A note on singular edges and hamiltonicity in claw-free graphs with locally disconnected vertices

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Abstract

An edge e of a graph G is called *singular* if it is not on a triangle; otherwise, e is *nonsingular*. A vertex is called *singular* if it is adjacent to a singular edge; otherwise, it is called *nonsingular*.

We prove the following. Let G be a connected claw-free graph such that every locally disconnected vertex $x \in V(G)$ satisfies the following conditions:

(i) if x is nonsingular of degree 4, then x is on an induced cycle of length at least 4 with at most 4 nonsingular edges,

(ii) if x is not nonsingular of degree 4, then x is on an induced cycle of length at least 4 with at most 3 nonsingular edges,

(iii) if x is of degree 2, then x is singular and x is on an induced cycle C of length at least 4 with at most 2 nonsingular edges such that $G[V(C) \cap V_2(G)]$ is a path or a cycle.

Then G is either hamiltonian, or G is the line graph of the graph obtained from $K_{2,3}$ by attaching a pendant edge to its each vertex of degree two. Some results on forbidden subgraph conditions for hamiltonicity in 3-connected claw-free graphs are also obtained as immediate corollaries

Keywords: Claw-free graph; Hamiltonian graph; Singular edge; Locally disconnected vertex; Closure; Contractible subgraph

1 Introduction

All graphs considered here are finite and undirected. For terminology and notation not defined here we refer to [2].

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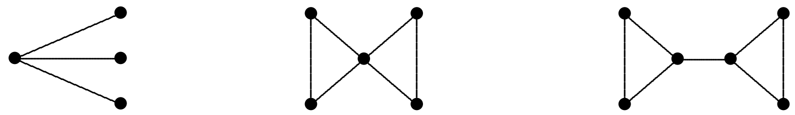
Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The *claw* is the graph $K_{1,3}$, the *0-hourglass* Γ_0 is the unique graph with degree sequence $4, 2, 2, 2, 2$ (i.e. two triangles with exactly one common vertex) and the *1-hourglass* Γ_1 is the unique simple non-2-edge-connected graph with degree sequence $3, 3, 2, 2, 2, 2$ (see Fig. 1).



The claw $K_{1,3}$ The 0-hourglass Γ_0 The 1-hourglass Γ_1

Figure 1: The graphs $K_{1,3}, \Gamma_0, \Gamma_1$

A graph is called *S-free* if it contains no induced subgraph isomorphic to S. Specifically, a graph is called *claw-free* for $S = K_{1,3}$ and *hourglass-free* for $S = \Gamma_0$, respectively.

For a vertex x of G, the set $N_G(x) = \{y \in V(G) : xy \in E(G)\}$ is called the *neighborhood* of x in G; the set $N_G[x] = N_G(x) \cup \{x\}$ is called the *closed neighborhood* of x in G. If F is a subgraph of a graph H, then a vertex x is said to be a *vertex of attachment of F in H* if $x \in V(F)$ and x has a neighbor in $V(H) \setminus V(F)$. The set of all vertices of attachment of a subgraph F in H is denoted by $A_H(F)$.

A vertex v of G is *locally connected* if $G[N_G(v)]$ is connected; otherwise, it is *locally disconnected*. We will use $V_{LC}(G)$ ($V_{LD}(G)$) to denote the set of all locally connected (locally disconnected) vertices of G, respectively. A graph G is called *locally connected* if every vertex of G is locally connected, i.e., $V_{LC}(G) = V(G)$.

Oberly and Sumner proved the following well-known result.

Theorem A [7]. *Every connected, locally connected claw-free graph on at least three vertices is hamiltonian.*

For presenting the following two extensions of Theorem A, we need some additional notations. We use $d(x, y)$ to denote the distance between vertices $x, y \in V(G)$. We say that a vertex v of a graph G is *N_2-locally connected* (*N^2-locally connected*), if the subgraph of G induced by the edge set $\{e = xy \in E(G) : v \notin \{x, y\}$ and $\{x, y\} \cap N(v) \neq \emptyset\}$ is connected (by the vertex set $\{x \in V(G) : 1 \leq d(x, v) \leq 2\}$ is connected), respectively. A graph G is said to be *N_2-locally connected* (N^2-locally connected) if every vertex of G is N_2-locally connected (N^2-locally connected), respectively. It is immediate to observe that every locally connected graph is N_2-locally connected, but the converse is not generally true, and also, every N_2-locally connected graph is N^2-locally connected, but the converse is not generally true. Throughout the paper, we denote $V_i(G) = \{x | d_G(x) = i\}$ and $V_{>i}(G) = \{x | d_G(x) > i\}$. An edge e is called a *pendant* edge if one of its vertices is of degree one; otherwise, it is *nonpendant*.

