On forbidden induced subgraphs for $K_{1,3}$-free perfect graphs*

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Abstract

Considering connected $K_{1,3}$-free graphs with independence number at least 3, Chudnovsky and Seymour (2010) showed that every such graph, say G, is 2ω-colourable where ω denotes the clique number of G. We study $(K_{1,3}, Y)$-free graphs, and show that the following three statements are equivalent.

(1) Every connected $(K_{1,3}, Y)$-free graph which is distinct from an odd cycle and which has independence number at least 3 is perfect.

(2) Every connected $(K_{1,3}, Y)$-free graph which is distinct from an odd cycle and which has independence number at least 3 is ω-colourable.

(3) Y is isomorphic to an induced subgraph of P_5 or Z_2 (where Z_2 is also known as hammer).

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Furthermore, for connected $(K_{1,3}, Y)$-free graphs (without an assumption on the independence number), we show a similar characterisation featuring the graphs P_4 and Z_1 (where Z_1 is also known as paw).

Keywords: perfect graphs, vertex colouring, forbidden induced subgraphs

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1 Introduction

We consider finite, simple, and undirected graphs but do not distinguish between isomorphic graphs. For terminology and notation not defined here, we refer the reader to [2].

We recall that an *induced subgraph* of a graph G is a graph on a vertex set $S \subseteq V(G)$ for which two vertices are adjacent if and only if these two are adjacent in G; in particular, we say that this subgraph is *induced by* S. Furthermore, a graph H is an *induced subgraph* of G if H is isomorphic to an induced subgraph of G. G is $(H_1, H_2, \ldots, H_k)$*-free* if none of its induced subgraphs is isomorphic to one of $\{H_1, H_2, \ldots, H_k\}$.

A graph is k*-colourable* if its vertices can be coloured with k colours so that adjacent vertices obtain distinct colours. The smallest integer k such that a given graph G is k-colourable is called the *chromatic number* of G, denoted by $\chi(G)$. Clearly, $\chi(G) \geq \omega(G)$ for every graph G, where $\omega(G)$, the so-called *clique number* of G, denotes the order of a maximum complete subgraph of G. For brevity, we shall say that a graph G is ω*-colourable* if $\chi(G) = \omega(G)$. Furthermore, G is *perfect* if $\chi(G') = \omega(G')$ for every induced subgraph G' of G.

Considering a relation between colourings and forbidden induced subgraphs, Gyárfás and, independently, Sumner conjectured the following:

Conjecture 1.1 (Gyárfás-Sumner Conjecture [8],[12])**.** *For every forest F, there is a function $f_F \colon \mathbb{N} \to \mathbb{N}$ such that if G is an F-free graph, then G is $f_F(\omega(G))$-colourable.*

If the graph F contains a cycle, then no such function f_F exists since Erdős [7] showed that there are graphs of arbitrarily large girth and chromatic number.

Considering forbidden induced forests, for example, Gyárfás proved $\chi(G) \leq R(\omega(G), 3)$ for every $K_{1,3}$-free graph G; here $R(\omega(G), 3)$ denotes the classical Ramsey number, that is, $R(\omega(G), 3)$ denotes the least integer k such that each graph of order at least k contains an independent set of size 3, or an induced subgraph which is complete and of order $\omega(G)$. This upper bound follows from Ramsey's Theorem [11] and from the trivial inequality $\chi(G) \leq \Delta(G)+1$, where $\Delta(G)$ denotes the maximum degree of G. However, $R(\omega(G), 3)$ is known to be non-linear in $\omega(G)$, cf. [9]. Additionally, every upper bound on the chromatic number in terms of the clique number of a $K_{1,3}$-free graph is non-linear since if G is a graph with independence number 2, then $\chi(G) \geq \frac{|V(G)|}{2}$ (since every colour can be used at most twice) but $\omega(G)$ might be at most $9\sqrt{|V(G)| \cdot \log |V(G)|}$ (by a result of Kim [9]). In particular, Brause et al. [3] showed that even in the class of $(2K_2, 3K_1)$-free graphs, we cannot bound the chromatic number of such a graph by a linear function depending on its clique number only. Excluding all graphs with independence number 2, Chudnovsky and Seymour [5] showed that every connected $K_{1,3}$-free graph G with independence number at least 3 is $2\omega(G)$-colourable.

