Closure for $\{K_{1,4}, K_{1,4} + e\}$-free graphs

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Abstract

We introduce a closure concept for hamiltonicity in the class of $\{K_{1,4}, K_{1,4} + e\}$-free graphs, extending the closure for claw-free graphs introduced by the first author [JCTB 70(1997), 217-224]. The closure of a $\{K_{1,4}, K_{1,4} + e\}$-free graph G with minimum degree at least 6 is uniquely determined, is a line graph of a triangle-free graph, and preserves hamiltonicity or non-hamiltonicity of G. As applications, we show that many results on claw-free graphs can be directly extended to the class of $\{K_{1,4}, K_{1,4} + e\}$-free graphs.

1 Introduction

We consider finite simple undirected graphs $G = (V(G), E(G))$, and for concepts and notations not defined here we refer to [2]. Specifically, we say that a graph G is *nontrivial* if $E(G) \neq \emptyset$; otherwise, G is *trivial.* If F, G are graphs, we write $F \subset G$ if F is a subgraph (not necessarily induced) of G, and $F \simeq G$ if F, G are isomorphic. For $x \in V(G)$, $N_G(x)$ denotes the *neighborhood* of x, for $F \subset G$, we set $N_F(x) = N_G(x) \cap V(F)$, and for $M \subset V(G)$, we denote $N_F(M) = \cup_{x \in M} N_F(x)$. For $x \in V(G)$, $d_G(x)$ denotes the *degree* of x, for $e = xy \in E(G)$, the integer $w_G(e) = d_G(x) + d_G(y)$ is called the *weight* of the edge e, the notation $\delta(G)$ stands for the minimum degree of G, and, for a positive integer k, we set $\sigma_k(G) = \min\{d_G(x_1) + \ldots + d_G(x_k) \mid \{x_1, \ldots, x_k\} \subset V(G) \text{ independent}\}$.

A path in G with endvertices x, y will be called an *(x, y)-path.* For a cycle C with a given orientation and $x, y \in V(C)$, the (x, y)-subpath of C, traversed in the given orientation, will be called a *segment* of C and denoted $x\overrightarrow{C}y$, and $y\overleftarrow{C}x$ will denote the same segment, traversed in the opposite orientation. A cycle (path) in G containing all vertices of G is called a *hamiltonian cycle* (*hamiltonian path*) in G. A graph G is *hamiltonian* if G contains

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a hamiltonian cycle, *Hamilton-connected* if G contains a hamiltonian (x, y)-path for any $x, y \in V(G)$, and *1-Hamilton-connected* if $G - x$ is Hamilton-connected for any $x \in V(G)$.

By a *clique* in G we mean a complete subgraph of G (not necessarily maximal). For $M \subset V(G)$, $\langle M \rangle_G$ denotes the induced subgraph on M in G. A vertex $x \in V(G)$ is *simplicial* if $\langle N_G(x) \rangle_G$ is a clique, and *locally connected* if $\langle N_G(x) \rangle_G$ is a connected graph. We say that a graph is *locally connected* if all its vertices are locally connected.

If $\mathcal{C}$ is a family of graphs, we say that a graph G is *$\mathcal{C}$-free*, if G does not contain a graph from $\mathcal{C}$ as an induced subgraph, and the graphs from $\mathcal{C}$ are in this context referred to as *forbidden subgraphs*. In the special case when $\mathcal{C} = \{K_{1,3}\}$, we simply say that G is *claw-free*. Further graphs that will be used as forbidden subgraphs are shown in Fig. 1. Note that whenever we list vertices of an induced subgraph, the vertices in the list are always ordered such that their degrees form a nonincreasing sequence (thus, e.g. the center of an induced $K_{1,4}$ is always the first vertex of the list).

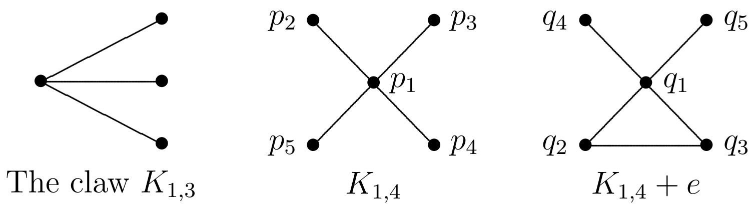

Figure 1: Forbidden subgraphs

If H is a graph, then the *line graph* of H is the graph $G = L(H)$ with $V(G) = E(H)$, in which two vertices are adjacent if and only if the corresponding edges of H share a vertex. If G is a line graph, different from the triangle K_3, then the graph H such that $G = L(H)$ (which is known to be uniquely determined), will be denoted $H = L^{-1}(G)$. It is well-known that a noncomplete line graph G is k-connected if and only if $H = L^{-1}(G)$ is *essentially k-edge-connected*, i.e., H contains no edge-cut R with $|R| < k$ such that $G - R$ has at least two nontrivial components. Also note that if $e \in E(H)$ and x_e is the corresponding vertex in $G = L(H)$, then $d_G(x_e) = w_H(e) - 2$, and if e is *pendant* (i.e., has a vertex of degree 1), then x_e is simplicial.

A closed trail T (i.e., an eulerian subgraph) in a graph H is said to be a *dominating closed trail* (abbreviated DCT) in H if every edge of H has at least one vertex on T (note that we admit a DCT to be trivial). The following classical result by Harary and Nash-Williams shows that a DCT in a graph H corresponds to a hamiltonian cycle in $L(H)$.

Theorem A [10]. *Let H be a graph with at least 3 edges. Then $L(H)$ is hamiltonian if and only if H has a DCT.*

The following concepts were introduced in [18]. For $x \in V(G)$, the *local completion* of G at x is the graph G'_x obtained from G by adding all edges with both vertices in $N_G(x)$ (thus, x is simplicial in G'_x). A locally connected nonsimplicial vertex is said to be *eligible*, and the set of all eligible vertices in G is denoted $V_{EL}(G)$. The *closure* $\mathrm{cl}(G)$ of a claw-free graph G is the graph obtained from G by recursively performing the local completion operation at eligible vertices, as long as this is possible (more precisely, there is a sequence of graphs $G_1, \dots, G_t$ such that $G_1 = G$, $G_{i+1} = (G_i)'_{x_i}$ for some $x_i \in V_{EL}(G_i)$, $i = 1, \dots, t-1$, $V_{EL}(G_t) = \emptyset$, and we set $\mathrm{cl}(G) = G_t$). The following theorem from [18] summarizes basic properties of the closure operation.

Theorem B [18]. *Let G be a claw-free graph. Then*
(*i*) $\mathrm{cl}(G)$ *is uniquely determined,*
(*ii*) $\mathrm{cl}(G)$ *is the line graph of a triangle-free graph,*
(*iii*) $\mathrm{cl}(G)$ *is hamiltonian if and only if G is hamiltonian.*

In this paper, we extend the closure for claw-free graphs introduced in [18] to the larger class of $\{K_{1,4}, K_{1,4}+e\}$-free graphs. It turns out that a direct extension is not possible, and we have to work in a slightly larger class $\mathcal{F}$, defined in Subsection 2.1. The closure operation and its main properties are then given in Subsections 2.2 and 2.3, and in Subsection 2.4, we give several applications of the closure to some famous conjectures, to a connectivity bound for hamiltonicity, and to degree and neighborhood conditions for hamiltonicity in $\{K_{1,4}, K_{1,4}+e\}$-free graphs.

2 Results

2.1 Class $\mathcal{F}$

We will use $\mathcal{F}$ to denote the class of all graphs G satisfying the following conditions (where we use the notation of vertices as in Fig. 1):
(1) G is $K_{1,4}$-free,
(2) $\delta(G) \geq 6$,
(3) if G is not $(K_{1,4}+e)$-free, then G contains a uniquely determined maximal clique $\mathcal{K}_G$ such that, for every induced $K_{1,4}+e$ in G, we have
(*i*) $\{q_1, q_2, q_3\} \subset V(\mathcal{K}_G)$,
(*ii*) $|N_{\mathcal{K}_G}(\{q_4, q_5\}) \setminus \{q_1\}| \geq 1$,
(*iii*) $|(N_{\mathcal{K}_G}(\{q_4, q_5\}) \setminus \{q_1\}) \cup (N_G(q_4) \cap N_G(q_5) \cap N_G(q_1))| \geq 3$.

The following lemma describes the structure of vertex neighborhoods in graphs from $\mathcal{F}$.

Lemma 1. *Let $G \in \mathcal{F}$ and let $x \in V(G)$. Then $\langle N_G(x)\rangle_G$ has at most two components, and, moreover, if $\langle N_G(x)\rangle_G$ has two components, then either they are both complete, or one of them is noncomplete and the second one is trivial.*

Proof. Clearly $\langle N_G(x)\rangle_G$ has at most three components for otherwise x is a center of an induced $K_{1,4}$. Suppose that $\langle N_G(x)\rangle_G$ has three components F_1, F_2, F_3. If one of them, say, F_1, is noncomplete, then, choosing nonadjacent vertices $v_1^1, v_1^2 \in V(F_1)$, and $v_i \in V(F_i)$, $i = 2, 3$, we have $\langle\{x, v_1^1, v_1^2, v_2, v_3\}\rangle_G \simeq K_{1,4}$, a contradiction. Hence all of F_1, F_2, F_3 are complete. Since $\delta(G) \geq 6$, one of them, say, F_1, is nontrivial. Choosing $v_i \in V(F_i)$, $i = 2, 3$, we have $\langle\{x, v_1^1, v_1^2, v_2, v_3\}\rangle_G \simeq K_{1,4}+e$ for any $v_1^1, v_1^2 \in V(F_1)$, implying that

$\langle V(F_1) \cup \{x\}\rangle_G = \mathcal{K}_G$. But then none of the subgraphs $\langle\{x, v_1^1, v_1^2, v_2, v_3\}\rangle_G$ can satisfy condition (3)(*ii*) from the definition of the class $\mathcal{F}$, a contradiction. Hence $\langle N_G(x)\rangle_G$ has at most two components.

Suppose that $\langle N_G(x)\rangle_G$ has two components F_1, F_2. If F_1, F_2 are both noncomplete, then, choosing two nonadjacent vertices in each of them, we again have an induced $K_{1,4}$. Hence one of the components, say, F_2, is complete. If F_1 is noncomplete and F_2 is nontrivial, then, choosing two nonadjacent vertices $v_1^1, v_1^2 \in V(F_1)$, we again have $\langle\{x, v_1^1, v_1^2, v_2^1, v_2^2\}\rangle_G \simeq K_{1,4}+e$ for any $v_2^1, v_2^2 \in V(F_2)$, implying $\langle V(F_2) \cup \{x\}\rangle_G = \mathcal{K}_G$, and none of the subgraphs $\langle\{x, v_1^1, v_1^2, v_2^1, v_2^2\}\rangle_G$ can satisfy condition (3)(*ii*) from the definition of $\mathcal{F}$. Hence either both F_1 and F_2 are complete, or, if F_1 is noncomplete, then F_2 is trivial. ■

Note that the class $\mathcal{F}$ contains all $\{K_{1,4}, K_{1,4}+e\}$-free graphs G with minimum degree $\delta(G) \geq 6$, since it is straightforward to observe that every $\{K_{1,4}, K_{1,4}+e\}$-free graph G with $\delta(G) \geq 6$ clearly satisfies conditions (1), (2) and (3). This observation shows that the class $\mathcal{F}$ is an extension of the class of $\{K_{1,4}, K_{1,4}+e\}$-free graphs with $\delta(G) \geq 6$.

2.2 Local completion

We say that a vertex $x \in V(G)$ is *eligible* if at least one component of $\langle N_G(x)\rangle_G$ is not complete. Note that, by Lemma 1, if $G \in \mathcal{F}$ and $x \in V_{EL}(G)$, then $\langle N_G(x)\rangle_G$ has exactly one noncomplete component, plus possibly another trivial component. We will use $V_{EL}(G)$ to denote the set of all eligible vertices in G.

For $x \in V_{EL}(G)$, the *local completion of* G *at* x is the graph G_x^*, obtained from G by adding to the noncomplete component of $\langle N_G(x)\rangle_G$ all missing edges (i.e., by replacing the noncomplete component of $\langle N_G(x)\rangle_G$ with a clique). The edges in $E(G_x^*) \setminus E(G)$ will be sometimes referred to as *new edges*.

Note that in the special case when G is claw-free, a vertex with a noncomplete component must be locally connected, and hence the local completion operation in claw-free graphs, as introduced in [18] and mentioned in the introduction, is a special case of the local completion operation as introduced here.

We present here two statements describing basic properties of the local completion operation which will be crucial for the main concept of this paper. The first of them shows that the local completion of a graph from the class $\mathcal{F}$ remains in $\mathcal{F}$.

