Thomassen's conjecture for line graphs of 3-hypergraphs

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Abstract

In 1986, Thomassen conjectured that every 4-connected line graph is hamiltonian. The conjecture is still wide open, and, as a possible approach to it, many statements that are equivalent or related to it have been studied. In this paper, we extend the statement to the class of line graphs of 3-hypergraphs, and generalize it to Tutte cycles and paths (note that a line graph of a 3-hypergraph is $K_{1,4}$-free but can contain induced claws $K_{1,3}$, and that a Tutte cycle/path is a cycle/path such that any component of its complement has at most three vertices of attachment). Among others, we formulate the following conjectures:

(i) every 2-connected line graph of a 3-hypergraph has a Tutte maximal cycle containing any two prescribed vertices,

(ii) every 3-connected line graph of a 3-hypergraph has a Tutte maximal cycle containing any three prescribed vertices,

(iii) every connected line graph of a 3-hypergraph has a Tutte maximal (a, b)-path for any two vertices a, b,

(iv) every 4-connected line graph of a 3-hypergraph is Hamilton-connected,

and we show that all these (seemingly much stronger) statements are equivalent with Thomassen's conjecture.

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1 Introduction

We use standard graph-theoretical terminology and notation, and for concepts and notations not defined here we refer the readers to [2]. Specifically, by a *graph* we always mean a simple finite undirected graph; whenever we admit multiple edges and/or loops, we always speak about a *multigraph.* An edge that is not a loop is referred to as an *open edge.* If F, G are graphs, we write $F \subset G$ if F is a *subgraph* of G (not necessarily induced), and, for $M \subset V(G)$, we use $\langle M \rangle_G$ to denote the *induced subgraph* of G on M. For $F \subset G$ and $M \subset V(G)$, we denote $e_G(F, M) = \{e = uv \in E(G) | \ u \in V(F), v \in M\}$. A graph G is *nontrivial* if $|E(G)| \geq 1$.

Throughout, $d_G(x)$ denotes the degree in G of a vertex $x \in V(G)$, $\delta(G)$ and $\Delta(G)$ denote the minimum degree and maximum degree of G, respectively, and, for a positive integer k, we set $V_k(G) = \{x \in V(G) | \ d_G(x) = k\}$ and $V_{\geq k}(G) = \{x \in V(G) | \ d_G(x) \geq k\}$. By a *clique* in G we mean a complete subgraph of G (not necessarily maximal).

A *walk* in a multigraph G is an alternating sequence of vertices and edges $Q = x_0e_1x_1 \dots x_{\ell-1}e_\ell x_\ell$ such that $e_i \in E(G)$ connects vertices x_{i-1} and x_i, $i = 1, \dots, \ell$. Note that any edge and vertex can occur more times in the sequence. For $a, b \in V(G)$, an (a, b)-*walk* in G is a walk such that $a = x_0$, $b = x_\ell$, and for $h, f \in E(G)$, an (h, f)-*walk* in G is a walk such that $e_1 = h$ and $e_\ell = f$. We will use $\mathrm{Int}(Q)$ to denote the set $\{x_1 \dots x_{\ell-1}\}$ of interior vertices of Q. A *trail* (*(a, b)-trail, (h, f)-trail*) in G is a walk (*(a, b)-walk, (h, f)-walk*) with no repeated edges, and a *path* (an *(a, b)-path*) is a trail (an (a, b)-trail) with no repeated vertices, respectively. In the special case when $a = b$, we say that Q is a *closed walk* (*closed trail, cycle*), respectively, and in this case, we consider all indices modulo ℓ and we set $\mathrm{Int}(Q) = V(Q)$.

A graph G is *hamiltonian* if it contains a hamiltonian cycle, *Hamilton-connected* if, for any $a, b \in V(G)$, G contains a hamiltonian (a, b)-path, and, for a positive integer k, G is *k-Hamilton-connected* if the graph $G - X$ is Hamilton-connected for any set $X \subset V(G)$ with $|X| = k$.

The *line graph* of a graph H is the graph $G = L(H)$ with $V(G) = E(H)$, in which two vertices are adjacent if and only if the corresponding edges of H share a vertex. If G is a line graph, different from the triangle K_3, then the graph H such that $G = L(H)$ (which is known to be uniquely determined) will be called the *preimage* of G and denoted by $H = L^{-1}(G)$. It is well-known that a noncomplete line graph G is k-connected if and only if $H = L^{-1}(G)$ is *essentially k-edge-connected,* i.e., H contains no edge-cut R with $|R| < k$ such that $H - R$ has at least two nontrivial components. Also note that if $e \in E(H)$ is pendant, then the corresponding vertex x_e in $G = L(H)$ is simplicial.

If $\mathcal{C}$ is a family of graphs, we say that a graph G is $\mathcal{C}$-free, if G does not contain any graph in $\mathcal{C}$ as an induced subgraph. In the special case when $\mathcal{C} = \{K_{1,3}\}$, we simply say that G is *claw-free.* Note that it is a well-known fact that every line graph is claw-free.

A graph G is *cyclically k-edge-connected* if G contains no edge-cut R with $|R| < k$ such that $G - R$ has at least two components containing a cycle. Finally, by a *snark* we mean a cubic cyclically 4-edge-connected graph of girth at least 5 which is not 3-edge-colorable.

We will be interested in the following conjectures. The first of them was established by Thomassen [17].

Conjecture A [17]. *Every 4-connected line graph is hamiltonian.*

A (seemingly) stronger version was established by Matthews and Sumner [12].

Conjecture B [12]. *Every 4-connected claw-free graph is hamiltonian.*

So far, the (seemingly) strongest version of these conjectures was established in [14] as follows.

Conjecture C [14]. *Every 4-connected claw-free graph is 1-Hamilton-connected.*

In another direction, the following conjecture on snarks has appeared independently at different places.

Conjecture D. *Every snark has a dominating cycle.*

Although all these conjectures seem to be quite different, they turn out to be all equivalent to Conjecture A. The equivalences were established in [13] for Conjecture B, in [14] for Conjecture C, and in [3] for Conjecture D.

Theorem E [13, 14, 3]. *Conjectures A, B, C and D are equivalent.*

Note that all these conjectures are wide open, and so far the strongest positive result in their direction shows that every 5-connected claw-free graph with minimum degree at least 6 is 1-Hamilton-connected [9]. More information on other equivalent versions of the conjectures can be found in the survey paper [4].

A *hypergraph* $\mathcal{H}$ consists of a finite set $V(\mathcal{H})$ of *vertices* of $\mathcal{H}$ and a (multi)set $E(\mathcal{H})$ of subsets of $V(\mathcal{H})$ that are called the *hyperedges* of $\mathcal{H}$. A hypergraph in which all hyperedges have at most k elements will be called a *k-hypergraph*. Note that, in hypergraphs, we admit parallel hyperedges, i.e., multiple copies of the same hyperedge, and also loops. A hyperedge of cardinality 2 will be sometimes also called an *edge* of $\mathcal{H}$. Thus, a hypergraph without hyperedges is a multigraph, and a multigraph without parallel edges and without loops is a graph.

The *line graph* (sometimes also called the *representative graph*) of a hypergraph $\mathcal{H}$ is the graph $G = L(\mathcal{H})$ with $V(G) = E(\mathcal{H})$, in which two vertices are adjacent if and only if the corresponding hyperedges of $\mathcal{H}$ have a vertex in common (note that $G = L(\mathcal{H})$ can be also viewed as the intersection graph of the set system $E(\mathcal{H})$). For $e \in E(\mathcal{H})$, we will use the notation $x = L(e)$ for the corresponding vertex $x \in V(L(\mathcal{H}))$. For an example of a 3-hypergraph and its line graph, see Fig. 2 (a), (b).

The following result characterizes graphs that are line graphs of an r-hypergraph.

Theorem F [1, 16]. *For every integer $r \geq 1$, a graph G is a line graph of an r-hypergraph if and only if $E(G)$ can be covered by a system of cliques $\mathcal{K}$ such that every vertex of G is in at most r cliques of $\mathcal{K}$.*

From Theorem F, we immediately observe that a line graph of an r-hypergraph is $K_{1,r+1}$-free, and, specifically, a line graph of a 3-hypergraph is $K_{1,4}$-free. Also note that, obviously, every line graph (of a graph) is a line graph of a 3-hypergraph.

The following fact shows that, in a sense, there are "many" line graphs of a 3-hypergraph that are not claw-free.

Theorem G [8]. *Let G be a graph with maximum degree $\Delta(G) \leq 4$. Then G is a line graph of a 3-hypergraph if and only if G is $K_{1,4}$-free.*

Proof. Obviously, if G is a line graph of a 3-hypergraph, then G is $K_{1,4}$-free by the above observation. The converse is a direct consequence of Theorem F and of Proposition 3 of [8]. ■

In this paper, we extend Conjectures A – D to the class of line graphs of 3-hypergraphs, and we show that these seemingly much stronger conjectures are in fact equivalent with Conjectures A – D.

Note that the first step beyond the class of claw-free graphs, or graphs obtained by some operations from the line graph preimages of their closure, was done in [15], where the closure operation from [13] was generalized to the class of $\{K_{1,4}, K_{1,4}+e\}$-free graphs with minimum degree at least 6, with some consequences related to Conjectures A – D. However, it can be seen that these classes are in fact independent: in one direction, the graph consisting of three cliques of order at least 7 sharing a vertex is a line graph of a 3-hypergraph (by Theorem F), but is not $(K_{1,4}+e)$-free, and, conversely, the existence of infinitely many $\{K_{1,4}, K_{1,4}+e\}$-free (even claw-free) graphs that are not a line graph of a 3-hypergraph follows from Theorem 1 of [8].

Also note that although in claw-free graphs, a connectivity bound implying hamiltonicity is known (recall the strongest known result showing that every 5-connected claw-free graph with minimum degree at least 6 is 1-Hamilton-connected [9]), the corresponding problem in the class of line graphs of 3-hypergraphs is still open.

2 Results

For our results, we will need some more terminology. For $Q, F \subset H$ with $V(Q) \cap V(F) = \emptyset$, a vertex $x \in V(Q)$ such that $N_H(x) \cap V(F) \neq \emptyset$ is called a *vertex of attachment of F in Q*. The set of all vertices of attachment of F in Q is denoted $V_A(F, Q)$. If C is a cycle in a graph G, then C is *maximal* if there is no cycle C' in G such that $V(C) \subsetneq V(C')$, a component F of $G-C$ is a *Tutte component* if $|V_A(F, C)| \leq 3$, and C is a *Tutte cycle of G* if either C is a hamiltonian cycle in G, or $|V(C)| \geq 4$ and every component of $G-C$ is a Tutte component. A maximal cycle which is also a Tutte cycle is called a *Tutte maximal cycle.*

Note that our definition of a Tutte cycle is the same as in [18], where results on Tutte cycles in the context of Conjectures A – D appeared for the first time. It would be also possible to define a Tutte cycle alternatively by the condition $|V_A(F, C)| \leq \min\{3, |V(C)|-1\}$ (admitting triangles), which would imply some changes in what follows, however, we prefer our definition to be consistent with [18].

Now, we state the following conjectures.

Conjecture 1. *Let G be a 2-connected line graph of a 3-hypergraph, and let $a, b \in V(G)$. Then G has a Tutte maximal cycle C such that $a, b \in V(C)$.*

Conjecture 2. *Let G be a 3-connected line graph of a 3-hypergraph, and let $a, b, c \in V(G)$. Then G has a Tutte maximal cycle C such that $a, b, c \in V(C)$.*

Note that if G is 4-connected, then a Tutte cycle becomes a hamiltonian cycle. Hence both Conjecture 1 and Conjecture 2 immediately imply Conjecture A.

Similarly, an (a, b)-path P in a graph G is *maximal* if there is no (a, b)-path P' in G such that $V(P) \subsetneq V(P')$, a component F of $G - P$ is a *Tutte component* if $|V_A(F, P)| \leq \min\{3, |V(P)| - 1\}$, and P is a *Tutte path of* G if either P is a hamiltonian path of G, or every component of $G - P$ is a Tutte component. A maximal (a, b)-path which is also a Tutte path is called a *Tutte maximal* (a, b)*-path.*

Conjecture 3. *Let G be a connected line graph of a 3-hypergraph, and let $a, b \in V(G)$. Then G has a Tutte maximal (a, b)-path.*

Obviously, if G is 4-connected, then a Tutte path becomes a hamiltonian path. Hence Conjecture 3 immediately implies the following conjecture.

Conjecture 4. *Every 4-connected line graph of a 3-hypergraph is Hamilton-connected.*

Since every line graph (of a graph) is a line graph of a 3-hypergraph, Conjecture 4 immediately implies Conjecture A.

The following theorem, which is the main result of this paper, shows that Conjectures 1, 2, 3 and 4, although seemingly much stronger, are in fact equivalent with all the previous conjectures.

Theorem 5. *Conjectures A, B, C, D and Conjectures 1, 2, 3 and 4 are equivalent.*

Example. Let G be the graph consisting of 3 cliques of order at least 5 sharing a 4-clique $\langle\{a, b, c, d\}\rangle_G$ (see Fig. 1). It is easy to verify that G is a 4-connected line graph of a 3-hypergraph, however, there is no hamiltonian (a, b)-path in $G - c$, hence G is not 1-Hamilton-connected. This example indicates that there is probably not much hope to establish an equivalence with a statement analogous to Conjecture C in line graphs of 3-hypergraphs since such an equivalence would immediately imply refuting the conjectures.

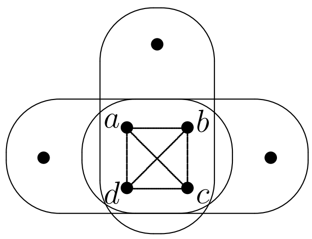

Figure 1: A 4-connected line graph of a 3-hypergraph that is not 1-Hamilton-connected.

3 Proof of Theorem 5

Crucial part of arguments in our proofs will be in a 3-hypergraph $\mathcal{H}$ for which $G = L(\mathcal{H})$, and in a corresponding graph, denoted $\mathrm{Gr}(\mathcal{H})$. For this, we will need some more technical concepts and statements that allow to translate the problem from G to $\mathcal{H}$ and $\mathrm{Gr}(\mathcal{H})$.