In addition, Brause et al., showed the following:

Theorem 1.2 (Brause et al. [3]). *Every connected $(K_{1,3}, 2K_2)$-free graph with independence number at least 3 is perfect.*

In this paper, we generalise the result of Theorem 1.2. We recall that, given a graph G, a *hole* in G is an induced cycle of length at least 4, and an *antihole* in G is an induced subgraph whose complement is a cycle of length at least 4. A hole (antihole) is *odd* if it has an odd number of vertices. As a main tool, we shall use the following result of Chudnovsky et al. [4].

Theorem 1.3 (The Strong Perfect Graph Theorem [4]). *A graph is perfect if and only if it contains neither an odd hole nor an odd antihole.*

With the aid of Theorem 1.3 and using results of Olariu and Ben Rebea (see Theorem 3.1 and Lemma 3.3, respectively), we present proofs of the following two results:

Theorem 1.4. *Let Y be a graph. If $\mathcal{G}$ denotes the class of all connected $(K_{1,3}, Y)$-free graphs that are distinct from an odd cycle, then the following statements are equivalent.*

(1) *Every graph of $\mathcal{G}$ is perfect.*

(2) *Every graph of $\mathcal{G}$ is ω-colourable.*

(3) *Y is an induced subgraph of P_4 or of Z_1.*

Furthermore, if P_4 and Z_1 are Y-free, then there exist infinitely many graphs of $\mathcal{G}$ which are not ω-colourable.

Theorem 1.5. *Let Y be a graph. If $\mathcal{G}_3$ denotes the class of all connected $(K_{1,3}, Y)$-free graphs that are distinct from an odd cycle and which have independence number at least 3, then the following statements are equivalent.*

(1) *Every graph of $\mathcal{G}_3$ is perfect.*

(2) *Every graph of $\mathcal{G}_3$ is ω-colourable.*

(3) *Y is an induced subgraph of P_5 or of Z_2.*

Furthermore, if P_5 and Z_2 are Y-free, then there exist infinitely many graphs of $\mathcal{G}_3$ which are not ω-colourable.

As a first step for proving these two theorems, we specify possible graphs Y in Section 2. The proofs of Theorems 1.4 and 1.5 are presented in Section 3. In addition, we study imperfect $(K_{1,3}, B)$-free graphs of independence number at least 3 in Section 4.

We conclude this section by recalling some further notation. Given two graphs G and H, we let $G \cup H$ denote the disjoint union of graphs G and H but write $2G$ for $G \cup G$. For brevity, we let $[k]$ denote the set $\{1, 2, \ldots, k\}$, and let P_k (C_k) denote the path (cycle) on k vertices. We use the notation $C_k\colon u_1u_2 \ldots u_ku_1$ to indicate the ordering of the vertices of the cycle C_k. When working with the vertices of C_k, all calculations using indices are considered modulo k. Finally, let us remark that the graphs, that are depicted in Fig. 1, are used throughout our paper.

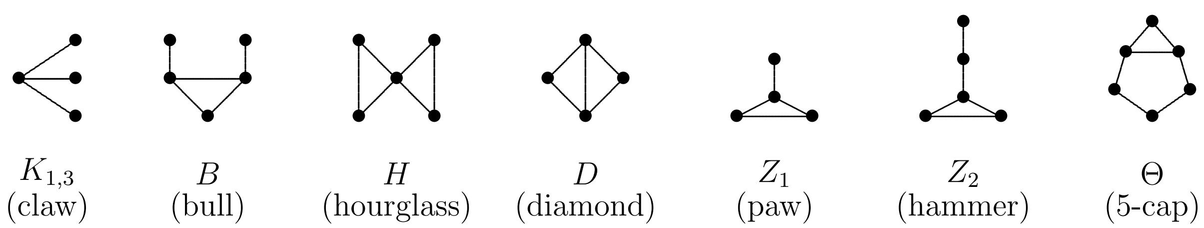

Figure 1: Some of the used forbidden subgraphs.

2 Auxiliary families of graphs

In this section, we show the following:

Lemma 2.1. *For every graph Y for which P_4 and Z_1 are Y-free, there exist infinitely many connected $(K_{1,3}, Y)$-free graphs that are distinct from an odd cycle but not ω-colourable. Furthermore, if P_5 and Z_2 are Y-free, then there are infinitely many such graphs whose independence number is at least* 3.

We define several graph families which will be used for proving Lemma 2.1.

