G. A graph G is *hamiltonian* if $c(G) = |V(G)|$, i.e., if G contains a *hamiltonian cycle*, and G is *Hamilton-connected* if, for any $x, y \in V(G)$, G contains a *hamiltonian (x,y)-path*, i.e., an (x,y)-path containing all vertices of G.

If $\mathcal{F}$ is a family of graphs, we say that G is *$\mathcal{F}$-free* if G does not contain an induced subgraph isomorphic to a member of $\mathcal{F}$, and the members of $\mathcal{F}$ are in this context referred to as *forbidden induced subgraphs*. Specifically, for $\mathcal{F} = \{K_{1,3}\}$, we say that G is *claw-free.*

Throughout, P_i denotes the path on i vertices. Further graphs often used as forbidden induced subgraphs are shown in Fig. 1; here the graph Γ_0 is called the *hourglass*, $B_{i,j}$ the
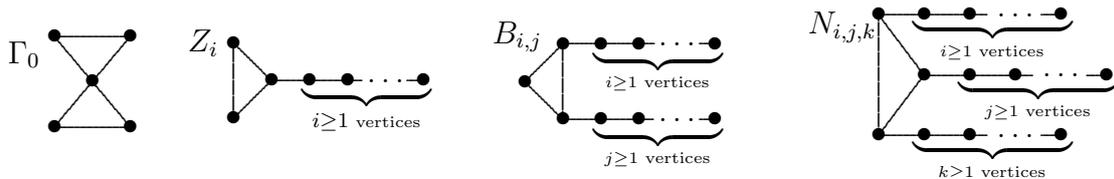

Figure 1: The graphs Γ_0, Z_i, $B_{i,j}$ and $N_{i,j,k}$

2

generalized bull and $N_{i,j,k}$ the *generalized net*.

If H is a graph (multigraph), then the *line graph* of H, denoted $L(H)$, is the graph with $E(H)$ as vertex set, in which two vertices are adjacent if and only if the corresponding edges have a vertex in common. Recall that every line graph is claw-free. It is well-known that if G is a line graph of a graph, then the graph H such that $G = L(H)$ is uniquely determined (with one exception of $G = K_3$). The graph H for which $L(H) = G$ will be called the *preimage* of G and denoted $H = L^{-1}(G)$. We will analogously write $v = L(e)$ and $e = L^{-1}(v)$ for a vertex $v \in V(G)$ and its corresponding edge $e \in E(H)$. However, note that in line graphs of multigraphs this is, in general, not true, as there can be nonisomorphic (multi)graphs with the same line graph. We will discuss this in more detail in Subsection 3.3, where this will be needed.

A vertex $x \in V(G)$ is said to be *eligible* if $\langle N_G(x) \rangle_G$ is a connected noncomplete graph. We will use $V_{EL}(G)$ to denote the set of all eligible vertices of G. For $x \in V(G)$, the *local completion of G at x* is the graph $G_x^* = (V(G), E(G) \cup \{uv \mid u, v \in N_G(x)\})$ (i.e., G_x^* is obtained from G by adding to $\langle N_G(x) \rangle_G$ all missing edges). The *closure* of a claw-free graph G is the graph $\mathrm{cl}(G)$ obtained from G by recursively performing the local completion operation at eligible vertices, as long as this is possible (more precisely, there is a sequence of graphs $G_1, \ldots, G_k$ such that $G_1 = G$, $G_{i+1} = (G_i)_x^*$ for some vertex $x \in V_{EL}(G_i)$, $i = 1, \ldots, k-1$, and $G_k = \mathrm{cl}(G)$). We say that G is *closed* if $G = \mathrm{cl}(G)$. The following result summarizes basic properties of the closure operation.

Theorem A [18].   *Let G be a claw-free graph. Then*
   (*i*) $\mathrm{cl}(G)$ *is uniquely determined,*
   (*ii*) $c(\mathrm{cl}(G)) = c(G)$,
  (*iii*) $\mathrm{cl}(G)$ *is the line graph of a triangle-free graph.*

Thus, the closure operation turns a claw-free graph G into a unique line graph of a triangle-free graph while preserving the length of a longest cycle (and hence also the hamiltonicity or nonhamiltonicity) of G.

There are many results on hamiltonian properties of graphs in classes defined in terms of forbidden induced subgraphs. In this paper, we will consider these questions in 3-connected graphs. We first summarize some known results.

Theorem B.   *Let G be a 3-connected claw-free graph.*
   (*i*) [**17**] *If G is P_{11}-free, then G is hamiltonian.*
   (*ii*) [**13**] *If G is Z_8-free, then G is hamiltonian.*
  (*iii*) [**7**] *If G is Z_9-free, then either G is hamiltonian, or G is isomorphic to the line graph of the graph obtained from the Petersen graph by adding one pendant edge to each vertex.*
  (*iv*) [**22, 10**] *If G is $N_{i,j,k}$-free with $i + j + k \leq 9$, then G is hamiltonian.*

3

Note that [22] announces an analogous result for 3-connected $\{K_{1,3}, B_{i,j}\}$-free graphs with $i + j \leq 9$ (with a family of exceptions), however, the proof in [22] is based on the statement that if a graph G is $\{K_{1,3}, B_{i,j}\}$-free, then so is $\mathrm{cl}(G)$, which is known not to be true. Since this is true for $\{K_{1,3}, N_{i,j,k}\}$-free graphs, the proof of (iv) in [22] can be trusted. (Moreover, note that the statement for $\{K_{1,3}, B_{i,j}\}$-free graphs with $i + j \leq 8$ is a direct consequence of (iv)).

