Every 3-connected $\{K_{1,3}, Z_7\}$-free graph of order at least 21 is Hamilton-connected

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Abstract

For an integer $i \geq 1$, Z_i is the graph obtained by attaching an endvertex of a path of length i to a vertex of a triangle. We prove that every 3-connected $\{K_{1,3}, Z_7\}$-free graph is Hamilton-connected, with one exceptional graph. The result is sharp.

Keywords: Hamilton-connected; closure; forbidden subgraph; claw-free; Z_i-free

1 Introduction

In this paper, we generally follow the most common graph-theoretical notation and terminology, and for notations and concepts not defined here we refer to [5]. Specifically, by a *graph* we always mean a simple finite undirected graph; whenever we admit multiple edges, we always speak about a *multigraph.* We use $d_G(x)$ to denote the *degree* of a vertex x in G, and for $i \geq 1$ we set $V_i(G) = \{x \in V(G)|\ d_G(x) = i\}$. If $x \in V_2(G)$ with $N_G(x) = \{y_1, y_2\}$, then the operation of replacing the path y_1xy_2 with the edge y_1y_2 is called *suppressing* the vertex x. The inverse operation is called *subdividing* the edge y_1y_2 with the vertex x. We write $F \subset H$ if F is a sub(multi)graph of H, $G_1 \simeq G_2$ if the (multi)graphs G_1, G_2 are isomorphic, and $\langle M\rangle_G$ to denote the *induced sub(multi)graph* on a set $M \subset V(G)$. The *line graph* of a multigraph H is the graph $G = L(H)$ with $V(G) = E(H)$, in which two vertices are adjacent if and only if the corresponding edges of H have at least one vertex in common. We say that a vertex $x \in V(G)$ is *simplicial* if $\langle N_G(x)\rangle_G$ is a complete graph, and we use $V_{SI}(G)$ to denote the set of all simplicial vertices of G.

For $x, y \in V(G)$, a path (trail) with endvertices x, y is referred to as an (x, y)*-path* ((x, y)*-trail*), a trail with terminal edges $e, f \in E(G)$ is called an (e, f)*-trail*, and $\mathrm{Int}(T)$ denotes the set of interior vertices of a trail T. A set of vertices $M \subset V(G)$ *dominates* an edge e, if e has at least one vertex in M, and a subgraph $F \subset G$ dominates e if $V(F)$ dominates e. A closed trail T is a *dominating closed trail* (abbreviated DCT) if T dominates all edges of G, and an (e, f)-trail is an *internally dominating* (e, f)*-trail* (abbreviated (e, f)-IDT) if $\mathrm{Int}(T)$

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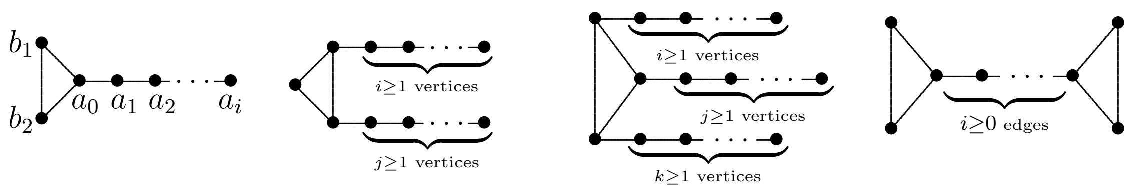
dominates all edges of G. A graph is *Hamilton-connected* if, for any $u, v \in V(G)$, G has a hamiltonian (u, v)-path, i.e., an (u, v)-path P with $V(P) = V(G)$.

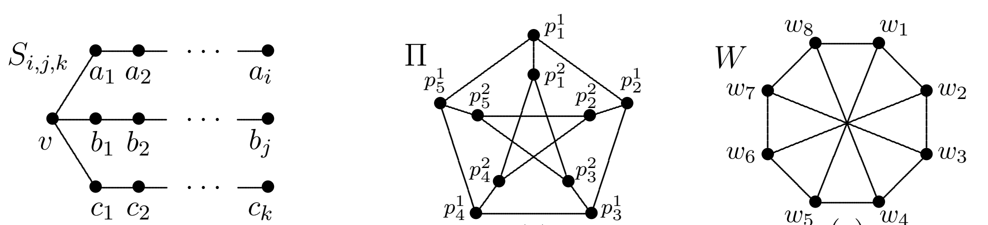
Finally, if $\mathcal{F}$ is a family of graphs, we say that G is $\mathcal{F}$*-free* if G does not contain an induced subgraph isomorphic to a member of $\mathcal{F}$, and the graphs in $\mathcal{F}$ are referred to in this context as *forbidden (induced) subgraphs*. If $\mathcal{F} = \{F\}$, we simply say that G is F*-free*. Here, the *claw* is the graph $K_{1,3}$, P_i denotes the path on i vertices, and Γ_i denotes the graph obtained by joining two triangles with a path of length i (see Fig. 1(d)). Several further graphs that will be used as forbidden subgraphs are shown in Fig. 1$(a), (b), (c)$. Whenever we will list vertices of an induced claw $K_{1,3}$, we will always list its center as the first vertex of the list, and when listing vertices of an induced subgraph $F \simeq Z_i$, we will always list first the vertices b_1, b_2, and then the vertices $a_0, a_1, \ldots, a_i$. Similarly, when listing vertices of an $S_{i,j,k}$ in a graph (see Fig. 2(a)), we will always write the list such that $i \le j \le k$, and we will use the notation $S_{i,j,k}(v; a_1a_2 \ldots a_i; b_1b_2 \ldots b_j; c_1c_2 \ldots c_k)$ (in the labeling of vertices as in Fig. 2(a)). The vertex v will be called the *center*, and the paths $va_1 \ldots a_i$, $vb_1 \ldots b_j$, $vc_1 \ldots c_k$ will be called the *branches* of the $S_{i,j,k}$.



(a) Z_i $(i \ge 1)$ (b) $B_{i,j}$ $(i, j \ge 1)$ (c) $N_{i,j,k}$ $(i, j, k \ge 1)$ (d) Γ_i $(i \ge 0)$

Figure 1: The graphs Z_i, $B_{i,j}$ and $N_{i,j,k}$

We also recall two well-known graphs that will occur as exceptions in some of the results, namely, the Petersen graph Π and the Wagner graph W (see Fig. 2$(b), (c)$). It is a well-known fact that the Wagner graph can be obtained from the Petersen graph by removing an arbitrary edge and suppressing the two created vertices of degree 2. We will often refer to these graphs using the labeling of their vertices as indicated in Fig. 2.



(a) (b) (c)

Figure 2: The graph $S_{i,j,k}$, the Petersen graph Π and the Wagner graph W

Theorem A lists the best known results on pairs of forbidden subgraphs implying Hamilton-connectedness of a 3-connected graph.

Theorem A [4, 7, 14, 15, 16, 21]. *Let G be a 3-connected $\{K_{1,3}, X\}$-free graph, where*

(i) **[7]** *$X = \Gamma_1$, or*

(ii) **[4]** *$X = P_9$, or*

(iii) **[21]** *$X = Z_6$, or*

(*iv*) **[21]** $X = B_{i,j}$ *for* $i + j \leq 7$, *or*
(*v*) **[14, 15, 16]** $X = N_{i,j,k}$ *for* $i + j + k \leq 7$.
Then G *is Hamilton-connected.*

Note that statement (*iii*) is an immediate corollary of (*iv*).

Let $\mathcal{W}$ be the family of graphs obtained by attaching at least one pendant edge to each of the vertices of the Wagner graph W (see Fig. 2(*c*)), and let $\mathcal{G} = \{L(H)| \ H \in \mathcal{W}\}$ be the family of their line graphs. Then any $G \in \mathcal{G}$ is 3-connected, non-Hamilton-connected, P_{10}-free, $B_{i,j}$-free for $i + j = 8$, and $N_{i,j,k}$-free for $i + j + k = 8$. Thus, this example shows that parts (*ii*), (*iv*) and (*v*) of Theorem A are sharp.

Let W^1 be the graph obtained from W by attaching exactly one pendant edge to each of its vertices. The following theorem is our main result.

Theorem 1. *Let* G *be a 3-connected* $\{K_{1,3}, Z_7\}$*-free graph such that* $G \not\simeq L(W^1)$*. Then* G *is Hamilton-connected.*

Proof of Theorem 1, consisting in direct case-distinguishing, is postponed to Section 4.

The exceptional graph $L(W^1)$ is 3-connected $\{K_{1,3}, Z_7\}$-free and not Hamilton-connected, showing that the assumption $G \not\simeq L(W^1)$ in Theorem 1 cannot be omitted. Also, for each graph $H \in \mathcal{W} \setminus \{W^1\}$, $L(H)$ is 3-connected $\{K_{1,3}, Z_8\}$-free and not Hamilton-connected, showing that Theorem 1 is sharp.

Since $|V(L(W^1))| = 20$, Theorem 1 has the following immediate corollary.

Corollary 2. *Let* G *be a 3-connected* $\{K_{1,3}, Z_7\}$*-free graph of order* $n \geq 21$*. Then* G *is Hamilton-connected.*

In Section 2, we collect necessary known results and facts on line graphs and on closure operations, and, in Subsection 2.5, we develop a method that allows to overcome the difficulty that the class of $\{K_{1,3}, Z_i\}$-free graphs is not stable under closure operations. In Section 3, we develop a technique that allows to significantly reduce the number of cases to be considered. Finally, in Section 5, we briefly update the discussion of remaining open cases in the characterization of forbidden pairs for Hamilton-connectedness from [15] and [21].

2 Preliminaries

In Subsections 2.1 – 2.4, we summarize some known facts that will be needed in our proof of Theorem 1, and in Subsection 2.5, we introduce a superclass of the class of $\{K_{1,3}, Z_i\}$-free graphs that is stable under the closure operations.

2.1 Line graphs of multigraphs and their preimages

While in line graphs of graphs, for a connected line graph G, the graph H such that $G = L(H)$ is uniquely determined with a single exception of $G = K_3$, in line graphs of multigraphs this is not true: a simple example are the graphs $H_1 = Z_1$ and H_2 a double edge with one pendant edge attached to each vertex — while $H_1 \not\simeq H_2$, we have $L(H_1) \simeq L(H_2)$. Using a modification of an approach from [23], the following was proved in [19].

Theorem B [19]. *Let G be a connected line graph of a multigraph. Then there is, up to an isomorphism, a uniquely determined multigraph H such that a vertex $e \in V(G)$ is simplicial in G if and only if the corresponding edge $e \in E(H)$ is a pendant edge in H.*

The multigraph H with the properties given in Theorem B will be called the *preimage* of a line graph G and denoted $H = L^{-1}(G)$. We will also use the notation $a = L(e)$ and $e = L^{-1}(a)$ for an edge $e \in E(H)$ and the corresponding vertex $a \in V(G)$.

An edge-cut $R \subset E(H)$ of a multigraph H is *essential* if $H - R$ has at least two nontrivial components, and H is *essentially k-edge-connected* if every essential edge-cut of H is of size at least k. It is a well-known fact that a line graph G is k-connected if and only if $L^{-1}(G)$ is essentially k-edge-connected. It is also a well-known fact that if X is a line graph, then a line graph G is X-free if and only if $L^{-1}(G)$ does not contain as a subgraph (not necessarily induced) a graph F such that $L(F) = X$. We give more details on this correspondence in Subsection 2.5 (Proposition 7).

Harary and Nash–Williams [10] established a correspondence between a DCT in H and a hamiltonian cycle in $L(H)$. A similar result showing that $G = L(H)$ is Hamilton-connected if and only if H has an (e_1, e_2)-IDT for any pair of edges $e_1, e_2 \in E(H)$, was given in [13] (in fact, part (ii) of the following theorem is slightly stronger than the result from [13], and its easy proof is given in [14]).

Theorem C [10, 13]. *Let H be a multigraph with $|E(H)| \geq 3$ and let $G = L(H)$.*

- (i) **[10]** *The graph G is hamiltonian if and only if H has a DCT.*
- (ii) **[13]** *For every $e_i \in E(H)$ and $a_i = L(e_i)$, $i = 1, 2$, G has a hamiltonian (a_1, a_2)-path if and only if H has an (e_1, e_2)-IDT.*

2.2 Strongly spanning trailable multigraphs

A multigraph H is *strongly spanning trailable* if for any $e_1 = u_1v_1, e_2 = u_2v_2 \in E(H)$ (possibly $e_1 = e_2$), the multigraph $H(e_1, e_2)$, which is obtained from H by replacing the edge e_1 by a path $u_1v_{e_1}v_1$ and the edge e_2 by a path $u_2v_{e_2}v_2$, has a spanning (v_{e_1}, v_{e_2})-trail.