For $s \geq 1$, we let F^0_s be the graph obtained from the cycle C_5 by adding a complete graph K_s, and by adding the edges connecting every vertex of the cycle to every vertex of the complete graph. (To be more precise, the complement of F^0_s is $sK_1 \cup C_5$.)

For $s \geq 3$, we let F^1_s be the graph obtained from $C_{2s+1}\colon u_1u_2 \dots u_{2s+1}u_1$ by adding s isolated vertices $x_1, x_2, \dots, x_s$, and by adding the edges x_iu_{2i-1}, x_iu_{2i} and x_iu_{2i+1} for every $i \in [s]$.

For $s \geq 2$, we let F^2_s be the graph obtained from the cycles $C_{2s+1}\colon u_1u_2 \dots u_{2s+1}u_1$ and $C_5\colon x_1x_2 \dots x_5x_1$ by identifying the vertex u_2 with x_2 and identifying u_3 with x_3, by adding a vertex z such that $N(z) = \{x_1, x_2, \dots, x_5\}$, and by adding the edges x_1u_1 and x_4u_4.

For $s \geq 1$, we let F^3_s be the graph obtained from $C_{6s+1}\colon u_1u_2 \dots u_{6s+1}u_1$ by adding $3s$ isolated vertices x^i_1, x^i_2, x^i_3 (for $i \in [s]$), and by adding the edges $x^i_1u_{6i-5}$, $x^i_1u_{6i-4}$, $x^i_1u_{6i-2}$, $x^i_1u_{6i-1}$ and $x^i_2u_{6i-4}$, $x^i_2u_{6i-3}$, $x^i_2u_{6i-1}$, $x^i_2u_{6i}$, and $x^i_3u_{6i-3}$, $x^i_3u_{6i-2}$, $x^i_3u_{6i}$, $x^i_3u_{6i+1}$ for every $i \in [s]$.

We define graphs F^4_s by considering their complements $\overline{F^4_s}$. For odd $s \geq 3$, we let $\overline{F^4_s}$ be the graph obtained from the cycles $C_{3s}\colon u_1u_2 \dots u_{3s}u_1$ and $C_3\colon x_1x_2x_3x_1$, by adding the edges x_1u_{3i-2}, x_2u_{3i-1} and x_3u_{3i} for every $i \in [s]$.

The structure of the graphs F^1_s, F^2_s, F^3_s and $\overline{F^4_s}$ is depicted in Fig. 2. In this section, we will use the above defined labelling of the vertices of F^1_s, F^2_s, F^3_s and F^4_s.

We let $\mathcal{F}^0$ denote the family which consists of all graphs F^0_s. Similarly, we let $\mathcal{F}^1$, $\mathcal{F}^2$, $\mathcal{F}^3$, $\mathcal{F}^4$ denote the family of all graphs F^1_s, F^2_s, F^3_s, F^4_s, respectively. We show the following:

Observation 2.2. *None of the graphs of $\mathcal{F}^0 \cup \mathcal{F}^1 \cup \mathcal{F}^2 \cup \mathcal{F}^3 \cup \mathcal{F}^4$ is ω-colourable.*

Proof. For the sake of a contradiction, we suppose that $F \in \mathcal{F}^0 \cup \mathcal{F}^1 \cup \mathcal{F}^2 \cup \mathcal{F}^3 \cup \mathcal{F}^4$ is ω-colourable.

If $F \in \mathcal{F}^0$, then it can be easily seen that $\omega(F) = s + 2$ and $\chi(F) = s + 3$, a contradiction.