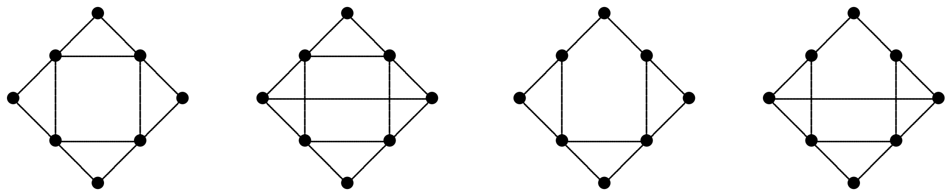
An edge of a graph G is called *singular* if it is not on a triangle of G; otherwise, it is called *nonsingular*. A vertex of a graph G is called *singular* if it is incident with at least one singular edge of G; otherwise, it is called *nonsingular*. The *line graph* of a graph H, denoted by $L(H)$, is the graph with $E(H)$ as its vertex set, in which two vertices are adjacent if and only if the corresponding edges have a vertex in common. Note that a nonpendant singular edge of a connected graph H of order at least three corresponds to a locally disconnected vertex of $L(H)$. If $G = L(H)$ is the line graph of a graph H, then we also say that H is the *line graph preimage* of G and denote $H = L^{-1}(G)$, and if $x \in V(G)$ is the vertex corresponding to an edge $e \in E(H)$, we will also denote $x = L(e)$ and $e = L^{-1}(x)$. It is well-known that for any connected line graph $G \neq K_3$, its line graph preimage is uniquely determined. For any induced subgraph C of a line graph G, we let $L^{-1}(C)$ denote the preimage of C.

Now we can state two results by Bielak [1] and by Tian and Xiong [11].

Theorem B [1]. *Let G be a connected, N_2-locally connected claw-free graph with $\delta(G) \geq 2$ such that*

(1) for every induced subgraph H of G isomorphic to either G_1 or G_2 in Fig. 2, at least one vertex in $V_3(H) \cup V_4(H)$ is locally connected in G.

Then G is hamiltonian.



G_1 G_2 G_3 G_4

Figure 2: The graphs G_1, G_2, G_3 and G_4

Theorem C [11]. *Let G be a connected, N^2-locally connected claw-free graph with $\delta(G) \geq 2$ satisfying*

(2) for every induced subgraph H of G isomorphic to one of $\{G_1, G_2, G_3, G_4\}$ in Fig. 2, at least one vertex in $V_3(H) \cup V_4(H)$ is locally connected in G.

Then G is hamiltonian.

The following result by Tian and Xiong [11] generalizes the results above and also the results in [6, 8].

Theorem D [11]. *Let G be a connected claw-free graph of order at least three satisfying the following conditions:*

(3) every locally disconnected vertex of degree at least 3 in G is on an induced cycle of length at least 4 with at most 3 nonsingular edges;

(4) every locally disconnected vertex of degree 2 in G is on an induced cycle C' with at most 2 nonsingular edges such that $G[V(C') \cap V_2(G)]$ is a path or a cycle.

Then G is hamiltonian.

In this paper, we further extend Theorem D. The following theorem is our main result.

Theorem 1. *Let G be a connected claw-free graph such that every vertex $x \in V_{LD}(G)$ satisfies the following conditions:*

(I) if x is nonsingular of degree 4 (i.e., $G[N_G[x]] \simeq \Gamma_0$), then x is on an induced cycle of length at least 4 with at most 4 nonsingular edges;

(II) if x is not nonsingular of degree 4 (i.e., $G[N_G[x]] \not\simeq \Gamma_0$), then x is on an induced cycle of length at least 4 with at most 3 nonsingular edges;

(III) if $d_G(x) = 2$, then x is singular (i.e., $G[N_G[x]] \simeq P_3$), and x is on an induced cycle C of length at least 4 with at most 2 nonsingular edges such that $G[V(C) \cap V_2(G)]$ is a path or a cycle.

Then G is either hamiltonian, or $G \simeq L(K'_{2,3})$, where $K'_{2,3}$ is the graph obtained from $K_{2,3}$ by attaching a pendant edge to every its vertex of degree two.

Proof of Theorem 1 is postponed to Section 3. Clearly, the exceptional graph $K'_{2,3}$ does not satisfy condition (3) of Theorem D. Furthermore, we have the following remark.

Remark 2. We have the following comments to our result.

(a) Note that condition (III) of Theorem 1 is equivalent to condition (4) of Theorem D.

Furthermore, Theorem 1 extends Theorem D. Clearly, any graph satisfying the assumptions of Theorem D also satisfies the assumptions of Theorem 1. Conversely, there are many graphs satisfying the assumptions of Theorem 1 but not those of Theorem D. For such an example, see Fig. 3 (here, since the unique induced cycle of length at least four containing x has exactly four nonsingular edges, the graph in Fig. 3 does not satisfy (3) in Theorem D).