We will need the following two results on "small" strongly spanning trailable multigraphs from [16]. Here, $\mathbb{W}$ is the set of multigraphs that are obtained from the Wagner graph W by subdividing one of its edges and adding at least one edge between the new vertex and exactly one of its neighbors.

Theorem D [16].

- (i) *Every 2-connected 3-edge-connected multigraph H with circumference $c(H) \leq 8$ other than the Wagner graph W is strongly spanning trailable.*
- (ii) *Every 3-edge-connected multigraph H with $|V(H)| \leq 9$ such that $H \notin \{W\} \cup \mathbb{W}$ is strongly spanning trailable.*

2.3 SM-closure

For $x \in V(G)$, the *local completion of* G *at* x is the graph $G_x^* = (V(G), E(G) \cup \{y_1y_2|\ y_1, y_2 \in N_G(x)\})$ (i.e., G_x^* is obtained from G by adding all the missing edges with both vertices in $N_G(x)$). Obviously, if G is claw-free, then so is G_x^*. Note that in the special case when G is a line graph and $H = L^{-1}(G)$, G_x^* is the line graph of the multigraph obtained from H by contracting the edge $L^{-1}(x)$ into a vertex and replacing the created loop(s) by pendant edge(s). Also note that clearly $x \in V_{SI}(G_x^*)$ for any $x \in V(G)$, and, more generally, $V_{SI}(G) \subset V_{SI}(G_x^*)$ for any $x \in V(G)$.

We say that a vertex $x \in V(G)$ is *eligible* if $\langle N_G(x)\rangle_G$ is a connected noncomplete graph, and we use $V_{EL}(G)$ to denote the set of all eligible vertices of G. Note that in the special case when G is a line graph and $H = L^{-1}(G)$, it is not difficult to observe that $x \in V(G)$ is eligible if and only if the edge $L^{-1}(x)$ is in a triangle or in a multiple edge of H. Based on the fact that if G is claw-free and $x \in V_{EL}(G)$, then G_x^* is hamiltonian if and only if G is hamiltonian, the *closure* $\mathrm{cl}(G)$ of a claw-free graph G was defined in [18] as the graph obtained from G by recursively performing the local completion operation at eligible vertices, as long as this is possible (more precisely: $\mathrm{cl}(G) = G_k$, where $G_1, \dots, G_k$ is a sequence of graphs such that $G_1 = G$, $G_{i+1} = (G_i)_{x_i}^*$ for some $x_i \in V_{EL}(G)$, $i = 1, \dots, k-1$, and $V_{EL}(G_k) = \emptyset$). The closure $\mathrm{cl}(G)$ of a claw-free graph G is uniquely determined, is a line graph of a triangle-free graph, and is hamiltonian if and only if so is G. However, as observed in [6], the closure operation does not preserve the (non-)Hamilton-connectedness of G.

To overcome this problem, the concept of an SM-closure G^M of a claw-free graph G was defined in [12] by the following construction.

(i) If G is Hamilton-connected, we set $G^M = \mathrm{cl}(G)$.

(ii) If G is not Hamilton-connected, we recursively perform the local completion operation at such eligible vertices for which the resulting graph is still not Hamilton-connected, as long as this is possible. We obtain a sequence of graphs $G_1, \dots, G_k$ such that
- $G_1 = G$,
- $G_{i+1} = (G_i)_{x_i}^*$ for some $x_i \in V_{EL}(G_i)$, $i = 1, \dots, k-1$,
- G_k has no hamiltonian (a, b)-path for some $a, b \in V(G_k)$,
- for any $x \in V_{EL}(G_k)$, $(G_k)_x^*$ is Hamilton-connected,

and we set $G^M = G_k$.

A resulting G^M is called a *strong M-closure* (or briefly an *SM-closure*) of the graph G, and a graph G equal to its SM-closure is said to be *SM-closed*. Note that for a given graph G, its SM-closure is not uniquely determined.

As shown in [19] and [12], if G is SM-closed, then $G = L(H)$, where H does not contain as a subgraph (not necessarily induced) any of the multigraphs shown in Fig. 3.

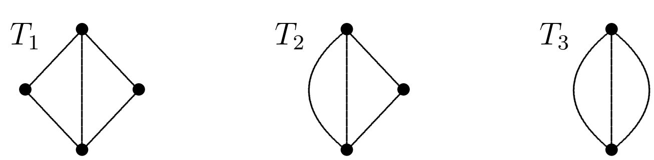


Figure 3: The diamond T_1, the multitriangle T_2 and the triple edge T_3

The following theorem summarizes basic properties of the SM-closure operation.

Theorem E [12]. *Let G be a claw-free graph and let G^M be its SM-closure. Then G^M has the following properties:*

(i) $V(G) = V(G^M)$ *and* $E(G) \subset E(G^M)$,

(ii) G^M *is obtained from* G *by a sequence of local completions at eligible vertices,*

(iii) G *is Hamilton-connected if and only if* G^M *is Hamilton-connected,*

(iv) *if* G *is Hamilton-connected, then* $G^M = \mathrm{cl}(G)$,

(v) *if* G *is not Hamilton-connected, then either*

- *(α)* $V_{EL}(G^M) = \emptyset$ *and* $G^M = \mathrm{cl}(G)$, *or*
- *(β)* $V_{EL}(G^M) \neq \emptyset$ *and* $(G^M)^*_x$ *is Hamilton-connected for any* $x \in V_{EL}(G^M)$,

(vi) $G^M = L(H)$, *where* H *contains either*

- *(α) at most 2 triangles and no multiedge, or*
- *(β) no triangle, at most one double edge and no other multiedge,*

(vii) *if* G^M *contains no hamiltonian* (a, b)*-path for some* $a, b \in V(G^M)$ *and*

- *(α)* X *is a triangle in* H, *then* $E(X) \cap \{L^{-1}_{G^M}(a), L^{-1}_{G^M}(b)\} \neq \emptyset$,
- *(β)* X *is a multiedge in* H, *then* $E(X) = \{L^{-1}_{G^M}(a), L^{-1}_{G^M}(b)\}$.

We will also need the following lemma on SM-closed graphs proved in [20].

Lemma F [20]. *Let G be an SM-closed graph and let $H = L^{-1}(G)$. Then H does not contain a triangle with a vertex of degree 2 in H.*

2.4 The core of the preimage of an SM-closed graph

The definition of the core is slightly problematic for multigraphs, therefore we restrict our observations to the case that we need, i.e., to preimages of 3-connected SM-closed graphs. The difficulties then do not occur since such a multigraph cannot have pendant multiedges by Theorem B, and cannot have pendant multitriangles (since there are no multitriangles at all).

Thus, let G be a 3-connected SM-closed graph and let $H = L^{-1}(G)$. The *core of H* is the multigraph $\mathrm{co}(H)$ obtained from H by removing all pendant edges and suppressing all vertices of degree 2.

Shao [22] proved the following properties of the core of a multigraph.

Theorem G [22]. *Let H be an essentially 3-edge-connected multigraph. Then*

(i) $\mathrm{co}(H)$ *is uniquely determined,*

(ii) $\mathrm{co}(H)$ *is 3-edge-connected,*

(iii) $V(\mathrm{co}(H))$ *dominates all edges of* H,

(iv) *if* $\mathrm{co}(H)$ *has a spanning closed trail, then* H *has a DCT.*

(v) *if* $\mathrm{co}(H)$ *is strongly spanning trailable, then* $L(H)$ *is Hamilton-connected.*

2.5 Closure operations and Z_i-free graphs

When applying closure techniques to $\{K_{1,3}, Z_i\}$-free graphs, we encounter a problem consisting in the fact that, for a $\{K_{1,3}, Z_i\}$-free graph G and $x \in V_{EL}(G)$, the local completion G_x^* is not necessarily Z_i-free. Although it can be shown [8] that $\mathrm{cl}(G)$ finally becomes Z_i-free, graphs that occur during the construction of $\mathrm{cl}(G)$, hence also an SM-closure, can contain an induced Z_i (in the terminology of [17], the class of $\{K_{1,3}, Z_i\}$-free graphs is weakly stable but not stable under the closure operation). In this paper, we overcome this difficulty by working in a slightly larger class of graphs which contains all $\{K_{1,3}, Z_i\}$-free graphs and is stable under the closure.

For a graph $F \simeq Z_i$, we will use T_F to denote the triangle of F and $V_2(T_F)$ to denote the (two-element) set of the vertices in T_F that are of degree 2 in F. We define a class $\mathcal{Z}_i^{SI}$ as follows.

For an integer $i \geq 1$, $\mathcal{Z}_i^{SI}$ is the class of all claw-free graphs G such that every induced subgraph $F \subset G$, $F \simeq Z_i$, satisfies $|V_2(T_F) \cap V_{SI}(G)| \geq 1$.

Clearly, $\mathcal{Z}_i^{SI}$ contains all $\{K_{1,3}, Z_i\}$-free graphs.

Throughout the rest of this subsection, we will keep the notation of vertices of an induced Z_i as in Fig. 1(a). For an induced $F \simeq Z_i$ in G_x^*, we will call the edges in $E(F) \setminus E(G)$ *new edges*, and we will denote $E(F) \setminus E(G) = \mathrm{new}(F)$.

Lemma 3. *Let $G \in \mathcal{Z}_i^{SI}$ and $x \in V(G)$. Then $G_x^* \in \mathcal{Z}_i^{SI}$.*

Proof. Let, to the contrary, $G \in \mathcal{Z}_i^{SI}$ and $x \in V(G)$ be such that G_x^* contains an induced subgraph $F \simeq Z_i$ with $V_2(T_F) \cap V_{SI}(G_x^*) = \emptyset$. Then also $V_2(T_F) \cap V_{SI}(G) = \emptyset$ (recall that $V_{SI}(G) \subset V_{SI}(G_x^*)$), and since $G \in \mathcal{Z}_i^{SI}$, we have $\mathrm{new}(F) \neq \emptyset$.

Suppose first that $\mathrm{new}(F) \cap E(T_F) = \emptyset$, and let, say, $e = a_j a_{j+1} \in \mathrm{new}(F)$ for some j, $0 \leq j \leq i-1$. Then we have $a_j x, a_{j+1} x \in E(G)$ since $e \in E(G_x^*) \setminus E(G)$, and $vx \notin E(G)$ for any $v \in V(F) \setminus \{a_j, a_{j+1}\}$ since F is induced in G_x^*. But then the graph $F' = \langle \{b_1, b_2, a_0, \ldots, a_j, x, a_{j+1}, \ldots, a_{i-1}\} \rangle_G$ is an induced Z_i in G with $V_2(T_{F'}) \cap V_{SI}(G) = \emptyset$, contradicting the assumption that $G \in \mathcal{Z}_i$.

Thus, we have $\mathrm{new}(F) \subset E(T_F)$. If $\mathrm{new}(F) = E(T_F)$, then $\langle \{x, b_1, b_2, a_0\} \rangle_G \simeq K_{1,3}$, a contradiction. Hence $1 \leq |\mathrm{new}(F)| \leq 2$.

Let first $|\mathrm{new}(F)| = 1$. If $\mathrm{new}(F) = \{b_1 b_2\}$, then $\langle \{a_0, b_1, b_2, a_1\} \rangle_G \simeq K_{1,3}$, a contradiction. Thus, up to a symmetry, $\mathrm{new}(F) = \{a_0 b_1\}$. Then necessarily $x \neq b_2$ (otherwise we would have $b_2 \in V_{SI}(G_x^*)$, contradicting the assumption that $V_2(T_F) \cap V_{SI}(G_x^*) = \emptyset$), and $b_2 x \in E(G)$, for otherwise $\langle \{a_0, b_2, x, a_1\} \rangle_G \simeq K_{1,3}$. Then G contains $F' = \langle \{x, b_2, a_0, a_1, \ldots, a_i\} \rangle_G \simeq Z_i$ with $V_2(T_{F'}) = \{x, b_2\}$, and $\{x, b_2\} \cap V_{SI}(G) = \emptyset$, a contradiction.

Thus, $|\mathrm{new}(F)| = 2$. Then $\{b_1, b_2, a_0\} \subset N_G(x)$ and, up to a symmetry, either $\mathrm{new}(F) = \{a_0 b_1, a_0 b_2\}$, or $\mathrm{new}(F) = \{b_1 b_2, a_0 b_2\}$, but in the first case $F' = \langle \{b_1, b_2, x, a_0, \ldots, a_{i-1}\} \rangle_G$, and in the second case $F' = \langle \{b_1, x, a_0, a_1, \ldots, a_i\} \rangle_G$ is an induced Z_i in G with $V_2(T_{F'}) \cap V_{SI}(G) = \emptyset$, a contradiction. ■

Next we define a class $\mathcal{Z}_i^T$ as follows.