If $\mathcal{H}$ is a 3-hypergraph, then $\mathrm{Gr}(\mathcal{H})$ denotes the graph obtained from $\mathcal{H}$ by subdividing every edge with a new vertex of degree 2, and by replacing every 3-hyperedge with a new vertex, adjacent to all three its vertices. The new added vertices will be referred to as *white vertices*, and the original vertices of $\mathcal{H}$ as *black vertices* of $\mathrm{Gr}(\mathcal{H})$. Thus, in $\mathrm{Gr}(\mathcal{H})$, black vertices correspond to the vertices of $\mathcal{H}$, and white vertices correspond to the edges and hyperedges of $\mathcal{H}$ (see Fig. 2(a) and (c)). We will use $V_b(\mathrm{Gr}(\mathcal{H}))$ and $V_w(\mathrm{Gr}(\mathcal{H}))$ to denote the set of black vertices and white vertices of $\mathrm{Gr}(\mathcal{H})$, respectively. Note that $\mathrm{Gr}(\mathcal{H})$ can contain some multiple edges, however, it is easy to see that the only multiedges in $\mathrm{Gr}(\mathcal{H})$ can be the double edges that appear from loops in $\mathcal{H}$. All other edges in $\mathrm{Gr}(\mathcal{H})$ are simple, therefore, we will keep using for $\mathrm{Gr}(\mathcal{H})$ the term "graph".

Note that $\mathrm{Gr}(\mathcal{H})$ is also sometimes called the *incidence graph* of the hypergraph $\mathcal{H}$, and denoted $IG(\mathcal{H})$.

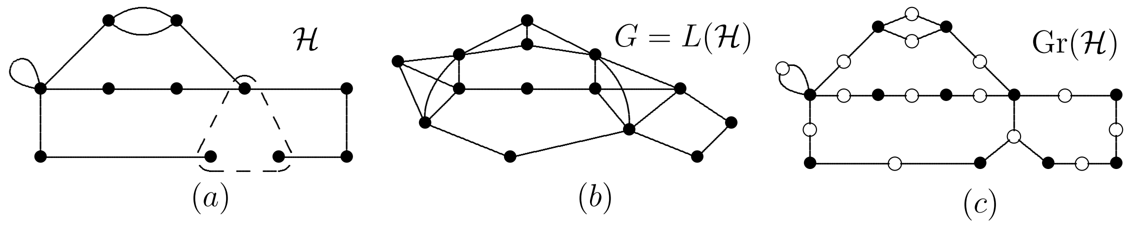

Figure 2: A 3-hypergraph $\mathcal{H}$, its line graph $G = L(\mathcal{H})$, and the graph $\mathrm{Gr}(\mathcal{H})$.

Observe that $\mathrm{Gr}(\mathcal{H})$ is a bipartite graph with bipartition $(V_b(\mathrm{Gr}(\mathcal{H})), V_w(\mathrm{Gr}(\mathcal{H})))$, and that $L(\mathcal{H})$ can be alternatively viewed as the graph obtained from $\mathrm{Gr}(\mathcal{H})$ by joining the (white) neighbors of every black vertex into a clique, and then removing all black vertices and possibly created multiple edges (see Fig. 2).

We first recall here some classical concepts and facts that are used to translate hamiltonian problems from a line graph $G = L(H)$ to H in the case when H is a (multi)graph.

Given a trail T and an edge e in a multigraph H, we say e is *dominated* by T if e is incident to a vertex in $\mathrm{Int}(T)$, and we use $D_H(T)$ to denote the set of all edges of H dominated by T (note that obviously $E(T) \subset D_H(T)$). A closed trail (an (a, b)-trail for $a, b \in V(T)$, an (h, f)-trail for $h, f \in E(T)$) T in H is said to be a *dominating closed trail*, abbreviated DCT (*dominating* (a, b)*-trail*, abbreviated (a, b)-DT; *dominating* (h, f)*-trail*, abbreviated (h, f)-DT), if $D_H(T) = E(H)$ ($D_H(\mathrm{Int}(T)) = E(H)$; $D_H(\mathrm{Int}(T)) = E(H)$, respectively). Note that in the special case of a cubic graph, a dominating closed trail becomes a dominating cycle.

A classical result by Harary and Nash-Williams [6] shows that a line graph $G = L(H)$ of order at least 3 is hamiltonian if and only if H contains a dominating closed trail.

Theorem H [6]. *Let H be a graph with at least three edges. Then $L(H)$ is hamiltonian if and only if H contains a DCT.*

The following result relates hamiltonian paths in a line graph to dominating trails in its preimage.

Theorem I [11]. *Let H be a graph with at least three edges. Then $L(H)$ is Hamilton-connected if and only if H has an (e_1, e_2)-DT for any pair of edges $e_1, e_2 \in E(H)$.*

In our proof, in order to translate the problem from $G = L(\mathcal{H})$ to $\mathcal{H}$ and $\mathrm{Gr}(\mathcal{H})$, we will need a refinement of the concept of a dominating (closed) trail based on walks.

If $Q = x_0e_1x_1 \ldots x_{\ell-1}e_\ell x_\ell$ is a walk and $e \in E(H)$ is an edge of Q, we say that e is *visited k-times* by Q if $|\{i = 0, \ldots \ell|\ e_i = e\}| = k$, and we say that Q is a *k-walk* if every its edge is visited at most k-times. Specifically, if Q is a 2-walk, then the edges visited twice (once) by Q will be refereed to as the *double* (*single*) *edges of* Q. Also note that a 1-walk is a trail. With a slight abuse of notation, we will sometimes also view a walk Q as a subgraph of H (also denoted Q) induced by the edges visited by the walk Q, and we will write $V(Q)$, $E(Q)$, $d_Q(x)$ etc.

We introduce the following notation: for a vertex $x_i \in V(Q)$, x_i^- denotes the set of predecessors of x_i, x_i^+ denotes the set of successors of x_i, x_i^{-E} denotes the set of preceding edges of x_i, and x_i^{+E} denotes the set of succeeding edges of x_i (thus, if x_i is a single vertex, then $x_i^-{=}\{x_{i-1}\}$ and $x_i^{-E}{=}\{e_{i-1}\}$; if x_i is visited more times, then $x_{i-1} \in x_i^-$ and $e_{i-1} \in x_i^{-E}$; similarly for x_i^+ and x_i^{+E}).

Now, let $W \subset V_2(H) \cup V_3(H)$ be an (arbitrary) set of vertices of degree 2 or 3 in H, and set $B = V(H) \setminus W$. Let $a, b \in V(H)$. We say that an (a, b)-walk Q in H is an (a, b, W)-*quasitrail in* H, if Q satisfies the following conditions:

- Q is a 2-walk,
- if $a \in W$, then $a^- = \emptyset$ and if $b \in W$, then $b^+ = \emptyset$ (i.e., if some endvertex of Q is in W, then it is visited only once),
- if $e \in E(Q)$ is a double edge of Q, then $e = vw$, where $w \in V(Q) \cap W$ and $w^{-E} = w^{+E} = \{e\}$,
- no edge in $a^{+E} \cup b^{-E}$ is a double edge of Q or a loop.

Similarly, if Q is a closed walk in H, we say that Q is a *closed W-quasitrail in H*, if Q is a closed 2-walk such that any double edge e of Q satisfies $e = vw$, where $w \in V(Q) \cap W$ and $w^{-E} = w^{+E} = \{e\}$.

A vertex $x \in V(H) \cap W$ which is incident to a double edge $e \in E(Q)$ such that $x^{-E} = x^{+E} = \{e\}$ will be called a *special vertex* of Q, and the set of all special vertices of Q will be denoted $V_s(Q)$.

From these definitions we immediately observe that if $W = \emptyset$, then $V_s(Q) = \emptyset$, and then an (a, b, W)-quasitrail is an (a, b)-trail, and a closed W-quasitrail is a closed trail.

Clearly, if Q is a quasitrail, then the walk $Q - V_s(Q)$ has only single edges, hence is a trail. This trail will be called the *support of* Q and denoted $S(Q)$. (Of course, if Q is an (a, b, W)-quasitrail, then $S(Q)$ is an (a, b)-trail, and if Q is a closed W-quasitrail, then $S(Q)$ is a closed trail.) We say that a quasitrail Q is *trivial* if $S(Q)$ is a trivial (i.e., edgeless) trail; otherwise Q is *nontrivial*.

An edge $e \in E(H)$ is *dominated* by an (a, b, W)-quasitrail Q (or by a closed W-quasitrail Q), if e has at least one vertex in $V(Q) \setminus (\{a, b\} \cap B)$ (or in $V(Q)$), respectively. We will use $D(Q)$ to denote the set of all edges of H that are dominated by Q, and we say that an (a, b, W)-quasitrail (a closed W-quasitrail) Q in H is *dominating* if $D(Q) = E(H)$. More generally, an (a, b, W)-quasitrail (a closed W-quasitrail) Q in H is said to be *domination-maximal* if there is no (a, b, W)-quasitrail (no closed W-quasitrail) Q' in H such that $D(Q) \subsetneq D(Q')$.

Finally, if Q is a closed W-quasitrail in H, then a nontrivial component F of $H - Q$ is said to be a *Tutte component* if $|e_H(F, V(Q) \cap B)| + |V_A(F, Q) \cap W| \leq 3$, and a closed W-quasitrail Q in H is called a *Tutte closed W-quasitrail in H* if either Q is a dominating closed W-quasitrail of H, or every nontrivial component F in $H - Q$ is a Tutte component. A domination-maximal closed W-quasitrail in H which is also Tutte is called a *Tutte domination-maximal closed W-quasitrail.*

The following result shows that a Tutte cycle in $G = L(\mathcal{H})$, where $\mathcal{H}$ is a 3-hypergraph, corresponds to a Tutte closed W-quasitrail in $\mathrm{Gr}(\mathcal{H})$.

Theorem 6. *Let $\mathcal{H}$ be a 3-hypergraph, and let $G = L(\mathcal{H})$ and $W = V_w(\mathrm{Gr}(\mathcal{H}))$.*

(i) *Let Q be a closed W-quasitrail in $\mathrm{Gr}(\mathcal{H})$, and let $A_Q \subset V(G)$ be the set of vertices that correspond to the white vertices of Q. Then there is a cycle C_Q in G such that $V(C_Q) = A_Q$.*
Moreover, if Q is a Tutte closed W-quasitrail with $|V_w(Q)| \geq 4$, then there is a cycle $\bar{C}_Q$ in G such that $V(\bar{C}_Q) \supset V(C_Q)$ and $\bar{C}_Q$ is a Tutte cycle.

(ii) *Let C be a cycle in G, and let $W_C \subset V_w(\mathrm{Gr}(\mathcal{H}))$ be the set of white vertices in $\mathrm{Gr}(\mathcal{H})$ that correspond to the vertices of C. Then there is a closed W-quasitrail Q_C in $\mathrm{Gr}(\mathcal{H})$ such that $V_w(Q_C) = W_C$.*
Moreover, if C is a Tutte maximal cycle, then Q_C is a Tutte closed W-quasitrail.

In the special case of a hamiltonian cycle in G, the following corollary of Theorem 6 can be considered as a generalization of Theorem H.

Corollary 7. *Let $\mathcal{H}$ be a 3-hypergraph, let $G = L(\mathcal{H})$, and set $W = V_w(\mathrm{Gr}(\mathcal{H}))$. Then the following statements are equivalent:*

(i) *G is hamiltonian,*
(ii) *$\mathrm{Gr}(\mathcal{H})$ contains a closed W-quasitrail Q such that $W \subset V(Q)$,*
(iii) *$\mathrm{Gr}(\mathcal{H})$ contains a dominating closed W-quasitrail.*

Proof of Theorem 6. (i). Recall that $G = L(\mathcal{H})$ can be obtained from $\mathrm{Gr}(\mathcal{H})$ by joining the (white) neighbors of every black vertex into a clique, and then removing all black vertices and possibly created multiple edges. The closed W-quasitrail Q in $\mathrm{Gr}(\mathcal{H})$ can be viewed as an alternating sequence of black and white vertices, in which consecutive white vertices have a black common neighbor. Since every white vertex has degree two or three in $\mathrm{Gr}(\mathcal{H})$, it appears at most once in Q. Thus, the sequence of white vertices of Q determines a cycle C_Q in $G = L(\mathcal{H})$ with $V(C_Q) = A_Q$ (for the hypergraph of Fig. 2, see Fig. 3(a), (b)).

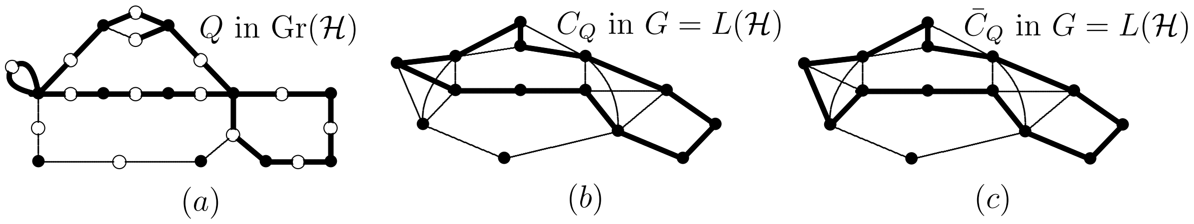

Figure 3: A Tutte closed W-quasitrail in Gr$(\mathcal{H})$ and a Tutte cycle in $G = L(\mathcal{H})$.

Moreover, suppose that Q is Tutte. Then the white vertices of Gr$(\mathcal{H})$ that have a (black) neighbor on Q but are not on Q, correspond in G to vertices that are not on C_Q but are contained in a clique containing some edge of C_Q. Extending C_Q through all such vertices (see Fig. 3(c)), we obtain a cycle $\bar{C}_Q$. By the construction of $\bar{C}_Q$, we have $V(\bar{C}_Q) \supset V(C_Q) = A_Q$, and the cycle $\bar{C}_Q$ corresponds to a closed W-quasitrail $\bar{Q}$ in Gr$(\mathcal{H})$ that is obtained from Q by inserting all white vertices that are not on Q but have a (necessarily black) neighbor on Q, as special vertices of $\bar{Q}$.