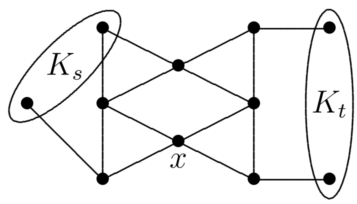


Figure 3: A graph satisfying the assumptions of Theorem 1 but not of Theorem D.

(b) Note that if $G = L(H)$, then any induced cycle in G corresponds to a cycle of H, and any cycle of H corresponds to an induced cycle of G. Also, a singular (nonsingular) edge of $G = L(H)$ corresponds to a path of length two in H whose inner vertex has degree exactly two (more than two), respectively. More generally, if C is an induced cycle of length at least 4 in a line graph $L(H)$, then C has exactly ℓ nonsingular edges if and only if its preimage $L^{-1}(C)$ has exactly ℓ vertices of degree greater than 2.

(c) Based on the above observations, it is easy to see that if G is a line graph, then the conditions (I), (II) and (III) of Theorem 1 in $G = L(H)$ can be translated to properties of the preimage H as follows:

Every singular edge $e \in E(H)$ satisfies the following:

(I') if both vertices of e are of degree exactly 3, then e is on a cycle C with $|V(C)| \geq 4$ and $|V(C) \cap V_{>2}(H)| \leq 4$,

(II') if e is nonpendant and at least one vertex of e is of degree different from 3, then e is on a cycle C with $|V(C)| \geq 4$ and $|V(C) \cap V_{>2}(H)| \leq 3$,

(III') if both vertices of e are of degree exactly 2, then e is on a cycle C with $|V(C)| \geq 4$ and $|V(C) \cap V_{>2}(H)| \leq 2$ such that all edges of C with both vertices of degree 2 in H determine a path or a cycle.

Theorem 1 has the following consequences, which are interesting on their own right. To state them, we need some additional notations.

A graph is said to have the *Hourglass k-property* if for every induced hourglass Γ_0, there exists a path $P(\Gamma_0)$ joining two vertices of Γ_0 whose distance is two in Γ_0 such that $P(\Gamma_0)$ has no inner vertex in Γ_0 and it has at most k nonsingular edges. Similarly, a graph is said to have the Γ_1 *k-property* if for every induced Γ_1, there exists a path $P(\Gamma_1)$ joining two vertices of Γ_1 whose distance is three in Γ_1 such that $P(\Gamma_1)$ has no inner vertex in Γ_1 and it has at most k nonsingular edges.

Corollary 3. *Let G be a connected claw-free graph satisfying the assumptions (II) and (III) of Theorem 1 such that G has the Hourglass 2-property. Then either G is hamiltonian, or G is the line graph of the graph $K'_{2,3}$.*

Proof. By the definition of the Hourglass 2-property, every locally disconnected nonsingular vertex v_0 of degree 4 is the center of an induced hourglass Γ_0 of G and hence there exists an induced path P_0 joining two vertices of the Γ_0 whose distance is two in Γ_0 such that P_0 has no inner vertex in Γ_0 and it has at most 2 nonsingular edges. Therefore v_0 lies on an induced cycle $C = G[E(P_0) \cup E(P_1)]$ (where P_1 is the path of length two of Γ_0 joining the endvertices of P_0) such that C has length at least 4 with at most 4 nonsingular edges (2 of which are from Γ_0). Therefore, G satisfies condition (I) of Theorem 1. Corollary 3 follows from Theorem 1. ■

Corollary 4. *Every connected claw-free graph with $\delta(G) \geq 3$ satisfying conditions (I) and (II) of Theorem 1 is hamiltonian.*

Proof. The assumption $\delta(G) \geq 3$ implies that G satisfies (III). ■

Note that there are graphs that satisfy the assumptions of Corollary 4 but are not 3-edge-connected. For such an example, see Fig. 3.

Corollary 5. *Every 3-edge-connected claw-free graph with Γ_i $(2-i)$-properties for $i \in \{0, 1\}$ is hamiltonian.*

Proof. Let G be a 3-edge-connected claw-free graph with Γ_i $(2-i)$-properties for $i \in \{0, 1\}$. Then, by the definition of the Γ_0 2-property and by the same arguments as in the proof of Corollary 3, G satisfies condition (I) of Theorem 1. Similarly, the condition that G has Γ_1 1-property implies G satisfies the condition (II) of Theorem 1. Now Corollary 5 follows from Corollary 4 since G is 3-edge-connected and hence $\delta(G) \geq 3$. ■

Corollary 6. *Every 3-edge-connected claw-free Γ_i-free graph for $i \in \{0, 1\}$ is hamiltonian.*

Remark 7. Specifically, Corollary 6 shows that every 3-connected $\{\text{claw}, \Gamma_0, \Gamma_1\}$-free graph is hamiltonian. This observation supports the results in [4], identifying all potential pairs of forbidden subgraphs for hamiltonicity in 3-connected claw-free graphs.