For an integer $i \geq 1$, $\mathcal{Z}_i^T$ is the class of all claw-free graphs G satisfying the following condition:

($*$) *for every induced subgraph $F \simeq Z_i$ in G, there is a vertex $x_F \in V_{EL}(G)$ such that $V(T_F) \subset N_G(x_F)$ and $\langle V(F)\rangle_{G^*_{x_F}} \not\simeq Z_i$.*

Clearly, $\mathcal{Z}_i^T$ contains all $\{K_{1,3}, Z_i\}$-free graphs.

Lemma 4. *Let $G \in \mathcal{Z}_i^T$ and $x \in V_{EL}(G)$. Then $G^*_x \in \mathcal{Z}_i^T$.*

Proof. Let $G \in \mathcal{Z}_i^T$ and $x \in V(G)$ be such that G^*_x contains an induced subgraph $F \simeq Z_i$ not satisfying condition ($*$). Since $G \in \mathcal{Z}_i^T$, necessarily $\mathrm{new}(F) \neq \emptyset$ (where, as in the proof of Lemma 3, we denote $\mathrm{new}(F) = E(F) \setminus E(G)$).

Suppose first that $\mathrm{new}(F) \cap E(T_F) = \emptyset$, and let, say, $a_j a_{j+1} \in \mathrm{new}(F)$ for some j, $0 \leq j \leq i-1$. Then we again have $N_G(x) \cap V(F) = \{a_j, a_{j+1}\}$, implying that $F' = \langle\{b_1, b_2, a_0, \ldots, a_j, x, a_{j+1}, \ldots, a_{i-1}\}\rangle_G \simeq Z_i$. Since $G \in \mathcal{Z}_i^T$, there is a vertex $x_{F'} \in V_{EL}(G)$ with the properties given by condition ($*$). Since $\langle V(F')\rangle_{G^*_{x_{F'}}} \not\simeq Z_i$, $x_{F'}$ has, besides $V(T_{F'}) = V(T_F)$, another neighbor in $V(F')$, and since F does not satisfy ($*$), $x_{F'}$ is adjacent to x in G, and x is in G the only neighbor of $x_{F'}$ in $V(F') \setminus V(T_{F'})$. But then $\langle\{x, x_F, a_j, a_{j+1}\}\rangle_G \simeq K_{1,3}$, a contradiction. Thus, $\mathrm{new}(F) \subset V(T_F)$.

Let first $x \in V(T_F)$. Then necessarily $|\mathrm{new}(F)| = 1$. If $x = a_0$, then $\mathrm{new}(F) = \{b_1 b_2\}$, and then $\langle\{a_0, b_1, b_2, a_1\}\rangle_G \simeq K_{1,3}$, a contradiction. Thus, up to a symmetry, $x = b_1$ and $\mathrm{new}(F) = \{a_0 b_2\}$. Let $a_0 u_1 \ldots u_k b_2$ be a shortest (a_0, b_2)-path in $\langle N_G(x)\rangle_G$ (it exists since $x \in V_{EL}(G)$). Necessarily $k \geq 1$ since $a_0 b_2 \notin E(G)$. If $u_1 a_j \in E(G)$ for some j, $1 \leq j \leq i$, then, observing that $u_1 \in V_{EL}(G)$ (otherwise u_1 is a center of a claw in G), we have also $u_1 \in V_{EL}(G^*_x)$, and then $\langle V(F)\rangle_{(G^*_x)^*_{u_1}} \not\simeq Z_i$, contradicting the assumption that F does not satisfy ($*$). Hence $u_1 a_j \notin E(G)$, $1 \leq j \leq i$, implying that $F' = \langle\{b_1, u_1, a_0, \ldots, a_i\}\rangle_G \simeq Z_i$. By the assumption, G satisfies ($*$), hence there is a vertex $x_{F'} \in V_{EL}(G)$ such that $\{b_1, u_1, a_0\} \subset N_G(x_{F'})$ and $\langle V(F')\rangle_{G^*_{x_{F'}}} \not\simeq Z_i$. But then $x_{F'} \in N_G(x)$, hence $b_2 x_{F'} \in E(G^*_x)$, and then also $\langle V(F)\rangle_{(G^*_x)^*_{x_{F'}}} \not\simeq Z_i$, contradicting the assumption that F does not satisfy ($*$). Hence $x \notin V(T_F)$.

Suppose that $|\mathrm{new}(F)| = 1$. If $\mathrm{new}(F) = \{b_1 b_2\}$, then $\langle\{a_0, b_1, b_2, a_1\}\rangle_G \simeq K_{1,3}$, a contradiction. Hence, up to a symmetry, we have $\mathrm{new}(F) = \{a_0 b_1\}$, implying that $\{a_0, b_1\} \subset N_G(x)$. Then also $b_2 x \in E(G)$, for otherwise $\langle\{a_0, b_2, x, a_1\}\rangle_G \simeq K_{1,3}$, and then $F' = \langle\{b_2, x, a_0, a_1, \ldots, a_i\}\rangle_G \simeq Z_i$. Since $G \in \mathcal{Z}_i^T$, there is a vertex $x_{F'} \in V_{EL}(G)$ such that $\{b_2, x, a_0\} \subset N_G(x_{F'})$ and $\langle V(F')\rangle_{G^*_{x_{F'}}} \not\simeq Z_i$. Then also $b_1 x_{F'} \in E(G^*_x)$ (since $\{x b_1, x x_{F'}\} \subset E(G)$), and $\langle V(F)\rangle_{(G^*_x)^*_{x_{F'}}} \not\simeq Z_i$, contradicting the assumption that F does not satisfy ($*$). Thus, we have $|\mathrm{new}(F)| = 2$.

Suppose that $\mathrm{new}(F) = \{a_0 b_1, a_0 b_2\}$. Then we have $\{a_0, b_1, b_2\} \subset N_G(x)$ and $F' = \langle\{b_1, b_2, x, a_0, \ldots, a_{i-1}\}\rangle_G \simeq Z_i$. Since G satisfies ($*$), there is a vertex $x_{F'} \in V_{EL}(G)$ such that $\{b_1, b_2, x\} \subset N_G(x_{F'})$ and $\langle V(F')\rangle_{G^*_{x_{F'}}} \not\simeq Z_i$, implying that $x_{F'} a_j \in E(G)$ for some j, $0 \leq j \leq i-1$ (note that we have $x_{F'} a_0 \in E(G^*_x)$, but not necessarily $x_{F'} a_0 \in E(G)$). If $x_{F'} a_j \in E(G)$ with $1 \leq j \leq i-1$, then $\langle V(F)\rangle_{(G^*_x)^*_{x_{F'}}} \not\simeq Z_i$, a contradiction. Hence $x_{F'} a_0 \in E(G)$, and then we have a contradiction by the same argument for the subgraph $F'' = \langle\{x, x_{F'}, a_0, \ldots, a_i\}\rangle_G \simeq Z_i$.

Thus, up to a symmetry, we have new$(F) = \{b_1b_2, a_0b_1\}$. Then again $\{a_0, b_1, b_2\} \subset N_G(x)$ and $F' = \langle\{b_2, x, a_0, \ldots, a_i\}\rangle_G \simeq Z_i$. By condition $(*)$ in G, there is a vertex $x_{F'} \in V_{EL}(G)$ such that $\{b_2, x, a_0\} \subset N_G(x_{F'})$ and $\langle V(F')\rangle_{G^*_{x_{F'}}} \not\simeq Z_i$, and then $b_1x_{F'} \in E(G^*_x)$ and $\langle V(F)\rangle_{(G^*_x)^*_{x_{F'}}} \not\simeq Z_i$, a contradiction. ■

Now we can define a class of graphs $\mathcal{Z}_i$, $i \geq 1$, by

$$\mathcal{Z}_i = \mathcal{Z}_i^{SI} \cap \mathcal{Z}_i^{T}.$$

By Lemmas 3 and 4, we immediately have the following fact.

Theorem 5. *Let $i \geq 1$ be an integer and let $G \in \mathcal{Z}_i$ and $x \in V_{EL}(G)$. Then $G^*_x \in \mathcal{Z}_i$.*

Theorem 5 has the following immediate corollary.

Corollary 6. *Let G be a $\{K_{1,3}, Z_i\}$-free graph and let G^M be an SM-closure of G. Then $G^M \in \mathcal{Z}_i$.*

In our proof of Theorem 1, we will work in the (multi)graph $H = L^{-1}(G^M)$, where G^M is an SM-closure of the 3-connected $\{K_{1,3}, Z_7\}$-free graph under consideration. For this, with respect to Corollary 6, we need to "translate" the properties of graphs from the class $\mathcal{Z}_i$ to the preimage $H = L^{-1}(G^M)$.

First of all, it is necessary to note that clearly $L(S_{1,1,i+1}) = Z_i$, but for the graph $S_{\bar{2},i+1}$, obtained by identifying a vertex of a double edge with an endvertex of a path of length $i+1$ (see Fig. 4), we also have $L(S_{\bar{2},i+1}) = Z_i$. Although apparently $L^{-1}(Z_i) = S_{1,1,i+1}$ by

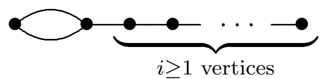


Figure 4: The multigraph $S_{\bar{2},i}$

Theorem B, it is still possible that, for an induced subgraph $X \simeq Z_i$ of a line graph G, the subgraph of $H = L^{-1}(G)$, corresponding to X, is isomorphic to $S_{\bar{2},i+1}$ (an easy example is the graph G obtained by replacing an edge of a sufficiently large cycle with a diamond, in which $H = L^{-1}(G)$ contains a double edge, and every induced Z_i in G corresponds to an $S_{\bar{2},i+1}$ in H). However, it turns out that this is not possible if $G \in \mathcal{Z}_i$.

Proposition 7. *Let $G \in \mathcal{Z}_i$, $i \geq 1$, be a line graph, and let $H = L^{-1}(G)$. Let X be an induced subgraph of G, and let $F \subset H$ be the corresponding subgraph of H. Then*

(i) H does not contain a subgraph (not necessarily induced) isomorphic to $S_{\bar{2},i+1}$,

(ii) $X \simeq Z_i$ if and only if $F \simeq S_{1,1,i+1}$,

(iii) every subgraph $F \subset H$, $F \simeq S_{1,1,i+1}$, satisfies the following conditions:

- *(α) at least one branch of length 1 of F is at a pendant edge of H, and*
- *(β) there is a triangle or a double edge in H containing the center of F and at least one further vertex on the branch of length $i+1$ of F.*

Proof. (i) If $S_{\bar{2},i+1} \subset H$, then G contains as an induced subgraph the graph $X = L(S_{\bar{2},i+1}) \simeq Z_i$ such that the vertices in $V_2(T_X)$ correspond to the two edges of the double edge in $S_{\bar{2},i+1}$, hence are nonsimplicial by Theorem B, contradicting the definition of the class $\mathcal{Z}_i^{SI}$.

(ii) It is straightforward to verify that there are exactly two (multi)graphs F such that $L(F) = Z_i$, namely, $S_{\bar{2},i}$ and $S_{1,1,i+1}$. Statement (ii) then follows from (i).

$(iii)(\alpha)$ Since $G \in \mathcal{Z}_i^{SI}$, every induced subgraph $X \simeq Z_i$ in G satisfies $|V_2(T_X) \cap V_{SI}(G)| \geq 1$ by the definition of the class $\mathcal{Z}_i^{SI}$. The rest follows from (ii) and from Theorem B.

$(iii)(\beta)$ As noted in Subsection 2.3, $x \in V_{EL}(G)$ if and only if the edge $L^{-1}(x)$ is in a triangle or in a double edge in H. The rest follows from (ii) and from condition $(*)$ in the definition of the class $\mathcal{Z}_i^T$. ∎

In the proof of Theorem 1, we will have to handle the exceptional graph $L(W^1)$. For this, we will need the following simple technical lemma.

Lemma 8. *Let G be a claw-free graph and let G^M be its SM-closure. If $G \not\simeq L(W^1)$, then $G^M \not\simeq L(W^1)$.*

Proof. Suppose, to the contrary, that $G^M \simeq L(W^1)$. Let $G_1, \ldots, G_k$ be the sequence of graphs that yields G^M, i.e., $G_1 = G$, $G_k = G^M$ and $G_{i+1} = (G_i)^*_{x_i}$ for some $x_i \in V_{EL}(G_i)$, $i = 1, \ldots, k-1$. We will use the labeling of vertices of the graph W as in Fig. 2(c), and we will further denote w_i' the neighbor of w_i in $V_1(W^1)$, $y_i = L(w_i w_i')$, and $y_{ij} = L(w_i w_j)$ for $i, j = 1, \ldots, 8$, $w_i w_j \in E(W)$. Then clearly $V_{SI}(L(W^1)) = \{y_i |\ i = 1, \ldots, 8\}$, and $V_{EL}(L(W^1)) = \emptyset$.