We show that every component of $G - \bar{C}_Q$ is a Tutte component. Let D be a component of $G - \bar{C}_Q$. Then D corresponds to (the white vertices of) some component F_D of Gr$(\mathcal{H}) - \bar{Q}$. By the construction, $V_A(F_D, \bar{Q}) \subset V_w(\bar{Q})$, implying $|V_A(D, \bar{C}_Q)| = |V_A(F_D, \bar{Q})|$. Since F_D contains at least one white vertex (otherwise D is empty), F_D has at least one edge.

Let $\bar{F}_D$ be the component of Gr$(\mathcal{H}) - Q$ containing F_D (i.e., $\bar{F}_D$ contains F_D plus the vertices in $V_A(F_D, \bar{Q})$ that are not on Q). Since every (special white) vertex in $V(\bar{Q}) \setminus V(Q)$ is connected to Q by at least one edge, and all such edges end in black vertices of Q, we have $|V_A(F_D, \bar{Q})| \leq |V_A(\bar{F}_D, Q) \cap W| + |e_{\mathrm{Gr}(\mathcal{H})}(\bar{F}_D, V(Q) \cap B)|$. Summarizing, we have $|V_A(D, \bar{C}_Q)| = |V_A(F_D, \bar{Q})| \leq |V_A(\bar{F}_D, Q) \cap W| + |e_{\mathrm{Gr}(\mathcal{H})}(\bar{F}_D, V(Q) \cap B)| \leq 3$, since Q is Tutte. Thus, D is a Tutte component.

Since $|V(\bar{C}_Q)| = |V_w(\bar{Q})| \geq |V_w(Q)| \geq 4$, $\bar{C}_Q$ is the requested Tutte cycle in G.

(ii). Let, conversely, $C = x_1 \dots x_k$ be a cycle in G, let $w_i \in V_w(\mathrm{Gr}(\mathcal{H}))$ be the white vertex corresponding to x_i, $i = 1, \dots, k$, and set $W_C = \{w_1, \dots, w_k\}$. Then $w_1, \dots, w_k$ is a sequence of white vertices in Gr$(\mathcal{H})$ such that each vertex occurs only once, and any two consecutive vertices have a black common neighbor (indices are considered modulo k). For every $i = 1, \dots, k$, let w_i^+ be a black common neighbor of w_i and w_{i+1}. Then clearly the sequence $Q_C = w_1 w_1^+ w_2 \dots w_{k-1}^+ w_k w_k^+$ determines a closed W-quasitrail in Gr$(\mathcal{H})$, in which w_i is a special vertex if an only if $w_{k-1}^+ = w_k^+$, $i = 1, \dots, k$. Clearly, $V_w(Q_C) = W_C$.

Moreover, suppose that C is a Tutte maximal cycle, let F be a nontrivial component of Gr$(\mathcal{H}) - Q_C$, and let D_F be the corresponding component of $G - C$. If some white vertex w in F has a black neighbor $w_{i_0}^+$ on Q for some $i_0 = 1, \dots, k$ (indices modulo k), and x is the vertex in G corresponding to w, then both $x_{i_0} x \in E(G)$ and $x_{i_0+1} x \in E(G)$, contradicting the maximality of C. Hence $|e_{\mathrm{Gr}(\mathcal{H})}(F, V(Q) \cap B)| = 0$. Since clearly $|V_A(F, Q) \cap W| = |V_A(D_F, C)|$, we have $|e_{\mathrm{Gr}(\mathcal{H})}(F, V(Q) \cap B)| + |V_A(F, Q) \cap W| = |V_A(D_F, C)| \leq 3$, since C is Tutte. Hence all components of Gr$(\mathcal{H}) - Q_C$ are Tutte components, and Q_C is a Tutte closed W-quasitrail. ■

Proof of Corollary 7. $(i) \Rightarrow (ii)$. If C is a hamiltonian cycle in G, then $W_C = V_w(\mathrm{Gr}(\mathcal{H}))$ (where W_C is the the set of white vertices in $\mathrm{Gr}(\mathcal{H})$ that correspond to vertices of C). Thus, the closed W-quasitrail G_C, given in Theorem 6(ii), contains all white vertices of $\mathrm{Gr}(\mathcal{H})$.

$(ii) \Rightarrow (iii)$. Any closed W-quasitrail in $\mathrm{Gr}(\mathcal{H})$ containing all white vertices is dominating.

$(iii) \Rightarrow (i)$. Let Q be a dominating closed W-quasitrail in $\mathrm{Gr}(\mathcal{H})$. If $V_w(Q) = V_w(\mathrm{Gr}(\mathcal{H}))$, then the corresponding cycle C_Q (see Theorem 6(i)) is a hamiltonian cycle in G and we are done. If there is a white vertex $w \in V(\mathrm{Gr}(\mathcal{H})) \setminus V(Q)$, then w has a (black) neighbor on Q since Q is dominating, and w can be inserted in Q as a special vertex. Inserting all such white vertices, we get a closed W-quasitrail $\bar{Q}$ with $V_w(\bar{Q}) = V_w(\mathrm{Gr}(\mathcal{H}))$, and then $C_{\bar{Q}}$ is a hamiltonian cycle in G. ■

3.1 Auxiliary conjectures

For the proof of the equivalences of Theorem 5, we will need the following equivalent conjectures.

Conjecture 8. *Let H be a 2-edge-connected multigraph and let $W \subset V_2(H) \cup V_3(H)$ and $e_1, e_2 \in E(H)$. Then H has a closed W-quasitrail Q such that*
- (i) $e_1, e_2 \in E(S(Q))$,
- (ii) *subject to (i), Q is a domination-maximal closed W-quasitrail,*
- (iii) *subject to (i) and (ii), Q is a Tutte closed W-quasitrail.*

Conjecture 9. *Let H be a 2-edge-connected multigraph and let $W \subset V_2(H) \cup V_3(H)$ and $e_1 \in E(H), v_1, v_2 \in V(H)$. Then H has a closed W-quasitrail Q such that*
- (i) $e_1 \in E(S(Q))$;
- (ii) *for $i = 1, 2$, either $v_i \in V(Q)$, or $v_i \in V(F_i)$, where F_i is a component of $H - Q$ such that $|e_H(F_i, V(Q) \cap B)| + |V_A(F_i, Q) \cap W| \leq 2$, and if both $v_1 \notin V(Q)$ and $v_2 \notin V(Q)$, then $F_1 \neq F_2$;*
- (iii) *subject to (i) and (ii), Q is a domination-maximal closed W-quasitrail;*
- (iv) *subject to (i), (ii) and (iii), Q is a Tutte closed W-quasitrail.*

If Q is an (a, b, W)-quasitrail in H, then a nontrivial component F of $H - Q$ is called a *Tutte component* if $|e_H(F, \mathrm{Int}(Q) \cap B)| + |V_A(F, Q) \cap W| \leq \min\{3, |D(\mathrm{Int}(S(Q)))| - 1\}$ if $\mathrm{Int}(S(Q)) \neq \emptyset$, or $|e_H(F, \mathrm{Int}(Q) \cap B)| + |V_A(F, Q) \cap W| \leq 1$ if $\mathrm{Int}(S(Q)) = \emptyset$, respectively. We say that Q is a *Tutte (a, b, W)-quasitrail in H* if either $D(Q) = E(H)$, or every component of $H - Q$ is a Tutte component. A domination-maximal (a, b, W)-quasitrail which is also Tutte is called a *Tutte domination-maximal (a, b, W)-quasitrail.*

Conjecture 10. *Let H be a 2-edge-connected multigraph, and let $W \subset V_2(H) \cup V_3(H)$ and $a, b \in W$, $a \neq b$, $ab \notin E(H)$. Then there is an (a, b, W)-quasitrail Q in H such that*
- (i) *Q is a domination-maximal (a, b, W)-quasitrail,*
- (ii) *subject to (i), Q is a Tutte (a, b, W)-quasitrail.*

Finally, we will also need the following conjecture posed in [10]. Let H be a graph with $\delta(H) = 2$ and $|V_2(H)| = 4$, and set $E^+(H) = \{uv|\ u, v \in V(H)\}$. Then H is said to be $V_2(H)$-*dominated* if for any two edges $e_1 = u_1v_1, e_2 = u_2v_2 \in E^+(H)$ with $\{u_1, v_1, u_2, v_2\} = V_2(H)$, the (multi)graph $H + \{e_1, e_2\}$ has a dominating closed trail containing e_1 and e_2, and H is said to be *strongly* $V_2(H)$-*dominated* if H is $V_2(H)$-dominated and for any $e = uv$ with $u, v \in V_2(H)$, the (multi)graph $H + \{e\}$ has a dominating closed trail containing e.

Conjecture J [10]. *Every subgraph H of an essentially 4-edge-connected cubic graph with $\delta(H) = 2$ and $|V_2(H)| = 4$ is strongly $V_2(H)$-dominated.*

As shown in [10], Conjecture J is also equivalent with the previous conjectures.

Theorem K [10]. *Conjecture J is equivalent with Conjectures A – D.*

It is easy to verify that Conjectures 1 and 2 imply Conjecture A, Conjecture 3 implies Conjecture 4, and Conjecture 4 implies Conjecture A. Moreover, we will prove the following implications.

Proposition 11. *Conjecture J implies Conjecture 8.*

Proposition 12. *Conjecture J implies Conjecture 9.*

Proposition 13. *Conjecture J implies Conjecture 10.*

Proposition 14. *Conjecture 8 implies Conjecture 1.*

Proposition 15. *Conjecture 9 implies Conjecture 2.*

Proposition 16. *Conjecture 10 implies Conjecture 3.*

These implications, together with the fact that Conjecture A and Conjecture J are equivalent by Theorems E and K, will establish the proof of Theorem 5 (see Fig. 4).

$$\begin{tikzcd}
& \boxed{\text{Conj. 8}} \arrow[rrr, Rightarrow, "\text{Prop. 14}"] & & & \boxed{\text{Conj. 1}} \arrow[d, Rightarrow, "\text{Easy}"] \\
\boxed{\text{Conj. A}} \arrow[r, Leftrightarrow, "\text{Thm. K}"] & \boxed{\text{Conj. J}} \arrow[u, Rightarrow, "\text{Prop. 11}"'] \arrow[r, Rightarrow, "\text{Prop. 12}"] \arrow[d, Rightarrow, "\text{Prop. 13}"'] & \boxed{\text{Conj. 9}} \arrow[r, Rightarrow, "\text{Prop. 15}"] & \boxed{\text{Conj. 2}} \arrow[r, Rightarrow, "\text{Easy}"] & \boxed{\text{Conj. A}} \\
& \boxed{\text{Conj. 10}} \arrow[r, Rightarrow, "\text{Prop. 16}"] & \boxed{\text{Conj. 3}} \arrow[rr, Rightarrow, "\text{Easy}"] & & \boxed{\text{Conj. 4}} \arrow[u, Rightarrow, "\text{Easy}"']
\end{tikzcd}$$

Figure 4: Scheme of proof of Theorem 5.

3.2 Lemmas

Let H be a connected graph, and let $e_1 = x_1x_2$ and $e_2 = x_3x_4$ (not excluding the possibility that some of x_1, x_2 coincides with some of x_3, x_4) be two edges in H. We construct the new (multi)graph, denoted by $H(e_1, e_2)$, from H by removing the edges e_1, e_2, by adding a new vertex $z \notin V(H)$, and by adding new edges $f_i = zx_i$, $i = 1, 2, 3, 4$, see Fig. 5 (note that if e.g. $x_1 = x_3$, then f_1, f_3 are parallel edges in $H(e_1, e_2)$).

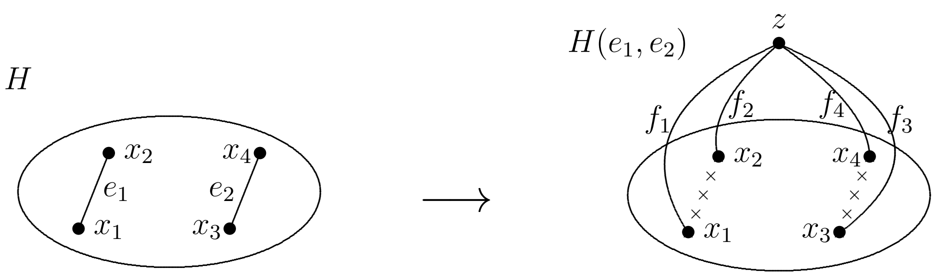

Figure 5: The (multi)graphs H and $H(e_1, e_2)$.

Lemma 17. *Let H be a 2-edge-connected multigraph, and let e_1, e_2 be two edges of H. Suppose that for any essential edge-cut R in H with $|R| \le 3$, each component of $H - R$ contains at least one of the edges e_1, e_2. Suppose further that the edges e_1, e_2 do not share a vertex of degree 3. Then $H(e_1, e_2)$ is essentially 4-edge-connected.*

Proof. Let, to the contrary, R' be an essential edge-cut in $H(e_1, e_2)$ with $|R'| \le 3$. Set $F' = \{f_1, f_2, f_3, f_4\}$. If $R' \cap F' = \emptyset$, then R' is also an essential edge-cut in H such that e_1, e_2 are contained in the same component of $H - R'$, a contradiction. Hence $|R' \cap F'| \ge 1$.

Recall that the assumption implies that there is no essential edge-cut R in H such that $|R| \le 3$ and either $e_1 \in R$ or $e_2 \in R$. If $|R' \cap F'| = 1$, say $f_1 \in R' \cap F'$, then $(R' - F') \cup \{e_1\}$ is an essential edge-cut in H, a contradiction. If $|R' \cap F'| = 3$, then $R' \subseteq F'$ and hence $H - e_1$ or $H - e_2$ is disconnected, a contradiction again.