Note that although the assumption on a graph to be 3-connected and $\{\text{claw}, \Gamma_0, \Gamma_1\}$-free seems to be too restrictive, there are still many such graphs: for example, every 3-connected graph G such that its complement $\overline{G}$ has girth $g(\overline{G}) \geq 5$, is $\{\text{claw}, \Gamma_0, \Gamma_1\}$-free since the complement of the claw contains a triangle and the complements of Γ_0 and Γ_1 contain a cycle of length 4.

Also note that none of the above consequences of Theorem 1 can be deduced directly from Theorem D. The following result by Tian, Xiong and Niu is another similar example for 3-connected graphs.

Theorem E [12]. *Let G be a 3-connected claw-free graph. If every locally disconnected vertex is on some induced cycle C of length at least 4 with at most 4 nonsingular edges, then G is hamiltonian.*

2 Preliminaries

Let T be a closed trail (i.e., an Eulerian subgraph) in G. We say that T is a *dominating closed trail* (abbreviated DCT), if $V(G) \setminus V(T)$ is an independent set in G (or, equivalently, if every edge of G has at least one vertex on T).

Harary and Nash-Williams [5] proved the following result, relating the existence of a DCT in a graph to the hamiltonicity of its line graph.

Theorem F [5]. *Let G be a graph with at least 3 edges. Then the line graph $L(G)$ is hamiltonian if and only if G has a DCT.*

Let x be a vertex of a claw-free graph G. If the subgraph induced by $N_G(x)$ is connected and noncomplete, x is called *eligible*; if the subgraph induced by $N_G(x)$ is complete, then x is called *simplicial*. The set of all eligible vertices in G will be denoted $V_{EL}(G)$. If the subgraph induced by $N_G(x)$ is connected, we add edges joining all pairs of nonadjacent vertices in $N_G(x)$ and obtain the graph G^*_x. This operation is called the *local completion of G at x*. The *closure* $\text{cl}(G)$ of a graph G is the graph obtained from G by recursively repeating the local completion operation, as long as this is possible. Clearly, if $V_{EL}(G) = \emptyset$, then $G = \text{cl}(G)$, and in this case we say that G is *closed*.

Theorem G [9]. *If G is a claw-free graph, then its closure $\text{cl}(G)$ satisfies the following:*

(i) *the closure $\text{cl}(G)$ is well-defined;*
(ii) *there is a triangle-free graph H such that $\text{cl}(G) = L(H)$;*
(iii) *G is hamiltonian if and only if $\text{cl}(G)$ is hamiltonian.*

We will also use the following strengthening of the closure operation introduced by Ryjáček and Schelp in [10].

If H is a graph and $F \subset H$ is a subgraph of H, then $H|_F$ denotes the graph obtained from H by identifying the vertices of F as a (new) vertex v_F, and by replacing the created loops by pendant edges. Note that $H|_F$ may contain multiple edges. If H is a graph, $X \subset V(H)$, and $\mathcal{A}$ is a partition of X into subsets, then $E(\mathcal{A})$ denotes the set of all edges a_1a_2 (not necessarily in H) such that a_1, a_2 are in the same element of $\mathcal{A}$. Further $H^{\mathcal{A}}$ denotes the graph with vertex set $V(H^{\mathcal{A}}) = V(H)$ and edge set $E(H^{\mathcal{A}}) = E(H) \cup E(\mathcal{A})$. Note that $E(H)$ and $E(\mathcal{A})$ are considered to be disjoint, i.e., if $e_1 = a_1a_2 \in E(H)$ and $e_2 = a_1a_2 \in E(\mathcal{A})$, then e_1, e_2 are parallel edges in $H^{\mathcal{A}}$.

Let F be a graph and let $A \subset V(F)$. We say that F is *A-contractible*, if for every even subset $X \subset A$ and for every partition $\mathcal{A}$ of X into two-element subsets, the graph $F^{\mathcal{A}}$ has a DCT containing all vertices of A and all edges of $E(\mathcal{A})$. Note that this definition allows X to be empty, in which case $F^{\mathcal{A}} = F$. Also, if F is A-contractible, then F is A'-contractible for any $A' \subset A$ (since every subset X of A' is a subset of A).

Specifically, if $F \subset H$ is a subgraph of H, we say that F is a *contractible subgraph* of H if F is $A_H(F)$-contractible. The following important property of the contractibility concept follows from the results in [10].