Since $x_{k-1} \in V_{SI}(G_k)$ and $G_k = G^M \simeq L(W^1)$, we can choose the notation such that $x_{k-1} = y_1$. Then $y_1 \in V_{EL}(G_{k-1})$, hence some of the edges in $\langle N_{G_k}(y_1) \rangle_{G_k}$ are new edges. Observe that $\langle N_{G_k}(y_1) \rangle_{G_k}$ is the triangle $\langle \{y_{12}, y_{15}, y_{18}\} \rangle_{G_k}$. If one edge, say, $y_{15}y_{18}$, is new, then $\langle \{y_{12}, y_{15}, y_{18}, y_{23}\} \rangle_{G_{k-1}} \simeq K_{1,3}$, a contradiction, and if two or three edges are new, then $\langle N_{G_{k-1}}(y_1) \rangle_{G_{k-1}}$ is not connected, a contradiction again. ∎

3 A special version of the "Nine-point-theorem"

The well-known "Nine-point-theorem" by Holton et al. [11] states that a 3-connected cubic graph contains a cycle passing through any 9 prescribed vertices, and its strengthened version by Bau and Holton [3] claims the same for cycles through 12 vertices, with the Petersen graph as an exception (proved with the help of a computer). For our purposes, we use a special version, developed in [14], based on another stronger version by Bau and Holton [2] that deals with a set of vertices and an edge (proved without computer). For this, we need some more terminology from [1].

Let G be a multigraph, $R \subset G$ a spanning subgraph of G, and let $\mathcal{R}$ be the set of components of R. Then G/R is the multigraph with $V(G/R) = \mathcal{R}$, in which, for each edge in $E(G)$ between two components of R, there is an edge in $E(G/R)$ joining the corresponding vertices of G/R (note that this means that G/R can have multiple edges even if G is a graph). The (multi-)graph G/R is said to be a *contraction* of G. (Roughly, in G/R, components of R are contracted to single vertices while keeping the adjacencies between them). Clearly, if R is connected, then $G/R = K_1$, and if R is edgeless, then $G/R = G$; these two contractions are called *trivial*.

The contraction operation maps $V(G)$ onto $V(G/R)$ (where vertices of a component of R are mapped on a vertex of G/R). If $G/R \simeq F$, then this defines a function $\alpha : G \to F$ which is called a *contraction of* G *on* F.

Throughout the rest of this section, Π denotes the Petersen graph.

The following special version of the "nine-point-theorem" was proved in [14].

Theorem H [14]. *Let H be a 3-edge-connected multigraph, $A \subset V(H)$, $|A| = 8$, and let $e \in E(H)$. Then either*

(*i*) *H contains a closed trail T such that $A \subset V(T)$ and $e \in E(T)$, or*

(*ii*) *there is a contraction $\alpha : H \to \Pi$ such that $\alpha(e) = xy \in E(\Pi)$ and $\alpha(A) = V(\Pi)\setminus\{x, y\}$.*

We will also need the following auxiliary result from [14].

Lemma I [14]. *Let H be a graph such that $\mathrm{co}(H) = W$. If there is a vertex $x \in V(\mathrm{co}(H))$ such that $N_H(x) = N_{\mathrm{co}(H)}(x)$, then $L(H)$ is Hamilton-connected.*

Theorem 9. *Let $G \in \mathcal{Z}_7$ be a 3-connected SM-closed graph such that $G \not\simeq L(W^1)$ and $\mathrm{co}(H)$, where $H = L^{-1}(G)$, is 2-connected, and let $e_1, e_2 \in E(H)$ be such that there is no (e_1, e_2)-IDT in H. Then for every set $A \subset V(\mathrm{co}(H))$, $|A| = 8$, there is an (e_1, e_2)-trail T in H such that $A \subset \mathrm{Int}(T)$.*

Proof. First of all, it should be noted here that some parts of the proof of Theorem 9 are (almost) the same as the corresponding parts of the proof of Theorem 9 in [14] and of Theorem 4 in [21]. Since the other parts are quite different, for the sake of completeness, we give a complete proof here (including the identical parts).

Let H be a graph satisfying the assumptions of the theorem. By Proposition 7, every subgraph (not necessarily induced) of H, isomorphic to $S_{1,1,8}$, has its center in a triangle or a double edge and at least one of its branches of length 1 at a pendant edge.

Let H' be the graph obtained from H by the following construction:

(*i*) if e_1, e_2 share a vertex of degree 2, say, $e_i = v_i v$, $i = 1, 2$ with $v \in V_2(H)$, we suppress v and set $h = v_1 v_2$,

(*ii*) otherwise, we subdivide e_i (or some edge in $\mathrm{co}(H)$ sharing a vertex with e_i if e_i is pendant) with a vertex v_i, $i = 1, 2$, and add a new edge $h = v_1 v_2$.

If there is no contraction $\alpha' : H' \to \Pi$ such that $\alpha'(h) = x_1 x_2 \in E(\Pi)$ and $\alpha'(A) = V(\Pi) \setminus \{x_1, x_2\}$, then, by Theorem H, there is a closed trail T' in H' such that $A \subset V(T')$ and $h \in E(T')$. Returning to H, i.e., subdividing h in case (*i*), or removing h and suppressing v_1, v_2 (and extending the trail to e_i if e_i is pendant) in case (*ii*), we obtain an (e_1, e_2)-trail T in H with $A \subset \mathrm{Int}(T)$.

Thus, we suppose that there is a contraction $\alpha' : H' \to \Pi$ such that $\alpha'(h) = x_1 x_2 \in E(\Pi)$ and $\alpha'(A) = V(\Pi) \setminus \{x_1, x_2\}$. In case (*i*), H contains a subgraph isomorphic to the Petersen graph with at least one subdivided edge which contains the graph $S_{1,1,8}$: in the labeling of vertices as in Fig. 2(*b*), if, say, the edge $p_1^1 p_2^1$ is subdivided with a vertex q, we have $S_{1,1,8}(p_1^1; q; p_5^1; p_2^1 p_3^1 p_4^1 p_4^2 p_1^2 p_3^2 p_5^2 p_2^2)$ as a subgraph of H with both branches of length 1 at nonpendant edges, a contradiction. Thus, for the rest of the proof, we suppose that H' is obtained by construction (*ii*).

Set $H_0 = \text{co}(H)$, and recall that H_0 is 3-edge-connected (since H is essentially 3-edge-connected). Let R' be the spanning subgraph of H' that defines α', and suppose that, say, the component $R_1 = (\alpha')^{-1}(x_1)$ of R' is nontrivial. Since $x_1 \in V(\Pi)$, the subgraph R_1 is separated from the rest of H' by a 3-edge-cut containing the edge h, implying that in H_0, the subgraph $R_1 - v_1$ is separated from the rest of H_0 by a 2-edge-cut, contradicting the fact that H_0 is 3-edge-connected. Hence $(\alpha')^{-1}(x_1)$, and symmetrically also $(\alpha')^{-1}(x_2)$, are trivial, i.e., $V((\alpha')^{-1}(x_i)) = \{v_i\}$, $i = 1, 2$. Removing from H' the edge h and suppressing v_1 and v_2, we obtain from R' the corresponding spanning subgraph R of H, and from R, in a standard way a spanning subgraph R_0 of H_0. Note that clearly every component of R' except $\{v_1\}$ and $\{v_2\}$ corresponds to a nonempty component of R_0 since α' maps H' on a cubic graph and hence every component of R' must contain a vertex of degree more than 2. Then the components of R_0 define a contraction $\alpha : H_0 \to W$, where W is the Wagner graph (see Fig. 2(c); recall that W can be obtained from Π by removing an edge and suppressing the created vertices of degree 2).

Case 1: *$\alpha^{-1}(w)$ is trivial for any $w \in V(W)$.*

Then we have $H_0 \simeq W$. By Lemma I, every vertex of H_0 is incident in H to a pendant edge or to a subdivided edge.

Subcase 1.1: *no edge of H_0 is subdivided in H.*

Then, by Lemma I, each vertex of H_0 is incident in H with at least one pendant edge, i.e., $H_0 \in \mathcal{W}$, and at least one vertex, say, w_1, is incident in H with at least two pendant edges since $G \not\simeq L(W^1)$ by the assumption of the theorem. Let w_1', w_1'' be two neighbors of w_1 of degree 1 in H, and let w_8' be a neighbor of w_8 of degree 1 in H. Then H contains $S_{1,1,8}(w_1; w_1'; w_1''; w_2w_3w_4w_5w_6w_7w_8w_8')$. By Proposition 7($iii$)($\beta$), w_1 is in a triangle or in a double edge; however, $H_0 \simeq W$, hence also H, contains neither a triangle nor a double edge, a contradiction.

Subcase 1.2: *at least one edge of H_0 is subdivided in H.*

Suppose first that some of the edges w_iw_{i+4} (indices modulo 8) is subdivided in H, say, w_1w_5 is subdivided with a vertex w_{15}. By Lemma I, w_3 has a pendant edge, or some edge incident to w_3 is subdivided. By symmetry, we have the following possibilities:

Case	Contradiction
Pendant edge w_3w_3'	$S_{1,1,8}(w_3; w_3'; w_4; w_2w_1w_{15}w_5w_6w_7w_8w_8')$
w_2w_3 subdivided with w_{23}	$S_{1,1,8}(w_2; w_{23}; w_6; w_1w_{15}w_5w_4w_3w_7w_8w_8')$
w_3w_7 subdivided with w_{37}	$S_{1,1,8}(w_3; w_{37}; w_4; w_2w_1w_{15}w_5w_6w_7w_8w_8')$

where w_8' is a neighbor of w_8 in $V(H) \setminus V(H_0)$ which exists by Lemma I (note that w_8' can be a vertex of degree 2, subdividing some of the edges incident to w_8, in which case the last two vertices of the long branch can occur in reverse order).

Thus, we can suppose that none of the edges w_iw_{i+4} is subdivided, thus, say, w_1w_2 is subdivided with a vertex w_{12}. Then similarly w_3 has a pendant edge or some of the edges w_2w_3, w_3w_4 is subdivided, and we have the following possibilities:

Case	Contradiction
Pendant edge w_3w_3'	$S_{1,1,8}(w_3; w_3'; w_4; w_2w_{12}w_1w_5w_6w_7w_8w_8')$
w_2w_3 subdivided with w_{23}	$S_{1,1,8}(w_2; w_{23}; w_6; w_{12}w_1w_5w_4w_3w_7w_8w_8')$
w_3w_4 subdivided with w_{34}	$S_{1,1,8}(w_3; w_{34}; w_7; w_2w_{12}w_1w_8w_4w_5w_6w_6')$

where again w_8' (w_6') is a neighbor of w_8 (of w_6) in $V(H)\setminus V(H_0)$, and the last two vertices of the long branch can occur in reverse order if w_8' (w_6') is of degree 2.

Since the graph H_0, hence also H, contains neither a triangle nor a double edge, each of the above subgraphs contradicts the fact that $G \in \mathcal{Z}_7$.

Case 2: *$\alpha^{-1}(w)$ is nontrivial for some $w \in V(W)$.*

Let $R_1^0, \ldots, R_8^0$ be the components of the graph R_0 that defines α, and choose the notation such that $R_i^0 = \alpha^{-1}(w_i)$, $i = 1, \ldots, 8$, and such that $R_1^0 = \alpha^{-1}(w_1)$ is nontrivial. Recall that $\cup_{i=1}^{8}(V(R_i^0)) = V(R_0) = V(H_0)$. Let R_i be the component of R that corresponds to R_i^0, $i = 1, \ldots, 8$ (i.e., $\cup_{i=1}^{8}(V(R_i)) = V(R) = V(H)$).

We observe that $e_1, e_2 \in E(H_0) \setminus E(R_0)$ since, by the construction of H', $\alpha^{-1}(x_i) = v_i$ are trivial and after deleting the edge h and suppressing the vertices v_1, v_2, each of the edges e_1, e_2 has its vertices in different components of R_0, hence also in different components of R. By Theorem E(vi),(vii), this implies that each R_i is a triangle-free (simple) graph. Moreover, each R_i^0 is 2-edge-connected since $R_i^0 = \alpha^{-1}(w_i)$ is separated from the rest of H_0 by a 3-edge-cut and a cut-edge in R_i^0 would create a 2-edge-cut in H_0.

We introduce the following notation. For any edge $w_iw_j \in E(W)$, we set $f_{ij} = \alpha^{-1}(w_iw_j)$ (i.e., f_{ij} joins R_i^0 and R_j^0), and we denote b_j^i its vertex in R_i^0 and b_i^j its vertex in R_j^0. Thus, we e.g. have $A_{H_0}(R_1^0) = \{b_2^1, b_5^1, b_8^1\}$, where $2 \leq |\{b_2^1, b_5^1, b_8^1\}| \leq 3$, and $\{f_{12}, f_{15}, f_{18}\}$ is the 3-edge-cut that separates R_1^0 from the rest of H_0.

