On exclusive sum labellings of hypergraphs

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Abstract

The class $\mathcal{E}_k$ of hypergraphs with a k-exclusive sum labelling is hereditary, but non-trivial to characterise even for $k = 1$. Our main result is a complete description of the minimal forbidden induced subhypergraphs of $\mathcal{E}_1$ that are 3-uniform with maximum vertex degree 2. We also show that every hypertree has a 1-exclusive sum labelling and every combinatorial design does not.

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1 Introduction

A *sum graph* $G(V, E)$ is a graph with $V \subset \mathbb{N}$ such that $\{x, y\} \in E$ if and only if there exists a vertex z such that $x + y = z$, in which case we say z is a *working vertex*. In an *exclusive* sum graph the working vertices form a set of isolated vertices. A graph G admits a *k-exclusive sum labelling* (k-ESL) if $G + \bar{K}_k$ is isomorphic to an exclusive sum graph. The class of such graphs is hereditary and well understood; Miller, Ryan and Ryjáček gave a characterisation in the form of a universal graph for every k [4].

The above notions may be generalised to hypergraphs by replacing pairs of vertices by arbitrary sets in the definitions. Let $\mathcal{E}_k$ be the class of hypergraphs that have a k-ESL, which we define formally in the next section. In contrast to graphs (2-uniform hypergraphs), characterising $\mathcal{E}_k$ is non-trivial even for $k = 1$. In fact, the minimal forbidden induced subhypergraphs of $\mathcal{E}_1$ that are 3-uniform and have maximum degree 2 are a rich, infinite family of hypergraphs that we describe completely in our main result.

We also study 1-exclusive sum labellings in two special classes of hypergraphs. On the one hand, we show that all hypertrees have such a labelling by a geometric argument. On the other, we show that any r-uniform hypergraph on n vertices that happens to be a t-(n, r, λ) design, for some t, λ, does not have a 1-exclusive sum labelling.

1.1 Related Work

Sum graphs were introduced by Harary [2], and have been widely studied; we omit a list of results but direct the interested reader to the dynamic survey by Gallian [1]. Exclusive sum graphs were first studied by Miller et al. [3] and, besides [1], there is also a survey on the topic by Ryan [5]. As previously mentioned, the class of graphs (*i.e.* 2-graphs) with a k-ESL is hereditary; Miller, Ryan and Ryjáček gave a complete characterisation of the class in the form of a universal graph.

Teichert generalised the notion of sum graphs to hypergraphs [8]. Results on sum hypergraphs can also be found in [1]. The exclusive variant was first studied by Sonntag and Teichert [6], whose results contrasted significantly with those obtained for graphs.

1.2 Preliminaries

A hypergraph with no empty edges, loops or repeated edges is said to be *simple* and the hypergraphs in this paper are simple unless stated otherwise. We may *delete* a vertex v from a hypergraph H by removing it and removing all the edges that contain it to obtain a new hypergraph, denoted by $H - v$. A hypergraph that can be obtained from H by deleting vertices is called an *induced subhypergraph* of H. A class of hypergraphs that is closed under vertex deletion is said to be *hereditary*. It is well known and easy to see that a hereditary class X of hypergraphs may be characterised by the set M of its minimal forbidden induced subhypergraphs. In such a case we use the notation $X = \mathrm{Free}(M)$ and $M = \mathrm{Forb}(X)$.

The *dual* $\mathcal{D}(H)$ of a hypergraph H is a hypergraph with $V(\mathcal{D}(H)) = E(H)$ and an edge $\eta_v = \{e \in E(H) : v \in e\}$ for each $v \in V(H)$. Note that the dual of a simple hypergraph is not necessarily simple and that $\mathcal{D}(\mathcal{D}(H)) = H$.

A *path* of *length* j in a hypergraph is an ordered list of distinct, alternating vertices and edges $(v_0, e_0, \ldots, e_{j-1}, v_j)$ such that, for $0 \leq i \leq j-1$, v_i and v_{i+1} are elements of e_i. We say that the path is *from* v_0 *to* v_j. The *distance* between two vertices u and u' is the shortest length of a path from u to u'. The distance between two edges e and e' in a hypergraph H is the distance between the vertices e and e' in $\mathcal{D}(H)$. A *cycle* of length j is a path in which $v_0 = v_j$. By a *hypertree* we mean a connected hypergraph with no cycles (i.e., a Berge-acyclic hypergraph). Note that any pair of edges sharing more than one vertex form a cycle of length 2; thus, a pair of edges in a hypertree share at most one vertex.

A *combinatorial design* is a (possibly non-simple) hypergraph with a certain highly symmetric structure. In particular, a t-(n, r, λ) *design* is an r-uniform hypergraph on n vertices such that every subset of vertices of size t is contained in exactly λ edges.

2 Exclusive sum hypergraphs

The definition of sum graphs can be generalised to hypergraphs in the following way. A (ρ, σ)-sum hypergraph $H(V, E)$ is a hypergraph with $V \subset \mathbb{N}$ such that if $S \subseteq V$ then $S \in E$ if and only if $\rho \leq |S| \leq \sigma$ and there exists a vertex z such that $\sum_{x \in S} x = z$, in which case we call z a working vertex as before. We say a (ρ, σ)-sum hypergraph is exclusive if all its working vertices are isolated, and a k-exclusive (ρ, σ)-sum labelling of a hypergraph H is an isomorphism from $H + \bar{K}_k$ to a (ρ, σ)-sum hypergraph.