Therefore, we may assume that $|R' \cap F'| = 2$. If $R' \cap F' = \{f_1, f_2\}$, then $(R' - F') \cup \{e_1\}$ is an essential edge-cut of H, a contradiction. Therefore, by symmetry, we may assume that $R' \cap F' = \{f_1, f_3\}$. Note that R' divides $H(e_1, e_2)$ into two components, say K_1 and K_2, such that, without loss of generality, K_1 contains x_1 and x_3, and K_2 contains z, x_2 and x_4. Note that $(R' - F') \cup \{e_1, e_2\}$ is an edge-cut of H that divides H into K_1 and $K_2 - z$. Since R' is an essential edge-cut in $H(e_1, e_2)$, $|V(K_1)| \ge 2$ and $|V(K_2)| \ge 2$. So, if $|V(K_2)| \ge 3$, then $R := (R' - F') \cup \{e_1, e_2\}$ is an essential edge-cut in H with $|R| = |R'| \le 3$, which contradicts the assumption. Hence we have $|V(K_2)| = 2$. This directly implies that $x_2 = x_4$ and the unique edge in $R' - F'$ is incident with it. However, this implies that x_2 has degree exactly three in $H(e_1, e_2)$ (and hence in H), a contradiction again. ■

Lemma 18. *Let H be an essentially 4-edge-connected multigraph with $\delta(H) \ge 3$. If Conjecture J is true, then all of the following hold:*

(*i*) *for any edge $e = xy$ in H, there exists a dominating closed trail T in H such that $x, y \in V(T)$ but $e \notin E(T)$,* (*ii*) *for any vertex z of degree four with $N_H(z) = \{x_1, x_2, x_3, x_4\}$, there exists a dominating closed trail T in $(H - z) + \{x_1x_2, x_3x_4\}$ such that $x_1x_2, x_3x_4 \in E(T)$,*

(*iii*) *for any vertex z of degree four with $N_H(z) = \{x_1, x_2, x_3, x_4\}$ and any $x_i, x_j \in N_H(z)$, $i \neq j$, there exists a dominating (x_i, x_j)-trail T in $H-z$ such that $x_k \in V(T)$ for any $x_k \in N_H(z)$ with $d_H(x_k) \geq 4$.*

In our proof, we will need the following operation, introduced in [5]. Let H be an essentially 4-edge-connected multigraph, and let $x \in V(H)$ be of degree $d_H(x) \geq 4$. The multigraph, obtained from H by deleting x, adding a cycle on $d_H(x)$ new vertices, and joining the new vertices to the original neighbors of x so that all new vertices have degree three and all neighbors of x have the same degree as in H, is called a *cubic inflation of H at x*. The operation is not unique, since it depends on the choice of the edges joining the new vertices to the original neighbors of x. Fleischner and Jackson [5] proved that by a suitable choice of these edges, some cubic inflation of H at x is essentially 4-edge-connected. By repeating this procedure, the resulting graph will eventually be cubic and still essentially (and hence cyclically) 4-edge-connected.

Theorem L [5]. *Let H be an essentially 4-edge-connected multigraph with $\delta(H) \geq 3$. Then some cubic inflation of H is also essentially 4-edge-connected.*

If $\delta(H) \geq 3$ and $\widetilde{H}$ is a cubic inflation of H, then, for any $x \in V(H)$, C_x will denote the cycle in $\widetilde{H}$ corresponding to x if $d_H(x) \geq 4$; otherwise (if $d_H(x) = 3$), C_x is considered to be trivial.

Proof of Lemma 18. Let H be an essentially 4-edge-connected multigraph with $\delta(H) \geq 3$. Suppose that Conjecture J is true. By Theorem L, there exists an essentially 4-edge-connected cubic inflation $\widetilde{H}$ of H.

(*i*) For an edge $e = xy$ in H, let $x'y'$ be the edge of $\widetilde{H}$ corresponding to the edge xy such that $x' \in V(C_x)$ and $y' \in V(C_y)$, let u_x and v_x be the neighbors of x' in $\widetilde{H}$ with $u_x, v_x \neq y$, let u_y and v_y be the neighbors of y' in $\widetilde{H}$ with $u_y, v_y \neq x$, and let $H' = \widetilde{H} - \{x', y'\}$. By the assumption that Conjecture J is true, the graph $H' + \{u_xv_x, u_yv_y\}$ has a dominating closed trail T' containing the edges u_xv_x and u_yv_y. By replacing the edges u_xv_x and u_yv_y with the paths $u_xx'v_x$ and $u_yy'v_y$, respectively, and contracting all the cycles C_v for all $v \in V_{\geq 4}(H)$, we obtain a dominating closed trail in H, say, T, such that $x, y \in V(T)$, however, since $x'y' \notin E(T')$, we have $e \notin E(T)$.

(*ii*) Let z be a vertex of degree four in H with neighbors x_1, x_2, x_3 and x_4. Note that C_z contains four vertices in $\widetilde{H}$. For $i \in \{1, 2, 3, 4\}$, let x'_i be the vertex in C_{x_i} such that x'_i has a neighbor in C_z, and let $H' = \widetilde{H} - V(C_z)$. By the assumption that Conjecture J is true, the graph $H' + \{x'_1x'_2, x'_3x'_4\}$ has a dominating cycle T' containing the edges $x'_1x'_2$ and $x'_3x'_4$. Contracting all the cycles C_v for all $v \in V_{\geq 4}(H) - \{z\}$, we obtain a dominating closed trail T in $(H - z) + \{x_1x_2, x_3x_4\}$ such that $x_1x_2, x_3x_4 \in E(T)$.

(*iii*) Similarly to the above, let z be a vertex of degree four in H with neighbors x_1, x_2, x_3 and x_4, let x'_i be the vertex in C_{x_i} such that x'_i has a neighbor in C_z for $i \in \{1, 2, 3, 4\}$, and let $H' = \widetilde{H} - V(C_z)$. By the assumption that Conjecture J is true, the graph $H' + \{x'_ix'_j\}$ has a dominating cycle T' containing the edge $x'_ix'_j$. Contracting all the cycles C_v for all

$v \in V_{\geq 4}(H) - \{z\}$, we obtain a dominating (x_i, x_j)-trail T in $H - z$ such that $x_k \in V(T)$ if $d_H(x_k) \geq 4$. ■

Before we state the next result, we first observe that the statement of Conjecture 8 is true for small graphs (this is easy to see for graphs on, say, at most four vertices).

Proposition 19. *Let H be a 2-edge-connected multigraph such that the statement of Conjecture 8 is true for every 2-edge-connected multigraph H' with $|V(H')| < |V(H)|$. Let $W \subset V_2(H) \cup V_3(H)$, let Q' be a Tutte closed W-quasitrail in H, and let Q'' be a closed W-quasitrail in H such that $D(Q'') \supsetneq D(Q')$. Then there is a Tutte closed W-quasitrail Q such that $D(Q) \supsetneq D(Q')$ and $D(Q') \cap S(Q'') \subset D(Q') \cap S(Q)$.*

Proof. Suppose that there is no such Q. Then for every closed W-quasitrail $\widetilde{Q}$ in H, satisfying $D(\widetilde{Q}) \supsetneq D(Q')$ and $D(Q') \cap S(Q'') \subset D(Q') \cap S(\widetilde{Q})$ (including $\widetilde{Q} = Q''$), at least one component of $H - \widetilde{Q}$ is non-Tutte. Thus, we can suppose that Q'' is chosen such that, among all closed W-quasitrails $\widetilde{Q}$ in H, satisfying $D(\widetilde{Q}) \supsetneq D(Q')$ and $D(Q') \cap S(Q'') \subset D(Q') \cap S(\widetilde{Q})$, the number of non-Tutte components of $H - Q''$ is smallest possible. Let $F''_1, \ldots, F''_{k''}$ be the nontrivial components of $H - Q''$, and let F''_{i_0} be a non-Tutte component. Let $F'_1, \ldots, F'_{k'}$ be components of $H - Q'$. Since $D(Q'') \supsetneq D(Q')$, every F''_i, $i = 1, \ldots, k''$, is a subgraph of some F'_{j_0} for some $j_0 = 1, \ldots, k'$. Since F''_{i_0} is non-Tutte while F'_{j_0} is Tutte, F''_{i_0} is a proper subgraph of F'_{j_0}, and F'_{j_0} contains at least one edge of Q'', implying that at least two edges of $e_H(F'_{i_0}, V(Q'))$ are in $E(Q'')$. Moreover, we have $|e_H(F'_{j_0}, V(Q') \cap B)| + |V_A(F'_{j_0}, Q') \cap W| \leq 3$ (where $B = V(H) \setminus W$), since F'_{j_0} is Tutte.

We make some observations concerning vertices in $V_A(F'_{j_0}, Q') \cap W$. First, if $x \in V_A(F'_{j_0}, Q') \cap W$ is in $V_s(Q')$, then, since $W \subset V_2(H) \cup V_3(H)$, there are, besides the edge $x^{-E} = x^{+E}$ of Q', one or two other edges incident to x. If both are incident to vertices in F'_{j_0}, then $|e_H(\{x\}, S(Q'))| = 1$, and if one of them is incident to a vertex in F'_{j_0}, we have $|e_H(F'_{j_0}, \{x\})| = 1$. Finally, if $x \in V_A(F'_{j_0}, Q') \cap W$ is in $S(Q')$, then $d_{Q'}(x) = 2$, hence $|e_H(F'_{j_0}, \{x\})| = 1$ (note that here and throughout the proof, we do not exclude the possibility that some of these edges under consideration are parallel edges of H).

We define a set of edges $R \subset E(H)$ as follows:

- for every $x \in V_A(F'_{j_0}, Q') \cap V_s(Q')$ with $|e_H(F'_{j_0}, \{x\})| = 1$, R contains this (only) edge in $e_H(F'_{j_0}, \{x\})$,
- for every $x \in V_A(F'_{j_0}, Q') \cap V_s(Q')$ with $|e_H(F'_{j_0}, \{x\})| = 2$, R contains the (only) double edge of Q' incident to x,
- for every $x \in V_A(F'_{j_0}, Q') \cap W \cap S(Q')$, R contains the (only) edge in $e_H(F'_{j_0}, \{x\})$,
- finally, R contains all edges in $e_H(F'_{j_0}, V(Q') \cap B)$.

Let $V_s^2(F'_{j_0}, Q')$ denote the set of all vertices $x \in V_s(Q')$ with $|e_H(F'_{j_0}, \{x\})| = 2$. Then R is an edge-cut separating $\bar{F} = \langle V(F'_{j_0}) \cup V_s^2(F'_{j_0}, Q')\rangle_H$ from (the rest of) Q'. By the construction, $|R| = |e_H(F'_{j_0}, V(Q') \cap B)| + |V_A(F'_{j_0}, Q') \cap W|$. Thus, we have $|R| \leq 3$ since F'_{j_0} is Tutte, and, on the other hand, $|R| \geq 2$ since F'_{j_0} contains at least one edge of Q''.

Let $u_1^Q u_1^F$ and $u_2^Q u_2^F$ (where $u_i^Q \in V(Q') \setminus V_s^2(F'_{j_0}, Q')$ and $u_i^F \in V(\bar{F})$, $i = 1, 2$), be the edges in $R \cap E(Q'')$, and if $|R| = 3$, then let $f_3 = u_3^Q u_3^F$ be the third edge of R. Let $\bar{F}_1$ be the multigraph obtained from $\bar{F}$ by adding a new black vertex z and the edges zu_1^F, zu_2^F, and, if $|R| = 3$, also zu_3^F. By the minimality of H, $\bar{F}_1$ contains a Tutte domination-maximal closed W-quasitrail T, containing the edges zu_1^F and zu_2^F.

If $|R| = 2$, or if $|R| = 3$ and either $Q'' - V(F'_{j_0})$ dominates $u_3^Q u_3^F$, or $u_3^Q u_3^F \notin D(Q'')$, then, concatenating $T - z$ with $Q'' - V(F'_{j_0})$ using the edges $u_1^Q u_1^F$ and $u_2^Q u_2^F$, we obtain a closed W-quasitrail Q''' in H such that $D(Q''') \supsetneq D(Q')$.

It remains to consider the case that $|R| = 3$ and $Q'' \cap V(F'_{j_0})$ dominates $u_3^Q u_3^F$. Let $f_1 = u_1^F u_2^F$ be a new added edge (if already $u_1^F u_2^F \in E(\bar{F}_1)$, then f_1 is a parallel edge), let f_2 be a new added loop at $(u_3^F)^+$, if u_3^F is a special vertex of degree 2 in Q'', or at u_3^F otherwise. Let $\bar{F}_2 = \bar{F}_1 - \{z\} + f_1 + f_2$. Since $Q'' \cap V(F'_{j_0})$ dominates $u_3^Q u_3^F$, the multigraph $\bar{F}_2$ (or $\bar{F}_2 - u_3^F$, if u_3^F is special), is 2-edge-connected.

Let $\bar{W}_2 = W \cap V(\bar{F}_2)$. By the minimality of H, $\bar{F}_2$ (or $\bar{F}_2 - u_3^F$, if u_3^F is special,) contains a Tutte domination-maximal closed $\bar{W}_2$-quasitrail T_{f_1,f_2}, containing f_1 and f_2. Concatenating $T_{f_1,f_2} - \{f_1, f_2\}$ with $Q'' - V(F'_{j_0})$, using the edges $u_1^Q u_1^F$ and $u_2^Q u_2^F$, we obtain a closed W-quasitrail Q''' in H such that $D(Q''') \supsetneq D(Q')$ and the number of non-Tutte components of $H - Q'''$ is smaller than the number of non-Tutte components of $H - Q''$, contradicting the choice of Q''. ■

3.3 Proof of Proposition 11

Proof. First recall that the statement of Conjecture 8 (and hence Proposition 11) is true for small graphs.

Suppose that Conjecture J is true, and let H be a counterexample to Conjecture 8. Then H is a 2-edge-connected multigraph such that, for some $W \subset V_2(H) \cup V_3(H)$ and $e_1, e_2 \in E(H)$, no domination-maximal closed W-quasitrail containing e_1 and e_2 is a Tutte closed W-quasitrail. We choose H such that

(i) H is a minimum counterexample to Conjecture 8 (i.e., $|V(H)|$ is minimum),

(ii) subject to (i), $|W|$ is minimum.

Claim 1. $W = \emptyset$.

Proof. Suppose that $W \neq \emptyset$ and let $u \in W$. Set $W' = W \setminus \{u\}$ and $B' = V(H) \setminus W' = B \cup \{u\}$. By the choice of H and W, H contains a Tutte domination-maximal closed W'-quasitrail Q' such that $e_1, e_2 \in S(Q')$. Clearly, Q' is also a closed W-quasitrail in H, and it is easy to observe that Q' is Tutte (since the only difference is in u, where, for any component F of $H - Q'$, for which u is a vertex of attachment, we have $|e_H(F, \{u\})| \geq |V_A(F, \{u\})| = 1$). Thus, Q' is not domination-maximal (among all closed W-quasitrails in H). We choose Q' such that is domination maximal Tutte among all closed W-quasitrails in H such that $e_1, e_2 \in S(Q')$.