<u>Claim 1.</u> *Let R_i^0 be a component of R_0, $1 \leq i \leq 8$, and let $A_{H_0}(R_i^0) = \{b_{j_1}^i, b_{j_2}^i, b_{j_3}^i\}$. Then there is a vertex $d^i \in V(R_i^0)$ and three internally vertex-disjoint (possibly trivial) $(d^i, b_{j_k}^i)$-paths $P_{j_k}^i$, $k = 1, 2, 3$.*

<u>Proof.</u> Let P be an arbitrary (possibly trivial) $(b_{j_1}^i, b_{j_2}^i)$-path in R_i^0, and let $P_{j_3}^i$ be a shortest $(d^i, b_{j_3}^i)$-path with $d^i \in V(P)$. Then the vertex d^i and the paths $P_{j_1}^i = d^iPb_{j_1}^i$ $P_{j_2}^i = d^iPb_{j_2}^i$ and $P_{j_3}^i$ have the required properties. □

<u>Claim 2.</u> *The component R_1 contains a cycle C of length at least 4, vertices $c_2, c_5, c_8 \in V(C)$ and paths Q_2^1, Q_5^1, Q_8^1 (possibly trivial) such that*

- (i) $2 \leq |\{c_2, c_5, c_8\}| \leq 3$,
- (ii) *Q_2^1 is a (c_2, b_2^1)-path, Q_5^1 is a (c_5, b_5^1)-path and Q_8^1 is a (c_8, b_8^1)-path,*
- (iii) *the paths Q_2^1, Q_5^1, Q_8^1 are internally vertex-disjoint.*

<u>Proof.</u> Let d^1 and P_2^1, P_5^1, P_8^1 be the vertex and paths in R_1^0 given by Claim 1. Since R_1^0 is nontrivial, at least one of P_2^1, P_5^1, P_8^1 is nontrivial. Suppose that, say, P_5^1 is nontrivial. We consider a (b_2^1, b_8^1)-path P and choose two edge-disjoint paths P_5', P_5'' such that

- P_5' is a (b_5^1, c_2)-path and P_5'' is a (b_5^1, c_8)-path for some $c_2, c_8 \in V(P')$,
- if $c_2 \neq c_8$, then c_2 is on P between c_8 and b_2^1, and

- c_2, c_8, P_5' and P_5'' are chosen such that $|E(P_5')| + |E(P_5'')|$ is smallest possible.

If $c_2 \neq c_8$, we choose c_5 as the last common vertex of P_5' and P_5'', and we set $C_0 = c_2Pc_8P_5''c_5P_5'c_2$, $Q_2^1 = c_2P_1b_2^1$, $Q_8^1 = c_8P_1b_8^1$, and, say, $Q_5^1 = c_5P_5'b_5^1$. If $c_2 = c_8$, we choose c_5 as the last common vertex of P_5' and P_5'' distinct from the vertex $c_2 = c_8$ (possibly $c_5 = b_5^1$), and set $C_0 = c_2P_5'c_5P_5''c_2$, $Q_2^1 = c_2P_1b_2^1$, $Q_8^1 = c_8P_1b_8^1$, and, say, $Q_5^1 = c_5P_5'b_5^1$.

If P_2 or P_8 is nontrivial, we get C_0, Q_2^1, Q_5^1 and Q_8^1 in the same way with the only difference that possibly $c_5 = c_8$ or $c_2 = c_5$.

We have obtained a cycle C_0 and paths Q_2^1, Q_5^1 and Q_8^1 in R_1^0 (note that C_0 can possibly be a triangle or a double edge). Now, let C be the cycle in R_1 that corresponds to the cycle C_0, and, with a slight abuse of notation, let Q_2^1, Q_5^1 and Q_8^1 be the corresponding paths in R_1. Then $|V(C)| \geq 4$ since R_1 is a triangle-free simple graph, and clearly C_0, Q_2^1, Q_5^1 and Q_8^1 have the requested properties. □

For the requested graph $S_{1,1,8}$, we describe a subgraph of H in which it is contained. Here, for integers i_0, j_0, k_0, $1 \leq i_0 \leq j_0 \leq k_0$, we use $S_{\geq i_0, \geq j_0, \geq k_0}$ to denote a graph containing an S_{i_0,j_0,k_0} as a subgraph. If a component R_i^0 contains the vertex of degree 3 of the $S_{\geq i_0, \geq j_0, \geq k_0}$, then it is located in the vertex d^i and uses the paths $P_{j_k}^i$, $k = 1, 2, 3$, given by Claim 1, and for any other component R_i^0, $2 \leq i \leq 8$, and $b_j^i, b_k^i \in A_{H_0}(R_i^0)$, we use $Q_{j,k}^i$ to denote an arbitrarily chosen (b_j^i, b_k^i)-path in R_i^0 (of course, if R_i^0 is trivial, all these paths collapse to a single vertex). If we relabel the vertices of the cycle C given by Claim 2 such that $C = u_1u_2 \dots u_{|V(C)|}$ with $u_1 = c_2$, then the requested subgraph, containing $S_{1,1,8}$, can be described as $S_{\geq 1, \geq 1, \geq 8}(d^4; P_3^4b_4^3; P_5^4b_4^5; P_8^4Q_{4,7}^8Q_{8,6}^7Q_{7,2}^6Q_{6,1}^2Q_2^1u_1u_2u_3u_4)$. Since $b_4^3, b_4^5 \in V(H_0)$, the branches of length 1 of the $S_{1,1,8}$ are at nonpendant edges, contradicting the fact that $G \in \mathcal{Z}_7$. ■

4 Proof of Theorem 1

The following lemma, combining techniques developed in the previous sections, will be crucial in our proof.

Lemma 10. *Let G be a 3-connected non-Hamilton-connected SM-closed claw-free graph. Then G has an induced subgraph $\tilde{G}$ (possibly $\tilde{G} = G$) such that $\tilde{G}$ is 3-connected, non-Hamilton-connected and SM-closed, and, moreover, $\tilde{H}_0 = \mathrm{co}(L^{-1}(\tilde{G}))$ is 2-connected, and either $c(\tilde{H}_0) \geq 9$ and $|V(\tilde{H})| \geq 10$, or $\tilde{H}_0 \in \{W\} \cup \mathbb{W}$.*

Proof. Let $H = L^{-1}(G)$, and set $H_0 = \mathrm{co}(H)$. By Theorem G(ii), H_0 is 3-edge-connected.

Suppose first that H_0 is not 2-connected, let $B_1^0, \dots, B_b^0$ be blocks of H_0, let $B_1, \dots, B_b$ be the corresponding subgraphs of H (i.e., $B_i^0 = \mathrm{co}(B_i)$, $i = 1, \dots, b$), and let B_i' be obtained from B_i by attaching a pendant edge to every vertex which is a cutvertex of H_0, $i = 1, \dots, b$. Then obviously $\mathrm{co}(B_i') = \mathrm{co}(B_i) = B_i^0$, and B_i^0 is 2-connected, $i = 1, \dots, b$. If every B_i' has an (f_1, f_2)-IDT for any $f_1, f_2 \in E(B_i')$, then an easy induction shows that $G = L(H)$ is Hamilton-connected, a contradiction. Hence there is a B_{i_0}' having no (f_1, f_2)-IDT for some $f_1, f_2 \in E(B_{i_0}')$.

Set $\tilde{H} = B'_{i_0}$ and $\tilde{G} = L(\tilde{H})$. Then $\tilde{G}$ is an induced subgraph of G (since $\tilde{H}$ is a subgraph of H), is 3-connected (since $\tilde{H}$ is essentially 3-edge-connected), non-Hamilton-connected (since $\tilde{H} = B'_{i_0}$ has no (f_1, f_2)-IDT) and SM-closed (since a local completion in $\tilde{G}$ is a local completion in G), and, by the construction, $\tilde{H}_0 = \mathrm{co}(\tilde{H}) = B^0_{i_0}$ is 2-connected. By Theorem G(v), $\tilde{H}_0$ is not strongly spanning trailable, implying that, by Theorem D, $c(\tilde{H}_0) \geq 9$ and $|V(\tilde{H}_0)| \geq 10$, unless $\tilde{H}_0 \simeq W$ or $\tilde{H}_0 \in \mathbb{W}$. ■

Proof of Theorem 1. Let G be a 3-connected $\{K_{1,3}, Z_7\}$-free graph, and suppose, to the contrary, that G is not Hamilton-connected. By Theorem E, by Corollary 6 and by Lemma 8, we can suppose that G is SM-closed, $G \in \mathcal{Z}_7$, and $G \not\simeq L(W^1)$. Let thus $H = L^{-1}(G)$. By Proposition 7, H contains no subgraph isomorphic to $S_{\bar{2},8}$, and every subgraph of H isomorphic to $S_{1,1,8}$ has its center in a triangle or a double edge and at least one of its branches of length 1 at a pendant edge.

Set $H_0 = \mathrm{co}(H)$. By Theorem G(ii), H_0 is 3-edge-connected. By Lemma 10, we can suppose that H_0 is 2-connected with $c(H_0) \geq 9$ and $|V(H_0)| \geq 10$, unless $H_0 \simeq W$ or $H_0 \in \mathbb{W}$. However, if $H_0 \simeq W$, then, by Theorem 9 and since $|V(H_0)| = 8$, H has an (e_1, e_2)-IDT for any $e_1, e_2 \in E(H_0)$ and hence also for any $e_1, e_2 \in E(H)$, implying that $G = L(H)$ is Hamilton-connected, a contradiction. So, let next $H_0 \in \mathbb{W}$, and let $\{e_1, e_2\}$ be a double edge in H_0. By symmetry, we can suppose that $V(e_1) = V(e_2) = \{w_1, v\}$, where $v \in V_2(H)$ subdivides either the edge w_1w_2 or the edge w_1w_5. If $\{e_1, e_2\}$ is a double edge also in H, then $e_1vw_1w_2w_3w_4w_5w_6w_7w_8e_2$ or $e_1w_1w_2w_3w_4w_5w_6w_7w_8e_2$ is an (e_1, e_2)-IDT in H, contradicting Theorem E(vii)(β). Thus, by Lemma F, both e_1 and e_2 are subdivided in H, say, e_i with a vertex $v_i \in V_2(H)$, $i = 1, 2$. Then, if v subdivides w_1w_2, H contains the subgraph $S_{1,1,8}(v; v_1; v_2; w_2w_3w_4w_5w_6w_7w_8w_1)$, and if v subdivides w_1w_5, H contains $S_{1,1,8}(v; v_1; v_2; w_5w_6w_7w_8w_1w_2w_3w_4)$. In both cases, we have an $S_{1,1,8}$ in H with both branches of length 1 at nonpendant edges, a contradiction.

Thus, we have $c(H_0) \geq 9$ and $|V(H_0)| \geq 10$. We consider the possible cases separately.

Throughout the proof, in each of the cases, C always denotes a cycle $C = x_1x_2 \ldots x_{c(H_0)}$ such that

(i) C is a longest cycle in H_0,

(ii) subject to (i), C dominates in H maximum number of edges.

We further denote $R = V(H) \setminus V(C)$, $N = \{y \in V(H_0) |\ N_R(y) = \emptyset\}$, $R_0 = R \cap V(H_0)$, and if $R_0 \neq \emptyset$, we set $R_0 = \{y_1, \ldots, y_{|R_0|}\}$ and we choose the notation such that $y_1x_1 \in E(H_0)$. An edge x_ix_j with $x_i, x_j \in V(C)$, $1 \leq i, j \leq |V(C)|$, will be called a *chord* of C, and we say that x_ix_j is a *k-chord* if the shorter one of the two subpaths of C determined by x_i and x_j has k interior vertices.

The proof of Theorem 1 consists in a thorough case analysis. In the proof, we will often list vertices of a subgraph $S_{i,j,k}$, and there are two general comments to all these situations.

- When some edge $e = x_ix_j$ of the $S_{i,j,k}$ is in $E(H_0)$, it can always happen that e is subdivided in H, i.e., formally, $e \notin E(H)$. However, it is immediate to see that if this happens, then the corresponding subgraph of H, which instead of $e = x_ix_j$ contains a path x_izx_j with $z \in V_2(H)$, also contains $S_{i,j,k}$ as a subgraph.
- When a vertex $x_i \in V(C)$ has a (potential) neighbor $z \in R$ and the vertex z occurs as the last vertex of a branch of the $S_{i,j,k}$, then such a vertex z can be an endvertex of a

pendant edge attached to x_i, or can be $z \in V_2(H)$ and z subdivides some of the edges incident to x_i. It should be noted that in the second case, the vertices x_i and z can occur in reverse order in the list (i.e., x_i being the last vertex of the branch).