It will be useful to state the definition of a k-exclusive (ρ, σ)-sum labelling in another way. Let H be a hypergraph with $\rho \leq |e| \leq \sigma$ for each edge e in $E(H)$ and let Y be a set of vertices disjoint from $V(H)$. Then a labelling $\phi : V(H) \cup Y \to \mathbb{N}^+$ is a k-exclusive (ρ, σ)-sum labelling of H if and only if $|Y| = k$ and ϕ satisfies the following properties:

(P1) $\phi(x) \neq \phi(y)$ for distinct vertices x, y in $V(H) \cup Y$.

(P2) $\sum_{x \in e} \phi(x)$ is in the image $\phi(Y)$ for each edge e in $E(H)$. We call $\phi(Y)$ the set of *isolate* values.

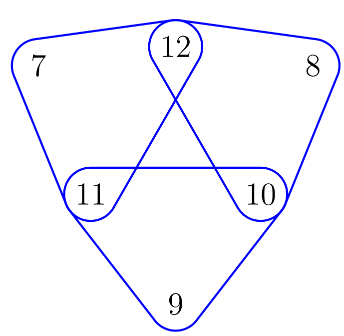

Figure 1: A 1-exclusive sum labelling of a hypergraph, with isolate value 30.

(P3) If $X \subseteq V(H) \cup Y$, $\rho \leq |X| \leq \sigma$, $X \notin E(H)$ then $\sum\limits_{x \in X} \phi(x) \neq \phi(v)$ for any vertex $v \in V(H)$.

(P4) If $X \subseteq V(H) \cup Y$, $\rho \leq |X| \leq \sigma$, $X \notin E(H)$ then $\sum\limits_{x \in X} \phi(x) \notin \phi(Y)$.

If $\rho = \sigma = r$ we simplify the terminology by referring to k-exclusive r-sum labellings; when $r = 2$ we recover the standard definition for graphs. If $\rho = 2, \sigma = \infty$, we refer simply to k-exclusive sum labellings.

Let $\mathcal{E}_k^{\rho,\sigma}, \mathcal{E}_k^r$ and $\mathcal{E}_k$ denote the classes of hypergraphs with, respectively, k-exclusive (σ, ρ)-sum, r-sum and sum labellings. It is easy to see that each of these classes is hereditary. Whereas $\mathcal{E}_k^2$ is well understood for all k, the situation is very different for general hypergraphs; the case $k = 1$ is already non-trivial.

Let $\mathcal{G}^{\rho,\sigma}$ denote the hypergraphs each edge of which have rank at least ρ and at most σ, and let $\mathcal{G}^r$ denote the r-uniform hypergraphs. We now make some observations about the relationships between the classes of hypergraphs we've defined.

Observation 1. *If $\rho' \leq \rho \leq \sigma \leq \sigma'$ then $\mathcal{E}_k^{\rho',\sigma'} \cap \mathcal{G}^{\rho,\sigma} \subseteq \mathcal{E}_k^{\rho,\sigma}$.*

Proposition 2. $\mathcal{E}_k^{\rho,\sigma} \subseteq \mathcal{E}_{(\sigma-\rho+1)k}$.

Proof. Let ϕ be a k-exclusive (ρ, σ)-sum labelling of a hypergraph H with isolate values $\Phi_1, \ldots, \Phi_k$. We may assume that $|V(H)| \geq \sigma$. Let ϕ' be the labelling of the vertices of H defined by $\phi'(x) = b\phi(x) + c$ where c and b are distinct primes strictly greater than $|V(H)|$. We claim that ϕ' is a $(\sigma-\rho+1)k$-exclusive sum labelling with isolate values given by $b\Phi_i+c(\rho+j)$ with $j = 0, \ldots, \sigma-\rho$.

For each edge e of H, we have $\sum_{x \in e} \phi'(x) = b \sum_{x \in e} \phi(x) + |e|c$. Clearly, $\sum_{x \in e} \phi(x) = \Phi_i$ for some i in $\{1, \ldots, k\}$ and $|e| = \rho+j$ for some j in $\{0, \ldots, \sigma-\rho\}$ as required. It is also clear that ϕ' satisfies (P1) and (P2).

Suppose that ϕ' does not satisfy (P3). Then there is some subset S of $V(H)$ such that $|S| > 1$ and $\sum_{x \in S} \phi'(x) = \phi'(y)$ for some vertex y in $V(H)$. Then:

$$b \sum_{x \in S} \phi(x) + c|S| = b\phi(y) + c$$
$$\sum_{x \in S} \phi(x) = \phi(y) + \frac{c}{b}(1 - |S|)$$

The left hand side is an integer by assumption. The right hand side is clearly not an integer unless $|S| = 1$, a contradiction.

Now suppose that ϕ' does not satisfy (P4). Then there is a subset S of $V(H)$ such that $S \notin E(H)$ and $\sum_{x \in S} \phi'(x) = b\Phi_i + c(\rho+j)$ for some i in $\{1, \ldots, k\}$ and j in $\{0, \ldots, \sigma-\rho\}$. Then:

$$b \sum_{x \in S} \phi(x) + c|S| = b\Phi_i + c(\rho + j)$$
$$\sum_{x \in S} \phi(x) = \Phi_i + \frac{c}{b}(\rho + j - |S|)$$

Again the left hand side is an integer. The right hand side is not an integer unless $|S| = \rho + j$. But then we have that $\sum_{x \in S} \phi(x) = \Phi_i$, which is a contradiction since $S \notin E(H)$. This completes the proof.