Lemma 2. *Let $G \in \mathcal{F}$, and let $x \in V_{EL}(G) \cap V(\mathcal{K}_G)$, or $x \in V_{EL}(G)$ if G is $(K_{1,4}+e)$-free. Then $G_x^* \in \mathcal{F}$.*

Proof. 1. Suppose, to the contrary, that G_x^* contains an induced subgraph $F \simeq K_{1,4}$, and denote its vertices as in Fig. 1. Since all new edges in G_x^* are in a clique, exactly one edge of F, say, p_1p_2, is new. Then $\langle\{p_1, x, p_3, p_4, p_5\}\rangle_G \simeq K_{1,4}$, a contradiction.

2. The condition $\delta(G_x^*) \geq 6$ is straightforward since $E(G) \subset E(G_x^*)$.

3. Suppose that G_x^* is not $(K_{1,4}+e)$-free. By Lemma 1, let X be the noncomplete component of $\langle N_G(x)\rangle_G$ (recall that $x \in V_{EL}(G)$). Then, again by Lemma 1, if G is not

$(K_{1,4}+e)$-free, we have $V(\mathcal{K}_G) \subset V(X) \cup \{x\}$. Set $\mathcal{K}_{G^*_x} = \langle V(X) \cup \{x\}\rangle_{G^*_x}$. We show that

- $\mathcal{K}_{G^*_x}$ satisfies conditions $(3)(i),(ii),(iii)$ from the definition of the class $\mathcal{F}$,
- no other maximal clique in G^*_x satisfies $(3)(i),(ii),(iii)$.

Let F be a new induced $K_{1,4}+e$ in G^*_x and denote its vertices as in Fig. 1. If the edge q_1q_4 is new, then $q_1x, q_4x \in E(G)$, and then $\langle\{q_1,q_2,q_3,x,q_5\}\rangle_G \simeq K_{1,4}+e$. Hence $\{q_1,q_2,q_3\} \subset V(\mathcal{K}_G)$, but then, since $xq_2, xq_3 \notin E(G)$, we have $x \notin V(\mathcal{K}_G)$, contradicting the assumption of Lemma 2. Hence $q_1q_4 \in E(G)$, and, symmetrically, $q_1q_5 \in E(G)$. Thus, the new edges in F are in the triangle $\langle\{q_1,q_2,q_3\}\rangle_{G^*_x}$. If q_2q_3 is the only new edge in F, then again $\langle\{q_1,q_2,q_3,q_4,q_5\}\rangle_G \simeq K_{1,4}$, a contradiction. Thus, up to symmetry, either q_1q_2 is the only new edge, or at least two edges are new, one of them q_1q_2.

Condition $(3)(i)$. If q_1q_2 is the only new edge, then for $x \neq q_3$ we have $xq_3 \in E(G)$ (otherwise $\langle\{q_1,x,q_3,q_4,q_5\}\rangle_G \simeq K_{1,4}$), hence $\{q_1,q_2,q_3\} \subset N_G(x)$, implying $\{q_1,q_2,q_3\} \subset V(\mathcal{K}_{G^*_x})$, and if $x = q_3$, then immediately $\{q_1,q_2,q_3\} \subset V(\mathcal{K}_{G^*_x})$, as requested. If at least two edges are new, then again $\{q_1,q_2,q_3\} \subset N_G(x)$, implying $\{q_1,q_2,q_3\} \subset V(\mathcal{K}_{G^*_x})$, as requested in condition $(3)(i)$.

Conditions $(3)(ii),(iii)$. Recall that all new edges are in the triangle $\langle\{q_1,q_2,q_3\}\rangle_{G^*_x}$, and q_2q_3 is not the only new edge. Up to symmetry, q_1q_2 is new. We distinguish two cases.

Case 1: *the edge q_1q_2 is new and $q_1q_3 \in E(G)$.*

Then $q_1, q_2 \in N_G(x)$. If $x \neq q_3$, then, considering $F_G = \langle\{q_1,x,q_3,q_4,q_5\}\rangle_G$, we have $xq_3 \in E(G)$, for otherwise $F_G \simeq K_{1,4}$. Then $F_G \simeq K_{1,4}+e$, implying $\{q_1,x,q_3\} \in V(\mathcal{K}_G)$. Since F_G satisfies $(3)(ii),(iii)$ in G and $V(\mathcal{K}_G) \subset V(\mathcal{K}_{G^*_x})$, F satisfies $(3)(ii),(iii)$ in G^*_x. Thus, it remains to consider the case $x = q_3$ (implying $q_2q_3 \in E(G)$).

Since $x \in V_{EL}(G)$, there is a (q_1,q_2)-path $Q = q_1v_1\ldots v_jq_2$ in $\langle N_G(x)\rangle_G$. Consider $F_Q = \langle\{q_1,v_1,q_3,q_4,q_5\}\rangle_G$. If $v_1q_4, v_1q_5 \notin E(G)$, then $F_Q \simeq K_{1,4}+e$, and we easily observe that F satisfies $(3)(ii),(iii)$ since F_Q satisfies $(3)(ii),(iii)$ (in G) and $V(\mathcal{K}_G) \subset V(\mathcal{K}_{G^*_x})$. Thus, by symmetry, we suppose that $v_1q_4 \in E(G)$.

Claim 1. *Let $n \in N_G(q_1) \setminus \{q_3,q_4,q_5,v_1\}$. If F does not satisfy conditions $(3)(ii),(iii)$, then $|N_G(n) \cap \{q_3,q_4,q_5\}| \geq 2$.*

Proof. Set $F_G = \langle\{q_1,q_3,n,q_4,q_5\}\rangle_G$. Since $F_G \not\simeq K_{1,4}$ and $\{q_3,q_4,q_5\}$ is independent, we have $N_G(n) \cap \{q_3,q_4,q_5\} \neq \emptyset$. Suppose, to the contrary, that $|N_G(n) \cap \{q_3,q_4,q_5\}| = 1$. If $nq_3 \in E(G)$, then $F_G \simeq K_{1,4}+e$, implying $\{q_1,q_3,n\} \subset V(\mathcal{K}_G)$, and since $V(\mathcal{K}_G) \subset V(\mathcal{K}_{G^*_x})$ and F_G satisfies $(3)(ii),(iii)$, F satisfies $(3)(ii),(iii)$ as well, a contradiction. If $nq_4 \in E(G)$, then $F_G \simeq K_{1,4}+e$, implying $\{q_1,n,q_4\} \subset V(\mathcal{K}_G)$, and since $V(\mathcal{K}_G) \subset V(\mathcal{K}_{G^*_x})$ and $x = q_3 \in V(\mathcal{K}_G)$, we have $q_3q_4 \in E(G^*_x)$, a contradiction. Finally, if $nq_5 \in E(G)$, then similarly $F_G \simeq K_{1,4}+e$, implying $\{q_1,n,q_5\} \subset V(\mathcal{K}_G)$, and then $q_3q_5 \in E(G^*_x)$, a contradiction again. Hence $|N_G(n) \cap \{q_3,q_4,q_5\}| \geq 2$. □

Now, since $\delta(G) \geq 6$, there are two distinct vertices $n_1, n_2 \in N_G(q_1) \setminus \{q_3, q_4, q_5, v_1\}$. If $n_1q_3, n_2q_3 \notin E(G)$, then, by Claim 1, we have $\{n_1, n_2\} \subset N_G(q_4) \cap N_G(q_5)$ and $v_1 \in N_{\mathcal{K}_{G_x^*}}(q_4)$ (recall that $x = q_3$, implying $v_1 \in V(\mathcal{K}_{G_x^*})$), and F satisfies $(3)(ii), (iii)$. Hence suppose that, say, $n_1q_3 \in E(G)$. If also $n_2q_3 \in E(G)$, then $\{n_1, n_2, v_1\} \subset V(\mathcal{K}_{G_x^*})$ and, by Claim 1, $\{n_1, n_2, v_1\} \subset N_{\mathcal{K}_{G_x^*}}(\{q_4, q_5\})$, and if $n_2q_3 \notin E(G)$, then, by Claim 1, $n_2 \in N_G(q_4) \cap N_G(q_5)$ and $\{n_1, v_1\} \subset N_{\mathcal{K}_{G_x^*}}(\{q_4, q_5\})$; in both cases, we have $(3)(ii), (iii)$.

<u>**Case 2:**</u> *both q_1q_2 and q_1q_3 are new.*

Then $q_1, q_2, q_3 \in N_G(x)$. Up to symmetry, there is a (q_1, q_2)-path $Q = q_1v_1 \dots v_jq_2$ in $\langle N_G(x)\rangle_G$. If $\langle\{q_1, v_1, x, q_4, q_5\}\rangle_G \simeq K_{1,4} + e$, then we observe that F satisfies $(3)(ii), (iii)$ in the same way as before. Thus, by symmetry, we suppose that $v_1q_4 \in E(G)$.

Similarly as in Case 1, we have the following claim.

<u>Claim 2.</u> *Let $n \in N_G(q_1) \setminus \{x, q_4, q_5, v_1\}$. If F does not satisfy conditions $(3)(ii), (iii)$, then $|N_G(n) \cap \{x, q_4, q_5\}| \geq 2$.*

<u>Proof.</u> Set $F_G = \langle\{q_1, x, n, q_4, q_5\}\rangle_G$, and observe that $N_G(n) \cap \{x, q_4, q_5\} \neq \emptyset$ for otherwise $F_G \simeq K_{1,4}$. Let, to the contrary, $|N_G(n) \cap \{x, q_4, q_5\}| = 1$. If $nx \in E(G)$, then $F_G \simeq K_{1,4} + e$, implying $\{q_1, x, n\} \subset V(\mathcal{K}_G)$, and F satisfies $(3)(ii), (iii)$ since F_G satisfies $(3)(ii), (iii)$ and $V(\mathcal{K}_G) \subset V(\mathcal{K}_{G_x^*})$. If $nq_4 \in E(G)$, then $F_G \simeq K_{1,4} + e$, implying $\{q_1, n, q_4\} \subset V(\mathcal{K}_G)$, and then $q_3q_4 \in E(G_x^*)$ since $xq_3 \in E(G)$; if $nq_5 \in E(G)$, then similarly $\{q_1, n, q_5\} \subset V(\mathcal{K}_G)$, implying $q_3q_5 \in E(G_x^*)$. In all cases, we have reached a contradiction, hence $|N_G(n) \cap \{x, q_4, q_5\}| \geq 2$. □

Since $\delta(G) \geq 6$, there are two distinct vertices $n_1, n_2 \in N_G(q_1) \setminus \{x, q_4, q_5, v_1\}$. If $n_1, n_2 \notin N_G(x)$, then, by Claim 2, we have $\{n_1, n_2\} \subset N_G(q_4) \cap N_G(q_5)$ and $v_1 \in N_{\mathcal{K}_{G_x^*}}(q_4)$; if $n_1, n_2 \in N_G(x)$, then $\{n_1, n_2, v_1\} \subset N_{\mathcal{K}_{G_x^*}}(\{q_4, q_5\})$; and if $n_1x \in E(G)$ but $n_2x \notin E(G)$, then $n_2 \in N_G(q_4) \cap N_G(q_5)$ and $\{n_1, v_1\} \subset N_{\mathcal{K}_{G_x^*}}(\{q_4, q_5\})$. In all cases, F satisfies $(3)(ii), (iii)$.

It remains to show that $\mathcal{K}_{G_x^*}$ is the only maximal clique in G_x^* satisfying $(3)(i), (ii), (iii)$. By Lemma 1, X is the only nontrivial component of $\langle N_G(x)\rangle_G$, hence $\mathcal{K}_{G_x^*}$ is the only nontrivial maximal clique in G_x^* containing x. Thus, if $\mathcal{K}'$ is another maximal clique in G_x^*, satisfying $(3)(i), (ii), (iii)$, then $x \notin V(\mathcal{K}')$. By $(3)(i)$, $\{q_1, q_2, q_3\} \subset V(\mathcal{K}')$, hence $x \notin \{q_1, q_2, q_3\}$. But then $\langle\{q_1, x, q_2, q_4, q_5\}\rangle_{G_x^*} \simeq K_{1,4} + e$, and since $\mathcal{K}'$ satisfies $(3)(i)$, we have $x \in V(\mathcal{K}')$, a contradiction.

We conclude that G_x^* satisfies all three conditions from the definition of the class $\mathcal{F}$, hence $G_x^* \in \mathcal{F}$. ■

Note that our intention in this paper is to develop a closure concept based on the local completion operation, in the class of $\{K_{1,4}, K_{1,4} + e\}$-free graphs. However, the following examples show that this is not possible, since in $\{K_{1,4}, K_{1,4} + e\}$-free graphs, an analogue of Lemma 2 is not true. This is why we have to work in the class $\mathcal{F}$ which slightly extends the class of $\{K_{1,4}, K_{1,4}+e\}$-free graphs, and, although more complicated, it has the important property that $G \in \mathcal{F}$ implies $G_x^* \in \mathcal{F}$.