Theorem C [1].   *Let G be a 3-connected $\{K_{1,3}, P_9\}$-free graph. Then G is Hamilton-connected.*

There are also many results on degree conditions for hamiltonian properties. We list here the best known ones in 3-connected claw-free graphs.

Theorem D [16].   *Let G be a 3-connected claw-free graph. Then*

$$c(G) \geq \min\{6\,\delta(G) - 15, n\}.$$

In [15], M. Li describes families of graphs $\mathcal{F}_1, \mathcal{F}_2$ with the following property.

Theorem E [15].   *Let G be a 3-connected claw-free graph of order $n \geq 363$ such that*

$$\delta(G) \geq \frac{n + 34}{12}.$$

Then either G is hamiltonian or $G \in \mathcal{F}_1 \cup \mathcal{F}_2$.

For detailed description of the classes $\mathcal{F}_1$ and $\mathcal{F}_2$ we refer the reader to [15]; here we only note that for every graph $G \in \mathcal{F}_1 \cup \mathcal{F}_2$, $L^{-1}(\mathrm{cl}(G))$ has a nontrivial contraction to the Petersen graph.

There are results indicating that some conditions for hamiltonian properties can be improved under an additional assumption that the graph under consideration is hourglass-free. For example, it is a well-known fact, observed independently by several authors (see e.g. [3]), that the Matthews-Sumner conjecture (every 4-connected claw-free graph is hamiltonian) is true in Γ_0-free graphs (and even for only 4-edge-connected graphs [21]); moreover, it was shown recently [11] that every 4-connected $\{K_{1,3}, \Gamma_0\}$-free graph is 1-Hamilton-connected, and 1-Hamilton-connectedness is polynomial in the class of $\{K_{1,3}, \Gamma_0\}$-free graphs.

In the present paper, we continue in this direction by showing that Theorems B, C, D and E can be substantially strengthened under an additional assumption that G is Γ_0-free.

2 Results.

In this section, we present our results. Their proofs are postponed to Section 3, and their sharpness will be discussed in Section 4.

Our first result strengthens Theorem B in the case of Γ_0-free graphs.

Theorem 1. Let G be a 3-connected $\{K_{1,3}, \Gamma_0\}$-free graph. If G is
  (i) P_{20}-free, or
  (ii) Z_{18}-free, or
  (iii) $N_{2i,2j,2k}$-free with $i + j + k \leq 9$,
then G is hamiltonian.

The situation with Hamilton-connectedness is more complicated since the closure operation is not applicable in this case (there are 3-connected claw-free graphs which are not Hamilton-connected while their closure is). We have to use another closure concept instead, and this allows to obtain the following strengthening of Theorem C.

Theorem 2. Let G be a 3-connected $\{K_{1,3}, \Gamma_0, P_{12}\}$-free graph. Then G is Hamilton-connected.

Our next result shows that the minimum degree bound on circumference given in Theorem D can be also substantially strengthened in the case of Γ_0-free graphs.

Theorem 3. Let G be a 3-connected $\{K_{1,3}, \Gamma_0\}$-free graph of order n. Then either

$$c(G) \geq \min\{\sigma_{12}(G), n\},$$

or $L^{-1}(\mathrm{cl}(G))$ has a nontrivial contraction to the Petersen graph.

Theorem 3 immediately implies the following corollary.

Corollary 4. Let G be a 3-connected $\{K_{1,3}, \Gamma_0\}$-free graph of order n. Then either

$$c(G) \geq \min\{12\,\delta(G), n\},$$

or $L^{-1}(\mathrm{cl}(G))$ has a nontrivial contraction to the Petersen graph. ∎

The proof technique of Theorem 3 also gives the following corollary, giving a weaker bound, but without any exception class.

Corollary 5. Let G be a 3-connected $\{K_{1,3}, \Gamma_0\}$-free graph of order n. Then

$$c(G) \geq \min\{\sigma_9(G), n\}.$$

Finally, as another consequence of the proof technique of Theorem 3, we have the following strengthening of Theorem E.

Theorem 6.   *Let G be a 3-connected $\{K_{1,3}, \Gamma_0\}$-free graph of order n such that*

$$\sigma_{13}(G) \geq n + 1.$$

Then either G is hamiltonian, or $L^{-1}(\mathrm{cl}(G))$ has a nontrivial contraction to the Petersen graph.

Theorem 6 has the following immediate consequence.

Corollary 7.   *Let G be a 3-connected $\{K_{1,3}, \Gamma_0\}$-free graph of order n such that*

$$\delta(G) \geq \frac{n+1}{13}.$$

Then either G is hamiltonian, or $L^{-1}(\mathrm{cl}(G))$ has a nontrivial contraction to the Petersen graph.

The proof technique also allows to obtain the following consequence of Theorem 6, having stronger assumptions, but no exception class.

Corollary 8.   *Let G be a 3-connected $\{K_{1,3}, \Gamma_0\}$-free graph of order n such that*

$$\sigma_{10}(G) \geq n + 1.$$

Then G is hamiltonian.

3   Proofs.

Before proving the results, we first summarize some known facts and give some auxiliary results that will be needed for our proofs.

3.1   Some auxiliary results and facts.

We say that a set $R \subset E(G)$ is an *essential edge-cut* of a connected graph G, if $G - R$ has at least two nontrivial (i.e., containing at least one edge) components. For $k \geq 1$, a graph G is *essentially k-edge-connected* if G has no essential edge-cut of size less than k. It is a well-known fact that if $G = L(H)$, then G is k-connected if and only if H is essentially k-edge-connected. We also recall that if $G = L(H)$, then a graph F is an induced subgraph of G if and only if $L^{-1}(F)$ is a subgraph (not necessarily induced) of H.