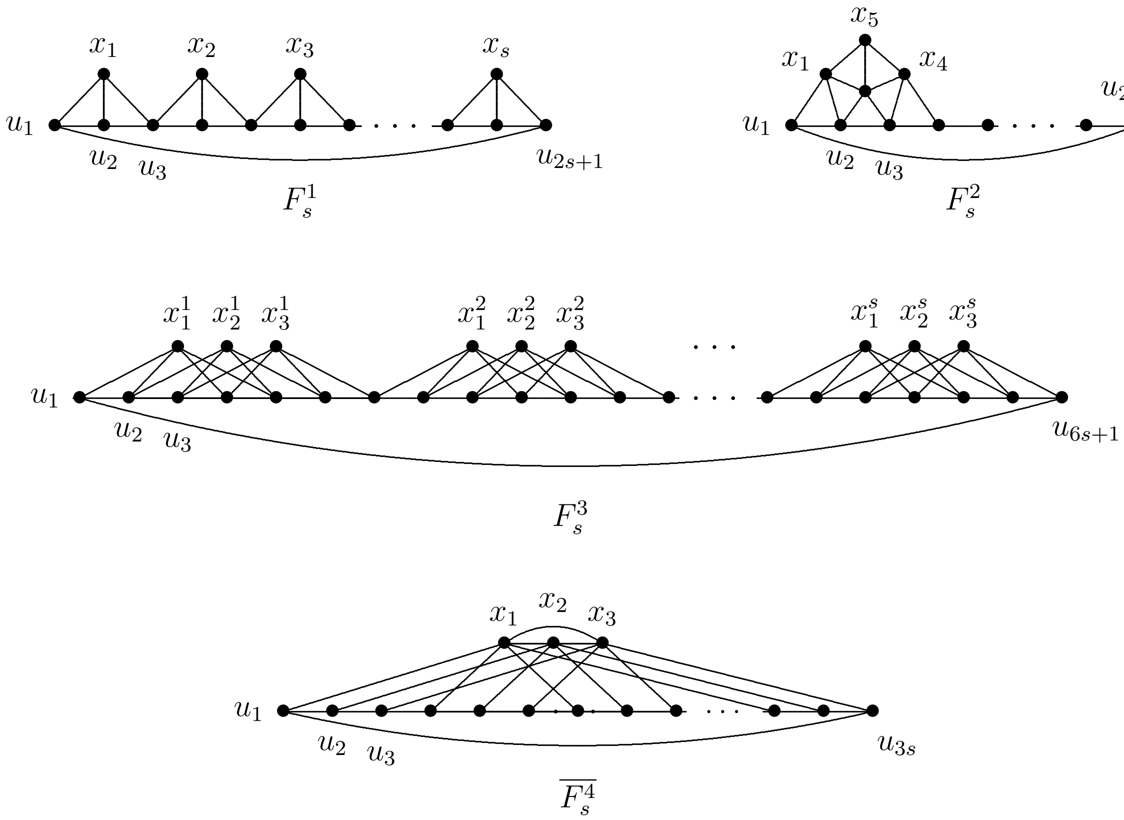

Figure 2: The structure of the graphs F_s^1, F_s^2, F_s^3 and $\overline{F_s^4}$.

If $F \in \mathcal{F}^1$, then clearly $\omega(F) = 3$. Since, by our supposition, F is 3-colourable, we consider a 3-colouring of the subgraph induced by $\{u_1, u_2, u_3, x_1\}$, and note that the vertices u_1 and u_3 must have the same colour. We apply a similar reasoning for u_3 and u_5, for u_5 and u_7, ..., and so on. In particular, we note that u_1 and u_{2s+1} have the same colour, which is a contradiction since both vertices are adjacent.

If $F \in \mathcal{F}^2$, then clearly $\omega(F) = 3$. But since F_1^0 is an induced subgraph of F, we have $\chi(F) \geq \chi(F_1^0) > \omega(F_1^0) = 3$, a contradiction.

If $F \in \mathcal{F}^3$, then we observe $\omega(F) = 3$. Since, by our supposition, F is 3-colourable, we consider a 3-colouring, say χ, of the subgraph induced by $\{u_1, u_2, \ldots, u_7, x_1^1, x_2^1, x_3^1\}$. We prove that the vertices u_1 and u_7 must have the same colour. Since $\{u_1, u_2, x_1^1\}$ induces K_3, we assume $\chi(u_1) = a$, $\chi(u_2) = b$, and $\chi(x_1^1) = c$ for discussing the two possible cases.

- If $\chi(u_3) = a$, then $\chi(u_4) = b$ and $\chi(x_2^1) = c$. This implies $\chi(u_5) = a$ and $\chi(x_3^1) = c$. Consequently, we get $\chi(u_6) = b$, and thus $\chi(u_7) = a$.

- If $\chi(u_3) = c$, then $\chi(x_2^1) = a$, and therefore $\chi(u_5) = b$. Hence, $\chi(u_4) = a$ and $\chi(u_6) = c$, and we get $\chi(x_3^1) = b$ and $\chi(u_7) = a$.

By a similar argument, we note that $\chi(u_{6i+1}) = a$ for every $i \in [s]$, contradicting the fact that u_1 and u_{6s+1} are adjacent.