Let Q'' be a domination-maximal closed W-quasitrail in H with $D(Q'') \supsetneq D(Q')$. To reach a contradiction, we show that Q'' can be chosen such that it is Tutte and still satisfies $D(Q'') \supsetneq D(Q')$. By Proposition 19, this is true. Thus, we have $W = \emptyset$. □

Claim 2. *For any edge-cut R in H with $|R| = 2$, each component of $H - R$ contains one of the edges e_1 and e_2. In particular, we have $\delta(H) \geq 3$.*

Proof. Let R be an edge-cut in H with $|R| = 2$, let K_1, K_2 be the components of $H - R$, and suppose that $\{e_1, e_2\} \subset R \cup E(K_1)$. Denote $R = \{u_1u_2, v_1v_2\}$ with $u_i, v_i \in V(K_i)$ for $i \in \{1, 2\}$ (not excluding the possibility that $u_1 = v_1$ or $u_2 = v_2$), and let H_1 be obtained from K_1 by adding the edge $f_1 = u_1v_1$ (if an edge u_1v_1 already exists in K_1, then f_1 is a parallel edge, and if $u_1 = v_1$, then f_1 is a loop). Then H_1 is 2-edge-connected and, by the minimality of H, H_1 contains a Tutte closed domination-maximal W-quasitrail Q_1 containing e_1 and e_2 (or the edge f_1, if some of e_1 and e_2 is contained in R). If possible, we choose Q_1 to contain the edge f_1.

If Q_1 does not contain f_1, then Q_1 is a Tutte domination-maximal closed W-quasitrail also in H, a contradiction. Hence Q_1 contains f_1, and f_1 is a single edge of Q_1 since $W = \emptyset$.

Similarly, let H_2 be obtained from K_2 by adding the edge $f_2 = u_2v_2$. Again by the minimality of H, H_2 contains a Tutte domination-maximal closed W-trail Q_2 containing f_2.

Let Q be the closed W-quasitrail obtained by concatenating $Q_1 - f_1$ and $Q_2 - f_2$ using the edges in R. Note that Q contains both e_1 and e_2.

Furthermore, since any component of $H - Q$ is either a component of $H_1 - Q_1$ or $H_2 - Q_2$, we see that Q is a Tutte domination-maximal closed W-quasitrail in H, containing both e_1 and e_2 exactly once, a contradiction. Therefore the first assertion holds.

If there exists a vertex of degree two, then the edge-cut formed by the two edges incident with this vertex violates the first assertion, a contradiction. Thus, we have $\delta(H) \geq 3$. $\square$

Claim 3. *For any essential edge-cut R in H with $|R| = 3$, each component of $H - R$ contains at least one of the edges e_1 and e_2.*

Proof. Similarly to the proof of Claim 2, let R be an essential edge-cut in H with $|R| = 3$, and let K_1, K_2 be the components of $H - R$. Suppose that $\{e_1, e_2\} \subset R \cup E(K_1)$, and let R be chosen such that, subject to this assumption, K_2 is as small as possible. Denote $R = \{u_1u_2, v_1v_2, w_1w_2\}$ with $u_i, v_i, w_i \in V(K_i)$ for $i \in \{1, 2\}$.

Let H_1 be obtained from K_1 by adding a new black vertex $x \notin V(K_1)$ and new edges u_1x, v_1x and w_1x. By the minimality of H, H_1 contains a Tutte domination-maximal closed trail Q_1 containing e_1 and e_2 (or, if some of e_1 and e_2 is contained in R, then Q_1 contains the corresponding "new" edge incident with x). If possible, we choose Q_1 to contain x.

If Q_1 does not contain x, then Q_1 is a Tutte domination-maximal closed W-trail in H, a contradiction. Hence Q_1 contains x. By symmetry, we may assume that $u_1x, v_1x \in E(Q_1)$ and $w_1x \notin E(Q_1)$.

Let H_2 be obtained from K_2 by adding a new vertex $y \notin V(K_2)$ and new edges yu_2, yv_2 and yw_2. By the minimality of K_2, the graph H_2 is essentially 4-edge-connected. Thus, by Lemma 18 (i), there exists a dominating closed trail Q_2 in H_2 such that Q_2 passes through both y and w_2, but $yw_2 \notin E(Q_2)$. Since $d_{H_2}(y) = 3$, we have $yu_2, yv_2 \in E(Q_2)$. Now, concatenating $Q_1 - x$ and $Q_2 - y$ using the edges u_1u_2 and v_1v_2, we obtain a closed W-quasitrail Q in H containing both e_1 and e_2. Since $u_2, v_2, w_2 \in V(Q_2) - \{y\} \subseteq V(Q)$, every component of $H - Q$ is a component of $H_1 - Q_1$ or of $H_2 - Q_2$. Hence the quasitrail Q is a Tutte domination-maximal closed W-quasitrail, a contradiction. $\square$

Claim 4. *The edges e_1 and e_2 do not share a vertex of degree three.*

Proof. Let, to the contrary, $x \in V_3(H)$ be incident to both e_1 and e_2, and let $f = xy$ be the third edge incident to x. Note that the graph H is essentially 4-edge-connected by Claims 2 and 3. By Lemma 18 (i), there exists a dominating closed trail Q in H such that Q passes through both x and y, but $xy \notin E(Q)$. Note that Q is a Tutte closed domination maximal W-quasitrail in H. Since $d_H(x) = 3$, we have $e_1, e_2 \in E(Q)$, a contradiction. □

Therefore, by Claims 2–4 and by Lemma 17, the graph $H' := H(e_1, e_2)$ is essentially 4-edge-connected. As in Figure 5, let $e_1 = x_1x_2$, $e_2 = x_3x_4$, and let z be the new vertex. By Lemma 18 (ii), there exists a dominating maximal closed W-quasitrail Q in $(H' - z) + \{x_1x_2, x_3x_4\}$ such that $x_1x_2, x_3x_4 \in E(Q)$. We immediately see that $H = (H' - z) + \{x_1x_2, x_3x_4\}$, and hence Q is a Tutte domination-maximal closed W-quasitrail in H with $e_1, e_2 \in E(Q)$ such that e_1, e_2 are used only once. ■

3.4 Proof of Proposition 12

Proof. First observe that the statement of Conjecture 9 (and hence Proposition 12) is true for small graphs (this is easy to see for graphs on, say, at most four vertices).

Suppose that Conjecture J is true, and let H be a counterexample to Conjecture 9. Then H is a 2-edge-connected multigraph such that, for some $W \subset V_2(H) \cup V_3(H)$ and $e_1 \in E(H)$, $v_1, v_2 \in V(H)$, no domination-maximal closed W-quasitrail Q such that $e_1 \in E(Q)$ and for each $i = 1, 2$, either $v_i \in V(Q)$ or v_i is in a component F_i of $H - Q$ such that $|e_H(F_i, V(Q) \cup B)| + |V_A(F_i, Q) \cap W| \leq 2$, and if both $v_1, v_2 \notin V(Q)$, then $F_1 \neq F_2$, is a Tutte closed W-quasitrail. We choose H such that

(i) H is a minimum counterexample to Conjecture 9 (i.e., $|V(H)|$ is minimum),

(ii) subject to (i), $|W|$ is minimum.

Claim 1. $W = \emptyset$.

Proof. Suppose that $W \neq \emptyset$ and let $u \in W$. Set $W' = W \setminus \{u\}$ and $B' = V(H) \setminus W' = B \cup \{u\}$. By the choice of H and W, H contains a Tutte domination-maximal closed W'-quasitrail Q' such that $e_1 \in S(Q')$ and for each $i = 1, 2$, either $v_i \in V(Q)$ or v_i is in a component F_i of $H - Q$ such that $|e_H(F_i, V(Q) \cup B)| + |V_A(F_i, Q) \cap W| \leq 2$, and if both $v_1, v_2 \notin V(Q)$, then $F_1 \neq F_2$. Clearly, Q' is also a closed W-quasitrail in H, and it is easy to observe that Q' is Tutte (since the only difference is in u, where, for any component F of $H - Q'$, for which u is a vertex of attachment, we have $|e_H(F, \{u\})| \geq |V_A(F, \{u\})| = 1$). Thus, Q' is not domination-maximal.

Let Q'' be a domination-maximal closed W-quasitrail in H with $D(Q'') \supsetneq D(Q')$. To reach a contradiction, we show that Q'' can be chosen such that it is Tutte. By Proposition 19, this is true. Thus, we have $W = \emptyset$. □

Claim 2. *For any edge-cut R in H with $|R| = 2$, each component of $H - R$ contains the edge e_1 or at least one of the vertices v_1, v_2.*

Proof. Let R be an edge-cut in H with $|R| = 2$, let K_1, K_2 be the components of $H - R$, and suppose that K_2 contains neither e_1 nor any of the vertices v_1, v_2. Denote $R = \{u_1u_2, z_1z_2\}$ with $u_i, z_i \in V(K_i)$ for $i \in \{1,2\}$ (not excluding the possibility that $u_1 = v_1$ or $u_2 = v_2$), and let H_1 be obtained from K_1 by adding the edge $f_1 = u_1z_1$ (if an edge u_1z_1 already exists in K_1, then f_1 is be a parallel edge, and if $u_1 = z_1$, then f_1 is a loop). Then H_1 is 2-edge-connected and, by the minimality of H, H_1 contains a Tutte closed domination-maximal W-quasitrail Q_1 containing e_1 (or the edge f_1, if the edge e_1 is contained in R) and satisfying the conditions for v_1, v_2. If possible, we choose Q_1 to contain the edge f_1.

If Q_1 does not contain f_1, then Q_1 is a Tutte domination-maximal closed W-quasitrail also in H, a contradiction. Hence Q_1 contains f_1, and f_1 is a single edge of Q_1 since $W = \emptyset$.

Similarly, let H_2 be obtained from K_2 by adding the edge $f_2 = u_2z_2$. Again by the minimality of H, H_2 contains a Tutte domination-maximal closed W-trail Q_2 containing f_2.

Let Q be the closed W-quasitrail obtained by concatenating $Q_1 - f_1$ and $Q_2 - f_2$ using the edges in R. Note that Q contains e_1 and satisfies the conditions for v_1 and v_2.

Furthermore, since any component of $H - Q$ is either a component of $H_1 - Q_1$ or $H_2 - Q_2$, we see that Q is a Tutte domination-maximal closed W-quasitrail in H, containing e_1 exactly once and satisfying the conditions for v_1 and v_2, a contradiction. □

Claim 3. *For any essential edge-cut R in H with $|R| = 3$, each component of $H - R$ contains the edge e_1 or at least one of the vertices v_1, v_2.*

Proof. Similarly to the proof of Claim 2, let R be an essential edge-cut in H with $|R| = 3$, and let K_1, K_2 be the components of $H - R$. Suppose that $\{e_1, v_1, v_2\} \subset R \cup E(K_1) \cup V(K_1)$, and let R be chosen such that, subject to this assumption, K_2 is as small as possible. Denote $R = \{u_1u_2, z_1z_2, w_1w_2\}$ with $u_i, z_i, w_i \in V(K_i)$ for $i \in \{1,2\}$.

Let H_1 be obtained from K_1 by adding a new black vertex $x \notin V(K_1)$ and new edges u_1x, z_1x and w_1x. By the minimality of H, H_1 contains a Tutte domination-maximal closed trail Q_1 containing e_1 and satisfying the conditions for v_1, v_2 (or, if some of e_1 and v_1, v_2 is contained in R, then Q_1 contains the corresponding "new" edge incident with x). If possible, we choose Q_1 to contain x.

If Q_1 does not contain x, then Q_1 is a Tutte domination-maximal closed W-trail in H, a contradiction. Hence Q_1 contains x. By symmetry, we may assume that $u_1x, z_1x \in E(Q_1)$ and $w_1x \notin E(Q_1)$.

Let H_2 be obtained from K_2 by adding a new vertex $y \notin V(K_2)$ and new edges yu_2, zv_2 and yw_2. By the minimality of K_2, the graph H_2 is essentially 4-edge-connected. Thus, by Lemma 18 (i), there exists a dominating closed trail Q_2 in H_2 such that Q_2 passes through both y and w_2, but $yw_2 \notin E(Q_2)$. Since the degree of y is three in H_2, note that $yu_2, yz_2 \in E(Q_2)$.