Examples. 1. Let G be the graph in Fig. 2(a), where the circle represents a clique of arbitrarily large order $p \geq 3$. It is easy to verify that G is $\{K_{1,4}, K_{1,4}+e\}$-free and $x \in V_{EL}(G)$. However, in G_x^*, we have $\langle\{q_1, q_2, x, q_4, q_5\}\rangle_G \simeq K_{1,4}+e$.

2. Let G_1, G_2 be the graphs shown in Fig. 2(b), where again the circles in G_2 represent cliques of arbitrarily large order $p_i \geq 3$, and let $\widetilde{G}$ be the graph obtained from G_1 and G_2 by joining each of the double-circled vertices of G_1 with all vertices of one of the cliques $K_{p_1}, K_{p_2}, K_{p_3}, K_{p_4}$ of G_2. Then again $\widetilde{G}$ is $\{K_{1,4}, K_{1,4}+e\}$-free, and it is easy to verify that, for any $x \in V_{EL}(\widetilde{G})$, $\widetilde{G}_x^*$ contains an induced $K_{1,4}+e$.

These examples show that in the class of $\{K_{1,4}, K_{1,4}+e\}$-free graphs, an analogue of Lemma 2 is not true. Note that G_x^* satisfies condition (3) from the definition of $\mathcal{F}$, and for $p \geq 7$, we moreover have $\delta(G_x^*) \geq 6$, hence $G_x^* \in \mathcal{F}$. Similarly, if $p_i \geq 6$, $i = 1,2,3,4$, then $\widetilde{G}_x^* \in \mathcal{F}$ for any $x \in V_{EL}(\widetilde{G})$.

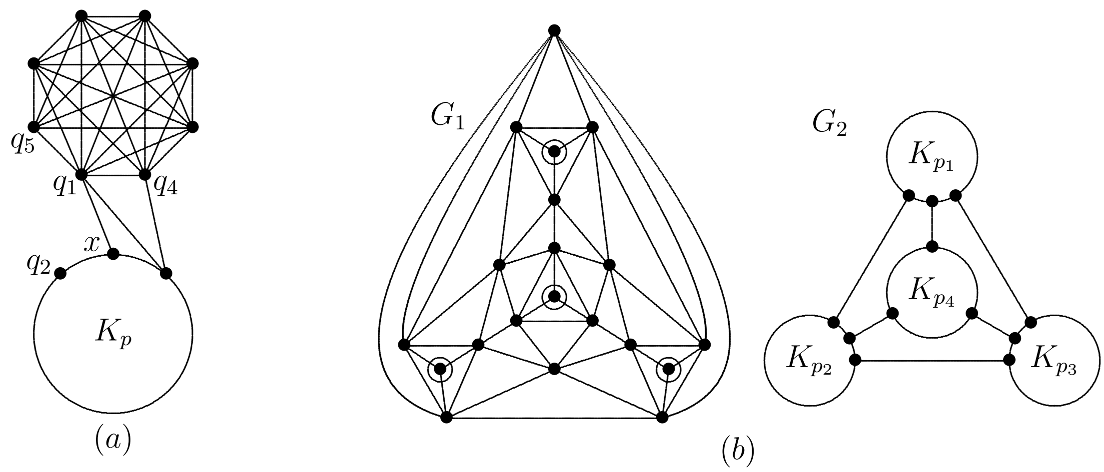

Figure 2: Two $\{K_{1,4}, K_{1,4}+e\}$-free graphs such that G_x^* is not $\{K_{1,4}, K_{1,4}+e\}$-free

The following proposition shows that the local completion operation preserves hamiltonicity or nonhamiltonicity of a graph $G \in \mathcal{F}$.

Proposition 3. *Let $G \in \mathcal{F}$, and let $x \in V_{EL}(G) \cap V(\mathcal{K}_G)$, or $x \in V_{EL}(G)$, if G is $(K_{1,4}+e)$-free. Then G_x^* is hamiltonian if and only if G is hamiltonian.*

Proof of Proposition 3 is postponed to Section 3.

2.3 Closure

Now we can introduce the main concept of this paper.

For a graph $G \in \mathcal{F}$, the *h-closure of* G, denoted $\mathrm{cl}^h(G)$, is the graph obtained from G by recursively performing the local completion operation at vertices $x \in V_{EL}(G) \cap V(\mathcal{K}_G)$, or $x \in V_{EL}(G)$ if G is $(K_{1,4}+e)$-free, as long as this is possible (more precisely, there is a sequence of graphs $G_1, \ldots, G_k$ such that (i) $G_1 = G$, (ii) $G_{i+1} = (G_i)^*_{x_i}$ for some $x_i \in V_{EL}(G_i) \cap V(\mathcal{K}_G)$, or $x_i \in V_{EL}(G_i)$ if G' is $(K_{1,4}+e)$-free, $i = 1, \ldots, k-1$, (iii) $V_{EL}(G_k) = \emptyset$; and we set $\mathrm{cl}^h(G) = G_k$). If a graph G is equal to its h-closure, we say that G is *h-closed*.

The following statement summarizes basic properties of the h-closure operation.

Theorem 4. *Let $G \in \mathcal{F}$. Then*

(i) $\mathrm{cl}^h(G)$ *is well-defined (i.e., uniquely determined),*

(ii) $\mathrm{cl}^h(G)$ *is the line graph of a triangle-free graph,*

(iii) $\mathrm{cl}^h(G)$ *is hamiltonian if and only if G is hamiltonian.*

Proof. (i) Let G^1, G^2 be two h-closures of a graph $G \in \mathcal{F}$, suppose that $E(G^1) \setminus E(G^2) \neq \emptyset$, and let $G_1, \ldots, G_t$ be the sequence of graphs that yields G^1. Let j be the smallest integer for which $E(G_j) \setminus E(G^2) \neq \emptyset$, and let $e = uv \in E(G_j) \setminus E(G^2)$. Then, since $e \in E(G_j)$, the vertices u, v are in the same component of $\langle N_{G_{j-1}}(x) \rangle_{G_{j-1}}$ for some vertex $x \in V_{EL}(G_{j-1})$. But then, since $E(\langle N_{G_{j-1}}(x) \rangle_{G_{j-1}}) \subset E(G_{j-1}) \subset E(G^2)$, the vertices u, v are in the same component of $\langle N_{G^2}(x) \rangle_{G^2}$, hence $e = uv \in E(G^2)$, a contradiction.

(ii) By the construction and by Lemma 1, the neighborhood in $\mathrm{cl}^h(G)$ of every vertex $u \in V(G)$ is either a clique (if $\langle N_G(u) \rangle_G$ is connected), or a disjoint union of two cliques (if $\langle N_G(u) \rangle_G$ is disconnected). It is straightforward to verify that such a graph is a line graph of a triangle-free graph (for easy details see e.g. Lemma 1 in [18]).

(iii) This follows immediately from Proposition 3 by induction. ■

Note that in the special case when G is claw-free, we have $\mathrm{cl}^h(G) = \mathrm{cl}(G)$.

2.4 Applications of the h-closure

In this section, we give several applications of the h-closure.

2.4.1 Thomassen's, Matthews-Sumner's and Bondy's conjectures

Thomassen [20] posed the following conjecture.

Conjecture C [20]. *Every 4-connected line graph is hamiltonian.*

Matthews and Sumner [16] stated the following, seemingly stronger conjecture.

Conjecture D [16]. *Every 4-connected claw-free graph is hamiltonian.*

So far, both these conjectures are wide open. However, it is known [18] that Conjectures C and D are equivalent. There are many further equivalent versions of these conjectures; among others, we mention here the "Dominating Cycle Conjecture", which appeared independently at several places, and which states that every snark (a 3-regular cyclically 4-edge-connected non-3-edge-colorable graph of girth at least 5) has a dominating cycle (for the equivalence, see [3]), or the statement that every 4-connected claw-free graph is 1-Hamilton-connected (see [19]). For more information on these equivalences, we refer the reader to the survey paper [4].

As a weaker version of Conjectures C and D, Bondy [9] suggested the following conjecture.

Conjecture E [9]. *There is a constant c_0 with $0 < c_0 \leq 1$ such that every cyclically 4-edge-connected cubic graph H of order n has a cycle of length at least $c_0 n$.*

It was known (see e.g. [4]) that Conjectures C and D imply Conjecture E, and, recently, Čada et al. [5] showed that Conjecture E implies the following weaker version of Conjecture C.

Conjecture F. *Every 4-connected line graph with minimum degree at least 5 is hamiltonian.*

We state here the following conjecture.

Conjecture 5. *Every 4-connected $\{K_{1,4}, K_{1,4}+e\}$-free graph with minimum degree at least 6 is hamiltonian.*

Comparing Conjecture 5 with Conjectures C – F, it seems that these should be independent, as Conjecture 5 deals with a larger class of graphs, but under an additional stronger assumption on the minimum degree. However, it turns out that Conjecture 5 is in fact equivalent with Conjecture F.

Theorem 6. *Conjecture 5 and Conjecture F are equivalent.*

Proof. (i) Suppose that Conjecture F is true, and let, to the contrary, G be a counterexample to Conjecture 5. Then $\mathrm{cl}^h(G)$ is a 4-connected nonhamiltonian line graph with $\delta(\mathrm{cl}^h(G)) \geq 6 > 5$, hence $\mathrm{cl}^h(G)$ is a counterexample to Conjecture F, a contradiction.

(ii) Conversely, let G be a counterexample to Conjecture F, i.e., a 4-connected nonhamiltonian line graph with $\delta(G) \geq 5$. Let $H = L^{-1}(G)$. Then H is an essentially 4-edge-connected graph such that $w_H(e) \geq 7$ for any $e \in E(H)$, and H has no DCT. Since $w_H(e) \geq 7$, every edge of H contains a vertex of degree at least 4. Let H' be the graph obtained from H by attaching three pendant edges to every vertex of degree at least 4. Then clearly H' is essentially 4-edge-connected (since the pendant edges were attached only to vertices of degree at least 4), has no DCT (since a DCT in H' would also be a DCT in H), and $w_{H'}(e) \geq w_H(e)+3 \geq 10 > 8$ for every edge $e \in E(H)$, and $w_{H'}(e) \geq 4+3+1 = 8$ for every pendant edge attached to H. Thus, the graph $G' = L(H')$ is a counterexample to Conjecture 5. ■

2.4.2 Hamiltonicity of graphs with high connectivity

The earliest positive results in the direction of Conjectures C and D, establishing a connectivity bound implying hamiltonicity in line graphs, were by Jackson [11], and independently by Zhan [21], who proved that every 7-connected line graph is hamiltonian [11], or Hamilton-connected [21], respectively. There were several improvements, decreasing the connectivity bound to 6, under some additional assumptions on vertices of degree 6 (for more details, see [4]). A substantial improvement was given by Kaiser and Vrána [12], who, using a new proof technique, showed the following.

Theorem G [12]. *Every 5-connected claw-free graph with minimum degree at least 6 is Hamilton-connected.*

Using a new closure technique [19] and reconsidering the proof, it was shown in [13] that the assumptions of Theorem G in fact imply 1-Hamilton-connectedness, and this is so far the strongest result in this direction. Here we extend Theorem G in another direction by extending the class of claw-free graphs to $\{K_{1,4}, K_{1,4}+e\}$-free graphs.

Theorem 7. *Every 5-connected $\{K_{1,4}, K_{1,4}+e\}$-free graph with minimum degree at least 6 is hamiltonian.*

Proof. If G is a counterexample to Theorem 7, then $\mathrm{cl}^h(G)$ is a counterexample to Theorem G. ■

2.4.3 Degree and neighborhood conditions for hamiltonicity

The first results, improving the classical Dirac's and Ore's degree conditions for hamiltonicity in the special case of claw-free graphs, were by Matthews and Sumner [17], who showed that every 2-connected claw-free graph G with $\delta(G) \geq \frac{n-2}{3}$ is hamiltonian, and by Zhang [22], who extended this result by showing that every κ-connected ($\kappa \geq 2$) claw-free graph G with $\sigma_{\kappa+1}(G) \geq n - \kappa$ is hamiltonian. Although both these results are sharp, there were many subsequent results, weakening the assumptions and describing families of "exceptional graphs". This process of consecutive improvements was concluded for 2-connected claw-free graphs by Favaron et al. [8], who gave a general method which, for arbitrary positive integer k, gives a finite number of finite families $\mathcal{F}_1, \ldots, \mathcal{F}_{r_k}$ of line graphs such that each $\mathcal{F}_i$ is generated by a single graph, and every "sufficiently large" claw-free graph G satisfying $\sigma_k(G) \geq n + k^2 - 4k + 7$ (or, as a corollary, $\delta(G) \geq \frac{n+k^2-4k+7}{k}$), is either hamiltonian, or $\mathrm{cl}(G) \in \cup_{i=1}^{r_k} \mathcal{F}_i$. The method was performed in [8] for $k = 6$, and in [14], with the help of a computer, for $k = 8$. Using a different approach, a similar result for minimum degree in 3-connected graphs was recently given by Chen, Lai and Xiong [6].