These facts will be always implicitly understood throughout the proof.

Claim 1. *Let $\{e_1, e_2\} \subset E(H_0)$ be a double edge in H_0. Then*

(i) $\{e_1, e_2\} \subset E(H)$,

(ii) $V(e_1) = V(e_2) \subset V(C)$,

(iii) if $|V(H_0)| = c(H_0)$, *then* $\{e_1, e_2\} \cap E(C) = \emptyset$.

Proof. Set $V(e_1) = V(e_2) = \{u_1, u_2\}$, let P be a shortest path from u_1 to C (possibly trivial if $u_1 \in V(C)$), and choose the notation such that P is a (u_1, x_1)-path (possibly $u_1 = x_1$ if P is trivial).

(i) If $\{e_1, e_2\} \not\subset E(H)$, then, by Lemma F, both e_1 and e_2 are subdivided in H, say, e_i with a vertex $v_i \in V_2(H)$, $i = 1, 2$. Then the graph $S_{1,1,\geq 8}(u_1; v_1; v_2; Px_1x_2x_3x_4x_5x_6x_7x_8x_9)$ contains a subgraph $S_{1,1,8}$ with both branches of length 1 at nonpendant edges, a contradiction. Hence $\{e_1, e_2\}$ is a double edge also in H.

(ii) If, say, $u_2 \notin V(C)$, then, for the same choice of P as above, H contains the subgraph $S_{\bar{2},\geq 8}(u_1; u_2; Px_1x_2x_3x_4x_5x_6x_7x_8x_9)$, containing an $S_{\bar{2},8}$, a contradiction.

(iii) If, say, $V(e_1) = V(e_2) = x_1x_2$, then $T = e_1x_2x_3 \ldots x_{c(H_0)}x_1e_2$ is an (e_1, e_2)-IDT in H, contradicting Theorem E$(vii)(\beta)$. □

Note that clearly a double edge in H is a double edge also in H_0; thus, by Claim 1(i), $\{e_1, e_2\}$ is a double edge in H if and only if $\{e_1, e_2\}$ is a double edge in H_0.

Claim 2. *If $c(H_0) \geq 10$, then no chord of C is subdivided in H.*

Proof. Let, say, $x_1x_i \in E(H_0)$ with $3 \leq i \leq c(H_0) - 1$ be subdivided in H with a vertex $v \in V_2(H)$. Then H contains the subgraph $S_{1,1,8}(x_1; v; x_{c(H_0)}; x_2x_3x_4x_5x_6x_7x_8x_9)$, a contradiction (note that the edges x_1v, $x_1x_{c(H_0)}$ are nonpendant). □

Case 1: *$c(H_0) = 9$ and $|V(H_0)| \geq 10$.*

Claim 3. *For any $u \in V(H_0)$, $|N_{R_0}(u)| \leq 1$.*

Proof. Let, to the contrary, $v_1, v_2 \in N_{R_0}(u)$ for some $u \in V(H_0)$. If $u \in V(C)$, say, $u = x_1$, then H contains $S_{1,1,8}(u; v_1; v_2; x_2x_3x_4x_5x_6x_7x_8x_9)$, a contradiction; and if u is at distance 1 from C, say, $ux_1 \in E(H_0)$, then H contains $S_{1,1,8}(u; v_1; v_2; x_1x_2x_3x_4x_5x_6x_7x_8)$, a contradiction again (note that the edges uv_1, uv_2 are nonpendant since $v_1, v_2 \in V(H_0)$, and none of the edges under consideration can be a double edge by Claim 1).

Thus, u is at distance at least 2 from C. Let P be a shortest path from u to C, and choose the notation such that P is a (u, x_1)-path and v_1 is the successor of u on P. Since $\delta(H_0) \geq 3$, u has, besides v_1 and v_2, another neighbor $v_3 \in V(H_0)$, and then H contains $S_{1,1,\geq 8}(u; v_2; v_3; v_1Px_1x_2x_3x_4x_5x_6x_7)$, a contradiction. □

By Claim 3, we have $\delta(\langle R_0 \rangle_{H_0}) \leq 1$.

Subcase 1.1: $E(\langle R_0\rangle_{H_0}) \neq \emptyset$.

Let $y_1y_2 \in E(\langle R_0\rangle_H)$. Since $\delta(H_0) \geq 3$, by Claim 3 and by Claim 1(ii), each of y_1, y_2 has two neighbors on C and these neighbors are distinct. Moreover, any two neighbors of any of y_1, y_2 must be at distance at least 2 on C, and any neighbor of y_1 must be from any neighbor of y_2 at distance at least 3 on C, for otherwise there is a cycle longer than C. However, this implies $|V(C)| \geq 3+3+2+2 = 10 > 9 = |V(C)|$, a contradiction.

Subcase 1.2: $E(\langle R_0\rangle_{H_0}) = \emptyset$.

Since $\delta(H_0) \geq 3$ and by Claim 1(ii), every vertex $y \in R_0$ has in H_0 three distinct neighbors on C. Since C is longest, no two neighbors of a $y \in R_0$ can be consecutive on C. Let $y_1 \in R_0$. By symmetry, we can choose the notation such that $N_C(y_1) \supset \{x_1, x_3, x_5\}$, $N_C(y_1) = \{x_1, x_4, x_7\}$, or $N_C(y_1) = \{x_1, x_3, x_6\}$.
We set $R_1 = R \setminus \{y_1\}$ and $N_1 = \{y \in V(H_0) |\ N_{R_1}(y) = \emptyset\}$.

Claim 4. *Let* $y_1 \in R_0$.
 (i) *If* $N_C(y_1) \supset \{x_1, x_3, x_5\}$, *then* $\{x_1, x_5, x_7, x_8\} \subset N_1$.
 (ii) *If* $N_C(y_1) = \{x_1, x_4, x_7\}$, *then* $\{x_2, x_3, x_5, x_6, x_8, x_9\} \subset N_1$.
 (iii) *If* $N_C(y_1) = \{x_1, x_3, x_6\}$, *then* $\{x_4, x_5, x_8\} \subset N_1$.

Proof. (i) If $x_1 \notin N_1$, then there is a vertex $x'_1 \in N_{R_1}(x_1)$, and H contains the subgraph $S_{1,1,8}(x_3; x_2; x_4; y_1x_5x_6x_7x_8x_9x_1x'_1)$, a contradiction; if $x_8 \notin N_1$, then there is a vertex $x'_8 \in N_{R_1}(x_8)$, and H contains $S_{1,1,8}(x_1; x_2; x_9; y_1x_3x_4x_5x_6x_7x_8x'_8)$, a contradiction again (note that here, and in all the following cases, the branches of length 1 are at nonpendant edges). The remaining cases are symmetric.

(ii) If $x_2 \notin N_1$, then there is a vertex $x'_2 \in N_{R_1}(x_2)$, and H contains the subgraph $S_{1,1,8}(x_4; x_3; y_1; x_5x_6x_7x_8x_9x_1x_2x'_2)$, a contradiction. The remaining cases are symmetric.

(iii) There are the following possibilities.

Neighbor of x_i in R_1	Contradiction
$x'_4 \in N_{R_1}(x_4)$	$S_{1,1,8}(x_6; x_5; y_1; x_7x_8x_9x_1x_2x_3x_4x'_4)$
$x'_5 \in N_{R_1}(x_5)$	$S_{1,1,8}(x_3; x_4; y_1; x_2x_1x_9x_8x_7x_6x_5x'_5)$
$x'_8 \in N_{R_1}(x_8)$	$S_{1,1,8}(x_6; x_7; y_1; x_5x_4x_3x_2x_1x_9x_8x'_8)$

In each of the cases, we have obtained a contradiction. □

Subcase 1.2.1: $|R_0| \geq 2$.

Let $y_1, y_2 \in R_0$. If $N_C(y_1) \supset \{x_1, x_3, x_5\}$, then, by Claim 3 and by Claim 4(i), $N_C(y_2) \subset \{x_2, x_4, x_6, x_9\}$. Since $|N_C(y_2)| \geq 3$, either $x_2, x_9 \in N_C(y_2)$, or $x_2, x_4 \in N_C(y_2)$ (in H_0), but in the first case the cycle $C' = x_1y_1x_3x_4x_5x_6x_7x_8x_9y_2x_2x_1$, and in the second case the cycle $C'' = x_1x_2y_2x_4x_3y_1x_5x_6x_7x_8x_9x_1$ is longer than C, a contradiction.

If $N_C(y_1) = \{x_1, x_4, x_7\}$, then, by Claim 3 and by Claim 4(ii), $N_C(y_2) = \emptyset$, a contradiction.

If $N_C(y_1) = \{x_1, x_3, x_6\}$, then, by Claim 3 and by Claim 4(iii), $N_C(y_2) \subset \{x_2, x_7, x_9\}$, and the cycle $C' = x_1x_2y_2x_9x_8x_7x_6x_5x_4x_3y_1x_1$ is longer than C, a contradiction.

Subcase 1.2.2: $|R_0| = 1$.

Then the set $V(C) \cup \{y_1\}$ dominates all edges of H.

Subcase 1.2.2.1: $N_C(y_1) \supset \{x_1, x_3, x_5\}$.

Recall that, by Claim 4(i), $\{x_1, x_5, x_7, x_8\} \subset N_1$. If $x_1x_7 \notin E(H_0)$, then the set $A_1 = (V(C) \cup \{y_1\}) \setminus \{x_1, x_7\}$ with $|A_1| = 8$ dominates all edges of H and $G = L(H)$ is Hamilton-connected by Theorem 9, a contradiction. Hence $x_1x_7 \in E(H_0)$. Similarly, considering the set $A_2 = (V(C) \cup \{y_1\}) \setminus \{x_1, x_8\}$ with $|A_2| = 8$, Theorem 9 implies $x_1x_8 \in E(H_0)$. Then the edges x_1x_7, x_1x_8, x_7x_8, x_8x_9 and x_1x_9 determine a diamond in H_0. If some of the edges x_1x_7, x_1x_8 is subdivided in H, say, x_1x_7 with a vertex $x_{17} \in V_2(H)$, then H contains $S_{1,1,8}(x_1; y_1; x_{17}; x_2x_3x_4x_5x_6x_7x_8x_9)$, a contradiction. Hence $x_1x_7 \in E(H)$, and, similarly, $x_1x_8 \in E(H)$. If some of the edges x_7x_8, x_8x_0, x_1x_9 is subdivided in H, say, x_8x_9 with a vertex $x_{89} \in V_2(H)$, then H contains $S_{1,1,8}(x_1; y_1; x_9; x_2x_3x_4x_5x_6x_7x_8x_{89})$. Hence $x_8x_9 \in E(H)$, and, similarly, $x_7x_8 \in E(H)$ and $x_1x_9 \in E(H)$. But then the edges x_1x_7, x_1x_8, x_7x_8, x_8x_9 and x_1x_9 determine a diamond also in H, a contradiction.

Subcase 1.2.2.2: $N_C(y_1) = \{x_1, x_4, x_7\}$.

Recall that, by Claim 4(ii), $\{x_2, x_5\} \subset N_1$. By Theorem 9 for the set $A = V(C) \cup \{y_1\} \setminus \{x_2, x_5\}$ with $|A| = 8$, we have $x_2x_5 \in E(H_0)$, but then the cycle $C' = x_1y_1x_4x_3x_2x_5x_6x_7x_8x_9x_1$ is longer than C, a contradiction.

Subcase 1.2.2.3: $N_C(y_1) = \{x_1, x_3, x_6\}$.

Recall that, by Claim 4(iii), $\{x_4, x_8\} \subset N_1$. Theorem 9 for the set $A = V(C) \cup \{y_1\} \setminus \{x_4, x_8\}$ with $|A| = 8$ then implies $x_4x_8 \in E(H_0)$. We observe that, moreover, $x_4 \in N$, since if $x_1y_1 \in E(H_0)$, then the cycle $C' = x_1x_2x_3y_1x_4x_5x_6x_7x_8x_9x_1$ is longer than C, a contradiction.

Then, if $N_{H_0}(y_1) = N_H(y_1)$, the set $A = V(C) \setminus \{x_4\}$ dominates all edges of H and $G = L(H)$ is Hamilton-connected by Theorem 9; hence y_1 is adjacent to some vertex $y_2 \in R \setminus R_0$. If $x_2 \in N$, then the cycle $C' = x_1y_1x_3x_4x_5x_6x_7x_8x_9x_1$ dominates more edges than C, contradicting the choice of C. Hence there is a vertex $x_2' \in N_R(x_2)$. But then H contains $S_{1,1,8}(x_6; x_5; x_7; y_1x_1x_9x_8x_4x_3x_2x_2')$, a contradiction.