If $F \in \mathcal{F}^4$, then let $F \cong F_s^4$ for some odd $s \geq 3$. For showing $\chi(F) > \omega(F)$, we consider an arbitrary independent set, say I, of $\overline{F}$. In the case where a vertex of $\{x_1, x_2, x_3\}$ belongs to I, we note that $|I| \leq s + 1$. Otherwise, the set I may contain up to $\frac{3s-1}{2}$ vertices since $V(F) \setminus \{x_1, x_2, x_3\}$ induces an odd cycle on $3s$ vertices in $\overline{F}$. Clearly, $\frac{3s-1}{2} \geq s + 1$ since $s \geq 3$, and thus $\omega(F) = \frac{3s-1}{2}$. Since F is $4K_1$-free and $\{x_1, x_2, x_3\}$ is the only independent set of size 3 in F, at most one colour can be used three times in a colouring assigning distinct colours to adjacent vertices. So, $\chi(F) \geq 1 + \frac{|V(F)|-3}{2}$. By recalling that $|V(F)| = 3s + 3$ and $|V(F)| - 3$ is odd, we have $\chi(F) \geq \frac{3s+3}{2} > \frac{3s-1}{2} = \omega(G)$, a contradiction. □

We apply Observation 2.2 and prove Lemma 2.1.

Proof of Lemma 2.1. We recall the families of graphs $\mathcal{F}^0, \mathcal{F}^1, \mathcal{F}^2, \mathcal{F}^3$ and $\mathcal{F}^3$ as defined above. We note that every graph of $\mathcal{F}^0 \cup \mathcal{F}^1 \cup \mathcal{F}^2 \cup \mathcal{F}^3 \cup \mathcal{F}^4$ is connected and distinct from an odd cycle. Furthermore, every such graph is not ω-colourable by Observation 2.2.

We note that all graphs of $\mathcal{F}^0$ are $(3K_1, 2K_2, K_1 \cup K_3)$-free (and thus $K_{1,3}$-free). Furthermore, it is not hard to see that all graphs of $\mathcal{F}^1$ and all graphs of $\mathcal{F}^3$ are $(K_{1,3}, K_4, C_4, C_5)$-free and $(K_{1,3}, D)$-free, respectively. Hence, if Y does not belong to the class of $(3K_1, 2K_2, K_1 \cup K_3, K_4, D, C_4, C_5)$-free graphs, then there are infinitely many connected $(K_{1,3}, Y)$-free graphs that are distinct from an odd cycle but not ω-colourable.

Consequently, we can assume that Y is $(3K_1, 2K_2, K_1 \cup K_3, K_4, D, C_4, C_5)$-free. We show that Y is an induced subgraph of P_4 or of Z_1. Since Y is $(3K_1, C_4, C_5)$-free, it is $(K_{1,3}, C_4, C_5, C_6, \ldots)$-free as well. In particular, if Y is K_3-free, then every component of Y is a path, and thus Y is an induced subgraph of P_4 since Y is $(3K_1, 2K_2)$-free.

We assume that Y contains a set, say T, of vertices that induces a complete subgraph of order 3. Since Y is $(K_1 \cup K_3, K_4, D)$-free, every vertex outside T has precisely one neighbour in T. Furthermore, every edge of Y is incident with a vertex of T since Y is $(2K_2, C_4)$-free. Consequently, we note that Y has at most one vertex outside T since Y is $3K_1$-free. Thus, Y is an induced subgraph of Z_1.

In order to show the "furthermore part" of the statement, we consider the families of graphs $\mathcal{F}^1$, $\mathcal{F}^2$, $\mathcal{F}^3$ and $\mathcal{F}^4$. We note that the families consist of graphs which all have independence number at least 3 (for graphs of $\mathcal{F}^1$ and $\mathcal{F}^3$, we take $\{u_1, u_3, u_5\}$ as an independent set; for graphs of $\mathcal{F}^2$, we take $\{u_1, u_3, x_5\}$ as an independent set; and for $\mathcal{F}^4$, we take $\{x_1, x_2, x_3\}$ as an independent set.).

Furthermore, we observe that all graphs of $\mathcal{F}^1$, all graphs of $\mathcal{F}^2$, and all graphs of $\mathcal{F}^3$, are $(K_{1,3}, B, K_4, C_4, C_5, C_6)$-free, $(K_{1,3}, H)$-free, and $(K_{1,3}, D)$-free, respectively. In addition, all graphs of $\mathcal{F}^4$ are $(K_{1,3}, 4K_1, 2K_1 \cup K_2, K_2 \cup K_3)$-free since their complements are $(K_1 \cup K_3, K_4, D, K_{2,3})$-free.