Theorem H [10]. *Let H be a graph and let $F \subset H$ be a contractible subgraph of H. Then H has a DCT if and only if $H|_F$ has a DCT.*

For any two sets $A, B \subset V(F)$, $A \cap B \neq \emptyset$, $A|_B$ denotes the set $(A \setminus B) \cup \{v_B\} \subset V(F|_B)$, and for $A \cap B = \emptyset$, we set $A|_B = A$. The next theorem from [10] shows that a contractible graph remains contractible after a partial contraction.

Theorem I [10]. *Let F be a graph and let $A, B \subset V(F)$. If F is A-contractible, then $F|_B$ is $A|_B$-contractible.*

Equivalently, Theorem I says that the family $\mathcal{F} = \{(F, A)|\ F \text{ is } A\text{-contractible}\}$ is closed under partial contraction.

Note that if F is a collapsible graph in the sense of Catlin [3], then F is $V(F)$-contractible. Similarly, the A-contractibility concept also generalizes the concept of X-collapsibility introduced by Veldman [13]. For more details, we refer to [10].

Several examples of A-contractible graphs are shown in Fig. 4 (where the vertices in the set A are double-circled).

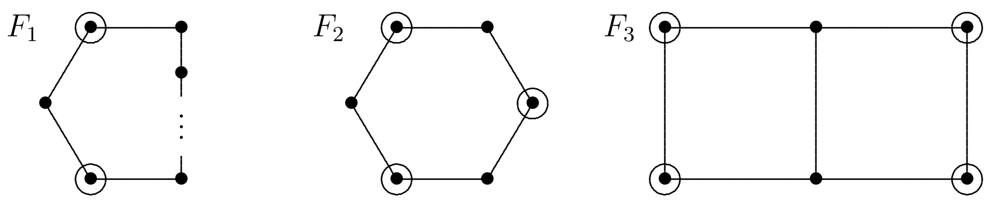


Figure 4: Examples of A-contractible graphs

It is easy to observe that if $F \subset H$, then $L(H|_F)$ is the graph obtained from $G = L(H)$ by replacing the (induced) subgraph $M = L(F)$ and its neighborhood with a complete graph, i.e., by the operation of *local completion* at the subgraph M, denoted G^*_M. An induced subgraph M of a graph $G = L(H)$ is said to be *eligible* if $F = L^{-1}(M)$ is a contractible subgraph in $H = L^{-1}(G)$ (i.e., if F is $A_H(F)$-contractible). This idea was used in [10] to introduce a closure concept, called the *contraction closure*, in the class of line graphs, as follows: the *$\mathcal{C}$-closure* of a line graph G is the graph $\mathrm{cl}^{\mathcal{C}}(G)$ for which there is a sequence of graphs $G_1, \ldots, G_t$ such that

(*i*) $G_1 = G$, $G_t = \mathrm{cl}^{\mathcal{C}}(G)$,
(*ii*) $G_{i+1} = (G_i)^*_M$ for some eligible induced subgraph $M \subset G_i$,
(*iii*) $G_t = \mathrm{cl}^{\mathcal{C}}(G)$ contains no eligible induced subgraph.

Theorem J [10]. *Let G be a line graph. Then*
(*i*) $\mathrm{cl}^{\mathcal{C}}(G)$ *is uniquely determined,*
(*ii*) G *is hamiltonian if and only if* $\mathrm{cl}^{\mathcal{C}}(G)$ *is hamiltonian.*

Thus, $\mathrm{cl}^{\mathcal{C}}(G)$ is a line graph which contains no eligible subgraphs, or, equivalently, if $\overline{H} = L^{-1}(\mathrm{cl}^{\mathcal{C}}(G))$, then $\overline{H}$ is obtained from $H = L^{-1}(G)$ by recursively contracting contractible subgraphs, as long as this is possible. Consequently, $\overline{H}$ contains no contractible subgraphs.

It should be noted here that in [10], the $\mathcal{C}$-closure operation is related to a certain family $\mathcal{C}$ of graphs, called a complete family, however, we do not give details here since for our proof it is sufficient to consider $\mathcal{C}$ to be the family of all line graphs of A-contractible graphs (in fact, we will only need to consider the family of the graphs in Fig. 4 and their partial contractions).

Finally, the concept of $\mathcal{C}$-closure can be extended to a closure in the class of all claw-free graphs by setting $\overline{\mathrm{cl}}(G) = \mathrm{cl}^{\mathcal{C}}(\mathrm{cl}(G))$. Since both $\mathrm{cl}(G)$ and $\mathrm{cl}^{\mathcal{C}}(G)$ are unique and preserve hamiltonicity/nonhamiltonicity, the same holds for $\overline{\mathrm{cl}}(G)$.

We will also need the following result by Tian and Xiong.