It is not difficult to observe that all these results can be directly extended to the class of $\{K_{1,4}, K_{1,4}+e\}$-free graphs with minimum degree at least 6 using the h-closure operation. We formulate this fact in the form of the following "metatheorem".

Theorem 8. *Let k an κ be positive integers, and let $f_k(n)$ be a function and $\mathcal{F}_k$ a family of line graphs such that every κ-connected claw-free graph G of order n satisfying $\sigma_k(G) \geq f_k(n)$ is either hamiltonian, or $\mathrm{cl}(G) \in \mathcal{F}_k$. Then every κ-connected $\{K_{1,4}, K_{1,4}+e\}$-free graph G of order n satisfying $\delta(G) \geq 6$ and $\sigma_k(G) \geq f_k(n)$ is either hamiltonian, or $\mathrm{cl}^h(G) \in \mathcal{F}_k$.*

Proof follows immediately from the fact that $\sigma_k(\mathrm{cl}^h(G)) \geq \sigma_k(G)$. ■

There are similar sufficient conditions which, instead of $\delta(G)$ or $\sigma_k(G)$, deal with the minimum of the neighborhood union $|N_G(x_1) \cup \ldots \cup N_G(x_k)|$ taken over all independent sets $\{x_1, \ldots, x_k\} \subset V(G)$ (see e.g. [1, 7, 15]). We conclude this subsection by observing that all these results also admit a direct extension to the class of $\{K_{1,4}, K_{1,4}+e\}$-free graphs with $\delta(G) \geq 6$ using the h-closure operation. We leave details to the reader.

3 Lemmas and proofs

3.1 Two lemmas on hamiltonian cycles in G_x^*

Lemma 9. *Let $G \in \mathcal{F}$ and $x \in V_{EL}(G)$. If G_x^* is hamiltonian, then there is a hamiltonian cycle C' in G_x^* such that C' uses at most two new edges.*

Proof. Let C be a hamiltonian cycle in G_x^* containing minimum number of new edges, and suppose, to the contrary, that C contains besides x at least three new edges u_1u_2, v_1v_2, z_1z_2. Choose the notation such that $C = u_1u_2\overrightarrow{C}v_1v_2\overrightarrow{C}z_1z_2\overrightarrow{C}x^-xx^+\overrightarrow{C}u_1$. If e.g. $x^+u_2 \in E(G)$, then $C' = u_1\overleftarrow{C}x^+u_2\overrightarrow{C}xu_1$ is a hamiltonian cycle in G_x^* containing less new edges, a contradiction. Hence $x^+u_2 \notin E(G)$, and similar contradictions show that $\{x^+, u_2, v_2, z_2\}$ is an independent set in G. But then $\langle\{x, x^+, u_2, v_2, z_2\}\rangle_G \simeq K_{1,4}$, a contradiction. ■

Lemma 10. *Let $G \in \mathcal{F}$ and $x \in V_{EL}(G)$. If G_x^* is hamiltonian, then there is a hamiltonian cycle C' in G_x^* such that C' uses at most one new edge.*

Proof. As above, let C be a hamiltonian cycle in G_x^* containing minimum number of new edges, and suppose, to the contrary, that C contains at least two new edges u_1u_2, v_1v_2. Choose the notation such that $C = u_1u_2\overrightarrow{C}v_1v_2\overrightarrow{C}x^-xx^+\overrightarrow{C}u_1$. First observe that $x^-x^+ \notin E(G)$, for otherwise, replacing in C the subpath x^-xx^+ by the edge x^-x^+ and the (new) edge u_1u_2 by the subpath u_1xu_2, we reduce the number of new edges in C. Similarly we have also $u_1u_2, v_1v_2, u_1v_1, u_1v_2, u_2v_2, u_2x^+, v_2x^+ \notin E(G)$ (note that, by Lemma 1, at least one of the vertices x^-, x^+, say, x^+, is in the same component of $\langle N_{G_x^*}(x)\rangle_{G_x^*}$ as the vertices u_1, u_2, v_1, v_2, hence $\langle\{x^+, u_1, u_2, v_1, v_2\}\rangle_{G_x^*}$ is a clique, and in each of the cases, we easily obtain a hamiltonian cycle in G_x^* with at most one new edge).

Now we distinguish (up to symmetry) two cases according to how many of the segments $x^+\overrightarrow{C}u_1$, $u_2\overrightarrow{C}v_1$, $v_2\overrightarrow{C}x^-$ are nontrivial.

Case 1: $u_2 = v_1$ *and* $x^- = v_2$ *(i.e., one segment, say,* x^+Cu_1*, is nontrivial).*

Since $\langle\{x, x^+, u_1, u_2, v_2\}\rangle_G$ cannot be a $K_{1,4}$, we have $x^+u_1 \in E(G)$. Then we have $\langle\{x, x^+, u_1, u_2, v_2\}\rangle_G \simeq K_{1,4} + e$, implying $\{x, x^+, u_1\} \subset V(\mathcal{K}_G)$.

The vertex x is eligible and hence $\langle N_G(x)\rangle_G$ contains a path joining the edge x^+u_1 and the vertices u_2 and v_2. Up to symmetry, there is a (u_2, u_1)-path or a (u_2, v_2)-path P in $\langle N_G(x)\rangle_G$. We choose the path P shortest possible.

Suppose first that P is of length at least 3. If $P = u_2p_1 \dots p_ku_1$, $k \geq 2$, then clearly $p_1u_1, p_1v_2 \notin E(G)$ (otherwise there is a shorter path), and then $\langle\{x, p_1, u_2, u_1, v_2\}\rangle_G \simeq K_{1,4} + e$, implying $\{x, p_1, u_2\} \subset V(\mathcal{K}_G)$, from which $u_1u_2 \in E(G)$, a contradiction. If $P = u_2p_1 \dots p_kv_2$, $k \geq 2$, then similarly $\langle\{x, p_1, u_2, u_1, v_2\}\rangle_G \simeq K_{1,4} + e$, implying $u_1u_2 \in E(G)$, a contradiction again.

Hence P is of length 2. Up to symmetry, either $P = u_2p_1u_1$ or $P = u_2p_1v_2$.

Subcase 1.a: $P = u_2p_1u_1$.

We have $p_1u_2 \notin E(C)$ since $u_1u_2, u_2v_2 \in E(C)$. If $u_1p_1 \in E(C)$, then $u_1 = p_1^+$, and then $C = u_1x^+\overrightarrow{C}p_1u_2v_2xu_1$ is a hamiltonian cycle in G_x^* using one new edge. Hence also $u_1p_1 \notin E(C)$, i.e., $p_1^-, p_1^+ \notin V(P)$. We consider the following possibilities.

Case	Hamiltonian cycle in G_x^* containing exactly one new edge
$p_1^-p_1^+ \in E(G)$	$C = u_1p_1u_2v_2xx^+\overrightarrow{C}p_1^-p_1^+\overrightarrow{C}u_1$
$p_1^-u_1 \in E(G)$	$C = u_1\overleftarrow{C}p_1u_2v_2xx^+\overrightarrow{C}p_1^-u_1$
$p_1^-u_2 \in E(G)$	$C = u_1x^+\overrightarrow{C}p_1^-u_2v_2xp_1\overrightarrow{C}u_1$
$p_1^+u_2 \in E(G)$	$C = u_1x^+\overrightarrow{C}p_1xv_2u_2p_1^+\overrightarrow{C}u_1$

In each of the cases we have reached a contradiction, hence $p_1^-p_1^+, p_1^-u_1, p_1^-u_2, p_1^+u_2 \notin E(G)$. Then $p_1^+u_1 \in E(G)$, for otherwise $\langle\{p_1, p_1^+, u_1, u_2, p_1^-\}\rangle_G \simeq K_{1,4}$. But then $\langle\{p_1, p_1^+, u_1, u_2, p_1^-\}\rangle_G \simeq K_{1,4} + e$, implying $\{p_1, p_1^+, u_1\} \subset V(\mathcal{K}_G)$. Since $\{x, x^+, u_1\} \subset V(\mathcal{K}_G)$, we have $p_1^+x^+ \in E(G)$, but then $C = u_1\overleftarrow{C}p_1^+x^+\overrightarrow{C}p_1u_2v_2xu_1$ is a hamiltonian cycle in G_x^* using one new edge, a contradiction.

Subcase 1.b: $P = u_2p_1v_2$.

Since $u_1u_2, u_2v_2, v_2x \in E(C)$, necessarily $u_2p_1, p_1v_2 \notin E(C)$, i.e., $p_1^-, p_1^+ \notin V(P)$. If $p_1^-p_1^+ \in E(G)$, then $C = u_1u_2p_1v_2xx^+\overrightarrow{C}p_1^-p_1^+\overrightarrow{C}u_1$ is a hamiltonian cycle in G_x^* using one new edge; hence $p_1^-p_1^+ \notin E(G)$. We further consider the following possibilities.

Case	Contradiction: a hamiltonian cycle in G
$p_1^-u_2 \in E(G)$	$C = u_1x^+\overrightarrow{C}p_1^-u_2xv_2p_1\overrightarrow{C}u_1$
$p_1^-v_2 \in E(G)$	$C = u_1x^+\overrightarrow{C}p_1^-v_2xu_2p_1\overrightarrow{C}u_1$
$p_1^+u_2 \in E(G)$	$C = u_1x^+\overrightarrow{C}p_1v_2xu_2p_1^+\overrightarrow{C}u_1$
$p_1^+v_2 \in E(G)$	$C = u_1x^+\overrightarrow{C}p_1u_2xv_2p_1^+\overrightarrow{C}u_1$

Hence $p_1^-u_2, p_1^-v_2, p_1^+u_2, p_1^+v_2 \notin E(G)$. But then $\langle\{p_1, p_1^+, p_1^-, u_2, v_2\}\rangle_G \simeq K_{1,4}$, a contradiction.

Case 2: $x^+ \neq u_1$ *and* $u_2 \neq v_1$ *(i.e., at least two segments are nontrivial).*

As in the previous case, $x^+u_1 \in E(G)$ (for otherwise $\langle\{x, x^+, u_1, u_2, v_2\}\rangle_G \simeq K_{1,4}$), and, symmetrically, $u_2v_1 \in E(G)$. Then $\langle\{x, x^+, u_1, u_2, v_2\}\rangle_G \simeq K_{1,4} + e)$, implying $\{x, x^+, u_1\} \subset V(\mathcal{K}_G)$, and, symmetrically, $\langle\{x, u_2, v_1, u_1, v_2\}\rangle_G \simeq K_{1,4} + e)$, implying $\{x, u_2, v_1\} \subset V(\mathcal{K}_G)$. But then $u_1u_2 \in E(G)$, a contradiction.

■

3.2 Proof of Proposition 3

Proof. Clearly, if G is hamiltonian, then G^*_x is hamiltonian as well. Let, conversely, G^*_x be hamiltonian, and suppose, to the contrary, that G is not hamiltonian. By Lemma 10, let C' be a hamiltonian cycle in G^*_x with one new edge e_1e_2, i.e., $C' = x^-xx^+C'e_1e_2C'x^-$. Thus, the vertices $\{x^-, x^+\}$ and $\{e_1, e_2\}$ divide C' into three segments, namely, $x^+C'e_1$, $e_2C'x^-$ and $\{x\}$, and at least one of the segments $x^+C'e_1$, $e_2C'x^-$ is nontrivial.

Case I: *the cycle C' can be chosen such that both $x^+C'e_1$ and $e_2C'x^-$ are nontrivial.*

Note that G contains no edge uv with $u \in \{x^+, e_1\}$ and $v \in \{e_2, x^-\}$ (otherwise there is a hamiltonian cycle in G). Thus, necessarily $x^+e_1 \in E(G)$ or $e_2x^- \in E(G)$ (or both), since otherwise $\langle\{x, x^+, x^-, e_1, e_2\}\rangle_G \simeq K_{1,4}$.

We denote the segments C_1, C_2, and their endvertices w_1, w_2 and z_1, z_2 by the following rules:

(*i*) if exactly one of the segments has adjacent endvertices (in G), we use C_1 to denote this segment, and w_1, w_2 to denote its endvertices (i.e., $w_1w_2 \in E(G)$),

(*ii*) if both segments have adjacent endvertices, we use C_1 and w_1, w_2 for that of them, for which, if possible, $\{w_1, w_2\} \cap V(\mathcal{K}_G) \neq \emptyset$.

Then C_2 denotes the second segment and z_1, z_2 denote its endvertices.

In the sequel, we say that a (u, v)-path P is a (C_1, C_2)*-path*, if P is a path in $\langle N_G(x)\rangle_G$ such that $u \in \{w_1, w_2\}$ and $v \in \{z_1, z_2\}$. We further fix the notation of the endvertices of C_1 and C_2 by the following rule.

(*iii*) If P is a shortest (C_1, C_2)-path, then we choose the notation of the endvertices of C_1 and C_2 such that P is a (w_2, z_2)-path (denoted $P(w_2, z_2)$).