Similarly as above, we assume that Y is $(K_{1,3}, 4K_1, 2K_1 \cup K_2, K_2 \cup K_3, B, K_4, D, H, C_4, C_5, C_6)$-free, and we show that Y is an induced subgraph of P_5 or of Z_2. We note that Y is $(K_{1,3}, C_4, C_5, C_6, C_7, \ldots)$-free since it is $(K_{1,3}, C_4, C_5, C_6, 2K_1 \cup K_2)$-free. In particular, if Y is K_3-free, then every component of Y is a path, and Y is an induced subgraph of P_5 since Y is $(4K_1, 2K_1 \cup K_2)$-free.

We assume that there is a set, say T, of vertices that induces a complete subgraph of order 3. We note that every vertex outside T has at most one neighbour in T since Y is (K_4, D)-free. Furthermore, at most one vertex outside T is adjacent to a vertex of T since Y is $(2K_1 \cup K_2, B, H, C_4)$-free, and at most one vertex outside T has no neighbour in T since Y is $(2K_1 \cup K_2, K_2 \cup K_3)$-free. In case Y has two vertices outside T, we note that these two vertices are adjacent since Y is $2K_1 \cup K_2$-free. Thus, we conclude that Y is an induced subgraph of Z_2. □

3 Proving the main results

In this section, we apply Theorem 1.3 and Lemma 2.1 and prove our main results, Theorem 1.4 and Theorem 1.5. For proving Theorem 1.4, we shall also use the following:

Theorem 3.1 (Olariu [10]). *If G is a connected Z_1-free graph, then G is K_3-free or complete multipartite.*

Proof of Theorem 1.4. By definition, every perfect graph is ω-colourable. So, (1) implies (2).

We note that the "furthermore part" of the statement follows by Lemma 2.1. In particular, we deduce from this lemma that (2) implies (3).

We show that (3) implies (1). Clearly, if Y is an induced subgraph of P_4, then the implication follows from Theorem 1.3 since every P_4-free graph contain neither an odd hole nor an odd antihole. If Y is an induced subgraph of Z_1, then, by Theorem 3.1, every Z_1-free graph is K_3-free or complete multipartite. We note that all complete multipartite graphs are perfect. Furthermore, every $(K_{1,3}, K_3)$-free graph has maximum degree at most 2. Thus, we conclude that every K_3-free graph of $\mathcal{G}$ is perfect as well since it is connected, distinct from an odd cycle, and has maximum degree at most 2. □

For further reference, we note the following:

Observation 3.2. *Let G be a $K_{1,3}$-free graph and let C be a set of its vertices such that C induces a cycle of length at least* 5*. If x is a vertex that does not belong to C but is adjacent to a vertex of C, then $N(x) \cap C$ induces K_2 or P_3 or P_4 or a C_5 or a $2K_2$.*

Proof. Since G is $K_{1,3}$-free, $N(x)$ cannot contain three pairwise independent vertices. Furthermore, the graph induced by $N(x) \cap C$ cannot contain K_1 as a component. □

In addition, we shall also use the following two Lemmas on $K_{1,3}$-free graphs for proving Theorem 1.5. The first one is due to Ben Rebea.

Lemma 3.3 (Ben Rebea's Lemma [1, 6])**.** *Let G be a connected $K_{1,3}$-free graph with independence number at least* 3*. If G contains an odd antihole, then it contains a hole of order* 5*.*

Let Θ be the graph obtained from C_6 by adding one edge between two vertices of distance 2 in C_6 (see Fig. 1).

Lemma 3.4. *Every connected $(K_{1,3}, \Theta)$-free graph with independence number at least* 3 *is C_5-free.*

Proof. We let G be a connected $(K_{1,3}, \Theta)$-free graph with independence number at least 3. For the sake of a contradiction, we suppose that G contains a set C of vertices inducing a cycle of length 5. Since G is connected and contains an independent set of size at least 3, there exists a vertex which does not belong to C but is adjacent to a vertex of C. Let x be an arbitrary one. We note that the graph induced by $N(x) \cap C$ is distinct from K_2 since G is Θ-free, and thus, by Observation 3.2, it is either P_3 or P_4 or C_5. In particular, x has two non-adjacent neighbours in C, and therefore a vertex which has no neighbour in C cannot be adjacent to x since G is

$K_{1,3}$-free. Since G is connected, it follows from the arbitrariness of x that every vertex of G is adjacent to a vertex of C.