Lemma K [11]. *Let G be a claw-free graph, s and ℓ nonnegative integers, and let C be an induced cycle in G with at most s nonsingular edges and with at least $s - \ell$ locally connected vertices. If $x \in V(C)$ is locally disconnected in* $\mathrm{cl}(G)$*, then there is an induced cycle C' of length at least 4 in* $\mathrm{cl}(G)$ *with $x \in V(C') \subseteq V(C)$ and with at most ℓ nonsingular edges.*

3 Proof of the main result

Proof of Theorem 1. Let G be a claw-free graph satisfying the assumptions of Theorem 1 and suppose, to the contrary, that G is not hamiltonian. Let $G' = \mathrm{cl}(G)$ and $H' = L^{-1}(G')$.

Claim 1. *The graph G' satisfies the assumptions of Theorem 1.*

Proof. Let $x \in V_{LD}(G')$. Clearly, $V_{LD}(G') \subset V_{LD}(G)$.

(*i*) If x satisfies the assumptions of (I) in G', then clearly x satisfies (I) in G, and the proof follows by Lemma K with $s = \ell = 4$.

(*ii*) If x satisfies the assumptions of (II) in G', then x satisfies in G the conditions (I) or (II). If x satisfies (II) in G, then the proof follows by Lemma K with $s = \ell = 3$; thus, suppose that x satisfies (I) in G. Then there is an edge in $E(G') \setminus E(G)$ containing x, implying that some neighbor of x on C is simplicial, hence locally connected, and the proof follows by Lemma K with $s = 4$ and $\ell = 3$.

(*iii*) If x satisfies the assumptions of (III) in G', then clearly x satisfies (III) also in G, and the proof follows by Lemma K with $s = \ell = 2$. □

Claim 2. *If $x \in V_{LD}(G')$ satisfies the assumptions of condition (III) in G', then the edge $e = L^{-1}(x) \in E(H')$ is in a contractible subgraph of H'.*

Proof. By condition (III) (see also condition (III') in Remark 2(c)), e is in a cycle C with $|V(C)| \geq 4$ and $|V(C) \cap V_{>2}(H')| \leq 2$ such that all edges of C with both vertices of degree 2 in H' determine a path or a cycle (see the graph F_1 in Fig. 4). It is straightforward to verify that $A_{H'}(C) = V(C) \cap V_{>2}(H')$ and C is a contractible subgraph of H'. □

Claim 3. *If $x \in V_{LD}(G')$ satisfies the assumptions of condition (II) in G', then the edge $e = L^{-1}(x) \in E(H')$ is in a contractible subgraph of H'.*

Proof. By condition (II) (see also condition (II') in Remark 2(c)), e is in a cycle C with $|V(C)| \geq 4$ and $|V(C) \cap V_{>2}(H')| \leq 3$, and at least one vertex of e is in $V_{>2}(H')$. We observe that, by condition (III), the cycle C can be chosen such that C contains no edge with both vertices in $V_2(H')$ (otherwise we replace the subpath containing such an edge with the rest of the cycle given by condition (III)). But then it is straightforward to verify that C is a contractible subgraph of H' (note that again $A_{H'}(C) = V(C) \cap V_{>2}(H')$; see also the graph F_2 in Fig. 4). □

Now set $\overline{G} = \mathrm{cl}^{\mathcal{C}}(G')$ and $\overline{H} = L^{-1}(\overline{G})$.

Claim 4. *Every vertex $x \in V_{LD}(\overline{G})$ satisfies condition (I).*

Proof. We have $x \in V_{LD}(\overline{G})$, implying $x \in V_{LD}(G')$, hence, by Claim 1, x satisfies (I), (II) or (III) in G'. However, if x satisfies (II) or (III) in G', then, by Claims 2 and 3, x is simplicial in $\overline{G}$, a contradiction. Hence x satisfies (I) in G', implying that, in H', the edge $e = L^{-1}(x)$ is in a cycle C with $|V(C)| \geq 4$ and $|V(C) \cap V_{>2}(H')| \leq 4$.

If C is contracted in $\overline{H}$, then x is simplicial in $\overline{H}$, a contradiction. If C is partially contracted in $\overline{H}$, then some two vertices in $V(C) \cap V_{>2}(H')$ are contracted into one vertex in $\overline{H}$, and the edges of C determine in $\overline{H}$ a contractible subgraph, a contradiction again. Hence the edges of C determine the same cycle also in $\overline{H}$.