For $i, j, k \geq 1$, $S_{i,j,k}$ will be the graph obtained from $K_{1,3}$ by subdividing its edges with $i - 1$, $j - 1$ and $k - 1$ vertices of degree 2, respectively (thus, $S_{1,1,1} = K_{1,3}$). It is easy

Figure 2: The graphs $L^{-1}(\Gamma_0)$ and $S_{i,j,k}$

to observe that $L^{-1}(N_{i,j,k}) = S_{i+1,j+1,k+1}$, $L^{-1}(B_{i,j}) = S_{1,i+1,j+1}$, and $L^{-1}(Z_i) = S_{1,1,i+1}$. Note that also $L^{-1}(P_i) = P_{i+1}$, and that $L^{-1}(\Gamma_0)$ is the unique graph with degree sequence $3, 3, 1, 1, 1, 1$ (see Fig. 2).

The following fact was proved in [4].

Theorem F [4]. Let G be a $\{K_{1,3}, F\}$-free graph, where $F \in \{P_i, N_{i,j,k}\}$ for some $i, j, k \geq 1$, and let $x \in V_{EL}(G)$. Then G_x^* is $\{K_{1,3}, F\}$-free.

Note that an analogue of Theorem F is not true in the case of $\{K_{1,3}, Z_i\}$-free and $\{K_{1,3}, \Gamma_0\}$-free graphs. However, it is still possible to prove the following weaker statement.

Theorem G [4, 5]. Let G be a $\{K_{1,3}, F\}$-free graph, where $F \in \{P_i, Z_i, N_{i,j,k}, \Gamma_0\}$, for some $i, j, k \geq 1$. Then $\mathrm{cl}(G)$ is $\{K_{1,3}, F\}$-free.

Recall that, as already noted, an analogue of Theorem G is not true in the case of $\{K_{1,3}, B_{i,j}\}$-free graphs, and this is why there is no corresponding result on $\{K_{1,3}, B_{i,j}\}$-free graphs so far.

A closed trail T (i.e., an eulerian subgraph) in a graph H is said to be a *dominating closed trail* (abbreviated DCT) in H if every edge of H has at least one vertex on T (note that we admit a DCT to be trivial). The following classical result by Harary and Nash-Williams shows that a DCT in a graph H is an analogue of a hamiltonian cycle in $L(H)$.

Theorem H [9]. Let H be a graph with at least 3 edges. Then $L(H)$ is hamiltonian if and only if H has a DCT.

Let $E_0, X \subset E(G)$, and let $G_1 = G|_X$. We say that G has G_1 as an E_0-*nontrivial contraction* if, for every vertex $v \in V(G_1)$, $\mathrm{con}^{-1}(v)$ either contains at least one edge $e \in E_0$, or $\mathrm{con}^{-1}(v)$ is incident with an edge $u'v' \in E_0$ such that $u' \in V(\mathrm{con}^{-1}(v))$ and $v' \notin V(\mathrm{con}^{-1}(v))$ with $d_G(v') = 2$.

Theorem I [6]. Let G be a 3-connected claw-free graph and let $H = L^{-1}(\mathrm{cl}(G))$. Let $S \subset V(G)$ be a vertex subset in G with $|S| \leq 12$, and let $X_S = L^{-1}(S)$. Then either G contains a cycle C with $S \subset V(C)$, or H has the Petersen graph as an X_S-nontrivial contraction.

7

As noted in [6], Theorem I immediately implies the following result which is originally by Győri and Plummer [8].

Theorem J [8]. *Let G be a 3-connected claw-free graph and let $S \subset V(G)$ with $|S| \leq 9$. Then G contains a cycle C with $S \subset V(C)$.*

We will also need the following fact describing the structure of preimages of closed $\{K_{1,3}, \Gamma_0\}$-free graphs.

Proposition 9. *Let G be a 3-connected closed $\{K_{1,3}, \Gamma_0\}$-free graph and let $H = L^{-1}(G)$. Then H is an essentially 3-edge-connected bipartite graph with bipartition $H = (X, Y)$ such that $X = V_{\geq 3}(H)$ and $Y = V_{\leq 2}(H)$.*

Proof. Obviously, H is essentially 3-edge-connected. Let $e = uv \in E(H)$. If $u, v \in V_{\geq 3}(H)$, then, since H is triangle-free, for some $u_1, u_2 \in N_H(u) \setminus \{v\}$ and $v_1, v_2 \in N_H(v) \setminus \{u\}$, the edges $uu_1, uu_2, uv, vv_1, vv_2$ determine in H a subgraph isomorphic to $L^{-1}(G)$, a contradiction. If $u, v \in V_{\leq 2}(H)$, then uv is separated from $H \setminus \{u, v\}$ by an edge-cut of size at most 2, a contradiction again. Hence every edge of H has one vertex in $V_{\geq 3}(H)$ and one vertex in $V_{\leq 2}(H)$, and the result follows. ∎

The following simple fact will be also useful.

Lemma 10. *Let G be a 3-connected closed $\{K_{1,3}, \Gamma_0\}$-free graph, and let T be a DCT in $H = L^{-1}(G)$. Then $V_{\geq 3}(H) \subset V(T)$.*

Proof. If some $x \in V_{\geq 3}(H)$ is not on T, then its neighbors are also not on T since $N_H(x) \subset V_{\leq 2}(H)$ by Proposition 9. Thus, the edges containing x have no vertex on T, a contradiction. ∎

Let G be a 3-connected closed $\{K_{1,3}, \Gamma_0\}$-free graph, and let $H = L^{-1}(G)$. We will use H_{sup} to denote the (multi)graph obtained from H by suppressing all vertices of degree 2 (i.e., for any $x \in V_2(H)$ with $N_H(x) = \{x_1, x_2\}$, by replacing the path $x_1 x x_2$ with the edge $x_1 x_2$), and H_{sup}^+ the (multi)graph obtained from H_{sup} by adding a pendant edge to every its vertex. (Note that if some vertices $x_1, x_2 \in V_{\geq 3}(H)$ have more common neighbors of degree 2, then we keep in H_{sup} the corresponding multiedge with endvertices x_1, x_2.)