Now, among all hamiltonian cycles in G^*_x with one new edge, we choose the cycle C' such that the length of the (C_1, C_2)-path $P(w_2, z_2)$ is minimum.

We first show that $P(w_2, z_2) = w_2v_1...v_jz_2$ has length at most 3. If $P(w_2, z_2) = w_2v_1...v_jz_2$ with $j \geq 3$, then necessarily $z_1z_2 \in E(G)$ (otherwise $\langle\{x, z_1, z_2, v_2, w_1\}\rangle_G \simeq K_{1,4}$), but then both $\langle\{x, w_1, w_2, z_2, v_2\}\rangle_G$ and $\langle\{x, z_1, z_2, w_1, v_2\}\rangle_G$ induce a $K_{1,4} + e$ such that the triangles are not in the same clique, a contradiction.

Subcase I.A: *the path $P(w_2, z_2) = w_2v_1v_2z_2$ has length* 3.

Note that in this subcase we do not exclude the possibility that $w_1 = w_2$, i.e. that C_1 is trivial (this remark, although redundant here, will be needed later on in the proof of Case II).

If $z_1z_2 \notin E(G)$, then $\langle\{x, v_1, w_2, z_1, z_2\}\rangle_G \simeq K_{1,4} + e$, implying $\{x, v_1, w_2\} \in V(\mathcal{K}_G)$, and then the condition (3)(iii) from the definition of the class $\mathcal{F}$ implies that some of z_1, z_2 is connected with v_1 and w_2 by a path of length 2, contradicting the assumption of this subcase. Hence $z_1z_2 \in E(G)$.

<u>Subcase I.A.1:</u> $v_2 \in V(C_1)$.

We have $v_2^-v_2^+ \notin E(G)$, since otherwise, for the cycle $C'' = w_1\overrightarrow{C_1}v_2^-v_2^+\overrightarrow{C_1}w_2z_1\overrightarrow{C_2}z_2v_2xw_1$ in G_x^*, we have $C_1 = w_1\overrightarrow{C_1}v_2^-v_2^+\overrightarrow{C_1}w_2$ and $C_2 = z_1\overrightarrow{C_2}z_2v_2$, hence there is a shorter (C_1, C_2)-path, contradicting the minimality of the (C_1, C_2)-path in the choice of C' (note that here, and in the following subcases, we have $v_1 \in V(C')$, hence $v_1 \in V(C'')$, and v_1 is not listed since its position on C' is not specified).

We further consider the following possibilities (see Fig. 3(a)).

Case	Contradiction: a hamiltonian cycle in G
$v_2^-z_2 \in E(G)$	$C = w_1\overrightarrow{C_1}v_2^-z_2\overleftarrow{C_2}z_1xv_2\overrightarrow{C_1}w_2w_1$
$v_2^+z_2 \in E(G)$	$C = w_1\overrightarrow{C_1}v_2xz_1\overrightarrow{C_2}z_2v_2^+\overrightarrow{C_1}w_2w_1$
$v_2^-x \in E(G)$	$C = w_1\overrightarrow{C_1}v_2^-xz_1\overrightarrow{C_2}z_2v_2\overrightarrow{C_1}w_2w_1$
$v_2^+x \in E(G)$	$C = w_1\overrightarrow{C_1}v_2z_2\overleftarrow{C_2}z_1xv_2^+\overrightarrow{C_1}w_2w_1$

Hence $v_2^-z_2, v_2^+z_2, v_2^-x, v_2^+x \notin E(G)$, and the subgraph $\langle\{v_2, x, z_2, v_2^-, v_2^+\}\rangle_G$ is an induced $K_{1,4} + e$, implying $\{v_2, x, z_2\} \subset V(\mathcal{K}_G)$. Thus $z_2 \in V(\mathcal{K}_G)$, but since $w_1, w_2 \notin N(z_2)$, neither w_1 nor w_2 is in $\mathcal{K}_G$, which contradicts condition (ii) in the choice of C_1.

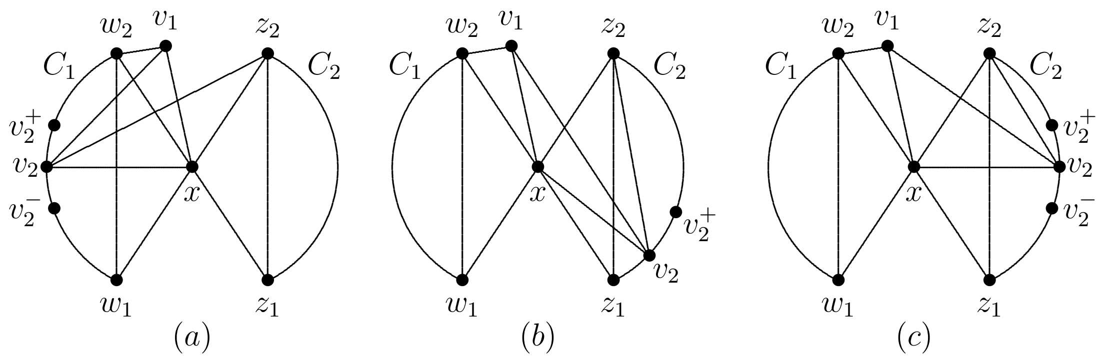

Figure 3: Subcases I.A.1, I.A.2.b) and I.A.2.c).

<u>Subcase I.A.2:</u> $v_2 \in V(C_2)$.

<u>Subcase I.A.2.a):</u> $v_2^- = z_1, v_2^+ = z_2$.

Since $z_1z_2 \in E(G)$, for the cycle $C'' = w_1\overrightarrow{C_1}w_2v_2z_2z_1xw_1$ in G_x^*, we have $C_2 = z_1z_2v_2$, contradicting the minimality of the (C_1, C_2)-path in the definition of C'.

<u>Subcase I.A.2.b):</u> $v_2^- = z_1, v_2^+ \neq z_2$ *(or, symmetrically,* $v_2^- \neq z_1, v_2^+ = z_2$*).*

Since $z_1z_2 \in E(G)$, there is a shorter (C_1, C_2)-path for $C_2 = v_2\overrightarrow{C_2}z_2z_1$ (see Fig. 3(b)).

Subcase I.A.2.c): $v_2^- \neq z_1, v_2^+ \neq z_2$.

If $v_2^-v_2^+ \in E(G)$ or $v_2^-z_2 \in E(G)$, then there is a shorter (C_1, C_2)-path for $C_2 = z_1\overrightarrow{C_2}v_2^-v_2^+\overrightarrow{C_2}z_2v_2$ or $C_2 = z_1\overrightarrow{C_2}v_2^-z_2\overleftarrow{C_2}v_2$, respectively; hence $v_2^-v_2^+, v_2^-z_2 \notin E(G)$ (see Fig. 3(c)).

Subcase I.A.2.c)(i): $v_1 \in V(C_1)$.

We consider the following possibilities (see Fig. 4(a)).

Case	Contradiction: a shorter (C_1, C_2)-path/a hamilt. cycle in G
$v_1^+ = w_2$	$P = v_1v_2z_2$ for $C_1 = w_2w_1\overrightarrow{C_1}v_1$ and $C_2 = z_1\overrightarrow{C_2}z_2$
$v_1^- = w_1$	$P = v_1v_2z_2$ for $C_1 = w_1w_2\overrightarrow{C_1}v_1$ and $C_2 = z_1\overrightarrow{C_2}z_2$
$v_1^-v_1^+ \in E(G)$	$P = v_1v_2z_2$ for $C_1 = w_1\overrightarrow{C_1}v_1^-v_1^+\overrightarrow{C_1}w_2v_1$ and $C_2 = z_1\overrightarrow{C_2}z_2$
$v_1^-x \in E(G)$	$P = v_1v_2z_2$ for $C_1 = v_1^-\overleftarrow{C_1}w_1w_2\overleftarrow{C_1}v_1$ and $C_2 = z_1\overrightarrow{C_2}z_2$
$v_1^+x \in E(G)$	$P = v_1v_2z_2$ for $C_1 = v_1^+\overrightarrow{C_1}w_2w_1\overrightarrow{C_1}v_1$ and $C_2 = z_1\overrightarrow{C_2}z_2$
$v_1^-w_2 \in E(G)$	$P = v_1v_2z_2$ for $C_1 = w_1\overrightarrow{C_1}v_1^-w_2\overleftarrow{C_1}v_1$ and $C_2 = z_1\overrightarrow{C_2}z_2$
$v_2^-v_1^- \in E(G)$	$C = w_1\overrightarrow{C_1}v_1^-v_2^-\overleftarrow{C_2}z_1xz_2\overleftarrow{C_2}v_2v_1\overrightarrow{C_1}w_2w_1$
$v_2^+v_1^- \in E(G)$	$C = w_1\overrightarrow{C_1}v_1^-v_2^+\overrightarrow{C_2}z_2xz_1\overrightarrow{C_2}v_2v_1\overrightarrow{C_1}w_2w_1$

Hence $v_1^+ \neq w_2$, $v_1^- \neq w_1$; $v_1^-v_1^+, v_1^-x, v_1^+x, v_1^-w_2, v_2^-v_1^-, v_2^+v_1^- \notin E(G)$.

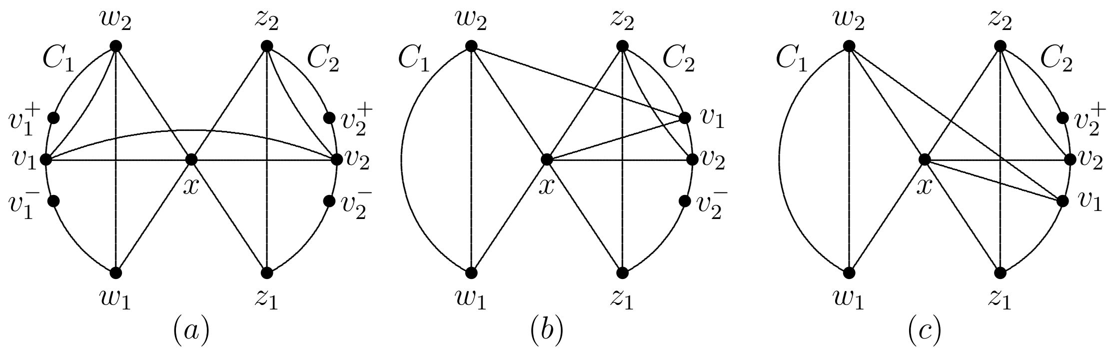

Figure 4: Subcases I.A.2.c)(i) and I.A.2.c)(ii).

We show that $v_1^-v_2 \notin E(G)$. Let, to the contrary, $v_1^-v_2 \in E(G)$. If $v_2^-v_1 \in E(G)$ or $v_2^+v_1 \in E(G)$, then the cycle $C = w_1\overrightarrow{C_1}v_1^-v_2\overrightarrow{C_2}z_2xz_1\overrightarrow{C_2}v_2^-v_1\overrightarrow{C_1}w_2w_1$ or $C = w_1\overrightarrow{C_1}v_1^-v_2\overleftarrow{C_2}z_1xz_2\overleftarrow{C_2}v_2^+v_1\overrightarrow{C_1}w_2w_1$ is a hamiltonian cycle in G, respectively; hence $v_2^-v_1, v_2^+v_1 \notin E(G)$. Then $\langle\{v_2, v_1, v_1^-, v_2^-, v_2^+\}\rangle_G$ is an induced $K_{1,4} + e$, hence $v_2, v_1, v_1^- \in V(\mathcal{K}_G)$, implying $v_1^-x \in E(G)$, a contradiction. Thus, $v_1^-v_2 \notin E(G)$.

Let us consider $\langle\{v_1, w_2, v_1^+, v_1^-, v_2\}\rangle_G$. If $v_2v_1^+ \notin E(G)$, then $v_1^+w_2 \in E(G)$ (otherwise $\langle\{v_1, w_2, v_1^+, v_1^-, v_2\}\rangle_G \simeq K_{1,4}$), but then $\langle\{v_1, w_2, v_1^+, v_1^-, v_2\}\rangle_G \simeq K_{1,4} + e$, implying $\{v_1, w_2, v_1^+\} \subset V(\mathcal{K}_G)$, contradicting the fact that $v_1^+x \notin E(G)$. Thus, we have $v_2v_1^+ \in E(G)$. Symmetrically also $v_2^+v_1 \in E(G)$, and then $C = w_1\overrightarrow{C_1}v_1v_2^+\overrightarrow{C_2}z_2xz_1\overrightarrow{C_2}v_2v_1^+\overrightarrow{C_1}w_2w_1$ is a hamiltonian cycle in G, a contradiction.

<u>**Subcase I.A.2.**$c)(ii)$**:**</u> $v_1 \in V(C_2)$.