Let $e = u_1u_2 = L^{-1}(x)$ be the edge corresponding to the vertex x in H', and, by the above observations, also in $\overline{H}$, and let $C = u_1u_2u_3u_4$ (where $\{u_1, u_2, u_3, u_4\} = V(C) \cap V_{>2}(\overline{H})$ and the edges u_2u_3, u_3u_4, u_4u_1 can be possibly subdivided by a vertex of degree 2). Since x satisfies (I) in G', we have $d_{H'}(u_1) = d_{H'}(u_2) = 3$. If also $d_{\overline{H}}(u_1) = d_{\overline{H}}(u_2) = 3$, then x satisfies (I) also in $\overline{G}$ and we are done. Thus, it remains to consider the case that, up to a symmetry, $d_{\overline{H}}(u_2) > 3$. Then there is a contractible subgraph $F \subset H'$ such that $u_2 \in V(F)$ but $u_1u_2 \notin E(F)$. Then $u_1u_2 \in E(H') \setminus E(F)$, implying $u_2 \in A_{H'}(F)$. By the definition of A-contractibility, F has a DCT T for $X = \emptyset$, and since $u_1u_2 \notin E(F)$ and $d_{H'}(u_2) = 3$, necessarily $u_3 \in V(T) \subset V(F)$. But then, in $\overline{H}$, u_2 and u_3 are contracted into one vertex, turning C into a contractible subgraph of $\overline{H}$, a contradiction. Hence $d_{\overline{H}}(u_1) = d_{\overline{H}}(u_2) = 3$ and the vertex x satisfies (I) in $\overline{G}$. □

Now, by Claim 4, $\overline{H}$ is a connected graph in which every vertex is of degree 1 or 3, and every nonpendant edge is in a cycle of length 4 with all vertices of degree 3. If $|V_3(\overline{H})| = 1$, then $\overline{H} = K_{1,3}$, $\overline{G} = C_3$ and there is nothing to do; hence let $|V_3(\overline{H})| \geq 2$, and let $C = u_1u_2u_3u_4$ be a cycle of length 4 in $\overline{H}$. Let u_i' be the neighbor of u_i outside C, $i = 1, 2, 3, 4$.

We distinguish, up to a symmetry, three cases (recall that $\overline{H}$ is triangle-free, and hence $u_i' \neq u_{i+1}'$, where $u_5' = u_1'$).

Case 1: $u_1' = u_3'$, $u_2' = u_4'$. If $u_1'u_2' \in E(\overline{H})$, then $V(\overline{H}) = \{u_1, u_2, u_3, u_4, u_1', u_2'\}$, $\overline{H}$ has a DCT and we are done. Hence suppose that $u_1'u_2' \notin E(\overline{H})$. Let u_i'' be the neighbor of u_i' distinct from u_i and u_{i+2}, $i = 1, 2$. If $d_{\overline{H}}(u_1'') = 3$, then it is straightforward to see that the edge $u_1'u_1''$ cannot be in a C_4; hence the edge $u_1'u_1''$ is pendant. Analogously, the edge $u_2'u_2''$ is also pendant, implying that $\overline{H}$ has a DCT.

Case 2: $u_1' = u_3'$, $u_2' \neq u_4'$. Let again u_1'' be the third neighbor of u_1' distinct from u_1 and u_3. Suppose that, say, $u_4' = u_1''$, and let u_4'' be the third neighbor of u_4', distinct from u_1' and u_4. Then neither of the edges $u_4'u_4''$, u_2u_2' can be in a C_4, hence $d_{\overline{H}}(u_4'') = d_{\overline{H}}(u_2') = 1$ and $\overline{H}$ has a DCT.

Thus, it remains to consider the case when u_1'', u_2', u_4'' are three distinct vertices. Then neither of the edges $u_1'u_1''$, u_2u_2', u_4u_4' can be in a C_4, implying that $\overline{H} \simeq K_{2,3}'$.

Case 3: $u_1' \neq u_3'$, $u_2' \neq u_4'$. If $V(\overline{H}) = \{u_1, u_2, u_3, u_4, u_1', u_2', u_3', u_4'\}$, then $\overline{H}$ has a DCT and we are done; thus, up to a symmetry, suppose that u_1u_1' is in a C_4. Then, up to a symmetry, we have $u_1'u_2' \in E(\overline{H})$, and both u_1' and u_2' have a third neighbor outside $\{u_1, u_2, u_3, u_4, u_1', u_2'\}$. But then the graph $F = \overline{H}[\{u_1, u_2, u_3, u_4, u_1', u_2'\}]$ has $A_{\overline{H}}(F) = \{u_1', u_2', u_3, u_4\}$ (see the graph F_3 in Fig. 4), and F is a contractible subgraph in the graph $\overline{H}$, a contradiction. ■

4 Concluding Remarks

4.1 Sharpness

In this section, we show that all conditions in Theorem 1 are sharp.