$\square$

Corollary 3. $\mathcal{E}_k^r = \mathcal{E}_k \cap \mathcal{G}^r$.

Proof. By Observation 1, $\mathcal{E}_k \cap \mathcal{G}^r \subseteq \mathcal{E}_k^r$. By Proposition 2, $\mathcal{E}_k^r \subseteq \mathcal{E}_k$. The result follows. $\square$

Since $\mathcal{E}_1$ is already not very well understood, the above corollary justifies our interest in the uniform case.

3 Minimal Forbidden Induced Subhypergraphs

We wish to characterise the minimal forbidden induced subhypergraphs of $\mathcal{E}_1$. We focus on 3-uniform hypergraphs with maximum degree 2. To simplify our discussion and our diagrams, we deal with the duals of such hypergraphs. If H is a 3-uniform hypergraph and $\Delta(H) \leq 2$ then the dual $\mathcal{D}(H)$ is simply a cubic graph, possibly with some edges of size 1 corresponding to vertices of degree 1. We will call an edge of size 1 a *half-edge*; where confusion may arise, we refer to an edge of size 2 as a *full* edge. It is convenient to consider labelling the edges of $\mathcal{D}(H)$ instead of the vertices of H. We define a *dual* 1-ESL of $\mathcal{D}(H)$ to correspond to a 1-ESL of H; in other words, $\phi : E(\mathcal{D}(H)) \to \mathbb{N}^+$ is a dual 1-ESL of $\mathcal{D}(H)$ if and only if the labelling ϕ' defined by $\phi'(v) = \phi(\eta_v)$ is a 1-ESL of H.

We must be careful to keep in mind that deleting a vertex v in H does not correspond directly to removing the corresponding edge in $\mathcal{D}(H)$. Rather, when we delete a vertex in H we remove its corresponding edge in $\mathcal{D}(H)$ and all the vertices it contains, possibly leaving behind some half-edges. Thus we will use the notation $\mathcal{D}(H) - v$ to indicate the hypergraph $\mathcal{D}(H - v)$.

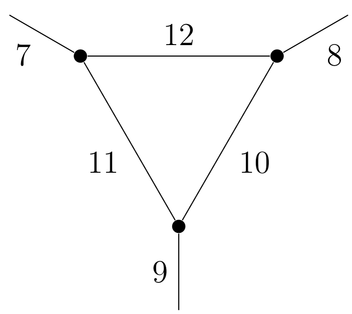

Figure 2: A dual 1-ESL of the dual of the hypergraph in Figure 1.

In order to state our main theorem, we introduce four types of cubic graphs equipped with a (not necessarily proper) 2-colouring. For a graph G with vertex set $B \cup R$, we give necessary and sufficient conditions for G to be of each type:

- Type 1: B and R are independent sets of equal size, G has exactly two half-edges
- Type 2: B and R have equal size, G has exactly one monochromatic edge of each colour and no half-edges
- Type 3: B and R are independent sets, $|B| = |R| + 1$, G has exactly three half-edges all incident to blue vertices
- Type 4: $|B| = |R| + 2$, G has exactly three monochromatic blue edges and no half-edges

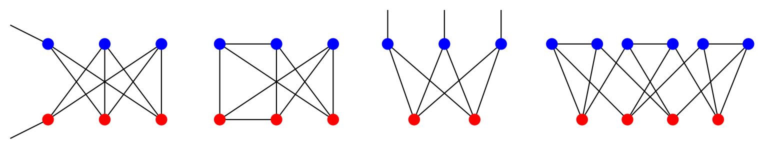

Figure 3: From left to right: an example of a graph of Type 1-4.

We are now ready to state our main theorem.

Theorem 4. *A 3-uniform hypergraph of maximum vertex degree 2 has a 1-ESL if and only if none of its induced subhypergraphs have a dual of Type 1-4.*

The following proposition gives one direction of the main theorem.

Proposition 5. *A graph of Type 1-4 does not have a dual 1-ESL.*

Proof. Let G be a graph of Type 1-4. Suppose ϕ is a dual 1-ESL of G with isolate value K. For a vertex v in G, let E_v be the set of edges incident with v. Then we have the following:

$$(|B| - |R|)K = \sum_{u \in B} \sum_{e \in E_u} \phi(e) - \sum_{u \in R} \sum_{e \in E_u} \phi(e) \tag{1}$$

In the first two types, the left hand side of this equation is 0 and on the right hand side, the labels of bichromatic full edges cancel out, leaving a red and a blue (half) edge that must have the same label. In the third type, the left hand side is K and the right hand side is the sum of the three half edges, giving a contradiction as they are not incident to the same vertex. In the fourth type, the left hand side is $2K$ and the right hand side is twice the sum of the monochromatic edges, again a contradiction. $\square$

We draw the reader's attention to an important set of hypergraphs. We denote by M the set of 3-uniform hypergraphs of vertex degree at most 2 that do not have a 1-ESL but all of whose induced subgraphs have a 1-ESL. Let $\mathcal{D}(M)$ be the set of duals of hypergraphs in M. It follows from Proposition 5 that every graph of Type 1-4 is the dual of a hypergraph with an induced subhypergraph in M. The rest of the section is devoted to proving that every graph in $\mathcal{D}(M)$ is itself of Type 1-4, which gives the theorem. In order to do this, we define a *potential labelling* λ to be an assignment of positive integers to the edges of a graph G such