Case 2: $c(H_0) = |V(H_0)| = 10$.

Since $\delta(H_0) \geq 3$, every vertex of C is in a chord.

Subcase 2.1: *C has a 1-chord.*

Let $x_1x_3 \in E(H_0)$. We observe that no edge of C except possibly x_1x_2 and x_2x_3 is subdivided in H, for if e.g. x_3x_4 is subdivided with a vertex $x_{34} \in V_2(H)$, then H contains $S_{1,1,8}(x_3; x_1; x_2; x_{34}x_4x_5x_6x_7x_8x_9x_{10})$, a contradiction. If there is an $x_{10}' \in N_R(x_{10})$, then H contains $S_{1,1,8}(x_3; x_1; x_2; x_4x_5x_6x_7x_8x_9x_{10}x_{10}')$, and if there is an $x_9' \in N_R(x_9)$, then H contains $S_{1,1,8}(x_1; x_{10}; x_2; x_3x_4x_5x_6x_7x_8x_9x_9')$, a contradiction. Hence $\{x_9, x_{10}\} \subset N$, and, symmetrically, $\{x_5, x_6\} \subset N$. Considering the set $A = V(C) \setminus \{x_6, x_9\}$ with $|A| = 8$, Theorem 9 implies $x_6x_9 \in E(H)$. Similarly, by Theorem 9, for the set $A = V(C) \setminus \{x_6, x_{10}\}$ we have $x_6x_{10} \in E(H)$, and for the set $A = V(C) \setminus \{x_5, x_{10}\}$ we have $x_5x_{10} \in E(H)$ (recall that none of these chords of C is subdivided in H by Claim 2). But then x_5, x_6, x_9 and x_{10} are vertices of a diamond in H, a contradiction.

Subcase 2.2: *C has a 3-chord.*

Let $x_1x_5 \in E(H_0)$. Since $\delta(H_0) \geq 3$ and $R_0 = \emptyset$, x_3 must be in a chord.

Subcase 2.2.1: *x_3 is in a 2-chord.*

By symmetry, let $x_3x_6 \in E(H_0)$. Then $\{x_5, x_7\} \subset N$, since if $x_5' \in N_R(x_5)$, then H contains $S_{1,1,8}(x_3; x_2; x_4; x_6x_7x_8x_9x_{10}x_1x_5x_5')$, and if $x_7' \in N_R(x_7)$, then H contains $S_{1,1,8}(x_3; x_2; x_4; x_6x_5x_1x_{10}x_9x_8x_7x_7')$. If $x_5x_7 \notin E(H)$, then the set $A = V(C) \setminus \{x_5, x_7\}$ dominates all edges of H and $G = L(H)$ is Hamilton-connected by Theorem 9, a contradiction. Hence $x_5x_7 \in E(H)$, and we are back in Subcase 2.1 (recall that, throughout the proof, we implicitly use Claim 2, i.e., the fact that for $|i - j| > 1$, $x_ix_j \in E(H)$ if and only if $x_ix_j \in E(H_0)$).

Subcase 2.2.2: *x_3 is in a 3-chord.*

Let $x_3x_7 \in E(H_0)$. Then, similarly as above, we have $x_6 \in N$ (otherwise H contains $S_{1,1,8}(x_3; x_2; x_4; x_7x_8x_9x_{10}x_1x_5x_6x_6')$), and also $x_8 \in N$ (otherwise H contains $S_{1,1,8}(x_3; x_2; x_4; x_7x_6x_5x_1x_{10}x_9x_8x_8')$). Theorem 9 for $A = V(C) \setminus \{x_6, x_8\}$ then implies $x_6x_8 \in E(H_0)$, and we are back in Subcase 2.1.

Subcase 2.2.3: *x_3 is in a 4-chord.*

Then $x_3x_8 \in E(H_0)$, and considering $S_{1,1,8}(x_3; x_2; x_4; x_8x_9x_{10}x_1x_5x_6x_7x_7')$ for an $x_7' \in N_R(x_7)$ and $S_{1,1,8}(x_3; x_2; x_4; x_8x_7x_6x_5x_1x_{10}x_9x_9')$ for an $x_9' \in N_R(x_9)$, we have $\{x_7, x_9\} \subset N$. Theorem 9 for $A = V(C) \setminus \{x_7, x_9\}$ then implies $x_7x_9 \in E(H_0)$, and we are back in Subcase 2.1.

Subcase 2.3: *C has only 4-chords.*

If some edge of C is subdivided in H, say, $x_1' \in V_2(H)$ with $N_H(x_1') = \{x_1, x_2\}$, then H contains $S_{1,1,8}(x_1; x_1'; x_{10}; x_6x_5x_4x_3x_2x_7x_8x_9)$, a contradiction. If some vertex of C is incident to a pendant edge, say, $x_1x_1' \in E(H)$ with $x_1' \in V_1(H)$, then H contains $S_{1,1,8}(x_1; x_1'; x_{10}; x_2x_3x_4x_5x_6x_7x_8x_9)$. By Proposition 7$(iii)(\beta)$, the vertex x_1 is in a triangle, but it is impossible to create a triangle using only edges of C and 4-chords. Thus, $V(C) = N$, i.e., $R = \emptyset$. By the assumption of the subcase, say, $x_1x_3 \notin E(H)$, implying that the set $A = V(C) \setminus \{x_1, x_3\}$ with $|A| = 8$ dominates all edges of H. Thus, $G = L(H)$ is Hamilton-connected by Theorem 9, a contradiction.

Subcase 2.4: *C has only 2-chords and 4-chords, and at least one 2-chord.*

Let T be a triangle in H_0. Then $V(T) \subset V(C) = V(H_0)$, implying that each edge of T is an edge of C, a 2-chord of C or a 4-chord of C. However, a 2-chord spans 3 edges of C, and a 4-chord spans 5 edges of C, implying that the sum of distances of vertices of T along C is odd, contradicting the fact that $|V(C)| = 10$. Thus, H_0 is triangle-free, and since a triangle in H is also a triangle in H_0 by Lemma F, H is also triangle-free.

Now, if, say, x_1 is incident to a pendant edge $x_1x_1' \in E(H)$ with $x_1 \in V_1(H)$, then H contains $S_{1,1,8}(x_1; x_1'; x_{10}; x_2x_3x_4x_5x_6x_7x_8x_9)$, hence x_1 is in a triangle, contradicting the fact that H is triangle-free. By symmetry, there are no pendant edges in H.

By the assumption, C has a 2-chord, let thus $x_1x_4 \in E(H_0)$. Since $x_5x_7 \notin E(H_0)$ by Subcase 2.1, if $x_5, x_7 \in N$, then $G = L(H)$ is Hamilton-connected by Theorem 9 for the set $A = V(C) \setminus \{x_5, x_7\}$, a contradiction. Hence at most one of the vertices x_5, x_7 is in N, i.e., at least one of the edges x_4x_5, x_5x_6, x_6x_7, x_7x_8 is subdivided in H. Applying the same argument to the 1-chords x_6x_8, x_7x_9 and x_8x_{10}, and to the 3-chords x_5x_9 and x_6x_{10}, we conclude that among the edges x_5x_6, x_6x_7, x_7x_8 x_8x_9 and x_9x_{10}, at least two of them are subdivided in H. Then, if, say, x_6x_7 is subdivided with $x'_6 \in V_2(H)$ and x_8x_9 is subdivided with $x'_8 \in V_2(H)$, we have $S_{1,1,8}(x_4; x_1; x_3; x_5x_6x'_6x_7x_8x'_8x_9x_{10})$ (other cases are analogous).

Case 3: *$c(H_0) \geq 10$ and $|V(H_0)| > c(H_0)$.*

Set $c(H_0) = t$. Then H contains $S_{1,1,8}(x_1; y_1; x_t; x_2x_3x_4x_5x_6x_7x_8x_9)$ (note that the edge x_1y_1 is nonpendant since $y_1 \in R_0$).

Case 4: *$c(H_0) = |V(H_0)| = 11$.*

Since $\delta(H_0) \geq 3$, every vertex of C is in a chord.

Subcase 4.1: *C has a 1-chord.*

Let $x_1x_3 \in E(H_0)$. Then H contains $S_{1,1,8}(x_3; x_1; x_2; x_4x_5x_6x_7x_8x_9x_{10}x_{11})$, a contradiction.

Subcase 4.2: *C has a 3-chord.*

Let $x_1x_5 \in E(H_0)$. Since $\delta(H_0) \geq 3$, x_3 is in a chord.

By symmetry, there are the following possibilities.

Chord containing x_3	Contradiction
2-chord x_3x_6	$S_{1,1,8}(x_3; x_2; x_4; x_6x_5x_1x_{11}x_{10}x_9x_8x_7)$
3-chord x_3x_7	$S_{1,1,8}(x_3; x_2; x_4; x_7x_6x_5x_1x_{11}x_{10}x_9x_8)$
4-chord x_3x_8	$S_{1,1,8}(x_3; x_2; x_4; x_8x_7x_6x_5x_1x_{11}x_{10}x_9)$

Subcase 4.3: *C has a 2-chord.*

Let $x_1x_4 \in E(H_0)$. By the previous subcases, C has only 2-chords and 4-chords. We consider the possible chords containing x_2.

Subcase 4.3.1: *x_2 is in the 2-chord x_2x_{10}.*

We show that $\{x_4, x_6, x_8\} \subset N$.

Neighbor of x_i in R	Contradiction
$x'_4 \in N_R(x_4)$	$S_{1,1,8}(x_2; x_1; x_3; x_{10}x_9x_8x_7x_6x_5x_4x'_4)$
$x'_6 \in N_R(x_6)$	$S_{1,1,8}(x_4; x_3; x_5; x_1x_{11}x_{10}x_9x_8x_7x_6x'_6)$
$x'_8 \in N_R(x_8)$	$S_{1,1,8}(x_{10}; x_9; x_{11}; x_2x_3x_4x_5x_6x_7x_8x'_8)$

Thus, $\{x_4, x_6, x_8\} \subset N$. By the previous subcases, $\{x_4, x_6, x_8\}$ is an independent set. Then the set $A = V(C) \setminus \{x_4, x_6, x_8\}$ with $|A| = 8$ dominates all edges of H and $G = L(H)$ is Hamilton-connected by Theorem 9, a contradiction.

Subcase 4.3.2: *x_2 is in the 2-chord x_2x_5.*

Since $\delta(H_0) \geq 3$, x_3 is in a chord. If x_3 is in a 2-chord, we are in a situation symmetric to Subcase 4.3.1, which implies a contradiction. Thus, by Subcases 4.1 and 4.2, x_3 is in a 4-chord, and, by symmetry, we can suppose that $x_3x_8 \in E(H_0)$ (recall that we already have $x_1x_4, x_2x_5 \in E(H_0)$, hence the second case $x_3x_9 \in E(H_0)$ is symmetric). We show that $\{x_1, x_3, x_{10}\} \subset N$.

Neighbor of x_i in R	Contradiction
$x'_1 \in N_R(x_1)$	$S_{1,1,8}(x_5; x_4; x_6; x_2x_3x_8x_9x_{10}x_{11}x_1x'_1)$
$x'_3 \in N_R(x_3)$	$S_{1,1,8}(x_5; x_4; x_6; x_2x_1x_{11}x_{10}x_9x_8x_3x'_3)$
$x'_{10} \in N_R(x_{10})$	$S_{1,1,8}(x_1; x_{11}; x_2; x_4x_5x_6x_7x_8x_9x_{10}x'_{10})$

Thus, $\{x_1, x_3, x_{10}\} \subset N$. Since the set $\{x_4, x_6, x_8\}$ is independent by the previous subcases, the set $A = V(C) \setminus \{x_1, x_3, x_{10}\}$ with $|A| = 8$ dominates all edges of H and $G = L(H)$ is Hamilton-connected by Theorem 9, a contradiction.

Subcase 4.3.3: *x_2 is in the 4-chord x_2x_7.*

Then $\{x_3, x_{11}\} \subset N$, since if there is a vertex $x'_3 \in N_R(x_3)$, then H contains the subgraph $S_{1,1,8}(x_7; x_6; x_2; x_8x_9x_{10}x_{11}x_1x_4x_3x'_3)$, and if there is an $x'_{11} \in N_R(x_{11})$, then H contains $S_{1,1,8}(x_4; x_1; x_3; x_5x_6x_7x_8x_9x_{10}x_{11}x'_{11})$.

We consider the set $A = V(C)\setminus\{x_3, x_6, x_{11}\}$. We have $x_3x_6 \notin E(H_0)$ and $x_3x_{11} \notin E(H_0)$ by the previous subcases. If A is independent, then $G = L(H)$ is Hamilton-connected by Theorem 9, a contradiction. Hence necessarily $x_6x_{11} \in E(H_0)$, and then H contains $S_{1,1,8}(x_2; x_1; x_3; x_7x_8x_9x_{10}x_{11}x_6x_5x_4)$.