1. Let H be the graph obtained from the graph $K_{2,3}$ by subdividing one edge with a vertex of degree 2 and attaching to each of its (four) vertices of degree 2 a pendant edge, and set $G = L(H)$ (see Fig. 5(a)). Then $V_{LD}(G) = V_4(G)$, G satisfies conditions (II) and (III) (since there are no such vertices), every its locally disconnected vertex is nonsingular of degree 4 and is on an induced cycle of length at least 4 with 5 nonsingular edges; however, G is nonhamiltonian. This example shows that, in the requirement "with at most 4 nonsingular edges" of condition (I), 4 cannot be relaxed to 5.

Moreover, an infinite family of such examples can be obtained by taking an arbitrary graph G_1 satisfying the assumptions of Theorem 1 and having two simplicial vertices y_1, y_2 of degree 2 with a common neighbor, and identifying x_i with y_1, $i = 1, 2$ (one such example is in Fig. 5(b)).

2. Let H be obtained from $K_{2,2t+1}$, $t \geq 1$, by attaching at least two pendant edges to every vertex of degree two, and let $G = L(H)$. Then G satisfies (I) and (III) (since there are no such vertices), and all locally disconnected vertices are not nonsingular of degree 4 and are on an induced cycle of length 4 with 4 nonsingular edges. However, G is nonhamiltonian. This example shows that, in the requirement "with at most 3 nonsingular edges" of condition (II), 3 cannot be relaxed to 4.

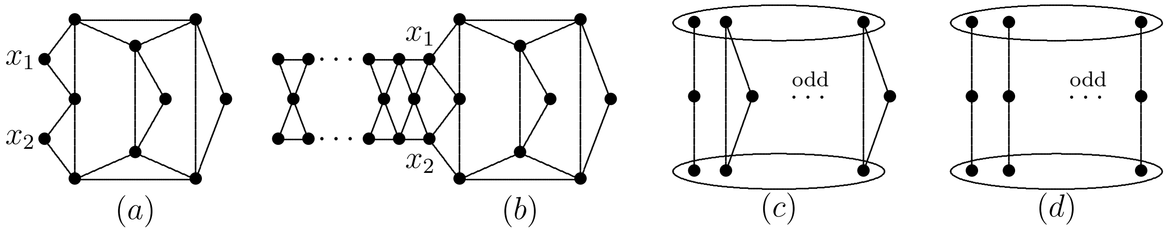


Figure 5: Sharpness examples

3. Let $H_1 = K_{2,2t+1}$, $t \geq 1$, choose a vertex $x \in V_2(H_1)$, let H be obtained from H_1 by attaching a pendant edge to each vertex of degree 2 distinct from x and by subdividing one of the edges incident to x, and set $G = L(H)$ (see Fig. 5(c)). Then G satisfies (I) (since there is no such vertex) and satisfies (II), but its vertex of degree 2 is in an induced cycle with 3 nonsingular edges. This example shows that, in the requirement "with at most 2 nonsingular edges" of condition (III), 2 cannot be relaxed to 3.

4. Let $H_1 = K_{2,2t+1}$, $t \geq 1$, let H be obtained from H_1 by subdividing each of the edges incident to one of its vertices of degree $2t + 1$ with one new vertex of degree 2, and set $G = L(H)$ (see Fig. 5(d)). Then G satisfies (I) and (II), each its vertex of degree 2 is on a cycle of length 6 with 2 nonsingular edges, but in each such cycle, the vertices of degree 2 induce neither a path nor a cycle. However, G is nonhamiltonian. This example shows that the requirement that $G[V(C) \cap V_2(G)]$ is a path or a cycle in condition (III) is necessary.

4.2 A generalization

It is easy to observe that, by Lemma K, Theorem 1 can be slightly extended as follows.

Theorem 8. *Let G be a connected claw-free graph of order at least three such that every vertex $x \in V_{LD}(G)$ satisfies the following conditions:*

(I") if x is nonsingular of degree 4 (i.e., $G[N_G[x]] \simeq \Gamma_0$), then x is on an induced cycle of length at least 4 with at most 4 nonsingular edges;

(II") if x is not nonsingular of degree 4 (i.e., $G[N_G[x]] \not\simeq \Gamma_0$), then there is an integer $s \geq 0$ such that x is on an induced cycle of length at least 4 with at most s nonsingular edges and with at least $s - 3$ locally connected vertices;

(III") if x is singular of degree 2 (i.e., $G[N_G[x]] \simeq P_3$), then there is an integer $s \geq 0$ such that x is on an induced cycle C with at most s nonsingular edges and with at least $s - 2$ locally connected vertices and such that $G[V(C) \cap V_2(G)]$ is a path or a cycle.

Then G is either hamiltonian, or $G \simeq L(K'_{2,3})$.

Proof. If G satisfies the assumptions of Theorem 8, then, by Lemma K, cl(G) satisfies the assumptions of Theorem 1. ■

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