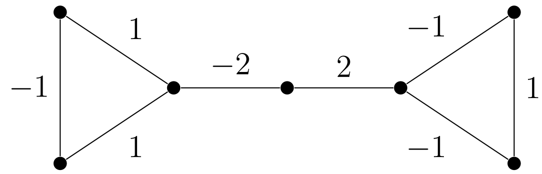

Figure 4: A barbell; note that the colours sum to 0 at vertices.

that the function $f(v) = \sum_{e \in E_v} \lambda(e)$ is a constant K. If $\lambda(x) = \lambda(y)$ we say that x, y is a *bad pair* for λ and if $\lambda(x) + \lambda(y) + \lambda(z) = K$ for x, y, z not incident with the same vertex, we say that x, y, z is a *bad triple*. Collectively, we refer to bad pairs and triples as *bad sets*. The number of bad sets for λ will be denoted by $\eta(\lambda)$. Clearly, λ is a dual 1-ESL of G if and only if $\eta(\lambda) = 0$. Observe that, by the proof of Proposition 5, a graph G of Type 1-4 has a set X of edges such that, for any potential labelling λ of G, X is a bad set. We call such a set of edges a *persistent* bad set.

Let x, y be a bad pair for a potential labelling λ. We observe that if x is in an even cycle C that does not contain y, we can find a potential labelling λ' that is strictly better than λ in the sense that $\eta(\lambda') < \eta(\lambda)$. We obtain this by giving the edges of C a proper 2-colouring with colours from $\{1, -1\}$ and setting:

$$\lambda'(z) = \begin{cases} 10\lambda(z) + 1 & \text{if } z \text{ has colour } 1 \\ 10\lambda(z) - 1 & \text{if } z \text{ has colour } -1 \\ 10\lambda(z) & \text{otherwise} \end{cases}$$

It is clear that λ' is a potential labelling, and that $\lambda'(x) \neq \lambda'(y)$. Furthermore, if $\lambda'(a) = \lambda'(b)$ for some a, b distinct from x, y, then it is easy to see that $\lambda(a) = \lambda(b)$. We call C an *improving* cycle. We can do the same operation with a path between two half-edges.

There are two other types of improving subgraphs. To describe them, let us define a *barbell* to be a pair of distinct odd cycles with a path between them. To obtain an improved potential labelling with a barbell, we give it a colouring. The edges on the path are given a proper 2-colouring with colours in $\{2, -2\}$. The edges of the cycles are coloured with colours in $\{1, -1\}$ according to their distance from the nearest edge of the path. If the nearest edge of the path has colour 2 (respectively, -2), the edges at even distance from it have colour 1 (respectively, -1) and those at odd distance have colour -1 (respectively, 1). See Figure 4 for an example of this colouring.

Now if x, y are a bad pair for λ and are on a barbell having different colours, then we can improve λ by setting:

$$\lambda'(z) = \begin{cases} 10\lambda(z) + 1 & \text{if } z \text{ has colour } 1 \\ 10\lambda(z) - 1 & \text{if } z \text{ has colour } -1 \\ 10\lambda(z) + 2 & \text{if } z \text{ has colour } 2 \\ 10\lambda(z) - 2 & \text{if } z \text{ has colour } -2 \\ 10\lambda(z) & \text{otherwise} \end{cases}$$

Again it is clear that λ' is a potential labelling that is strictly better than λ. We can do the same operation with an odd cycle and a half-edge with a path between them; we call such a subgraph a *half-barbell*. We also observe that if x, y, z are a bad triple for λ, and if some subset of x, y, z are in an improving subgraph such that their colours do not add up to zero, then we have $\lambda'(x) + \lambda'(y) + \lambda'(z) \neq 10K$, and therefore x, y, z are not a bad triple for λ'. We summarise the above discussion in the following lemma.

Lemma 6. *Let G be a graph with a set of edges S that induce either:*

1. *an even cycle*
2. *a path between two half-edges*
3. *a barbell*
4. *a half-barbell*

and let S be coloured as described above. Let x, y, z be edges of G. If x is in S but y isn't, or if x and y are in S with different colours, then x, y is not a persistent bad pair. Furthermore if some subset of x, y, z are in S but their respective colours do not add up to 0, then x, y, z is not a persistent bad triple.

We now prove several lemmas that restrict the behaviour of persistent bad sets in graphs in $\mathcal{D}(M)$. The following simple observation will be useful.

Observation 7. *If G is a simple graph with degree sequence $2, 3, 3, 3, \ldots$ then G has an odd cycle.*

Proof. Suppose there is a proper 2-colouring B, R of G. The sum of the degrees of B must be equal to the sum of the degrees of R. But (without loss of generality) the sum of the degrees of B is 2 mod 3, and the sum of the degrees of R is 0 mod 3, giving a contradiction. □

Lemma 8. *Let G be in $\mathcal{D}(M)$. Then G is connected.*

Proof. Suppose G is disconnected and let X be a persistent bad set in G. If there is some component G_X of G containing all members of X then G_X has no dual 1-ESL which contradicts the definition of M. Let $x, y \in X$ and let G_x and G_y be the components of G containing x and y respectively. Without loss of generality we may assume that x is the only element of X in G_x.

If G_x has more than one half-edge, there is a path between them that includes x which is improving. If G_x has one half-edge, then by Observation 7 there is an odd cycle C in G_x and a path from the half-edge to C that includes x (either on the path or in C) will be improving. Thus G_x has no half-edges.

If x is a cut edge, then both sides of the cut have an odd cycle by Observation 7 and the path between them goes through x and forms an improving barbell.

If x is not a cut edge, then it must be on a cycle C, which must be odd. If G_x has a cut edge e, then there is an odd cycle C' on the side of the cut not containing x and then C, C' and the path between them form an improving barbell.