First observe that if $w_1v_1 \notin E(G)$, then $\langle\{x, z_1, z_2, w_1, v_1\}\rangle_G \simeq K_{1,4} + e$, implying $\{x, z_1, z_2\} \in V(\mathcal{K}_G)$, which contradicts the choice of C_1 (condition (ii) in the definition of C_1); hence $w_1v_1 \in E(G)$. Moreover, v_1, v_2 cannot be consecutive on C_2: if $v_1 = v_2^+$, then $C = w_1\overrightarrow{C_1}w_2xz_1\overrightarrow{C_2}v_2z_2\overleftarrow{C_2}v_1w_1$ is a hamiltonian cycle in G (see Fig. 4(b)), and if $v_2 = v_1^+$, then, for $C_1 = z_1\overrightarrow{C_2}v_1w_2\overleftarrow{C_1}w_1$ and $C_2 = z_2\overleftarrow{C_2}v_2$, $P = w_1v_1v_2$ is a shorter (C_1, C_2)-path (see Fig. 4(c)).

We now consider the following possibilities.

Case	Contradiction: a shorter (C_1, C_2)-path/a hamilt. cycle in G
$v_1^-v_1^+ \in E(G)$	$P = v_1v_2z_2$ for $C_1 = w_1\overrightarrow{C_1}w_2v_1$ and $C_2 = z_1\overrightarrow{C_2}v_1^-v_1^+\overrightarrow{C_2}z_2$
$v_1^-w_2 \in E(G)$	$C = w_1\overrightarrow{C_1}w_2v_1^-\overleftarrow{C_2}z_1xz_2\overleftarrow{C_2}v_1w_1$
$v_1^+w_2 \in E(G)$	$C = w_1\overrightarrow{C_1}w_2v_1^+\overrightarrow{C_2}z_2xz_1\overrightarrow{C_2}v_1w_1$
$v_1^-z_2 \in E(G)$	$C = w_1\overrightarrow{C_1}w_2xz_1\overrightarrow{C_2}v_1^-z_2\overleftarrow{C_2}v_1w_1$

We show that $v_2v_1^-, v_2v_1^+ \in E(G)$ (not excluding the possibility that $v_1^+ = v_2^-$ or $v_2^+ = v_1^-$). Consider $F = \langle\{v_1, v_2, v_1^-, v_1^+, w_2\}\rangle_G$. If $v_2v_1^-, v_2v_1^+ \notin E(G)$, then $F \simeq K_{1,4}$; hence v_2 is adjacent to at least one of v_1^-, v_1^+. If $v_2v_1^- \in E(G)$ but $v_2v_1^+ \notin E(G)$, then $F \simeq K_{1,4} + e$, implying $\{v_1, v_2, v_1^-\} \subset V(\mathcal{K}_G)$ and $xv_1^- \in E(G)$; but then $C = w_1\overrightarrow{C_1}w_2xv_1^-\overleftarrow{C_2}z_1z_2\overleftarrow{C_2}v_1w_1$ is a hamiltonian cycle in G, a contradiction. Similarly, if $v_2v_1^+ \in E(G)$ but $v_2v_1^- \notin E(G)$, then $\langle\{v_1, v_2, v_1^+, v_1^-, w_2\}\rangle_G \simeq K_{1,4} + e$, hence $\{v_1, v_2, v_1^+\} \subset V(\mathcal{K}_G)$ and $xv_1^+ \in E(G)$, and then $C = w_1\overrightarrow{C_1}w_2xv_1^+\overrightarrow{C_2}z_2z_1\overrightarrow{C_2}v_1w_1$ is a hamiltonian cycle in G, a contradiction again. Thus, $v_2v_1^-, v_2v_1^+ \in E(G)$ (see Fig. 5(a); note that our argument works both if $v_1 \in z_1\overrightarrow{C_2}v_2$ and if $v_1 \in v_2\overrightarrow{C_2}z_2$, and does not exclude the possibility that $v_1^+ = v_2^-$ or $v_2^+ = v_1^-$).

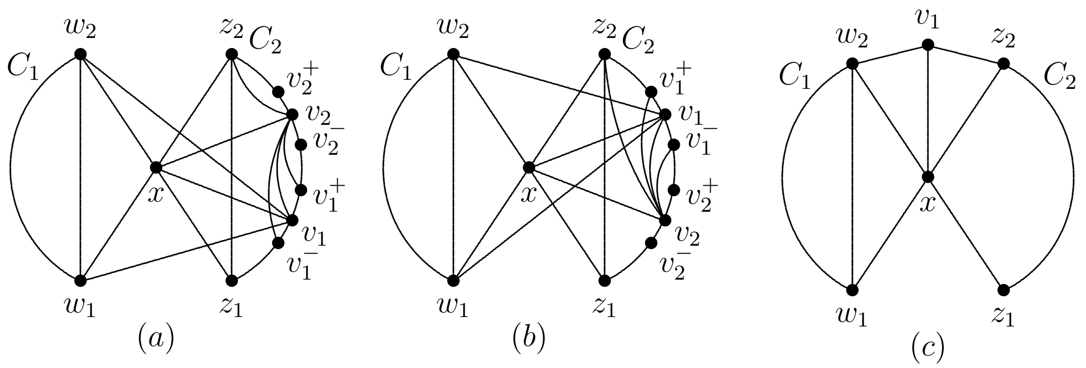

Figure 5: Subcases I.A.2.$c)(ii)$, I.A.2.$c)(ii)(\beta)$ and I.B.

<u>**Subcase I.A.2.**$c)(ii)(\alpha)$**:**</u> $v_1 \in z_1\overrightarrow{C_2}v_2$.

We consider the following possibilities.

Case	Contradiction: a shorter (C_1, C_2)-path/a hamilt. cycle in G
$v_2^+ v_1^+ \in E(G)$	$P = v_1 v_2 z_2$ for $C_1 = w_1 \overrightarrow{C_1} w_2 v_1$, $C_2 = z_1 \overrightarrow{C_2} v_1^- v_2 \overleftarrow{C_2} v_1^+ v_2^+ \overrightarrow{C_2} z_2$
$v_2^+ v_1^- \in E(G)$	$C = w_1 \overrightarrow{C_1} w_2 x z_1 \overrightarrow{C_2} v_1^- v_2^+ \overrightarrow{C_2} z_2 v_2 \overleftarrow{C_2} v_1 w_1$
$v_1^- v_2^- \in E(G)$	$P = v_1 v_2 z_2$ for $C_1 = w_1 \overrightarrow{C_1} w_2 v_1$, $C_2 = z_1 \overrightarrow{C_2} v_1^- v_2^- \overleftarrow{C_2} v_1^+ v_2 \overrightarrow{C_2} z_2$

Hence $v_2^+ v_1^+, v_2^+ v_1^-, v_1^- v_2^- \notin E(G)$.

Suppose first that $v_1^+ = v_2^-$, and consider $F_1 = \langle \{v_2, z_2, v_2^+, v_2^-, v_1^-\} \rangle_G$ (note that $v_2^+ \neq z_2$ by Subcase I.A.2.b)). If $v_2^+ z_2 \notin E(G)$, then $F_1 \simeq K_{1,4}$; hence $v_2^+ z_2 \in E(G)$ and then $F_1 \simeq K_{1,4} + e$. This specifically implies $z_2 \in V(\mathcal{K}_G)$, which contradicts condition (ii) in the definition of $\mathcal{C}_1$. Thus, $v_1^+ \neq v_2^-$ (see Fig. 5(a)).

We consider $F_2 = \langle \{v_2, v_2^-, v_1^+, v_1^-, v_2^+\} \rangle_G$. If $v_1^+ v_2^- \notin E(G)$, then $F_2 \simeq K_{1,4}$; hence $v_1^+ v_2^- \in E(G)$ and then $F_2 \simeq K_{1,4} + e$, which implies $\{v_2, v_2^-, v_1^+\} \subset V(\mathcal{K}_G)$ and $v_2^- x \in E(G)$. Then $C = w_1 \overrightarrow{C_1} w_2 v_1 \overleftarrow{C_2} z_1 z_2 \overleftarrow{C_2} v_2 v_1^+ \overrightarrow{C_2} v_2^- x w_1$ is a hamiltonian cycle in G.

<u>**Subcase I.A.2.c)$(ii)(\beta)$:**</u> $v_1 \in v_2 \overrightarrow{C_2} z_2$.

If $v_2^- v_1^- \in E(G)$, then, for $C_1 = w_1 \overrightarrow{C_1} w_2 v_1$ and $C_2 = z_1 \overrightarrow{C_2} v_2^- v_1^- \overleftarrow{C_2} v_2 v_1^+ \overrightarrow{C_2} z_2$, there is a shorter (C_1, C_2)-path, if $v_2^- v_1^+ \in E(G)$, then $C = w_1 \overrightarrow{C_1} w_2 x z_1 \overrightarrow{C_2} v_2^- v_1^+ \overrightarrow{C_2} z_2 v_2 \overrightarrow{C_2} v_1 w_1$, and if $v_2^- z_2 \in E(G)$, then $C = w_1 \overrightarrow{C_1} w_2 v_1 \overleftarrow{C_2} v_2 v_1^+ \overrightarrow{C_2} z_2 v_1^- \overleftarrow{C_2} z_1 x w_1$ is a hamiltonian cycle in G. Hence $v_2^- v_1^-, v_2^- v_1^+, v_2^- z_2 \notin E(G)$ (see Fig. 5(b)).

We consider $F = \langle \{v_2, z_2, v_1^+, v_1^-, v_2^-\} \rangle_G$ (note that $v_1^+ \neq z_2$ since otherwise there is a shorter (C_1, C_2)-path). If $v_1^+ z_2 \notin E(G)$, then $F \simeq K_{1,4}$; hence $v_1^+ z_2 \in E(G)$ and $F \simeq K_{1,4} + e$, implying $\{v_2, z_2, v_1^+\} \subset V(\mathcal{K}_G)$ and $v_1^+ x \in E(G)$, but then $C = w_1 \overrightarrow{C_1} w_2 x v_1^+ \overleftarrow{C_2} z_1 z_2 \overleftarrow{C_2} v_1 w_1$ is a hamiltonian cycle in G.

<u>**Subcase I.B:**</u> *the path* $P(w_2, z_2) = w_2 v_1 z_2$ *has length* 2.
(See Fig. 5(c).)

<u>**Subcase I.B.1:**</u> $v_1 \in V(C_1)$.

We consider the following possibilities (see Fig. 6(a)).

Case	Contradiction: a hamiltonian cycle in G
$v_1^- = w_1$	$C = w_1 w_2 \overleftarrow{C_1} v_1 z_2 \overleftarrow{C_2} z_1 x w_1$
$v_1^+ = w_2$	$C = w_1 \overrightarrow{C_1} v_1 z_2 \overleftarrow{C_2} z_1 x w_2 w_1$
$v_1^- v_1^+ \in E(G)$	$C = w_1 \overrightarrow{C_1} v_1^- v_1^+ \overrightarrow{C_1} w_2 v_1 z_2 \overleftarrow{C_2} z_1 x w_1$
$v_1^+ z_2 \in E(G)$	$C = w_1 \overrightarrow{C_1} v_1 x z_1 \overrightarrow{C_2} z_2 v_1^+ \overrightarrow{C_1} w_2 w_1$
$v_1^- z_2 \in E(G)$	$C = w_1 \overrightarrow{C_1} v_1^- z_2 \overleftarrow{C_2} z_1 x v_1 \overrightarrow{C_1} w_2 w_1$
$v_1^+ x \in E(G)$	$C = w_1 \overrightarrow{C_1} v_1 z_2 \overleftarrow{C_2} z_1 x v_1^+ \overrightarrow{C_1} w_2 w_1$
$v_1^- w_2 \in E(G)$	$C = w_1 \overrightarrow{C_1} v_1^- w_2 \overleftarrow{C_1} v_1 z_2 \overleftarrow{C_2} z_1 x w_1$

Hence $v_1^- \neq w_1$, $v_1^+ \neq w_2$, and $v_1^- v_1^+, v_1^+ z_2, v_1^- z_2, v_1^+ x, v_1^- w_2 \notin E(G)$.

We consider $F = \langle\{v_1, w_2, z_2, v_1^-, v_1^+\}\rangle_G$. If $v_1^+w_2 \notin E(G)$, then $F \simeq K_{1,4}$; hence $v_1^+w_2 \in E(G)$. Then $F \simeq K_{1,4} + e$, implying $\{v_1, v_1^+, w_2\} \subset V(\mathcal{K}_G)$, contradicting the fact that $v_1^+x \notin E(G)$.