We let I be a set of 3 independent vertices of G. We consider the graph induced by $C \cup I$, and note that no vertex of C is adjacent to all vertices of I since G is $K_{1,3}$-free. Thus, there are only three options of interconnecting C and I (see Fig. 3). We conclude that the graph induced

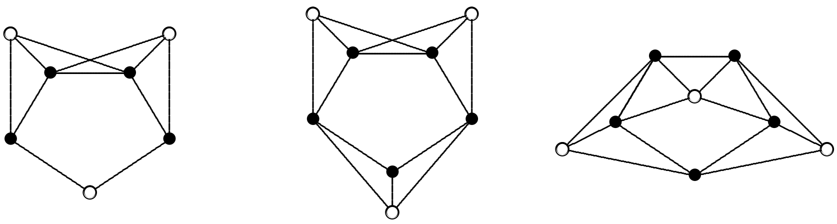

Figure 3: The only possible realisations of the graph induced by $C \cup I$, where the white vertices depict the vertices of I.

by $C \cup I$ is not Θ-free, a contradiction. □

We combine the ingredients and finally prove Theorem 1.5.

Proof of Theorem 1.5. Similarly as in the proof of Theorem 1.4, we note that (1) implies (2), and (2) implies (3), and that the "furthermore part" is satisfied.

We show that (3) implies (1). We consider a graph G of $\mathcal{G}_3$, and proceed in two steps.

First of all, we show that G is $(C_7, C_9, \dots)$-free. Clearly, the claim is satisfied in case Y is an induced subgraph of P_5. We assume that Y is an induced subgraph of Z_2. For the sake of a contradiction, we suppose that G contains a set C of vertices inducing an odd cycle of length at least 7. Since G is connected and distinct from an odd cycle, there is a vertex x that does not belong to C but is adjacent to a vertex of C. By Observation 3.2, the graph induced by $N(x) \cap C$ is either K_2 or P_3 or P_4 or $2K_2$. In all cases, the graph induced by $C \cup \{x\}$ is not Z_2-free, a contradiction.

For our second step, we note that Θ is neither P_5-free nor Z_2-free. Hence G is $(K_{1,3}, \Theta)$-free, and thus G is C_5-free by Lemma 3.4. In particular, G contains no odd antihole by Lemma 3.3, which implies that G is perfect by Theorem 1.3. □

4 Characterising exceptional graphs

For every graph Y, we considered connected $(K_{1,3}, Y)$-free graphs that are distinct from an odd cycle but which have an independence number of at least 3, and we showed that either all such graphs are perfect or there are infinitely many such graphs which are not ω-colourable. We note that for some graphs Y, not ω-colourable $(K_{1,3}, Y)$-free graphs that are distinct from an odd cycle but which have an independence number of at least 3 could be described precisely.

In this section, we study $(K_{1,3}, B)$-free graphs but need the terminology of an inflation of a cycle. For $k \geq 4$ and $n_1, n_2, \ldots, n_k \geq 1$, we let $C[n_1, n_2, \ldots, n_k]$ denote the graph whose vertex set can be partitioned into k sets $W_1, W_2, \ldots, W_k$ of sizes $n_1, n_2, \ldots, n_k$, respectively, such that

- $W_i \cup W_j$ induces a complete graph (of order $n_i + n_j$) if either $|i - j| = 1$ or $|i - j| = k - 1$,
- $W_i \cup W_j$ induces a disjoint union of two complete graphs if $1 < |i - j| < k - 1$

for all distinct $i, j \in [k]$. For brevity, we say that the graph $C[n_1, n_2, \ldots, n_k]$ is an *inflation of the cycle C_k*. Two examples are depicted in Fig. 4.

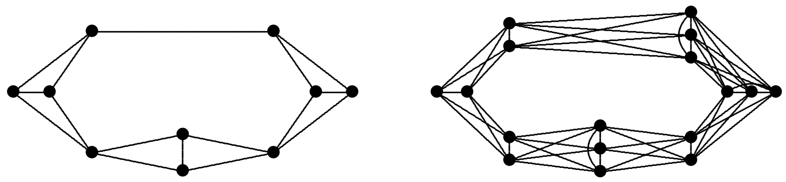

Figure 4: Two examples of an inflation of C_7.