We have shown that G_x is a bridgeless cubic graph. By Petersen's Theorem, there is a perfect matching in G_x. But now we can improve any potential labelling ϕ of G. Let ϕ' be a potential labelling of G defined as follows $\phi'(e) = 10\phi(e)$ for each edge e not in G_x; $\phi'(e) = 10\phi(e) + 2$ for each matched edge in G_x $\phi'(e) = 10\phi(e) - 1$ for each unmatched edge in G_x. It is easy to see that ϕ' is a potential labelling and that X is not a bad set for ϕ' which is a contradiction. □

From now on we assume that a graph G in $\mathcal{D}(M)$ is connected.

Lemma 9. *Let G be in $\mathcal{D}(M)$ and let X be a persistent bad set. Then no two edges of X are adjacent.*

Proof. First, let $X = \{x, y\}$. If X is a pair of half-edges, the lemma follows immediately from Lemma 6. If X consists of one half-edge x and one full edge y, then observe that G can have no other half edge and must be bipartite, otherwise there is an improving subgraph. But this is a contradiction, as it is easy to see that there is no cubic bipartite graph with exactly one half-edge.

Suppose X is a pair of full edges x, y. Consider the case in which x is a cut edge. Let G' be the graph induced by those edges that are separated from y by x. By Observation 7, there is an odd cycle in G'. If y is in an even cycle, it is improving and we are done, but if y is in an odd cycle then there is clearly an improving barbell. Thus y is a cut-edge, and it is easy to see that there is an improving barbell by Observation 7. Since we have that x and (by symmetry) y are not cut-edges, they must be in a cycle C (because G is cubic). Clearly C must be odd. Let v be the vertex incident with x and y and let e be the third edge incident with v. If there is a path from e to C that does not include v, there is an improving even cycle. So there is no such cycle and it is easy to see that e is a cut-edge. Let G_{xy} be the component of $G - e$ containing x and y; let G' be the rest of G. There must be an odd cycle C in G' and therefore G_{xy} is bipartite. But now G_{xy} is of Type 1.

Now, let $X = \{x, y, z\}$. Clearly, x, y and z cannot all be incident with the same vertex. If x, y and z are half-edges and (say) x and y are adjacent, then clearly G is bipartite, or there is an improving half-barbell. But then $G - \{x, y\}$ is of Type 1.

Suppose X has two half-edges x, y and one full edge z. Consider the case in which x, y are incident with a vertex v, and let e be the third edge incident with v. Then if G has an odd cycle C, every path from C to v must go through z to avoid an improving half-barbell. In this case, let G_{xy} be the graph induced by the edges that are on paths from x or y to z

(including x, y and z as half-edges). Clearly G_{xy} is bipartite and has no half-edges except x, y (or there is an improving half-barbell or path respectively), but then $G_{xy} - \{x, y, e\}$ is of Type 3. We deduce that G is itself bipartite, and the same argument shows that z is not a cut-edge. Therefore G has no half-edge apart from x, y, since otherwise there is an improving path. We conclude that $G - \{x, y, e\}$ is of Type 1.

We now consider the case in which x and y are half-edges, z is a full edge, and x and z are incident with a vertex v. Let e be the third edge incident with v; clearly e is not a half-edge. Any path from x to y must avoid z otherwise it is improving and so z is a cut-edge. Let G_{xy} be the graph induced by the edges that are on paths from x or y to z (including x, y and z as half-edges). By Observation 7 there is an odd cycle in $G \setminus G_{xy}$ and so G_{xy} is bipartite, otherwise there is an improving barbell. But now $G_{xy} - \{x, z, e\}$ is Type 3.

In the remaining cases, x is a half-edge and y, z are full edges. In the first of these cases, x and y are incident with a vertex v; let e be the third edge incident with v. If y is in an even cycle without z then it is improving, but if y is in an odd cycle C without z then C together with x forms an improving half-barbell. Similarly, z cannot be in a cycle without y. Now consider the subgraph G' induced by the edges that are on paths from e to z that avoid y (including e and z as half-edges). If G' is non-empty, it must not have a half edge e' different from e and z, otherwise a path from e' to x is improving. Similarly, it cannot contain an odd cycle. But it is easy to see that G' is cubic, and therefore it is of Type 1. We conclude that G' is empty; that is, y separates e and z. It is easy to see that e is a cut edge; let G_0 be the side of the cut that does not include x. By Observation 7, G_0 must have an odd cycle or a half-edge, but then there is an improving subgraph. This completes the proof for this case.

Finally, we consider the case in which y and z are incident with a vertex v. Let e be the third edge incident with v. If y is in an even cycle without z then it is improving, but if y is in an odd cycle C without z there is a path from C to x that forms an improving half-barbell. This implies that e is a cut-edge. Let G_{yz} be the side of the cut containing y and z, and let G' be the other side. Suppose first that x is in G_{yz}. Then there cannot be another half-edge in G_{yz} or there is an improving path. If there is an odd cycle C there is a path from C to x that forms an improving half-barbell. But since G_{yz} is bipartite, $G_{yz} \cup \{e\}$ is of Type 1. So x must be in G'. As before, there cannot be another half-edge or an odd cycle in G'. But then $G' \cup \{e\}$ is of Type 1. $\square$

Lemma 10. *Let G be in $\mathcal{D}(M)$. If x, y are a persistent bad pair then either both x and y are full edges, or both x and y are half edges. Similarly, if x, y, z are a persistent bad triple, they are either all full edges or all half edges.*

Proof. Suppose that x, y and G are as stated in the first part of the lemma. Without loss of generality, suppose that x is a half edge and y is a full edge. We first assume that y is a cut edge. Let G_x be the component of $G \setminus y$ that contains x. If G_x has a half edge e different from x then there is a path from e to x which is improving. Additionally, if G_x has an odd cycle C then there is a path from C to x which forms an improving half barbell. Thus G_x is bipartite, and since $G_x \cup \{y\}$ has exactly two half edges it is of Type 1. We may therefore assume that y is not a cut edge, but then it must be in some cycle C. If C is even we are done, but if C is odd there is a path from C to x which forms an improving half barbell.