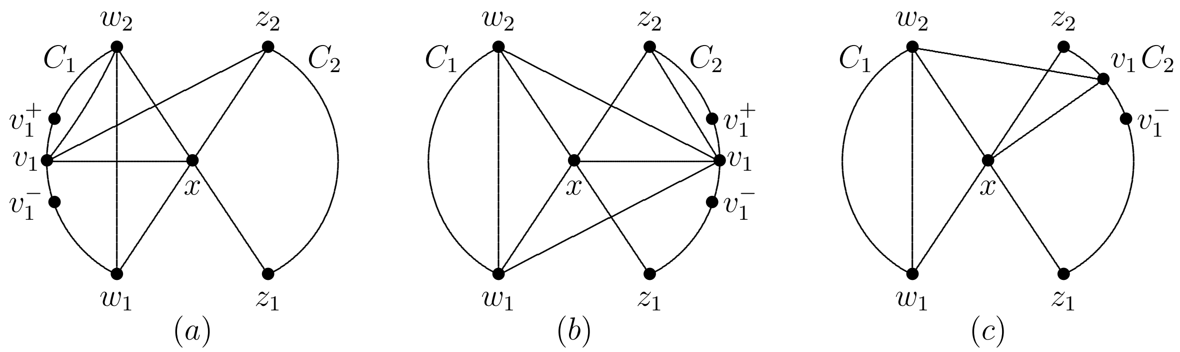

Figure 6: Subcases I.B.1, I.B.2.a) and I.B.2.b).

<u>**Subcase I.B.2:**</u> $v_1 \in V(C_2)$.

We have $z_1z_2 \notin E(G)$, since otherwise the situation is symmetric to that in Subcase I.B.1. Therefore, $\langle\{x, w_1, w_2, z_1, z_2\}\rangle_G \simeq K_{1,4} + e$, implying $\{x, w_1, w_2\} \subset V(\mathcal{K}_G)$. By condition (3)($ii$) from the definition of the class $\mathcal{F}$, v_1 can be chosen such that $v_1 \in V(\mathcal{K}_G)$, implying $w_1v_1 \in E(G)$.

<u>**Subcase I.B.2.**a**):**</u> $v_1^- \neq z_1, v_1^+ \neq z_2$.

We consider the following possibilities (see Fig. 6(b)).

Case	Contradiction: a hamiltonian cycle in G
$v_1^-v_1^+ \in E(G)$	$C = w_1\overrightarrow{C_1}w_2v_1z_2\overleftarrow{C_2}v_1^-v_1^+\overleftarrow{C_2}z_1xw_1$
$v_1^+w_2 \in E(G)$	$C = w_1\overrightarrow{C_1}w_2v_1^+\overrightarrow{C_2}z_2v_1\overleftarrow{C_2}z_1xw_1$
$v_1^-w_2 \in E(G)$	$C = w_1\overrightarrow{C_1}w_2v_1^-\overleftarrow{C_2}z_1xz_2\overleftarrow{C_2}v_1w_1$
$v_1^-z_2 \in E(G)$	$C = w_1\overrightarrow{C_1}w_2v_1\overrightarrow{C_2}z_2v_1^-\overleftarrow{C_2}z_1xw_1$

Hence $v_1^-v_1^+, v_1^+w_2, v_1^-w_2, v_1^-z_2 \notin E(G)$.

We consider $F = \langle\{v_1, z_2, v_1^+, v_1^-, w_2\}\rangle_G$. If $v_1^+z_2 \notin E(G)$, then $F \simeq K_{1,4}$; hence $v_1^+z_2 \in E(G)$, and then $F \simeq K_{1,4} + e$, implying $\{v_1, z_2, v_1^+\} \subset V(\mathcal{K}_G)$. Since also $\{x, w_1, w_2\} \subset V(\mathcal{K}_G)$, we have $w_2z_2 \in E(G)$, a contradiction.

<u>**Subcase I.B.2.**b**):**</u> $v_1^+ = z_2$

We consider $F = \langle\{x, w_1, w_2, z_1, z_2\}\rangle_G \simeq K_{1,4} + e$ (see Fig. 6(c)). By condition (3)(iii) from the definition of $\mathcal{F}$, there are three internally vertex-disjoint paths P_1', P_2', P_3' of length 2 in $\langle N_G(x)\rangle_G$ such that each P_i is a (w_2, z_2)-path, a (w_2, z_1)-path, or a (z_1, z_2)-path. Since each of the paths P_1', P_2', P_3' has its interior vertex in $V(C_1) \setminus \{w_1, w_2\}$ or in $V(C_2) \setminus \{z_1, z_2\}$, one of them, denoted P', has its interior vertex $p \in N_G(x) \setminus \{z_1^+, z_2^-\}$.

<u>**Subcase I.B.2.**b**)(**i**):**</u> *P' is a (w_2, z_2)-path.*

Then, relabeling $P := P'$ and $v_1 := p$, we are back in Subcase I.B.1 if $p \in V(C_1)$, or in Subcase I.B.2.a) if $p \in V(C_2)$ (note that $p \in V(\mathcal{K}_G)$ by condition (3)(ii)).

Subcase I.B.2.b)(ii): *P' is a (w_2, z_1)-path.*

Then, reversing the orientation of C_2 and relabeling $z_1 \leftrightarrow z_2$, we are in the previous subcase.

Subcase I.B.2.b)(iii): *P' is a (z_1, z_2)-path.*

Let first $p \in V(C_2)$. If $p^-p^+ \in E(G)$, then $C = w_1\overrightarrow{C_1}w_2v_1\overleftarrow{C_2}p^+p^-\overleftarrow{C_2}z_1pz_2xw_1$, and if $p^-z_2 \in E(G)$, then $C = w_1\overrightarrow{C_1}w_2v_1\overleftarrow{C_2}pz_2p^-\overleftarrow{C_2}z_1xw_1$ is a hamiltonian cycle in G; hence $p^-p^+, p^-z_2 \notin E(G)$. Symmetrically, $p^+z_1 \notin E(G)$, and we also know that $z_1z_2 \notin E(G)$ (see Fig. 7(a)). If $p^+z_2 \notin E(G)$, then $p^-z_1 \in E(G)$, for otherwise $\langle\{p, p^-, z_1, p^+, z_2\}\rangle_G \simeq K_{1,4}$, but then $\langle\{p, p^-, z_1, p^+, z_2\}\rangle_G \simeq K_{1,4} + e$, implying $z_1 \in V(\mathcal{K}_G)$, a contradiction. Hence $p^+z_2 \in E(G)$, and then $C = w_1\overrightarrow{C_1}w_2v_1\overleftarrow{C_2}p^+z_2p\overleftarrow{C_2}z_1xw_1$ is a hamiltonian cycle in G, a contradiction.

Thus, $p \in V(C_1)$. We consider the following possibilities (see Fig. 7(b)).

Case	Contradiction: a hamiltonian cycle in G
$p^-p^+ \in E(G)$	$C = w_1\overrightarrow{C_1}p^-p^+\overrightarrow{C_1}w_2xz_2pz_1\overrightarrow{C_2}v_1w_1$
$p^+z_1 \in E(G)$	$C = w_1\overrightarrow{C_1}pz_2\overleftarrow{C_2}z_1p^+\overrightarrow{C_1}w_2xw_1$
$p^+z_2 \in E(G)$	$C = w_1\overrightarrow{C_1}pz_1\overrightarrow{C_2}z_2p^+\overrightarrow{C_1}w_2xw_1$

Hence $p^-p^+, p^+z_1, p^+z_2 \notin E(G)$, and, symmetrically, also $p^-z_1, p^-z_2 \notin E(G)$. Since also $z_1z_2 \notin E(G)$, we have $\langle\{p, p^-, p^+, z_1, z_2\}\rangle_G \simeq K_{1,4}$, a contradiction.

Subcase I.B.2.c): *$v_1^- = z_1, v_1^+ \neq z_2$.*

Then, reversing the orientation of C_2 and relabeling $z_1 \leftrightarrow z_2$, we are in Subcase I.B.2.b) (see Fig. 7(c)).

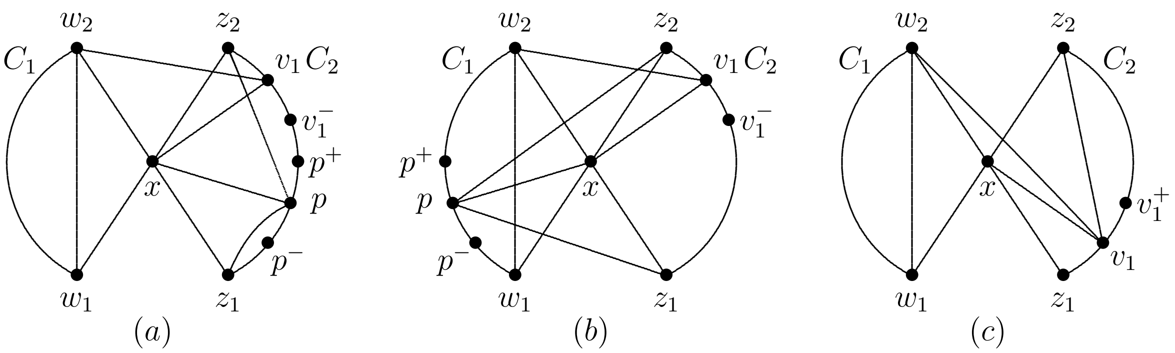

Figure 7: Subcases I.B.2.b)(iii) and I.B.2.c).

Case II: *for any choice of C', one of $x^+C'e_1$, $e_2C'x^-$ is trivial.*

Choose the notation such that $e_2 = x^-$ and $C' = x^-xx^+C_1e_1x^-$, and, as in Case I, rename by w_1, w_2 the endvertices of C_1 and set $z_1 = x^-$. Take a shortest path $P(w_2, z_1) = w_2v_1 \ldots v_kz_1$

in $\langle N_G(x)\rangle_G$, and choose C' such that $P(w_2, z_1)$ is shortest possible. Of course, C' is a hamiltonian cycle, hence $v_1, \ldots, v_k \in V(C_1) \setminus \{w_1, w_2\}$. Also observe that if $k \geq 3$, then $w_1w_2 \in E(G)$, for otherwise $\langle\{x, w_1, w_2, v_2, z_1\}\rangle_G \simeq K_{1,4}$. Then $\langle\{x, w_1, w_2, v_2, z_1\}\rangle_G \simeq K_{1,4} + e$, implying that, if $k \geq 3$, then $\{x, w_1, w_2\} \subset V(\mathcal{K}_G)$.

Now, if $k \geq 4$, then $\langle\{x, v_k, z_1, w_2, v_{k-2}\}\rangle_G \simeq K_{1,4} + e$, implying $z_1 \in V(\mathcal{K}_G)$, thus $w_2z_1 \in E(G)$, a contradiction. Hence $k \leq 3$, i.e., $P(w_2, z_1)$ is of length at most 4. Moreover, if $k = 2$, then we obtain a contradiction by Subcase I.A. (recall that, in the proof, we admitted $w_1 = w_2$, which, by symmetry, gives the necessary proof for this case). Hence $k \in \{1, 3\}$, i.e., $P(w_2, z_1)$ is of length 4 or 2.

Subcase II.A: *the path* $P(w_2, z_1) = w_2v_1v_2v_3z_1$ *has length* 4.
If $v_3^-x \in E(G)$, then $C = w_1\overrightarrow{C_1}v_3^-xz_1v_3\overrightarrow{C_1}w_2w_1$ is a hamiltonian cycle in G; hence $v_3^-x \notin E(G)$, and symmetrically also $v_3^+x \notin E(G)$. This specifically also implies that $v_3^-, v_3^+ \notin P(w_2, z_1)$. If $v_3^-v_3^+ \in E(G)$, then, setting $C_1 = w_1\overrightarrow{C_1}v_3^-v_3^+\overrightarrow{C_1}w_2$ and $C_2 = z_1v_3$, we are back in Case I; hence $v_3^-v_3^+ \notin E(G)$. If $v_3^-z_1 \in E(G)$, then $C = w_1\overrightarrow{C_1}v_3^-z_1v_3\overrightarrow{C_1}w_2xw_1$ is a hamiltonian cycle in G; hence $v_3^-z_1 \notin E(G)$, and symmetrically also $v_3^+z_1 \notin E(G)$. Then $\langle\{v_3, x, z_1, v_3^-, v_3^+\}\rangle_G \simeq K_{1,4} + e$ (see Fig. 8(*a*)). Since $\{x, w_1, w_2\} \subset V(\mathcal{K}_G)$, we have $w_2z_1 \in E(G)$, a contradiction.

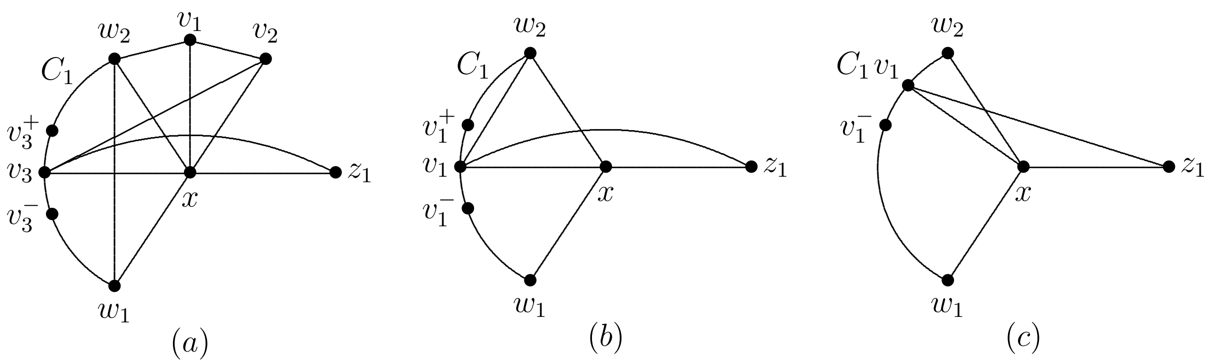

Figure 8: Subcases II.A, II.B.1 and II.B.2.