Lemma 11. *Let G be a 3-connected closed $\{K_{1,3}, \Gamma_0\}$-free graph, and let $H = L^{-1}(G)$. Then H has a DCT if and only if H_{sup}^+ has a DCT.*

Proof. If T is a DCT in H, then, by Lemma 10, T contains all vertices of H of degree at least 3. Hence the corresponding closed trail in H_{sup}^+ (obtained from T in the obvious way by suppressing all vertices in $V(T) \cap V_2(H)$) is a DCT in H_{sup}^+. Conversely, if T is a DCT in H_{sup}^+, then T is also a closed trail in H_{sup} and T contains all nonpendant vertices of H_{sup}. Hence the corresponding closed trail in H (obtained by subdividing every edge in T) is a DCT in H. ∎

3.2 Proof of Theorem 1.

Let G be a nonhamiltonian 3-connected $\{K_{1,3}, \Gamma_0\}$-free graph. For the proof of Theorem 1, we need to show that G contains each of the graphs P_{20}, Z_{18} and $N_{2i,2j,2k}$ with $i+j+k=9$ as an induced subgraph.

By Theorems A and G, we can suppose that G is closed. Thus, let $H = L^{-1}(G)$. We need to show that H contains as a subgraph (not necessarily induced) each of the graphs $L^{-1}(P_{20}) = P_{21}$, $L^{-1}(Z_{18}) = S_{1,1,19}$ and $L^{-1}(N_{2i,2j,2k}) = S_{2i+1,2j+1,2k+1}$ for $i+j+k=9$. By Lemma 11, by Theorem H and by Theorem B (i), (iii) and (iv), the graph H_{sup}^+ contains as a subgraph each of the graphs $P_{12} = L^{-1}(P_{11})$, $S_{1,1,10} = L^{-1}(Z_9)$ (or is isomorphic to the Petersen graph with added pendant edges), and $S_{i+1,j+1,k+1} = L^{-1}(N_{i,j,k})$ for $i+j+k=9$. We consider these cases separately.

Let first $P = x_1x_2\ldots x_{12}$ be a P_{12} in H_{sup}^+. Then $P' = x_2\ldots x_{11}$ is a P_{10} in H_{sup} (note that the edges x_1x_2 and $x_{11}x_{12}$ can possibly be pendant in H_{sup}^+). For any $i = 2,\ldots,10$, choose a vertex $x_i^+ \in N_H(x_i) \cap N_H(x_{i+1})$ (note that, by Proposition 9, $x_i^+ \in V_2(H)$). Then $P'' = x_2x_2^+x_3x_3^+\ldots x_{10}^+x_{11}$ is a P_{19} in H. Moreover, by the construction, $x_2, x_{11} \in V_{\geq 3}(H)$. Choose $x_2^- \in N_H(x_2) \setminus \{x_2^+\}$ and $x_{11}^+ \in N_H(x_{11}) \setminus \{x_{10}^+, x_2^-\}$. Then $x_2^-, x_{11}^+ \notin V(P'')$ and $P''' = x_2^-x_2x_2^+x_3x_3^+\ldots x_{10}^+x_{11}x_{11}^+$ is a P_{21} in H.

Secondly, if H_{sup}^+ is the Petersen graph with one pendant edge added to each its vertex, then H is the subdivision of the Petersen graph, i.e., the graph obtained by adding a vertex of degree 2 to each of its edges (see Fig. 4(a)), and it is straightforward to verify that H contains an $S_{1,1,19}$.

Thus, we can suppose that there is an $S \subset H_{\mathrm{sup}}^+$ with $S \simeq S_{1,1,10}$. Set $V(S) = \{c, b_1, b_2, a_1, \ldots, a_{10}\}$, where c is the center, cb_1 and cb_2 are the two short branches, and $P = ca_1\ldots a_{10}$ is the long branch of S. Then P is a P_{11} in H_{sup}^+. Since clearly $c \in V_{\geq 3}(H_{\mathrm{sup}}^+)$, hence also $c \in V_{\geq 3}(H_{\mathrm{sup}})$, but a_{10} can possibly be pendant in H_{sup}^+, $P' = ca_1\ldots a_9$ is a P_{10} in H_{sup} with $c, a_9 \in V_{\geq 3}(H_{\mathrm{sup}})$. Then, similarly as in the first part, we have a path $P'' = cc^+a_1a_1^+\ldots a_8^+a_9$ which is a P_{19} in H with $c, a_9 \in V_{\geq 3}(H)$. Now observe that if $N_H(c) \setminus \{c^+\} = N_H(a_9) \setminus \{a_8^+\}$, then, by Proposition 9, $\{cc^+, a_9a_8^+\}$ is an essential edge-cut of size 2 in H, contradicting the connectivity assumption. Thus, by symmetry, we can suppose that there is a vertex $b' \in N_H(c) \setminus N_H(a_9)$. Then, choosing $b'' \in N_H(c) \setminus \{b', c^+\}$ and $a_9^+ \in N_H(a_9) \setminus \{a_8^+, b''\}$, and adding the edges cb', cb'' and $a_9a_9^+$ to P'', we have an $S_{1,1,19}$ in H.