Now suppose that x, y, z and G are as stated in the second part of the lemma. There are two possibilities. Suppose first that, without loss of generality, x is a half edge and y and z are full edges. Assume that y is a cut edge, and let G_x and G_z be the components of $G \setminus y$ containing x and z respectively. If G_x and G_z are distinct, an identical argument to that above gives the lemma. If z is in an even cycle we are done, but if z is in an odd cycle C there is a path from C to x which forms an improving half barbell. So z is a cut edge. Let G' be the connected component of $G_x \cup \{y\}$ that contains x. We know that G' must also contain y or we can repeat the above argument again. If G' contains an odd cycle there is an improving half barbell, but if G' is bipartite then $G' \cup \{z\}$ is of Type 3. This contradiction shows that y is not a cut edge, and by symmetry neither is z. By an analogous argument, $\{y, z\}$ is not a cutset. Therefore y is in a cycle C that does not contain z. If C is even we are done, but if C is odd there is a path from C to x which forms an improving half barbell (even if the path goes through z).

The second possibility in this case is that, without loss of generality, x and y are half edges and z is a full edge. It is easy to see that the same argument as before shows that z is not a cut edge and must be in some cycle C. If C is even we are done, otherwise there is a path from C to x which forms an improving half barbell. □

Lemma 11. *If G is in $\mathcal{D}(M)$ and X is a persistent bad set in G, then X is not a cutset.*

Proof. If X is a set of half edges then there is nothing to prove. Suppose X is a cutset and let G_0 and G_1 be the two sides of the cut. If there is a half-edge on both sides of the cut then there is an improving path between them. Suppose G_0 has no half-edges. Then it must contain an odd cycle C, else $G_0 \cup X$ is of Type 1 or Type 3 which is a contradiction. If G_1 contains a half-edge y then there is a path from C to y which forms an improving half-barbell. So G_1 also has no half-edges, and contains an odd cycle C'. But there is a path from C to C' that forms an improving barbell. □

Corollary 12. *If x is an element of a persistent bad set X in a graph G in $\mathcal{D}(M)$ and x is not a half-edge, then x is in a cycle that does not contain any other element of X.*

Proof. This is true of any set of at least two edges that is not a cut set, the proof is by induction on the size of X. □

Lemma 13. *Let G be in $\mathcal{D}(M)$, and let X be a persistent bad set in G. There is no half-edge in G outside of X.*

Proof. In the case where X is a set of half-edges, the lemma obviously holds as there will be an improving path. Suppose X is a persistent bad set of full edges, and that e is a half-edge in G. Let x be an element of X. By Corollary 12, x is in a cycle C that does not contain any other element of X. If C is even then it is improving, but if C is odd there is a path from C to e that forms an improving half-barbell (even if the path contains some element of X, it is still improving). □

We now know that a graph G in $\mathcal{D}(M)$ either has a persistent bad pair of half-edges, a bad triple of half-edges, a bad pair of full edges or a bad triple of full edges, and that G has no half-edges outside of X. It remains to prove that these four cases correspond exactly to the four types of graphs given above.

Proposition 14. *Let G be in $\mathcal{D}(M)$ and let X be a persistent bad set of half-edges. Then G is of Type 1 (when $|X| = 2$) or of Type 3 (when $|X| = 3$).*

Proof. It is enough in both cases to show that the graph is bipartite. Let x be an element of X. Suppose there is an odd cycle C in G. Since G is connected there is a path from C to x. This is a half-barbell that clearly cannot include any element of X besides x and is therefore improving. By Lemma 6, X cannot be a persistent bad set which is a contradiction. Therefore there is no odd cycle in G and so G is in Type 1 (if $|X| = 2$) or Type 3 (if $|X| = 3$) as required. □

Proposition 15. *Let G be in $\mathcal{D}(M)$ and let x and y be a persistent bad pair of full edges. Then G is of Type 2.*

Proof. Let G, x, y be as stated and suppose G is not of Type 2. We first deal with the case in which G is 2-edge-connected. Then we claim that there is no path from x to y containing an even number of edges. Suppose to the contrary that there is such a path P. Let v be a vertex on P that splits P into two paths of equal parity b, and let z be the edge of v not on P. Since G is 2-edge-connected, there is a path from x to z, and a path from y to z, both of which are disjoint from P. In order to avoid an improving even cycle, both of these paths must have parity $\bar{b}$. It is easy to see that there must be a path P' from x to y with an even number of edges that is disjoint from P. But then $\{x, y\} \cup P \cup P'$ is an improving cycle which is a contradiction.

Now suppose G has a bridge z. By Lemma 11, z is not equal to x or y. There are two cases to consider. In the first case, there is a path from x to y that does not include z. Let G_z be the graph induced by edges that are separated from x and y by z, and let G' be the graph induced by the remaining edges. By Corollary 12, x is in a cycle C that does not contain y. Clearly C is odd. By Observation 7 there is an odd cycle C' in G_z, and since there is a path from C to C' the graph has an improving barbell.