Subcase II.B: *the path* $P(w_2, z_1) = w_2v_1z_1$ *has length* 2.

Subcase II.B.1: $w_1^+ \neq v_1, v_1^+ \neq w_2$.
We consider the following possibilities (see Fig. 8(*b*)).

Case	Contradiction: a hamiltonian cycle in G
$v_1^-v_1^+ \in E(G)$	$C = w_1\overrightarrow{C_1}v_1^-v_1^+\overrightarrow{C_1}w_2v_1z_1xw_1$
$v_1^-z_1 \in E(G)$	$C = w_1\overrightarrow{C_1}v_1^-z_1v_1\overrightarrow{C_1}w_2xw_1$
$v_1^+z_1 \in E(G)$	$C = w_1\overrightarrow{C_1}v_1z_1v_1^+\overrightarrow{C_1}w_2xw_1$
$v_1^-w_2 \in E(G)$	$C = w_1\overrightarrow{C_1}v_1^-w_2\overleftarrow{C_1}v_1z_1xw_1$

Hence $v_1^-v_1^+, v_1^-z_1, v_1^+z_1, v_1^-w_2 \notin E(G)$.

We consider $F = \langle\{v_1, v_1^+, w_2, v_1^-, z_1\}\rangle_G$. If $v_1^+w_2 \notin E(G)$, then $F \simeq K_{1,4}$; hence $v_1^+w_2 \in E(G)$, and then $F \simeq K_{1,4} + e$, implying $\{v_1, v_1^+, w_2\} \subset V(\mathcal{K}_G)$. Therefore also $v_1^+x \in E(G)$. Then, for $C_1 = w_1\overrightarrow{C_1}v_1z_1$ and $C_2 = w_2\overleftarrow{C_1}v_1^+$, we are back in Case I.

Subcase II.B.2: $v_1^+ = w_2$.

If $w_1w_2 \in E(G)$, then $C = w_1\overrightarrow{C_1}v_1z_1xw_2w_1$ is a hamiltonian cycle in G; hence $w_1w_2 \notin E(G)$ (see Fig. 8(c)). By the assumption that $\delta(G) \geq 6$, there is a vertex $n \in N_G(x)$ such that $n \notin \{w_1, w_2, v_1, z_1, w_1^+\}$. Since C_2 is trivial and C' is a hamiltonian cycle, necessarily $n \in V(C_1)$. We consider the subgraph $F = \langle\{x, n, w_1, w_2, z_1\}\rangle_G$. Since $F \not\simeq K_{1,4}$ and $\{w_1, w_2, z_1\}$ is independent, necessarily $N_G(n) \cap \{w_1, w_2, z_1\} \neq \emptyset$.

First suppose that $|N_G(n) \cap \{w_1, w_2, z_1\}| \geq 2$.

(α) If $w_1n, w_2n \in E(G)$, we relabel $C_1 := z_1v_1\overleftarrow{C_1}w_1$, $w_1 := z_1$, $w_2 := w_1$, $z_1 := w_2$ and $v_1 := n$;

(β) if $w_2n, z_1n \in E(G)$, we set $v_1 := n$;

(γ) if $w_1n, z_1n \in E(G)$, we set $v_1 := n$, $w_1 \leftrightarrow w_2$, and reverse the orientation of C_1;

and, in all three cases, we are back in Subcase II.B.1. Hence $|N_G(n) \cap \{w_1, w_2, z_1\}| = 1$.

Subcase II.B.2.a): $w_1n \in E(G)$.

Then $F = \langle\{x, n, w_1, w_2, z_1\}\rangle_G \simeq K_{1,4} + e$, implying $\{x, n, w_1\} \subset V(\mathcal{K}_G)$ (see Fig. 9(a)). As in Subcase I.B.2.b), by condition (3)(iii) from the definition of $\mathcal{F}$, there is a path P' in $\langle N_G(x)\rangle_G$ of length 2 with interior vertex $p \in N_G(x) \setminus \{w_1^+, w_2^-\}$ such that P' is a (w_2, z_1)-path, a (w_1, z_1)-path, or a (w_1, w_2)-path.

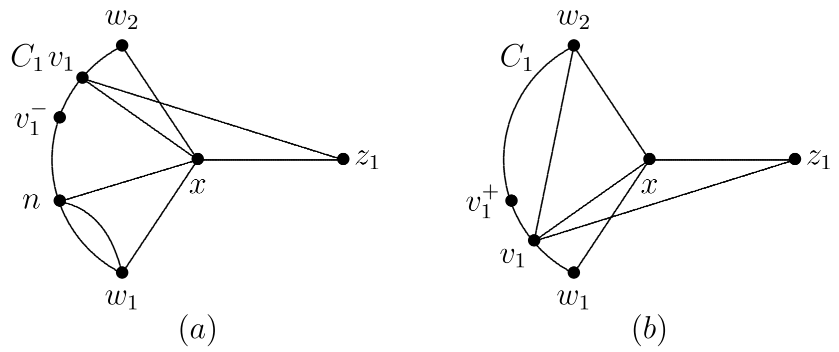

Figure 9: Subcases II.B.2.(a) and II.B.3.

Subcase II.B.2.a)(i): *P' is a (w_2, z_1)-path.*

We relabel $P := P'$, $v_1 := p$, and we are in Subcase II.B.1.

Subcase II.B.2.a)(ii): *P' is a (w_1, z_1)-path.*

We reverse the orientation of C_1 and relabel $w_1 \leftrightarrow w_2$, and we are in the previous subcase.

Subcase II.B.2.a)(iii): *P' is a (w_1, w_2)-path.*

We relabel $C_1 := z_1v_1\overleftarrow{C_1}w_1$, $w_1 := z_1$, $w_2 := w_1$, $z_1 := w_2$, $P := P'$ and $v_1 := p$, and we are again in Subcase II.B.1.

Subcase II.B.2.b): $w_2n \in E(G)$.

Then similarly $F = \langle\{x, n, w_2, w_1, z_1\}\rangle_G \simeq K_{1,4} + e$, implying $\{x, n, w_2\} \subset V(\mathcal{K}_G)$, and, by condition (3)($iii$), there is a path P' in $\langle N_G(x)\rangle_G$ of length 2 with interior vertex $p \in N_G(x) \setminus \{w_1^+, w_2^-\}$ which is a (w_2, z_1)-path, a (w_1, z_1)-path, or a (w_1, w_2)-path.

Subcase II.B.2.b)(i): *P' is a (w_2, z_1)-path.*

We set $P := P'$, $v_1 := p$, and we are in Subcase II.B.1.

Subcase II.B.2.b)(ii): *P' is a (w_1, z_1)-path.*

We reverse the orientation of C_1, relabel $w_1 \leftrightarrow w_2$, and we are in the previous subcase.

Subcase II.B.2.b)(iii): *P' is a (w_1, w_2)-path.*

We relabel $C_1 := z_1 v_1 \overleftarrow{C_1} w_1$, $w_1 := z_1$, $w_2 := w_1$, $z_1 := w_2$, $P := P'$ and $v_1 := p$, and we are in Subcase II.B.1.

Subcase II.B.2.c): $z_1n \in E(G)$.

We relabel $C_1 := w_1 \overrightarrow{C_1} v_1 z_1$ and $w_2 \leftrightarrow z_1$, and we are in Subcase II.B.2.b).

Subcase II.B.3: $w_1^+ = v_1$.

We reverse the orientation of C_1, relabel $w_1 \leftrightarrow w_2$, and we are in Subcase II.B.2 (see Fig. 9(b)).

■

4 Concluding remarks

1. The graph G in Fig. 10(a) is nonhamiltonian and $\{K_{1,4}, K_{1,4} + e\}$-free; however, since G is locally connected, $\mathrm{cl}^h(G)$ is complete, thus hamiltonian. This example shows that Proposition 3 and Theorem 4 cannot be true without the minimum degree assumption. An infinite family of graphs with similar properties (nonhamiltonian $\{K_{1,4}, K_{1,4} + e\}$-free with hamiltonian h-closure) is shown in Fig. 10(b).

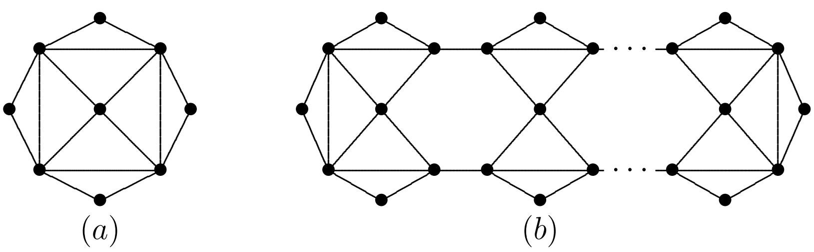

Figure 10: Nonhamiltonian $\{K_{1,4}, K_{1,4} + e\}$-free graphs with hamiltonian h-closure

2. Let H be the graph in Fig. 11, let H^+ be obtained from H by attaching at least two pendant edges to each of its black vertices, let $\overline{G} = L(H^+)$, and let G be the graph obtained from $\overline{G}$ by removing the edge $x_{e_3}x_{e_4}$, where x_{e_i} is the vertex of $\overline{G} = L(H^+)$ corresponding to the edge e_i of H^+, $i = 1, 2, 3, 4$. Since H^+ is triangle-free, $\overline{G}$ is a closed line graph. Moreover, in G, the only claw centers are the vertices x_{e_1}, x_{e_2}, and it is easy to see that G is $\{K_{1,4}, K_{1,4} + e\}$-free. One component of $\langle N_G(x_{e_1})\rangle_G$ is the path $x_{e_3}x_{e_2}x_{e_4}$ and, in $G^*_{x_{e_1}}$, the edge $x_{e_3}x_{e_4}$ is added, hence $\mathrm{cl}^h(G) = \overline{G}$.

It is straightforward to verify that none of the edges f_1, f_2, f_3 can be contained in a DCT in H^+, hence any DCT in H^+ must pass through the edges e_1, e_2, and the edges e_3, e_4 are dominated, but not passed. This means that H^+ has a DCT, hence $\overline{G} = \mathrm{cl}^h(G)$ is hamiltonian, but every hamiltonian cycle in $\overline{G}$ must contain the edge $x_{e_3}x_{e_4}$. Consequently, the graph G is nonhamiltonian. We have $\delta(G) = 4$ (and, moreover, G is 3-connected), hence this example shows that Proposition 3 and Theorem 4 cannot be true even if the minimum degree assumption is weakened to $\delta(G) \geq 4$.

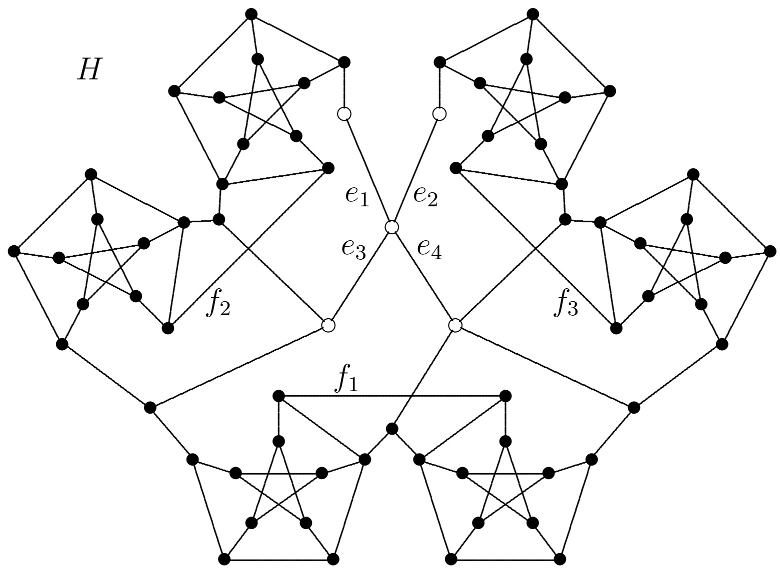

Figure 11: A nonhamiltonian $\{K_{1,4}, K_{1,4} + e\}$-free graph with hamiltonian h-closure

3. We admit that our results could be true for $\delta(G) \geq 5$, but, since our proof heavily relies on the condition $\delta(G) \geq 6$, the proof of such an improvement would require a new idea, and we leave this as an open question.

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