Finally, let $S \subset H_{\mathrm{sup}}^+$ be an $S_{i+1,j+1,k+1}$ in H_{sup}^+ for some $i, j, k \geq 1$, $i+j+k=9$. Then, similarly as before, removing from S the endvertices of the three branches, we have an $S_{i,j,k}$ in H_{sup}. Subdividing all its edges, we get an $S_{2i,2j,2k}$ in H such that endvertices of all three branches are in $V_{\geq 3}(H)$. Adding to each of them an edge to a neighbor of degree 2 (it is easy to see that it is always possible), we get an $S_{2i+1,2j+1,2k+1}$ in H. $\blacksquare$

3.3   Proof of Theorem 2.

We now turn our attention to the Theorem 2. Its proof will need some additional concepts and notations. As these will be needed only for the proof of Theorem 2, we give them here.

The situation is different here since there are 3-connected claw-free graphs G such that $\mathrm{cl}(G)$ is Hamilton-connected while G is not (for an example, see e.g. [19]). To overcome this difficulty, the concept of a strong multigraph closure of a claw-free graph G was introduced in [12] as follows.

For a given claw-free graph G, we construct a graph G^M by the following construction.

(*i*) If G is Hamilton-connected, we set $G^M = \mathrm{cl}(G)$.

(*ii*) If G is not Hamilton-connected, we recursively perform the local completion operation at such eligible vertices for which the resulting graph is still not Hamilton-connected, as long as this is possible. We obtain a sequence of graphs $G_1, \ldots, G_k$ such that

    (1) $G_1 = G$,

    (2) $G_{i+1} = (G_i)^*_{x_i}$ for some $x_i \in V_{EL}(G_i), i = 1, \ldots, k$,

    (3) G_k has no hamiltonian (a, b)-path for some $a, b \in V(G_k)$,

    (4) for any $x \in V_{EL}(G_k), (G_k)^*_x$ is Hamilton-connected,

and we set $G^M = G_k$.

A graph G^M obtained by the above construction is called a *strong multigraph closure* (or briefly an *SM-closure*) of the graph G, and a graph G equal to its SM-closure is said to be *SM-closed*. If G^M is an SM-closure of a claw-free graph G, then clearly G^M is Hamilton-connected if and only if so is G. Note that, for a given graph G, its SM-closure is not necessarily uniquely determined, however, any SM-closure of a claw-free graph G is a line graph of a multigraph.

As already mentioned, for simple graphs, it is well-known that if G is a line graph (of some graph), then the graph H such that $G = L(H)$ is uniquely determined (with one exception of the graphs C_3 and $K_{1,3}$, for which both $L(C_3)$ and $L(K_{1,3})$ are isomorphic to C_3). However, this is not true for multigraphs, as it is easy to construct infinitely many examples of nonisomorphic multigraphs with isomorphic line graphs. This drawback can be overcome by imposing an additional requirement that if $G = L(H)$, then simplicial vertices in G correspond to pendant edges in H. It can be shown [19] that for any line graph G there is a uniquely determined multigraph H such that $G = L(H)$ and simplicial vertices in G correspond to pendant edges in H. This graph H will be called the *multigraph preimage* of G and denoted $H = L_M^{-1}(G)$. Note that for a given line graph G, $L^{-1}(G)$ and $L_M^{-1}(G)$ can be different (for an example, see Fig. 3).

It is an easy observation that in the special case when G is a line graph and $H = L_M^{-1}(G)$, a nonsimplicial vertex $x \in V(G)$ is locally connected if and only if the edge $e = L_M^{-1}(x)$ is in a triangle or in a multiedge in H.
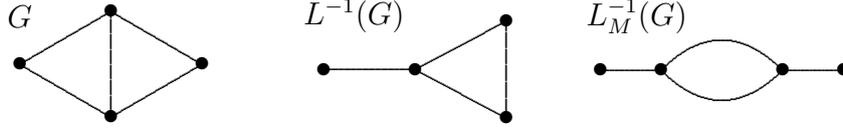
Figure 3: A line graph G and its two preimages

Given a trail T and an edge e in a multigraph G, we say that e is *dominated* (*internally dominated*) by T if e is incident to a vertex (to an interior vertex) of T, respectively. A trail T in G is called an *internally dominating trail*, shortly IDT, if T internally dominates all the edges in G. The following result is an analogue of Theorem H for Hamilton-connectedness.

Theorem K [14]. *Let H be a multigraph with $|E(H)| \geq 3$. Then $G = L(H)$ is Hamilton-connected if and only if, for any pair of edges $e_1, e_2 \in E(H)$, H has an internally dominating (e_1, e_2)-trail.*

It can be shown (see [12]) that if G is SM-closed and if $H = L_M^{-1}(G)$, then H contains no multiple edge of multiplicity more than 2, no multitriangle (a triangle with a multiple edge) and no diamond (a pair of triangles with a common edge). The following two theorems summarize further basic properties of the SM-closure operation which will be of importance for our proof.

Theorem L [12]. *Let G be a claw-free graph and let G^M be its SM-closure. Then*
  (a) G is Hamilton-connected if and only if G^M is Hamilton-connected;
  (b) G^M is a line graph of a multigraph, and $H = L_M^{-1}(G^M)$ satisfies one of the following conditions:
      (i) H is a triangle-free simple graph;
      (ii) There are $e, f \in E(H)$ such that there is no (e, f)-IDT and either
          (α) H is triangle-free and $\{e, f\}$ is the only multiedge in H, or
          (β) H is a simple graph containing at most 2 triangles, each triangle in H contains at least one of e, f, and if H contains 2 triangles, then the triangles have no common edge.

Theorem M [20]. *Let G be an SM-closed graph and let $H = L_M^{-1}(G)$. Then H does not contain a triangle with a vertex of degree 2.*

The following lemma is implicit in the proof of Lemma 3 of [11], however, for the sake of completeness, we include it here with its (easy) proof.