In the final case there is a bridge z that separates x from y. Let G_x be the graph induced by edges that are separated from y by z, and let G_y be the graph induced by the edges that are separated from x by z. We give a 2-colouring to the vertices of G in the following way: the vertices incident with y are blue, the vertices incident with x are red; the vertices of G_y that are at odd distance from the vertices incident with y and the vertices of G_x at even distance from the vertices incident with x are red; all other vertices are blue. Observe that there cannot be a monochromatic edge in G_x other than x, otherwise x is in an even cycle without y. Similarly, y is the only monochromatic edge in G_y. It remains only to show that z is not monochromatic. Clearly y is in an odd cycle C_y and x is in an odd cycle C_x. There is a path from C_y to C_x that includes z, and this is a barbell. If z is monochromatic then x and y are at even distance in this barbell, and it is improving. This contradiction completes the proof. □

Proposition 16. *Let G be in $\mathcal{D}(M)$ and let x, y and z be a persistent bad triple of full edges. Then G is of Type 4.*

Proof. Let G, x, y and z be as stated and suppose G is not of Type 4. Suppose x, y and z are all on a cycle C. Clearly, C is an odd cycle, otherwise it is improving. By Lemma 11, $\{x, y\}$ is not a cutset, so C has a chord-path P that separates C into two arcs A_x and A_y, which contain x and y respectively. Without loss of generality, let z be an edge of A_y. Now, $A_x \cup P$ is an odd cycle or else it is improving. Since C is odd, we have that $A_y \cup P$ is an even cycle. This cycle is improving unless y, z are at odd distance in C. But $\{y, z\}$ is not a cutset (again by Lemma 11) so C has another chord-path P' which separates C into two arcs A'_y and A'_z, which contain y and z respectively. If x is in A'_y, then by the same reasoning as above, x and y are at odd distance in C. But then x and z must also be at odd distance in C, because C is an odd cycle. By symmetry we deduce that all three pairs are at odd distance in C; in other words, we may assume that in any cycle C containing x, y and z, each of the three paths obtained from C by removing x, y and z have an even number of edges.

Now suppose that there is a path P with an odd number of edges connecting x and y such that z is not in P. Since x, y is not a cutset, there must be another path P' such that $P \cup P' \cup \{x, y\}$ is a cycle, which we will denote by C. By the above argument, z is not on C. To avoid an improving cycle, we must have that P' has even length. Since x, y, z is not a cutset, there must be a path P'' from P to P' that does not go through x, y, z. This is a chord-path for C; let A_x and A_y be its arcs containing x and y respectively. Since C is an odd cycle, A_x and A_y have different parities. But then either $A_x \cup P''$ or $A_y \cup P''$ is an even cycle, which is a contradiction. Therefore any path connecting x to y is odd if and only if it contains z, and so on by symmetry.

To complete the proof, we give a colouring of the vertices of G according to the parity of their distances from the vertices incident with x, y or z. The vertices at even distance from the vertices incident with x, y or z are blue; the vertices at odd distance are red. It remains to show that there is no monochromatic edge other than x, y, z. Assume the contrary, and let uv be the closest monochromatic edge to x, y, z. Let P_u, P_v be the shortest paths connecting x, y, z to u and v respectively. These cannot be disjoint, as otherwise we have either an even cycle including exactly one of x, y, z or we have an odd path from (say) x to y that does not include z. Since they are shortest paths, they are properly 2-coloured by the colouring given above. We may therefore assume that the symmetric difference of P_u and P_v is an odd cycle C, and the intersection is a path with one end in C and one end in x. But x is an odd cycle C' that does not include y or z. So $C' \cup P_u \cup P_v$ is an improving barbell. This contradiction shows that G is in Type 4 as required. □

4 Additional results

4.1 Combinatorial designs

In this section we study (simple) hypergraphs that happen to be combinatorial designs. Recall that an r-uniform hypergraph on n vertices is a t-(n, r, λ) design if every subset of vertices of size t is a subset of exactly λ edges. Such a design is *non-trivial* if $t > 1$ and $n > r$. We will show that a non-trivial t-design does not have a 1-ESL. We make use of the following standard result.

Lemma 17. *([7]) Let H be a hypergraph and suppose that H is a t-(n, r, λ)-design. Suppose that $X \subseteq V(H)$, where $|X| = s \leq t$. Then there are exactly $\lambda_s = \lambda \frac{\binom{v-s}{t-s}}{\binom{k-s}{t-s}}$ edges in H that contain all the vertices of X.*

Theorem 18. *Let H be a simple hypergraph. Suppose H is a non-trivial t-(n, r, λ) design. Then H has no 1-ESL.*

Proof. Let $x_1, \ldots, x_n$ be the vertices of H and suppose that H has a 1-ESL ψ with isolate value K. Consider the set M of edges, that contain the vertices $x_1, \ldots, x_{t-1}$. By Lemma 17, we have that $|M| = \lambda_{t-1}$, and therefore:

$$\sum_{e \in M} \sum_{x \in e} \psi(x) = K\lambda_{t-1}$$

Since every subset of vertices of size t appears in λ edges, there are λ edges in the set M containing the vertex x_i for each i greater than $t-1$. So we can write:

$$K\lambda_{t-1} = \lambda_{t-1} \sum_{i=1}^{t-1} \psi(x_i) + \lambda \sum_{i=t}^{n} \psi(x_i)$$