Subcase 4.3.4: *x_2 is in the 4-chord x_2x_8.*

Since this is the only remaining subcase, by symmetry, x_3 is in the 4-chord x_3x_8. Then H contains $S_{1,1,8}(x_8; x_2; x_3; x_9x_{10}x_{11}x_1x_4x_5x_6x_7)$.

Subcase 4.4: *C has only 4-chords.*

By parity, some vertex of C is in two 4-chords. Choose the notation such that $x_1x_6, x_1x_7 \in E(H_0)$. The possible 4-chords containing x_2 are x_2x_7 and x_2x_8. However, if $x_2x_7 \in E(H_0)$, then the edges x_1x_2, x_6x_7, x_1x_6, x_1x_7 and x_2x_7 determine a diamond in H_0. If, say, x_1x_2 is subdivided in H with a vertex x'_1, then H contains $S_{1,1,8}(x_2; x'_1; x_7; x_3x_4x_5x_6x_1x_{11}x_{10}x_9)$. Hence $x_1x_2 \in E(H)$, and, symmetrically, $x_6x_7 \in E(H)$. Since also $x_1x_6, x_1x_7, x_2x_7 \in E(H)$ by Claim 2, the chord x_2x_7 implies a diamond in H, a contradiction. Thus, $x_2x_8 \in E(H_0)$. Then the possible 4-chords containing x_{10} are x_4x_{10} and x_5x_{10}, however, if $x_4x_{10} \in E(H_0)$, then H contains $S_{1,1,8}(x_{10}; x_9; x_{11}; x_4x_5x_6x_1x_7x_8x_2x_3)$, and if $x_5x_{10} \in E(H_0)$, then H contains $S_{1,1,8}(x_{10}; x_9; x_{11}; x_5x_6x_1x_7x_8x_2x_3x_4)$.

Case 5: $c(H_0) = |V(H_0)| = 12$.

Since $\delta(H_0) \geq 3$, every vertex of C is in a chord. If $x_1x_3 \in E(H_0)$, then H contains the subgraph $S_{1,1,8}(x_3; x_1; x_2; x_4x_5x_6x_7x_8x_9x_{10}x_{11})$, and if $x_1x_4 \in E(H_0)$, then H contains $S_{1,1,8}(x_4; x_1; x_3; x_5x_6x_7x_8x_9x_{10}x_{11}x_{12})$. By symmetry, C has no 1-chords and no 2-chords.

Subcase 5.1: *C has a 3-chord.*

Let $x_1x_5 \in E(H_0)$. We consider possible chords containing x_3. By symmetry, there are the following possibilities.

Chord containing x_3	Contradiction
3-chord x_3x_7	$S_{1,1,8}(x_3; x_2; x_4; x_7x_6x_5x_1x_{12}x_{11}x_{10}x_9)$
4-chord x_3x_8	$S_{1,1,8}(x_3; x_2; x_4; x_8x_7x_6x_5x_1x_{12}x_{11}x_{10})$
5-chord x_3x_9	$S_{1,1,8}(x_3; x_2; x_4; x_9x_8x_7x_6x_5x_1x_{12}x_{11})$

Subcase 5.2: *C has a 4-chord.*

Let $x_1x_6 \in E(H_0)$.Then, for a chord containing x_3, we have the following possibilities.

Chord containing x_3	Contradiction
4-chord x_3x_8	$S_{1,1,8}(x_3; x_2; x_4; x_8x_9x_{10}x_{11}x_{12}x_1x_6x_7)$
5-chord x_3x_9	$S_{1,1,8}(x_3; x_2; x_4; x_9x_{10}x_{11}x_{12}x_1x_6x_7x_8)$
4-chord x_3x_{10}	$S_{1,1,8}(x_3; x_2; x_4; x_{10}x_{11}x_{12}x_1x_6x_7x_8x_9)$

Subcase 5.3: *C has only 5-chords.*

Then H contains $S_{1,1,8}(x_1; x_2; x_{12}; x_7x_6x_5x_4x_3x_9x_{10}x_{11})$.

Case 6: $c(H_0) = |V(H_0)| = 13$.

Since $\delta(H_0) \geq 3$, every vertex of C is in a chord.

Subcase 6.1: *C has a k-chord for some k, $1 \leq k \leq 3$.*

By symmetry, we can suppose that $x_1x_{k+2} \in E(H_0)$, $1 \leq k \leq 3$. Then H contains the subgraph $S_{1,1,8}(x_1; x_2; x_{13}; x_{k+2}x_{k+3} \dots x_{k+9})$.

Subcase 6.2: *C has a 4-chord.*

Let $x_1x_6 \in E(H_0)$. By the previous subcases and by symmetry, possible chords containing x_{10} are $x_{10}x_2$ or $x_{10}x_3$, and then H contains $S_{1,1,8}(x_6; x_5; x_7; x_1x_{13}x_{12}x_{11}x_{10}x_2x_3x_4)$ if $x_{10}x_2 \in E(H_0)$, or $S_{1,1,8}(x_3; x_2; x_4; x_{10}x_{11}x_{12}x_{13}x_1x_6x_7x_8)$ if $x_{10}x_3 \in E(H_0)$.

Subcase 6.3: *C has only 5-chords.*

Let $x_1x_7 \in E(H_0)$. By symmetry, we have $x_4x_{10} \in E(H_0)$, and then H contains $S_{1,1,8}(x_4; x_3; x_5; x_{10}x_{11}x_{12}x_{13}x_1x_7x_8x_9)$.

Case 7: $c(H_0) = |V(H_0)| = 14$.

Since $\delta(H_0) \geq 3$, every vertex of C is in a chord.

Subcase 7.1: *C has a k-chord for some k, $1 \leq k \leq 4$.*

By symmetry, we can suppose that $x_1x_{k+2} \in E(H_0)$, $1 \leq k \leq 4$. Then H contains the subgraph $S_{1,1,8}(x_1; x_2; x_{14}; x_{k+2}x_{k+3} \dots x_{k+9})$.

Subcase 7.2: *C has a 5-chord.*

Let $x_1x_7 \in E(H_0)$. The vertex x_4 is in a chord and, by the previous subcases and by symmetry, $x_4x_{10} \in E(H_0)$ or $x_4x_{11} \in E(H_0)$. However, in the first case H contains the subgraph $S_{1,1,8}(x_4; x_3; x_5; x_{10}x_{11}x_{12}x_{13}x_{14}x_1x_7x_8)$, and in the second case H contains $S_{1,1,8}(x_4; x_3; x_5; x_{11}x_{12}x_{13}x_{14}x_1x_7x_8x_9)$.

Subcase 7.3: *C has only 6-chords.*

Then H contains $S_{1,1,8}(x_1; x_2; x_{14}; x_8x_7x_6x_5x_4x_3x_{10}x_{11})$.

Case 8: $c(H_0) = |V(H_0)| = 15$.

Since $\delta(H_0) \geq 3$, every vertex of C is in a chord.

Subcase 8.1: *C has a k-chord for some k, $1 \le k \le 5$.*

By symmetry, we can suppose that $x_1x_{k+2} \in E(H_0)$, $1 \le k \le 5$. Then H contains the subgraph $S_{1,1,8}(x_1; x_2; x_{15}; x_{k+2}x_{k+3}\dots x_{k+9})$.

Subcase 8.2: *C has only 6-chords.*

Let $x_1x_8 \in E(H_0)$. Up to a symmetry, the only possibility for a 6-chord containing x_{12} is x_5x_{12}, and then H contains $S_{1,1,8}(x_1; x_8; x_{15}; x_2x_3x_4x_5x_{12}x_{11}x_{10}x_9)$.

Case 9: *$c(H_0) = |V(H_0)| = 16$.*

Subcase 9.1: *C has a k-chord for some k, $1 \le k \le 6$.*

By symmetry, we can suppose that $x_1x_{k+2} \in E(H_0)$, $1 \le k \le 6$. Then H contains the subgraph $S_{1,1,8}(x_1; x_2; x_{16}; x_{k+2}x_{k+3}\dots x_{k+9})$.

Subcase 9.2: *C has only 7-chords.*

Then H contains $S_{1,1,8}(x_1; x_2; x_{16}; x_9x_8x_7x_6x_5x_4x_3x_{11})$.

Case 10: *$c(H_0) = |V(H_0)| \ge 17$.*

Set $c(H_0) = t$. By symmetry, we can choose the notation such that $x_1x_i \in E(H_0)$ for some i, $3 \le i \le \lfloor \frac{t}{2} \rfloor + 1$, and then H contains $S_{1,1,8}(x_1; x_2; x_i; x_t; x_{t-1}x_{t-2}x_{t-3}x_{t-4}x_{t-5}x_{t-6}x_{t-7})$. ■

5 Concluding remarks

1. Throughout the proof of Theorem 1, whenever we reached a contradiction by finding in H a subgraph $F \simeq S_{1,1,8}$, we always (often implicitly) used the fact that F does not satisfy the conditions of Proposition 7, or, equivalently, that $G = L(H)$ fails to satisfy the conditions of the class $\mathcal{Z}_7$. This means that we have in fact proved the following slightly stronger result.

Theorem 11. *Let G be a 3-connected claw-free graph such that $G \not\simeq L(W^1)$ and every induced subgraph $F \simeq Z_7$ in G satisfies the following conditions:*
(i) $|V_2(T_F) \cap V_{SI}(G)| \ge 1$,
(ii) there is a vertex $x_F \in V_{EL}(G)$ such that $V(T_F) \subset N_G(x_F)$ and $\langle V(F)\rangle_{G^*_{x_F}} \not\simeq Z_7$.
Then G is Hamilton-connected.

2. Similarly as the main results of [14], [15] and [21], Theorem 1 admits another slight extension. For $s \ge 0$, a graph G is *s-Hamilton-connected* if the graph $G - M$ is Hamilton-connected for any set $M \subset V(G)$ with $|M| \le s$. Obviously, an s-Hamilton-connected graph must be $(s+3)$-connected. Since an induced subgraph of a $\{K_{1,3}, Z_7\}$-free graph is also $\{K_{1,3}, Z_7\}$-free, we immediately have the following fact, which extends Corollary 2 and shows that, in $\{K_{1,3}, Z_7\}$-free graphs, the obvious necessary condition is also sufficient.

Corollary 12. *Let $s \ge 0$ be an integer, and let G be a $\{K_{1,3}, Z_7\}$-free graph of order $n \ge s + 21$. Then G is s-Hamilton-connected if and only if G is $(s+3)$-connected.*

Note that it would be possible to replace the condition $n \geq s + 21$ with an assumption involving the exceptional graph; however, the resulting conditions would be, in our opinion, too technical and therefore not interesting. We leave details to the reader.

3. We can now update the discussion of potential pairs X, Y of connected graphs that might imply Hamilton-connectedness of a 3-connected $\{X, Y\}$-free graph, summarized in [15] and [21].

As shown in [7], up to a symmetry, necessarily $X = K_{1,3}$, and, summarizing the discussions from [4], [7], [9] and [15], there are the following possibilities for Y (see Fig. 1 for the graphs Z_i, $B_{i,j}$, $N_{i,j,k}$ and Γ_i):

(i) $Y \in \{\Gamma_1, \Gamma_3\}$, *or* $Y = \Gamma_5$ *for* $n = |V(G)| \geq 21$,

(ii) $Y = P_i$ *with* $4 \leq i \leq 9$,

(iii) $Y = Z_i$ *with* $i \leq 6$, *or* $Y = Z_7$ *for* $n = |V(G)| \geq 21$,

(iv) $Y = B_{i,j}$ *with* $i + j \leq 7$,

(v) $Y = N_{i,j,k}$ *with* $i + j + k \leq 7$.

Best known results in the direction of each of these subgraphs are summarized in Theorem A, and we summarize the current status of the problem in the following table.

The graph Y	Possible	Best known	Reference	Open
Γ_i	Γ_1, Γ_3, Γ_5 for $n \geq 21$	Γ_1	[7]	Γ_3; Γ_5 for $n \geq 21$
P_i	$4 \leq i \leq 9$	P_9	[4]	—
Z_i	$i \leq 7$	Z_7	This paper	—
$B_{i,j}$	$i + j \leq 7$	$i + j \leq 7$	[21]	—
$N_{i,j,k}$	$i + j + k \leq 7$	$i + j + k \leq 7$	[14, 15, 16]	—

Thus, the only remaining open cases are the pairs $\{K_{1,3}, \Gamma_3\}$ (for all graphs), and $\{K_{1,3}, \Gamma_5\}$ for $n \geq 21$ (or, possibly, for $G \not\simeq L(W^1)$).

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