Now, concatenating $Q_1 - x$ and $Q_2 - y$ using the edges u_1u_2 and z_1z_2, we obtain a closed W-quasitrail Q in H containing e_1 and satisfying the conditions for v_1, v_2. Since $u_2, v_2, w_2 \in V(Q_2) - \{y\} \subseteq V(Q)$, every component of $H - Q$ is a component of $H_1 - Q_1$ or of $H_2 - Q_2$. Hence the quasitrail Q is a Tutte domination-maximal closed W-quasitrail, a contradiction. □

Claim 4. *For any essential edge-cut R in H with $|R| = 2$, each component of $H - R$ contains either the edge e_1 or both the vertices v_1, v_2.*

Proof. Let R be an essential edge-cut in H with $|R| = 2$, set $R = \{u_1u_2, z_1z_2\}$, and let K_1, K_2 be the components of $H - R$. Suppose, to the contrary, that K_2 contains only the vertex v_1 (recall that K_2 must contain at least one of e_1, v_1, v_2 by Claim 2). We choose K_2 smallest possible and contract it to a vertex z. We denote the resulting graph H_1. By the minimality of H, H_1 contains a Tutte domination-maximal closed W-quasitrail Q_1 containing e_1 and satisfying the conditions for z, v_2 (or, if some of e_1 and v_1, v_2 is contained in R, then Q_1 contains the corresponding "new" edge incident with z). If possible, we choose Q_1 to contain z. If Q_1 does not contain z, then Q_1 is a Tutte domination-maximal closed W-trail in H, a contradiction. Hence Q_1 contains z. Similarly, let H_2 be obtained from K_2 by adding the edge $f_2 = u_2z_2$. Again by the minimality of H, H_2 contains a Tutte domination-maximal closed W-trail Q_2 containing f_2 and a loop on the vertex v_1. Now, concatenating $Q_1 - z$ and $Q_2 - f_1$ using the edges u_1u_2 and z_1z_2, we obtain a closed W-quasitrail Q in H containing e_1 and satisfying the conditions for v_1, v_2. Since $u_2, z_2, v_1 \in V(Q_2) - \{f_2\} \subseteq V(Q)$, every component of $H - Q$ is a component of $H_1 - Q_1$ or of $H_2 - Q_2$. Hence the quasitrail Q is a Tutte domination-maximal closed W-quasitrail, a contradiction. □

Claim 5. *Neither v_1 nor v_2 is incident with e_1.*

Proof. Let, to the contrary, $e_1 = v_1x$ for some $x \in V(H)$. Let H' be obtained from H by adding a loop f at the vertex v_2. By Proposition 11, H' contains a Tutte domination-maximal closed W-quasitrail Q' containing e_1 and f. Then $Q = Q' - f$ is a Tutte domination-maximal closed W-quasitrail containing e_1, v_1 and v_2, a contradiction. □

Claim 6. *H is essentially 3-edge-connected.*

Proof. Let, to the contrary, $R = \{u_1u_2, w_1w_2\}$ be an essential edge-cut in H, and let K_1, K_2 be components of $H - R$. By Claim 4, we can choose the notation such that $e_1 \in E(K_1)$, $v_1, v_2 \in V(K_2)$, and $u_i, w_i \in V(K_i)$, $i = 1, 2$, Let H_i be obtained from K_i by adding a new edge $f_i = u_iw_i$, $i = 1, 2$. By Proposition 11, H_1 contains a Tutte domination-maximal closed W-quasitrail Q_1 containing e_1 and f_1. By the minimality of H, H_2 contains a Tutte domination-maximal closed W-quasitrail Q_2 containing f_2 and satisfying the conditions for v_1, v_2. Concatenating $Q_1 - f_1$ and $Q_2 - f_2$, using the edges u_1u_2 and w_1w_2, we obtain a Tutte domination-maximal closed W-quasitrail in H containing e_1 and satisfying the conditions for v_1, v_2, a contradiction. □

Set $H^+ = H + v_1v_2$. By Claims 2–5 and by Lemma 17, the graph $H' := H^+(e_1, v_1v_2)$ is essentially 4-edge-connected. As in Figure 5, let $e_1 = x_1x_2$, $e_2 = v_1v_2$, and let z be the new vertex. By Lemma 18 (*iii*), there exists a dominating maximal closed W-quasitrail Q in $(H' - z) + \{x_1x_2\}$ such that $x_1x_2 \in E(Q)$. We immediately see that $H = (H' - z) + \{x_1x_2\}$. If both $v_1 \notin V(Q)$ and $v_2 \notin V(Q)$, then $F_1 \neq F_2$, since otherwise we have a contradiction with Claim 6. Hence Q is a Tutte domination-maximal closed W-quasitrail in H with $e_1 \in E(Q)$ and satisfying the conditions for v_1, v_2 such that e_1 is used only once. ■

3.5 Proof of Proposition 13

Proof. Suppose that Conjecture J is true. By Propositions 11 and 12 (which are already proved), this assumption implies that Conjectures 8 and 9 are also true. Let H be a counterexample to Conjecture 10. Then H is a 2-edge-connected multigraph such that, for some $W \subset V_2(H) \cup V_3(H)$ and $a, b \in W$, $ab \notin E(H)$, there exists no Tutte domination-maximal (a, b, W)-quasitrail in H.

Claim 1. *H contains a Tutte (a, b, W)-quasitrail.*

Proof. We distinguish three cases.

Case 1: $d_H(a) = d_H(b) = 2$. Set $e = ab$ and $\bar{H} = H + e$. By Conjecture 8, $\bar{H}$ contains a Tutte closed W-quasitrail $\bar{Q}$ such that $e = ab \in S(\bar{Q})$ and $\bar{Q}$ is domination-maximal. Let Q be the (a, b, W)-quasitrail in H obtained by removing the edge e from $\bar{Q}$.

If $\bar{Q}$ can be chosen such that $|D(\mathrm{Int}(S(Q)))| \geq 4$, then, since $a, b \in W$, since a component of $H - Q$ is also a component of $H - \bar{Q}$, and since $\bar{Q}$ is Tutte, for any component F of $H - Q$ we have $|e_H(F, \mathrm{Int}(Q) \cap B)| + |V_A(F, Q) \cap W| = |e_H(F, \bar{Q} \cap B)| + |V_A(F, \bar{Q}) \cap W| \leq 3 = \min\{3, |D(\mathrm{Int}(S(Q)))| - 1\}$, implying that Q is Tutte. Thus, it remains to consider the case that $|D(\mathrm{Int}(S(Q)))| \leq 3$.

Suppose first that there is an edge $f = xy \in E(H)$ such that $\{x, y\} \cap \{a, b\} = \emptyset$, and, additionally to the above choice, choose $\bar{Q}$ such that $S(\bar{Q})$ contains e and f (which is possible by Conjecture 8). Then $|V(Q)| \geq 4$ and $|D(\mathrm{Int}(S(Q)))| \geq 3$. If $d_H(u) \geq 3$ for some $u \in \mathrm{Int}(Q)$, then $|D(\mathrm{Int}(S(Q)))| \geq 4$ and we are in the previous case. Hence if $\mathrm{Int}(S(Q)) \neq \emptyset$, then all vertices in $\mathrm{Int}(Q)$ are of degree 2, and then, for any nontrivial component F of $H - Q$, we have $V_A(F, Q) \subset \{a, b\}$, implying that $|e_H(F, \mathrm{Int}(Q) \cap B)| + |V_A(F, Q) \cap W| \leq 2 = \min\{3, |D(\mathrm{Int}(S(Q)))| - 1\}$ and Q is Tutte. If $\mathrm{Int}(S(Q)) = \emptyset$, then $Q = ab$, and since H is 2-edge-connected and $d_H(a) = d_H(b) = 2$, for any nontrivial component F of $H - Q$, we have $V_A(F, Q) = \{a, b\}$ and, since F is nontrivial, Q is not domination-maximal, a contradiction.

Hence $H - \{a, b\}$ is edgeless, implying that $H - Q$ has no nontrivial component and Q is Tutte.

Case 2: $d_H(a) = 3$ and $d_H(b) = 2$. Choose an edge $e_a = aw$ incident to a and set $H^- = H - e_a$ and $\bar{H}^- = H^- + e$, where $e = ab$. Then $d_{H^-}(a) = d_{H^-}(b) = 2$ and, if $\bar{H}^-$ is 2-edge-connected, we are in Case 1. Thus, suppose that $\bar{H}^-$ is not 2-edge-connected.

Let H_1 be a maximal 2-edge-connected subgraph of $\bar{H}^-$ containing e, and let $v \in V(H_1)$ be the cutvertex of $\bar{H}^-$ separating w from (the rest of) H_1. Let H_1^+ be obtained from H_1 by adding a loop f at the vertex v. By Conjecture 8, H_1^+ contains a Tutte closed W-quasitrail Q^+ with $e, f \in S(Q^+)$. Let Q be the (a, b, W)-quasitrail in H obtained from Q^+ by removing e and f. By the construction, obviously $|D(\mathrm{Int}(S(Q)))| \geq 3$, and by the fact that $V_A(H - H_1^+, H_1^+) = \{a, v\}$, and by analogous arguments as in Case 1, we observe that Q is Tutte.

Case 3: $d_H(a) = d_H(b) = 3$. Choose edges $e_a = aw_1$ and $e_b = bw_2$, and set $H^- = H - e_a - e_b$ and $\bar{H}^- = H^- + e$. Then again $d_{H^-}(a) = d_{H^-}(b) = 2$.

Suppose first that $\bar{H}^-$ is disconnected, i.e., $\{e_a, e_b\}$ is an edge-cut of H. Let H_1 be the component of $\bar{H}^-$ containing e, and $H_2 = \bar{H}^- - H_1$. Then H_1 is 2-edge-connected, and if Q^+ can be chosen such that $|D(\mathrm{Int}(S(Q)))| \geq 3$, then in H_1 we are in Case 1. The resulting Tutte (a, b, W)-quasitrail in H_1 is Tutte also in H since $V_A(H_2, H_1) = \{a, b\}$. The remaining small cases are solved by a similar easy argument as in Case 1.

Hence we suppose that $\bar{H}^-$ is connected. Let H_1 be a maximal 2-edge-connected subgraph of $\bar{H}^-$ containing the edge e. We distinguish four subcases.

Subcase 3.1: $V(H_1) \subsetneq V(\bar{H}^-)$ and w_1, w_2 are in the same component of $\bar{H}^- - H_1$. Let $v \in V(H_1)$ be the cutvertex of $\bar{H}^-$ separating w_1 and w_2 from (the rest of) H_1, let f be a loop on v, and set $H_1^+ = H_1 + f$. As in Case 2, let Q^+ be a Tutte closed W-quasitrail in H_1^+ with $e, f \in S(Q^+)$, and let $Q = Q^+ - e - f$. Then Q is a Tutte (a, b, W)-quasitrail in H_1, and Q is Tutte also in H since $V_A(H - H_1, H_1) = \{a, b, v\}$.

Subcase 3.2: $V(H_1) \subsetneq V(\bar{H}^-)$ and w_1, w_2 are in different components of $\bar{H}^- - H_1$. Let $v_i \in V(H_1)$ be the cutvertex of $\bar{H}^-$ separating w_i from (the rest of) H_1, $i = 1, 2$. If $v_1 = v_2$, then we set $v = v_1 = v_2$ and $H_1^+ = H_1 + f$, where f is a loop on v, and proceed in the same way as in Subcase 3.1. Thus, we suppose that $v_1 \neq v_2$. By Conjecture 9, H_1 has a Tutte closed W-quasitrail Q^+ such that $e \in S(Q^+)$ and v_1, v_2 satisfy the conditions given in Conjecture 9. Then $Q = Q^+ - e$ is a Tutte (a, b, W)-quasitrail in H_1, and, by the properties of v_1 and v_2 given in Conjecture 9, Q is Tutte also in H.

Subcase 3.3: $V(H_1) \subsetneq V(\bar{H}^-)$, $w_1 \in V(H_1)$ and $w_2 \in V(\bar{H}^- - H_1)$. Let $v_2 \in V(H_1)$ be the cutvertex of $\bar{H}^-$ separating w_2 from (the rest of) H_1, and set $H_1^+ = H_1 + e$ (where $e = ab$). If $v_2 = w_1$, we set $H_1^{++} = H_1^+ + f$, where f is a loop on $v_2 = w_1$, and, by Conjecture 8, we have a Tutte closed W-quasitrail Q^+ in H_1^{++} with $e, f \in S(Q^+)$, from which $Q = Q^+ - e - f$ is a requested Tutte (a, b, W)-quasitrail in H. Thus, we suppose that $v_2 \neq w_1$. By Conjecture 9, H_1^+ has a Tutte closed W-quasitrail Q^+ such that $e \in S(Q^+)$ and v_1, v_2 satisfy the conditions of Conjecture 9, and then $Q = Q^+ - e$ is a requested Tutte (a, b, W)-quasitrail in H.

Subcase 3.4: $V(H_1) = V(\bar{H}^-)$. If $w_1 = w_2$, we proceed in the same way as in Subcase 3.1, using Conjecture 8, and if $w_1 \neq w_2$, we proceed in the same way as in Subcase 3.2, using Conjecture 9. $\square$

Now, by Claim 1, we can choose Q such that Q is domination-maximal among all Tutte (a, b, W)-quasitrails in H. Thus, Q is not Tutte domination-maximal.

Let Q' be a domination-maximal closed (a, b, W)-quasitrail in H with $D(Q') \supsetneq D(Q)$. To reach a contradiction, we show that Q' can be chosen such that it is Tutte. So, suppose the opposite, and choose Q' such that the number of non-Tutte components of $H - Q'$ is smallest possible.

Let $F'_1, \ldots, F'_{k'}$ be the components of $H - Q'$. By the assumption, some F'_i, $i = 1, \ldots, k'$, are not Tutte components. We choose Q' such that the number of non-Tutte components among $F'_1, \ldots, F'_{k'}$ is smallest possible, and we show that Q' is Tutte. Let, to the contrary, F'_{i_0} be a non-Tutte component. Let $F_1, \ldots, F_k$ be components of $H - Q$. Since $D(Q') \supsetneq D(Q)$, every F'_i, $i = 1, \ldots, k'$, is a subgraph of some F_{j_0} for some $j_0 = 1, \ldots, k$. Since F'_{i_0} is non-Tutte while F_{j_0} is Tutte, F'_{i_0} is a proper subgraph of F_{j_0}, and F_{j_0} contains at least one edge of Q', implying that at least one edge of $e_H(F_{i_0}, V(Q))$ is in $E(Q')$. Moreover, we have $|e_H(F_{j_0}, V(\mathrm{Int}(Q)) \cap B)| + |V_A(F_{j_0}, Q) \cap W| \leq 3$ since F_{j_0} is Tutte.

We make some observations concerning vertices in $V_A(F_{j_0}, Q) \cap W$. First, if $x \in V_A(F_{j_0}, Q) \cap W$ is in $V_s(Q)$ or $x \in \{a, b\}$, then, since $W \subset V_2(H) \cup V_3(H)$, there are, besides the edge $x^{-E} = x^{+E}$ of Q', one or two other edges incident to x. If both are incident to vertices in F_{j_0}, then $|e_H(\{x\}, S(Q))| = 1$, and if one of them is incident to a vertex in F_{j_0}, we have $|e_H(F_{j_0}, \{x\})| = 1$. Finally, if $x \in V_A(F_{j_0}, Q) \cap W$ is in $\mathrm{Int}(S(Q))$, then $d_Q(x) = 2$, hence $|e_H(F_{j_0}, \{x\})| = 1$ (note that here and throughout the proof, we do not exclude the possibility that some of these edges under consideration are parallel edges of H).

We define a set of edges $R \subset E(H)$ as follows:

- for every $x \in V_A(F_{j_0}, Q) \cap V_s(Q)$ with $|e_H(F_{j_0}, \{x\})| = 1$, R contains this (only) edge in $e_H(F_{j_0}, \{x\})$,
- for every $x \in V_A(F_{j_0}, Q) \cap V_s(Q)$ with $|e_H(F_{j_0}, \{x\})| = 2$, R contains the (only) double edge of Q incident to x,
- for every $x \in V_A(F_{j_0}, Q) \cap W \cap S(\mathrm{Int}(Q))$, R contains the (only) edge in $e_H(F_{j_0}, \{x\})$,
- for every $x \in \{a, b\}$ with $|e_H(F_{j_0}, \{x\})| = 1$, R contains this (only) edge in $e_H(F_{j_0}, \{x\})$,
- for every $x \in \{a, b\}$ with $|e_H(F_{j_0}, \{x\})| = 2$, R contains the first or the last edge on Q,
- finally, R contains all edges in $e_H(F_{j_0}, V(\mathrm{Int}(Q)) \cap B)$.