Lemma 12. *Let G be a claw-free graph such that every induced hourglass in G is centered at an eligible vertex, and let $x \in V_{EL}(G)$. Then every induced hourglass in G_x^* is centered at an eligible vertex.*

Proof. Let $F \overset{\text{IND}}{\subset} G_x^*$, $F \simeq \Gamma_0$, be an induced hourglass centered at a vertex $u_0 \in V(G_x^*)$, and suppose that u_0 is locally disconnected in G_x^*. Denote $V(F) = \{u_0, u_1, u_2, u_3, u_4\}$ such that $E(F) = \{u_0u_1, u_0u_2, u_0u_3, u_0u_4, u_1u_2, u_3u_4\}$. Since clearly u_0 is locally disconnected also in G, $E(F) \not\subset E(G)$. If G contains all the edges of F containing u_0, then u_0 centers a claw in G; hence we can choose the notation such that $u_0u_1 \notin E(G)$. Then $u_0, u_1 \in N_G(x)$. Let v_1 be the first vertex on a shortest (u_0, u_1)-path in $\langle N_G(x) \rangle_G$. Clearly $xu_3, xu_4 \notin E(G)$ for otherwise F is not induced in G_x^*, and $v_1u_3, v_1u_4 \notin E(G)$ for otherwise u_0 is not locally disconnected in G_x^*. But then $\langle \{u_0, x, v_1, u_3, u_4\} \rangle_G$ is an induced hourglass in G, centered at the vertex u_0, which is locally disconnected in G, a contradiction. ∎

Proof of Theorem 2. The proof basically follows the strategy of proof of Theorem 1, but instead of the closure $\text{cl}(G)$, we use the SM-closure G^M.

Let G be a 3-connected $\{K_{1,3}, \Gamma_0\}$-free graph, and suppose that G is not Hamilton-connected. We need to show that G contains an induced P_{12}. By Theorems L (a) and F and by Lemma 12, we can suppose that G is SM-closed and every induced hourglass in G is centered at an eligible vertex. Let $H = L_M^{-1}(G)$. For the proof, we need to show that H contains a (not necessarily induced) $L_M^{-1}(P_{12}) = P_{13}$.

By Theorem L (b), H can contain either at most 2 triangles (and these are edge-disjoint), or one double edge. Let T_H be the subgraph of H the components of which are the triangles or the double edge (T_H can have at most 2 components, and can be empty). Note that a vertex $v \in V(G)$ is eligible in G if and only if $L_M^{-1}(v) \in E(T_H)$.

<u>Claim 1.</u> $V(T_H) \subset V_{\geq 3}(H)$.

<u>Proof.</u>   The claim follows easily by the connectivity assumption, by Theorem M, and by the definition of $L_M^{-1}(G)$. □

By Claim 1, we specifically have $V(T_H) \subset V(H_{\sup})$.

<u>Claim 2.</u>   Let $e = uv \in E(H) \setminus E(T_H)$. Then $|\{u,v\} \cap V_{\geq 3}(H)| = |\{u,v\} \cap V_{\leq 2}(H)| = 1$.

<u>Proof.</u>   By the connectivity assumption, no two vertices in $V_{\leq 2}(H)$ can be adjacent, hence e has at most one vertex in $V_{\leq 2}(H)$. By Lemma 12 and by the fact that $L_M^{-1}(x) \in E(T_H)$ for $x \in V_{EL}(G)$, e cannot be a central edge of an $L^{-1}(\Gamma_0)$ (note that it is easy to see that $L^{-1}(\Gamma_0) \simeq L_M^{-1}(\Gamma_0)$). Hence e has at most one vertex in $V_{\geq 3}(H)$. Thus, e has one vertex in $V_{\geq 3}(H)$ and one vertex in $V_{\leq 2}(H)$. □

<u>Claim 3.</u>   The graph $L(H_{\sup}^+)$ is not Hamilton-connected.

<u>Proof.</u>   Choose $e, f \in E(H)$ such that there is no (e, f)-IDT in H, and let e', f' be the corresponding edges of $H_{\sup}^+$. Let, to the contrary, T' be an (e', f')-IDT in $H_{\sup}^+$. By the definition of $H_{\sup}^+$, T' contains all vertices in $V_{\geq 3}(H_{\sup})$ as internal vertices. By

Claims 1 and 2, the corresponding trail T in H (obtained by subdividing each edge in $E(T) \cap (E(H) \setminus E(T_H))$ with a vertex of degree 2), is an (e, f)-IDT in H, a contradiction. $\square$

Now, by Claim 3 and by Theorem C, there is a path $P \subset H_{\text{sup}}^+$ such that $P \simeq L_M^{-1}(P_9) = P_{10}$. Removing the endvertices of P (which can be pendant in H_{sup}^+), we have a path $P' \subset H_{\text{sup}}$, $P' \simeq P_8$. Set $\ell = |E(P') \cap E(T_H)|$. Then clearly $0 \le \ell \le 4$. Since $|E(P')| = 7$, $7 - \ell$ edges of P' are in $E(H_{\text{sup}}) \setminus E(T_H)$, and, subdividing each of these edges with a vertex of degree 2, we get a path P'' in H with $|V(P'')| = |V(P')| + 7 - \ell = 15 - \ell$ with endvertices in $V_{\ge 3}(H)$.

If $\ell = 4$, then T_H consists of 2 triangles and $V(T_H) \subset V(P'')$, implying that both endvertices of P'' are in $V(H) \setminus V(T_H)$. Adding to each of them an edge to a neighbor of degree 2, we have in H a path P''' with $|V(P''')| = 15 - \ell + 2 = 13$.