Our main result of this section characterises connected imperfect $(K_{1,3}, B)$-free graphs with independence number at least 3.

Theorem 4.1. *Every connected $(K_{1,3}, B)$-free graph with independence number at least 3 is either perfect or an inflation of an odd cycle of length at least 7.*

We note that Theorem 4.1 follows from the combination of Theorem 1.3 and Lemma 3.3 and the following lemma.

Lemma 4.2. *Let G be a connected $(K_{1,3}, B)$-free graph.*

(1) *If G contains an independent set of size 3, then G is C_5-free.*

(2) *If G contains an induced cycle of length k with $k \geq 6$, then G is an inflation of C_k.*

Proof. We note that G is $(K_{1,3}, \Theta)$-free since it is $(K_{1,3}, B)$-free, and thus statement (1) follows by Lemma 3.4.

It remains to show (2). We let C be a set of vertices inducing a cycle of length $k \geq 6$ in G. Clearly, if $V(G) = C$, then the statement is trivially satisfied. Since G is connected, we can assume that there is a vertex which does not belong to C but is adjacent to a vertex of C. Let x be an arbitrary one. We consider the graph induced by $N(x) \cap C$, and note that this graph is distinct from K_2, $2K_2$, and P_4 since G is B-free. Thus, by Observation 3.2, the graph induced by $N(x) \cap C$ is a path of 3 vertices, which implies that every vertex outside C is adjacent to a vertex of C since G is connected but also $K_{1,3}$-free. We let $C_k\colon v_1v_2 \dots v_kv_1$ be the cycle induced by C, and divide $V(G) \setminus C$ into k disjoint sets $M_1, M_2, \dots, M_k$ so that if w belongs to M_i, then $N(w) \cap C = \{v_i, v_{i+1}, v_{i+2}\}$ for every $i \in [k]$.

Next, let us consider two vertices of $V(G)\setminus C$, say w and w'. We let $c := |N(w) \cap N(w') \cap C|$, that is, c denotes the number of common neighbours of w and w' in C. We deduce the following:

- If $c = 3$, then w and w' are adjacent since G is $K_{1,3}$-free.
- If $c = 2$, then w and w' are adjacent since G is B-free.
- If $c = 1$, then w and w' are non-adjacent since G is B-free.
- If $c = 0$, then w and w' are non-adjacent since G is $K_{1,3}$-free.

Consequently, we note that G is an inflation of C_k. □

References

[1] A. Ben Rebea, Étude des stables dans les graphes quasi-adjoints, Thèse, Université de Grenoble, France, 1981.

[2] J. A. Bondy and U. S. R. Murty, Graph Theory, Springer, 2008.

[3] C. Brause, B. Randerath, I. Schiermeyer, and E. Vumar, On the chromatic number of $2K_2$-free graphs, **Discrete Appl. Math.** 253 (2019), 14–24.

[4] M. Chudnovsky, N. Robertson, P. Seymour, and R. Thomas, The strong perfect graph theorem, **Ann. of Math.** 164 (2006) 51–229.

[5] M. Chudnovsky and P. Seymour, Claw-free graphs VI. Colouring, **J. Combin. Theory B** 100 (6) (2010) 560–572.

[6] V. Chvátal and N. Sbihi, Recognizing claw-free perfect graphs, **J. Combin. Theory B** 44 (1988) 154–176.

[7] P. Erdős, Graph theory and probability, **Canad. J. Math.** 11 (1959), 34–38.

[8] A. Gyárfás, Problems from the world surrounding perfect graphs, in **Proc. Int. Conf. on Comb. Analysis and Applications** (Pokrzywna, 1985), Zastos. Mat. 19 (1987) 413–441.

[9] J.H. Kim, The Ramsey number $R(3,t)$ has order of magnitude $t^2/\log t$ (English summary), **Random Structures and Algorithms** 7 (1995) 173–207.

[10] S. Olariu, Paw-free graphs, **Inform. Process. Letters** 28 (1988) 53–54.

[11] F. P. Ramsey, On a problem of formal logic, **Proc. London Math. Soc. Ser. 2** 30 (1930) 264–285.

[12] D. P. Sumner, Subtrees of a graph and the chromatic number, In: The theory and applications of graphs, Wiley, 1981, 557–576.