Let $V_s^2(F_{j_0}, Q)$ denote the set of all vertices $x \in V_s(Q) \cup \{a, b\}$ with $|e_H(F_{j_0}, \{x\})| = 2$. Then R is an edge-cut separating $\bar{F} = \langle V(F_{j_0}) \cup V_s^2(F_{j_0}, Q)\rangle_H$ from (the rest of) Q. By the construction, $|R| = |e_H(F_{j_0}, \mathrm{Int}(Q) \cap B)| + |V_A(F_{j_0}, \mathrm{Int}(Q)) \cap W| + |V_A(F_{j_0}, \{a, b\})| = |e_H(F_{j_0}, \mathrm{Int}(Q) \cap B)| + |V_A(F_{j_0}, Q) \cap W| \leq \min\{3, |D(\mathrm{Int}(Q))| - 1\}$ since F_{j_0} is Tutte. Thus, we have $|R| \leq 3$. On the other hand, $|R| \geq 2$ since H is 2-edge-connected.

Let $u_1^Q u_1^F$ and $u_2^Q u_2^F$ (where $u_i^Q \in V(Q) \setminus V_s^2(F_{j_0}, Q)$ and $u_i^F \in V(\bar{F})$, $i = 1, 2$), be the edges in $R \cap E(Q')$, and if $|R| = 3$, then let $f_3 = u_3^Q u_3^F$ be the third edge of R. Let $\bar{F}_1$ be the multigraph obtained from $\bar{F}$ by adding a new black vertex z and the edges zu_1^F, zu_2^F, and, if $|R| = 3$, also zu_3^F. By Conjecture 8, $\bar{F}_1$ contains a Tutte domination-maximal closed W-quasitrail T, containing the edges zu_1^F and zu_2^F.

Suppose first that either $|R| = 2$, or $|R| = 3$ and either $Q' - V(F_{j_0})$ dominates $u_3^Q u_3^F$ and $u_3^F \notin \{a, b\}$, or $u_3^Q u_3^F \notin D(Q')$. Then, concatenating $T - z$ with $Q' - V(F_{j_0})$ using the edges $u_1^Q u_1^F$ and $u_2^Q u_2^F$, we obtain a closed W-quasitrail Q'' in H such that $D(Q'') \supsetneq D(Q)$.

Next we consider the case that $|R| = 3$ and $Q' \cap V(F_{j_0})$ dominates $u_3^Q u_3^F$. Let $f_1 = u_1^F u_2^F$ be a new added edge (if already $u_1^F u_2^F \in E(\bar{F}_1)$, then f_1 is a parallel edge), let f_2 be a new added loop at $(u_3^F)^+$, if u_3^F is a special vertex of degree 2 in Q', or at u_3^F otherwise. Let $\bar{F}_2 = \bar{F}_1 - \{z\} + f_1 + f_2$. Since $Q' \cap V(F_{j_0})$ dominates $u_3^Q u_3^F$, the multigraph $\bar{F}_2$ (or $\bar{F}_2 - u_3^F$, if u_3^F is special), is 2-edge-connected. Let $\bar{W}_2 = W \cap V(\bar{F}_2)$. By Conjecture 8, $\bar{F}_2$ (or $\bar{F}_2 - u_3^F$, if u_3^F is special), contains a Tutte domination-maximal closed $\bar{W}_2$-quasitrail T_{f_1,f_2}, containing f_1 and f_2. Concatenating $T_{f_1,f_2} - \{f_1, f_2\}$ with $Q' - V(F_{j_0})$, using the edges $u_1^Q u_1^F$ and $u_2^Q u_2^F$, we obtain an (a, b, W)-quasitrail Q'' in H such that $D(Q'') \supsetneq D(Q)$.

Finally, it remains to consider the case that $|R| = 3$, $Q' - V(F_{j_0})$ dominates $u_3^Q u_3^F$ and $u_3^F \in \{a, b\}$. By the definition of F_{j_0}, $u_3^Q u_3^F$ and Q', we have $|e_H(F_{j_0}, \{u_3^F\})| = 2$ and $u_3^F \in V(\bar{F}) \setminus V(F_{j_0})$. Let w_1, w_2 be the neighbors of u_3^F in F_{j_0}, and set $e_1 = u_1^F u_2^F$ and $Q'_F = Q' \cap V(F_{j_0})$. Then $Q'_F + e_1$ is a nontrivial closed W-quasitrail in $F_{j_0} + e_1$. Let F_Q be a maximal 2-edge-connected subgraph of $F_{j_0} + e_1$ containing the closed trail $S(Q'_F + e_1)$. If $w_i \notin V(F_Q)$, then we denote by v_i the vertex in F_Q which is a cutvertex of F_{j_0} separating w_i from F_Q; otherwise, we set $v_i = w_i$, $i = 1, 2$. By Conjecture 9, there is a closed W-quasitrail Q_F in F_Q such that Q_F contains e_1 and has the properties given by Conjecture 9 with respect to v_1 and v_2. Concatenating $Q_F - e_1$ with $Q' - V(F_{j_0})$, using the edges $u_1^Q u_1^F$ and $u_2^Q u_2^F$, we again obtain an (a, b, W)-quasitrail Q'' in H such that $D(Q'') \supsetneq D(Q)$.

In each of the cases, we have obtained an (a, b, W)-quasitrail Q'' in H satisfying $D(Q'') \supsetneq D(Q)$ such that the number of non-Tutte components og $H - Q''$ is smaller than the number of non-Tutte components of $H - Q'$, contradicting the choice of Q'. ■

3.6 Proof of Proposition 14

Proof. Suppose that Conjecture 8 is true, and let G be a 2-connected line graph of a 3-hypergraph $\mathcal{H}$ and $a, b \in V(G)$. We choose such a hypergraph $\mathcal{H}$ such that it has minimum number of hyperedges. Since a 3-hyperedge with one or two vertices of degree one can be replaced by an edge without changing the line graph, every 3-hyperedge of $\mathcal{H}$ has all three vertices of degree at least two. Secondly, we replace all pendant edges in $\mathcal{H}$ by loops (which can be done without changing the line graph). By these two choices, we have $\delta(\mathcal{H}) \geq 2$. If all edges of $\mathcal{H}$ are loops, then $\mathcal{H}$ is a star, G is a clique and the statement is trivially true. Thus, we can suppose that $\mathcal{H}$ contains open edges. Additionally, subject to the above choice of $\mathcal{H}$, we choose $\mathcal{H}$ such that $\mathrm{Gr}(\mathcal{H})$ has minimum number of black vertices.

Recall that $G = L(\mathcal{H})$ can be obtained from $\mathrm{Gr}(\mathcal{H})$ by joining the (white) neighbors of every black vertex into a clique, and then removing all black vertices and possibly created multiple edges. Thus, if $\mathrm{Gr}(\mathcal{H})$ contains a bridge e, then the white vertex of e is a cutvertex of G, contradicting the assumption that G is 2-connected. Hence $\mathrm{Gr}(\mathcal{H})$ is 2-edge-connected.

Let e_1 and e_2 be the (3-hyper)edges of $\mathcal{H}$ for which $L(e_1) = a$ and $L(e_2) = b$ (not excluding the possibility that e_1 or e_2 is a loop). For $i \in \{1, 2\}$, the (hyper)edge e_i corresponds to a white vertex w_i in $\mathrm{Gr}(\mathcal{H})$. We define a Tutte cycle C in G as follows. By Conjecture 8 (which is supposed to be true), $\mathrm{Gr}(\mathcal{H})$ contains a Tutte domination-maximal closed W-quasitrail Q such that $e_1, e_2 \in E(S(Q))$, where W is the set of white vertices of $\mathrm{Gr}(\mathcal{H})$.

Set $C = \bar{C}_Q$, where $\bar{C}_Q$ is the cycle given in Theorem 6(i). Then we have $a, b \in V(C)$, and all components of $G - C$ are Tutte components (note that to show that C is Tutte, we need to verify that $|V(C)| \geq 4$).

Claim 1. *The cycle C is maximal among all cycles such that all components of their complement are Tutte and the corresponding closed W-quasitrail in $\mathrm{Gr}(\mathcal{H})$ contains the edges e_1 and e_2.*

Proof. We recall that the edges of $\mathrm{Gr}(\mathcal{H})$ that are dominated by Q but not on Q correspond in G to vertices that are not on C' but are contained in a clique containing some edge of C'. The rest follows from the definition of Q. □

Recall that to show that C is a Tutte cycle, it remains to show that $|V(C)| \geq 4$. Thus, suppose that $|V(C)| = 3$, and let Q be the corresponding closed W-quasitrail in $\mathrm{Gr}(\mathcal{H})$ by Theorem 6. Then $|V_w(Q)| = 3$. Since $e_1, e_2 \in E(S(Q))$, we have $E(S(Q)) \neq \emptyset$.

If Q has one black vertex, then Q is a star with black center and possibly multiple edges to the three white vertices, and then Q has at least one multiple edge since $E(S(Q)) \neq \emptyset$. But then, choosing e_1 in a multiple edge of Q, and e_2 outside Q, we get a closed W-quasitrail Q' in $\mathrm{Gr}(\mathcal{H})$ contradicting the maximality of Q. Thus, Q has at least two black vertices.

Then it is straightforward to verify that there are three possible cases for the structure of Q, shown in Fig. 6, where

- the dashed lines indicate possible edges to vertices outside Q, and
- the double-circled vertices cannot have any neighbors outside Q, which for the rightmost vertex in case (b) follows by structural properties of $\mathrm{Gr}(\mathcal{H})$, and for all other double-circled vertices follows from the fact that a neighbor outside Q would create a clique in G allowing to extend C, contradicting its maximality.

(Note that case (a) also includes the possibility that the right white vertex is connected to the left black vertex by an edge that is in $\mathrm{Gr}(\mathcal{H})$ but not in Q.)

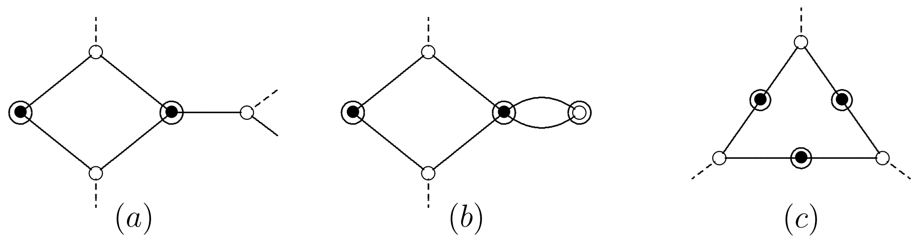

Figure 6: Three possible structures of Q if $V_w(Q) = 3$ and $V_b(Q) \geq 2$.

Then, in cases (a) and (b), we can remove the leftmost black vertex, and in case (c), we can replace the three black vertices by one black vertex adjacent to the three white vertices, without changing the line graph. In all three cases, we have reduced the number of black vertices of $\mathrm{Gr}(\mathcal{H})$, contradicting the choice of $\mathcal{H}$. This contradiction proves that $|V(C)| \geq 4$. Thus, C is a Tutte cycle containing a and b.

It remains to show that C can be chosen to be maximal. Let $\bar{C}$ be a cycle in G which is maximal among all Tutte cycles containing a and b such that the corresponding closed W-quasitrail in $\mathrm{Gr}(\mathcal{H})$ is Tutte. If $\bar{C}$ is maximal among all cycles containing a and b, then $\bar{C}$ is a requested Tutte maximal cycle containing a and b and we are done. Thus, suppose that there is a maximal cycle $\bar{C}'$ containing a and b such that $V(\bar{C}') \supsetneq V(\bar{C})$, and $\bar{C}'$ is not a Tutte cycle.

Let $\bar{Q}$ and $\bar{Q}'$ be the closed W-quasitrail in $\mathrm{Gr}(\mathcal{H})$ that corresponds to $\bar{C}$ and $\bar{C}'$, respectively, by Theorem 6. Then $D(\bar{Q}') \supsetneq D(\bar{Q})$ since $V(\bar{C}') \supsetneq V(\bar{C})$. By Proposition 19, there is a Tutte closed W-quasitrail $\bar{Q}''$ in $\mathrm{Gr}(\mathcal{H})$ such that $D(\bar{Q}'') \supsetneq D(\bar{Q})$. Consequently, the corresponding cycle $\bar{C}''$ in G is a Tutte cycle containing a and b such that $V(\bar{C}'') \supsetneq V(\bar{C})$, contradicting the choice of $\bar{C}$. ■

3.7 Proof of Proposition 15

Proof. Suppose that Conjecture 9 is true, and let G be a 3-connected line graph of a 3-hypergraph $\mathcal{H}$ and $a, b, c \in V(G)$. Similarly as in the proof of Proposition 14, we choose $\mathcal{H}$ such that it has minimum number of hyperedges, and replace all pendant edges in $\mathcal{H}$ by loops (which can be both done without changing the line graph), and get $\delta(\mathcal{H}) \geq 2$. If all edges of $\mathcal{H}$ are loops, then $\mathcal{H}$ is a star, G is a clique and the statement is trivially true; hence we suppose that $\mathcal{H}$ contains open edges.

Set $\bar{H} = \mathrm{Gr}(\mathcal{H})$, let v_1, v_2, v_3 be the vertices in $\bar{H}$ corresponding to a, b and c, respectively, and let e_1 be an arbitrary edge in $\bar{H}$ containing the vertex v_3. Let Q be a Tutte closed W-quasitrail in $\bar{H}$ with the properties given in Conjecture 9 (which is supposed to be true).