If $\ell = 3$, then T_H also consists of 2 triangles, and $|V(T_H) \setminus V(P)| \le 1$. Thus, one endvertex of P'' is in $V(H) \setminus V(T_H)$, and adding to it an edge to a neighbor of degree 2, we have in H a path P''' with $|V(P''')| = 15 - \ell + 1 = 13$.

Finally, if $\ell \le 2$, then already $|V(P'')| = 15 - \ell \ge 13$. $\blacksquare$

3.4 Proofs of Theorems 3 and 6 and of Corollaries 5 and 8.

We again begin with some definitions and lemmas. A *clique covering* of a graph G is a collection of cliques $\mathcal{K} = \{K_1 \ldots, K_s\}$ such that $V(K_1) \cup \ldots \cup V(K_s) = V(G)$. A clique covering is *minimum* if the number $s = |\mathcal{K}|$ of cliques to cover $V(G)$ is smallest possible. It is a well-known fact that if $G = L(H)$, then a clique in G corresponds to a star or to a triangle in H. Thus, if G is closed claw-free and $H = L^{-1}(G)$, then H is triangle-free, and hence a minimum clique covering of G corresponds to a minimum covering of the edges of H with stars.

Lemma 13. *Let G be a 3-connected closed $\{K_{1,3}, \Gamma_0\}$-free graph and let $\mathcal{K}$ be a minimum clique covering of G, $\mathcal{K} = \{K_1, \ldots, K_s\}$. Then*
   (i) *all cliques in $\mathcal{K}$ have at least three vertices and are pairwise vertex-disjoint,*
   (ii) *there is an independent set $I = \{z_1, \ldots, z_s\}$ in G such that $z_i \in V(K_i)$, $i = 1, \ldots, s$.*

Proof. (i) By Proposition 9, H is bipartite with bipartition (X, Y) such that $X = V_{\ge 3}(G)$ and $Y = V_{\le 2}(G)$. Then it is straightforward to see that $\mathcal{K}$ corresponds to the covering of $E(H)$ with the system of stars that are centered at the vertices of X. Since $X = V_{\ge 3}(G)$, each of the cliques has at least three vertices, and since X is independent, the cliques are vertex-disjoint.

(ii) By Proposition 9 and by Hall's theorem, H has a matching which covers all vertices of X. The corresponding vertices of $G = L(H)$ form the requested independent set. $\blacksquare$

Lemma 14. *Let G be a 3-connected closed $\{K_{1,3}, \Gamma_0\}$-free graph, $\mathcal{K} = \{K_1, \ldots, K_s\}$ a minimum clique covering of G, and $I = \{z_1, \ldots, z_s\} \subset V(G)$ an independent set such that $z_i \in V(K_i)$, $i = 1, \ldots, s$. Let $1 \leq t \leq s$, and let C be a cycle in G such that $\{z_1, \ldots, z_t\} \subset V(C)$. Then there is a cycle $\overline{C}$ in G such that $V(K_1) \cup \ldots \cup V(K_t) \subset V(\overline{C})$ and $|V(\overline{C})| \geq \sum_{i=1}^{t} d_G(z_i)$.*

Proof.   Let C be a cycle containing the vertices $z_1, \ldots, z_t$. We first observe that C contains at least one edge from each K_i, $i = 1, \ldots, t$: indeed, if z_i is simplicial, then the two edges of C containing z_i are both in K_i, and if z_i is nonsimplicial, then z_i has at most one neighbor in $V(G) \setminus V(C)$, implying that at least one of the two edges of C containing z_i must be in K_i. Thus, C contains at least one edge of each of the cliques $K_1, \ldots, K_t$, and it is straightforward to see that C can be extended to a cycle $\overline{C}$ such that $V(K_1) \cup \ldots \cup V(K_t) \subset V(\overline{C})$.

Now, if z_i is simplicial, then z_i has all neighbors in K_i, hence $|V(K_i)| = d_G(z_i) + 1$, and if z_i is nonsimplicial, then $|V(K_i)| = d_G(z_i)$ since z_i has exactly one neighbor outside K_i. Choose the notation such that $z_1, \ldots, z_\ell$ are simplicial and $z_{\ell+1}, \ldots, z_t$ are nonsimplicial, $0 \leq \ell \leq t$. Then we have $|V(\overline{C})| \geq \sum_{i=1}^{t} |V(K_i)| = \sum_{i=1}^{\ell} (d_G(z_i) + 1) + \sum_{i=\ell+1}^{t} d_G(z_i) = \sum_{i=1}^{t} d_G(z_i) + \ell \geq \sum_{i=1}^{t} d_G(z_i)$. ∎

Proof of Theorem 3.   Let G be a 3-connected $\{K_{1,3}, \Gamma_0\}$-free graph of order n. By Theorems A and G, we can suppose that G is closed. Set $H = L^{-1}(G)$ and suppose that H does not have a nontrivial contraction to the Petersen graph.

Let $\mathcal{K} = \{K_1 \ldots, K_s\}$ be a minimum clique covering of G and, by Lemma 13, let $I = \{z_1, \ldots, z_s\}$ be an independent set in G with $z_i \in V(K_i)$, $i = 1, \ldots, s$. For an integer t, $1 \leq t \leq s$, set $\mathcal{K}_t = \{K_1 \ldots, K_t\} \subset \mathcal{K}$ and $I_t = \{z_1, \ldots, z_t\} \subset I$. Set $X_{I_t} = L^{-1}(I_t)$, and observe that, by our assumption, H cannot have the Petersen graph as an X_{I_t}-nontrivial contraction for any t, $1 \leq t \leq s$.