Analogously, we consider the λ_{t-1} edges containing the vertices $x_2, x_3, \ldots, x_t$ and we obtain:

$$K\lambda_{t-1} = \lambda_{t-1} \sum_{i=2}^{t} \psi(x_i) + \lambda \sum_{i=t+1}^{n} \psi(x_i) + \lambda\psi(x_1)$$

The difference of the previous two equations gives $(\lambda - \lambda_{t-1})\psi(x_1) = (\lambda - \lambda_{t-1})\psi(x_t)$. Since $n \neq r$, and therefore $\lambda \neq \lambda_{t-1}$, we have that $\psi(x_1) = \psi(x_t)$, which is a contradiction. □

4.2 Hypertrees

Observation 19. *Let $H \in \mathcal{E}_1$. If a component of H has an edge of size 2 then it is isomorphic to K_2.*

Proof. Let ϕ be a 1-ESL of H and let $e = \{u, v\}$ be an edge in H. Suppose the statement does not hold. Then there is an edge e' that shares a vertex with e, say u. But then $\sum_{x \in e' \setminus u} \phi(x) = K - \phi(u) = \phi(v)$. If $|e'| = 2$ this contradicts (P1); otherwise, it contradicts (P3). □

Theorem 20. *Every hypertree in $\mathcal{G}^{3,\infty}$ has a 1-ESL.*

Proof. Let H be such a hypertree with n vertices and m edges. We define a vertex labelling scheme that assigns to each vertex a linear combination of variables $\{x_0, \ldots, x_{n-m}\}$ which are interpreted as variables over $\mathbb{Z}$. We call such a label *generic*. For clarity we surround the generic labels with angle brackets e.g. $\langle x_0 - x_1 - x_2 \rangle$.

Let the edges of H be denoted $e_1, \ldots, e_m$ according to a breadth-first search in the edge-intersection graph starting at an arbitrary edge e_1.

We start by labelling the vertices of e_1 with labels $\langle x_1 \rangle, \ldots, \langle x_{|e_1|-1} \rangle, \langle x_0 - (x_1 + \cdots + x_{|e_1|-1}) \rangle$. For each subsequent edge e_i, we continue in the following way. While there are multiple unlabelled vertices in e_i we choose one and give it a label $\langle x_{j+1} \rangle$ where x_j was the previous variable to appear in the list of labels. When there is exactly one unlabelled vertex u in e_i we give it the label $\langle x_0 - \sum_{v \in e_i \setminus \{u\}} f(v) \rangle$ where $f(v)$ denotes the label of the vertex v. Note that the sum of the labels in each edge is equal to x_0, and that there are indeed $n - m + 1$ distinct variables in the list of labels. Therefore each point $P = (x_0, \ldots, x_{n-m})$ in $\mathbb{Z}^{n-m+1}$ generates a potential labelling ϕ_P of H, with x_0 as the isolate value. We claim that we can choose P such that ϕ_P is a 1-ESL.

To ensure that $\phi_P(v)$ is positive for each vertex v, we simply choose each x_i so that $x_i >> \sum_{j>i} x_j$. This follows from the fact that in the generic label of each vertex, the variable with the smallest index appears positively, which is easy to verify.

We must also ensure that ϕ_P gives unique labels to satisfy (P1). Let the generic label of a vertex u be denoted $\langle \sum c_i^u x_i \rangle$. For each pair u, v, we need to pick P to avoid the hyperplane defined by:

$$\sum c_i^u x_i = \sum c_i^v x_i$$

To satisfy (P3), we must ensure that $\sum_{u \in S} \phi_P(u) \neq \phi_P(v)$ for any multiple subset of vertices S and any vertex v. This is equivalent to avoiding the hyperplane defined by:

$$\sum_{u \in S} \sum c_i^u x_i = \sum c_i^v x_i$$

Finally, to satisfy (P4), we must ensure that $\sum_{u \in S} \phi_P(u) \neq x_0$ whenever $S \notin E(H)$. This is also equivalent to avoiding a hyperplane, namely the one defined by:

$$\sum_{u \in S} \sum c_i^u x_i = x_0$$

It is clear that we can pick P to avoid this finite set of lower dimensional subspaces. This completes the proof. $\square$

References

[1] Gallian, J. A. A dynamic survey of graph labeling. *Electronic Journal of Combinatorics 1*, DynamicSurveys (2018), DS6.

[2] Harary, F. Sum graphs and difference graphs. *Congressus Numerantium 72* (1990), 101–108.

[3] Miller, M., Patel, D., Ryan, J., Sugeng, K., Slamin, and Tuga, M. Exclusive sum labeling of graphs. *JCMCC 55* (2005), 149–158.

[4] Miller, M., Ryan, J., and Ryjáček, Z. Characterisation of graphs with exclusive sum labelling. *Australasian Journal of Combinatorics 69*, 3 (2017), 334–348.

[5] Ryan, J. Exclusive sum labeling of graphs: A survey. *AKCE International Journal of Graphs and Combinatorics 6*, 1 (2009), 113–126.

[6] Sonntag, M., and Teichert, H.-M. Some results on exclusive sum labelings of hypergraphs. *Graphs and Combinatorics 31*, 6 (2015), 2401–2412.

[7] Stinson, D. *Combinatorial designs: constructions and analysis*. Springer Science & Business Media, 2007.

[8] Teichert, H.-M. The sum number of d-partite complete hypergraphs. *Discussiones Mathematicae Graph Theory 19*, 1 (1999), 79–91.