We show that $v_1, v_2 \in V(Q)$. Suppose, to the contrary, that e.g. $v_1 \notin V(Q)$. Then $v_1 \in V(F)$, where F is a component of $\bar{H} - Q$ such that $|e_{\bar{H}}(F, V(Q) \cap B)| + |V_A(F, Q) \cap W| \leq 2$. Observe that, by the choice of Q, $|e_{\bar{H}}(F, V(Q) \cap B)| = 0$: if $f \in e_{\bar{H}}(F, V(Q) \cap B)$, then f is an edge joining some black vertex b in Q to a (necessarily white) vertex w in F, but then, adding to Q the walk bwb, we increase the number of white vertices in Q. Thus, $|e_{\bar{H}}(F, V(Q) \cap B)| = 0$. But then the white vertices in $V_A(F, Q)$ determine a vertex cut of size 2 in G, contradicting the 3-connectedness assumption. Hence $v_1 \in V(Q)$. Analogously $v_2 \in V(Q)$, and obviously also $v_3 \in V(Q)$ since Q contains the edge e_1.

Rest of the proof follows the same arguments as those in the proof of Proposition 14, therefore we only sketch the remaining steps and leave details to the reader:

- the sequence of white vertices in Q determines a cycle C' in G, containing a, b and c, which can be extended to a cycle C by adding all vertices in cliques containing an edge of C',
- we observe that all components of $G - C$ are Tutte components, and that C is maximal among all cycles that contain a, b and c, and all components of their complement are Tutte components,
- we show that $|V(C)| \geq 4$, using the three possibilities shown in Fig. 6, implying that C is a Tutte cycle containing a, b and c,
- using Proposition 19 and Theorem 6, we show that C can be chosen to be maximal.

■

3.8 Proof of Proposition 16

Proof. Suppose that Conjecture 10 is true, let G be a connected line graph of a 3-hypergraph $\mathcal{H}$, and let $u, v \in V(G)$. As in the proofs of Propositions 14 and 15, we choose $\mathcal{H}$ with minimum number of hyperedges, replace all pendant edges by loops, and suppose that $\mathcal{H}$ contains open edges (for otherwise G is a clique and the statement is trivially true).

Case 1: G is 2-connected. Set $\bar{H} = \mathrm{Gr}(\mathcal{H})$, and let $a, b \in V(\bar{H})$ be the vertices corresponding to u and v, respectively. Note that $a, b \in V_w(\bar{H})$, and that $\bar{H}$ is 2-edge-connected (since otherwise the white vertex of a bridge in $\bar{H}$ corresponds to a cutvertex of G, contradicting the 2-connectedness assumption).

By Conjecture 10, let Q be an (a, b, W)-quasitrail in $\bar{H}$, having the properties given in Conjecture 10. As in the proof of Theorem 6, Q can be viewed as an alternating sequence of black and white vertices, which starts at $a \in V_w(\bar{H})$ and ends at $b \in V_w(\bar{H})$, and in which consecutive white vertices have a black common neighbor. The sequence of white vertices of Q then determines a (u, v)-path P_Q in G. Note that, by the construction, $V(P_Q)$ corresponds to $V_w(Q)$, hence $|V(P_Q)| = |V_w(Q)|$.

If $V(P') \supsetneq V(P_Q)$ for some (u, v)-path P' in G, then for the corresponding (a, b, W)-quasitrail Q' in $\bar{H}$ we have $V_w(Q') \supsetneq V_w(Q)$, contradicting (i) and (ii) of Conjecture 10. Thus, P_Q is maximal.

Let F be a component of $G - P_Q$, and let F_H be the corresponding nontrivial component of $\bar{H} - Q$. By (i) and (ii) of Conjecture 10, we have $|e_{\bar{H}}(F_H, V(Q) \cap B)| = 0$ and $V_A(F_H, Q) \subset V_w(Q)$. Hence $|V_A(F, P_Q)| = |e_{\bar{H}}(F_H, V(Q) \cap B)| + |V_A(F_H, Q) \cap W| \leq \min\{3, |D(\mathrm{Int}(S(Q)))| - 1\}$ since Q is Tutte. If $|V(P_Q)| \geq 4$, then also $|D(\mathrm{Int}(S(Q)))| \geq 4$, and then $\min\{3, |D(\mathrm{Int}(S(Q)))| - 1\} = 3 = \min\{3, |V(P_Q)| - 1\}$, implying that P_Q is a Tutte path. The small cases for $|V(P_Q)| \leq 3$ will be considered separately.

If $|V(P_Q)| = 2$, then also $|V_w(Q)| = 2$, and $V(Q) = \{a, w, b\}$, where w is a black common neighbor of a and b. By (i) and (ii) of Conjecture 10, $d_{\bar{H}}(w) = 2$. By the 2-connectedness assumption of Case 1, there is a nontrivial component F_H of $\bar{H} - Q$ with $V_A(F_H, Q) = \{a, b\}$. Thus, $|V_A(F_H, Q) \cap W| = 2$, while $|D(\mathrm{Int}(S(Q)))| - 1 = 1$, contradicting the fact that Q is Tutte.

It remains to consider the case $|V(P_Q)| = 3$. Set $V_w(Q) = \{a, c, b\}$. Then we have the following three possibilities (see also Fig 7, where the dashed lines indicate edges that are possible but not necessary):

(a) a, b, c have one black common neighbor w on Q and c is a special vertex of Q,

(b) a, b, c have one black common neighbor w on Q and c is not a special vertex of Q (implying that wc is a double edge of $\bar{H}$),

(c) $b^+ = c^- = w_1$ and $c^+ = b^- = w_2$ with $w_1 \neq w_2$ and $w_1, w_2 \in V_b(Q)$.

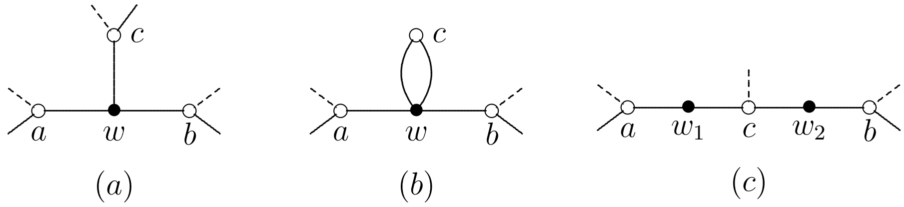

Figure 7: Three possible structures of Q if $V_w(Q) = 3$.

In case (a), by (i) and (ii) of Conjecture 10, we have $d_{\bar{H}}(w) = 3$, implying that $\min\{3, |D(\mathrm{Int}(S(Q)))| - 1\} = 2$. If, for some nontrivial component F_H of $\bar{H} - Q$, we have $V_A(F_H, Q) = \{a, b, c\}$, then $|(V_A(F_H, Q) \cap W)| = 3$ and F_H is not Tutte, a contradiction. Hence, by the 2-connectedness assumption, $|(V_A(F_H, Q) \cap W)| = 2$ for every nontrivial component F_H of $\bar{H} - Q$, implying $|V_A(F, P_Q)| = 2$ for every component F of $G - P_Q$. Since $\min\{3, |V(P_Q)| - 1\} = 2$, F is Tutte in G.

In case (b), first observe that $d_{\bar{H}}(w) = 4$ by (i) and (ii) of Conjecture 10, and that $d_{\bar{H}}(c) = 2$ by the choice of $\mathcal{H}$ (since if $d_{\bar{H}}(c) = 3$, then the double edge wc can be replaced by a single edge, i.e., the hyperedge of $\mathcal{H}$ corresponding to c can be replaced by an edge, without changing $G = L(\mathcal{H})$). Thus, by the 2-connectedness assumption, we have $V_A(F_H, Q) = \{a, b\}$ for every nontrivial component F_H of $\bar{H} - Q$. Consequently, $|V_A(F, P_Q)| = 2$, and, since also $\min\{3, |V(P_Q)| - 1\} = 2$, P_Q is Tutte.

In case (c) similarly observe that $d_{\bar{H}}(w_1) = d_{\bar{H}}(w_2) = 2$ by (i) and (ii) of Conjecture 10. Suppose that c has a neighbor outside Q. By the 2-connectedness assumption and by symmetry, we can suppose that there is a (c, b)-path P_c in $\bar{H} - w_2$. Let Q' be the (a, b)-path obtained by replacing in Q the subpath cw_2b by P_c. If $|V(P_c)| \geq 4$, then $D(Q') \supsetneq D(Q)$, contradicting (i) of Conjecture 10, hence $|V(P_c)| = 3$. Set $V(P_c) = \{c, w_c, b\}$. Obviously, $w_c \in V_b(\bar{H})$. If $d_{\bar{H}}(w_c) \geq 3$, then similarly $D(Q') \supsetneq D(Q)$; hence $d_{\bar{H}}(w_c) = 2$. By the 2-connectedness assumption, there is an (a, b)-path in $\bar{H} - w_1$, hence $d_{\bar{H}}(b) = 3$. Then b and c correspond in $\mathcal{H}$ to two 3-hyperedges sharing two vertices of degree 2, but then these hyperedges can be replaced by edges without changing the line graph, contradicting the choice of $\mathcal{H}$. This contradiction proves that $d_{\bar{H}}(c) = 2$. Then, by the 2-connectedness assumption, for any nontrivial component F_H of $\bar{H} - Q$, we have $V_A(F_H, Q) = \{a, b\}$, hence $|V_A(F, P_Q)| = 2$. Since also $\min\{3, |V(P_Q)| - 1\} = 2$, P_Q is a Tutte path.

Case 2: G has a cutvertex.

Let P' be a shortest (u, v)-path in G, and, for each nontrivial block B of G with $|V(B) \cap V(P')| \geq 2$, let P'_B be the Tutte maximal path between the two cutvertices of G which are in $V(B) \cap V(P')$, obtained by Case 1 (or, for the first/last block, between u/v and the first/last cutvertex of G which is on P', respectively). The requested Tutte maximal (u, v)-path P is obtained by replacing the subpath $P' \cap B$ with the Tutte maximal path P'_B, for every nontrivial block B with $|V(B) \cap V(P')| \geq 2$. ■

4 Concluding remarks

1. Motivated by Theorem G, we can also state the following conjecture on 4-regular graphs, which turns out to be also equivalent to all the previous ones.

Conjecture 20. *Every 4-connected 4-regular $K_{1,4}$-free graph is Hamilton-connected.*

Theorem 21. *Conjecture 20 is equivalent with Conjectures A, B, C, D, 1, 2, 3 and 4.*

Proof. Suppose that Conjecture 20 is true. Then, by Theorem G, every 4-connected 4-regular line graph of a 3-hypergraph, hence also every 4-connected 4-regular line graph (of a graph) is Hamilton-connected. Specifically, every line graph of a snark is Hamilton-connected, hence hamiltonian, and, by Theorem H, every snark has a dominating cycle, implying that Conjecture D is true.

Conversely, if Conjecture A is true, then, by Theorem 5, every 4-connected line graph of a 3-hypergraph is Hamilton-connected, and Theorem G implies Conjecture 20. ■

2. We would also like to recall the following question, raised in [7].

Question M [7]. *If $r \geq 4$ and G is r-connected and $K_{1,r}$-free, is G hamiltonian?*

Specifically, for $r = 4$, we obtain the question whether every 4-connected $K_{1,4}$-free graph is hamiltonian. Although this statement is much stronger than the other conjectures considered in this paper, it still remains wide open, and no progress on Question M is known so far.

References

[1] C. Berge: Hypergraphs. Combinatorics of finite sets. Translated from the French. North-Holland Mathematical Library, 45. North-Holland Publishing Co., Amsterdam (1989).

[2] J.A. Bondy, U.S.R. Murty: Graph Theory. Springer, 2008.

[3] H.J. Broersma, G. Fijavž, T. Kaiser, R. Kužel, Z. Ryjáček, P. Vrána: Contractible subgraphs, Thomassen's conjecture and the Dominating cycle conjecture for snarks. Discrete Math. 308 (2008), 6064-6077.

[4] H.J. Broersma, Z. Ryjáček, P. Vrána: How many conjectures can you stand? - a survey. Graphs Combin. 28 (2012), 57-75.

[5] H. Fleischner, B. Jackson: A note concerning some conjectures on cyclically 4-edge-connected 3-regular graphs. In: L.D. Andersen, I.T. Jakobsen, C. Thomassen, B. Toft, P.D. Vestergaard (eds.): Graph Theory in Memory of G.A. Dirac. Annals of Discrete Mathematics, vol. 41, pp. 171-177, North-Holland, Amsterdam (1989).

[6] F. Harary, C.St.J.A. Nash-Williams: On eulerian and hamiltonian graphs and line graphs. Canad. Math. Bull. 8 (1965) 701-710.

[7] B. Jackson, N.C. Wormald: k-walks in graphs. Australas. J. Comb. 2 (1990), 135-146.

[8] R. Javadi, Z. Maleki, B. Omoomi: Local Clique Covering of Claw-Free Graphs. J. Graph Theory 81 (2016), 92-104.

[9] T. Kaiser, Z. Ryjáček, P. Vrána: On 1-Hamilton-connected claw-free graphs. Discrete Math. 321 (2014), 1-11.

[10] R. Kužel, Z. Ryjáček, P. Vrána: Thomassen's conjecture implies polynomiality of 1-Hamilton-connectedness in line graphs. J. Graph Theory 69 (2012), 241-250.

[11] D. Li, H.-J. Lai, M. Zhan: Eulerian subgraphs and Hamilton-connected line graphs. Discrete Appl. Math. 145 (2005), 422-428.

[12] M.M. Matthews, D.P. Sumner: Hamiltonian results in $K_{1,3}$-free graphs. J. Graph Theory 8 (1984), 139-146.

[13] Z. Ryjáček: On a closure concept in claw-free graphs. J. Combin. Theory Ser. B 70 (1997), 217-224.

[14] Z. Ryjáček, P. Vrána: A closure for 1-Hamilton-connectedness in claw-free graphs. J. Graph Theory 75 (2014), 358-376.

[15] Z. Ryjáček, P. Vrána, S. Wang: Closure for $\{K_{1,4}, K_{1,4}+e\}$-free graphs. J. Combin. Theory Ser. B 134 (2019), 239-263.

[16] P.V. Skums, S.V. Suzdal, R.I. Tyshkevich: Edge intersection graphs of linear 3-uniform hypergraphs. Discrete Math. 309 (2009), 3500-3517.

[17] C. Thomassen: Reflections on graph theory. J. Graph Theory 10 (1986), 309-324.

[18] R. Čada, S. Chiba, K. Ozeki, P. Vrána, K. Yoshimoto: Equivalence of Jackson's and Thomassen's conjectures. J. Combin. Theory Ser. B 114 (2015), 124-147.