Now, if $s \leq 12$, then, by Theorem I, G contains a cycle C with $I \subset V(C)$, which, by Lemma 14, can be extended to a cycle $\overline{C}$ with $V(\overline{C}) \supset V(K_1) \cup \ldots \cup V(K_s) = V(G)$, i.e., to a hamiltonian cycle in G, implying $c(G) = n$. If $s > 12$, then similarly, for $t = 12$, G contains a cycle C with $I_{12} \subset V(C)$, which can be extended to a cycle $\overline{C}$ with $|V(\overline{C})| \geq \sum_{i=1}^{12} d_G(z_i)$. Since I is independent, we have $\sum_{i=1}^{12} d_G(z_i) \geq \sigma_{12}(G)$. Thus, we conclude that $c(G) \geq \min\{\sigma_{12}(G), n\}$. ∎

Proof of Corollary 5.   Let G be a 3-connected $\{K_{1,3}, \Gamma_0\}$-free graph of order n, and let $H, \mathcal{K}, I, \mathcal{K}_t$ and I_t be as in the proof of Theorem 3.

If $s \leq 9$, then, by Theorem J, we similarly have a cycle containing I, which, by Lemma 14, can be extended to a hamiltonian cycle in G, implying $c(G) = n$. If $s > 9$, then, for $t = 9$, we have a cycle C containing I_9, which is extendable to a cycle $\overline{C}$ with $|V(\overline{C})| \geq \sum_{i=1}^{9} d_G(z_i) \geq \sigma_9(G)$. Thus, we conclude that $c(G) \geq \min\{\sigma_9(G), n\}$. ∎

Proof of Theorem 6.   Let $G, n, H, \mathcal{K}, I, \mathcal{K}_t$ and I_t be the same as in the proof of Theorem 3, and, additionally, suppose that $\sigma_{13}(G) \geq n + 1$. By the previous arguments,

it is sufficient to show that $s = |\mathcal{K}| \leq 12$, i.e., that G can be covered by at most 12 cliques. Let, to the contrary, $s = |\mathcal{K}| \geq 13$. Then $V(K_1) \cup \ldots \cup V(K_{13}) \subset V(G)$, implying that $n \geq \sum_{i=1}^{13} |V(K_i)| = \sum_{i-1}^{\ell} (d_G(z_i) + 1) + \sum_{i=\ell+1}^{13} d_G(z_i) \geq \sum_{i=1}^{13} d_G(z_i) \geq \sigma_{13}(G)$, a contradiction. ∎

Proof of Corollary 8 is analogous for $|\mathcal{K}| \leq 9$ and $|\mathcal{K}| > 9$, respectively. Details are left to the reader. ∎

4 Concluding remarks.

Let H be the subdivision of the Petersen graph (see Fig. 4(a)), and let $G = L(H)$. Then clearly H is essentially 3-edge-connected and has no DCT, hence G is a 3-connected $\{K_{1,3}, \Gamma_0\}$-free nonhamiltonian graph. Since H contains no P_{22}, no $S_{1,1,20}$ and no $S_{i,j,k}$ for $i + j + k \geq 22$, G is P_{21}-free, Z_{19}-free and $N_{i,j,k}$-free for $i + j + k \geq 19$. This example shows that parts (i), (ii) and (iii) of Theorem 1 are sharp.

Note that the sharpness example can be extended to an infinite family by adding arbitrary number of pendant edges to vertices of degree 3 in H.
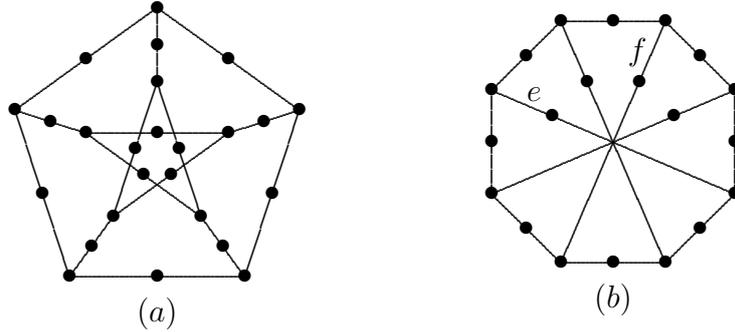

Figure 4: Subdivisions of the Petersen graph (a) and of the Wagner graph (b)

We think that Theorem 2 is not sharp, and we believe that the following is true.

Conjecture 15. *Let G be a 3-connected $\{K_{1,3}, \Gamma_0, P_{16}\}$-free graph. Then G is Hamilton-connected.*

Let H be the subdivision of the Wagner graph (see Fig. 4(b)), let $e, f \in E(H)$ be the edges indicated in Fig. 4(b), and let $G = L(H)$. Then it is easy to verify that H has no (e, f)-IDT, hence G is not Hamilton-connected. However, it is easy to check that G is 3-connected and $\{K_{1,3}, \Gamma_0, P_{17}\}$-free. This example shows that Conjecture 15, if true, is sharp.

Note that this example can be also extended to an infinite family by adding pendant edges to vertices of degree 3.

Let again H be the subdivision of the Petersen graph, and let $G = L(H)$. It is easy to check that a closed trail in H that dominates maximum number of edges is any cycle passing through any 9 of the total 10 vertices of degree 3, and it dominates 27 edges. Thus, $c(G) = 27$ (while $n = 30$). Since also $\sigma_9(G) = 27$, we have $c(G) = \sigma_9(G) < n$, which shows that Corollary 5 is sharp.

Similarly, $\sigma_{10}(G) = 30 = n$ and G is nonhamiltonian, which shows that also Corollary 8 is sharp.

Note that, in this case, adding pendant edges to vertices of degree 3 in H creates in G a new independent set with smaller degree sum, namely, the set of simplicial vertices that correspond to the added pendant edges. Thus, adding pendant edges does not create an infinite family of sharpness examples, and we admit that Corollaries 5 and 8 could be possibly improved under an additional assumption that G is sufficiently large.

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