# Hamilton-connected {claw,net}-free graphs, I

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September 29, 2021

**Abstract**

This is the first one in a series of two papers, in which we complete the characterization of forbidden generalized nets implying Hamilton-connectedness of a 3-connected claw-free graph. In this paper, we first develop the necessary techniques that allow to handle the problem, namely:

($i$) we strengthen the closure concept for Hamilton-connectedness in claw-free graphs, introduced by the second and third authors, such that not only the line graph preimage of a closure, but also its core has certain strong structural properties,

($ii$) we prove a special version of the "nine-point-theorem" by Holton et al. that allows to handle Hamilton-connectedness of "small" $\{K_{1,3}, N_{i,j,k}\}$-free graphs (where $N_{i,j,k}$ is the graph obtained by attaching endvertices of three paths of lengths $i, j, k$ to a triangle),

($iii$) by combination of these techniques, as an application, we prove that every 3-connected $\{K_{1,3}, N_{1,3,3}\}$-free graph is Hamilton-connected.

The paper is followed by its second part in which we show that every 3-connected $\{K_{1,3}, X\}$-free graph, where $X \in \{N_{1,1,5}, N_{2,2,3}\}$, is Hamilton-connected. All the results on Hamilton-connectedness are sharp.

Keywords: Hamilton-connected; closure; forbidden subgraph; claw-free; net-free

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[7]Research supported by project GA20-09525S of the Czech Science Foundation

[8]Research supported by Natural Science Foundation of China (Nos. 11871099, 12131013)

[9]Research supported by Natural Science Foundation of China (No. 12101187)

# 1 Introduction

In this paper, by a *graph* we always mean a simple finite undirected graph; if we admit multiple edges, we always speak about a *multigraph*. We follow the most common graph-theoretical terminology and notation, and for notations and concepts not defined here we refer the reader e.g. to [6]. Specifically, we use $d_G(x)$ to denote the *degree* of a vertex $x$ in $G$, and for $i \geq 1$ we set $V_i(G) = \{x \in V(G)|\ d_G(x) = i\}$. If $x \in V_2(G)$ with $N_G(x) = \{y_1, y_2\}$, then the operation of replacing the path $y_1xy_2$ by the edge $y_1y_2$ is called *suppressing* the vertex $x$. The inverse operation is called *subdividing* the edge $y_1y_2$ with the vertex $x$.

We write $F \subset G$ if $F$ is a sub(multi)graph of $G$ (not excluding the possibility $F = G$), and $\langle M\rangle_G$ to denote the *induced sub(multi)graph* on a set $M \subset V(G)$. A vertex $x \in V(G)$ is *simplicial* if $\langle N_G(x)\rangle_G$ is a complete graph. For $F \subset G$, a vertex $x$ is said to be a *vertex of attachment of* $F$ *in* $G$ if $x \in V(F)$ and $N_G(x) \cap (V(G) \setminus V(F)) \neq \emptyset$. The set of all vertices of attachment of a sub(multi)graph $F$ in $G$ is denoted $A_G(F)$.

By a *closed trail* in $G$ we mean an eulerian subgraph of $G$, and a connected subgraph with exactly two vertices of odd degree is called a *trail* in $G$. Its vertices of odd degree are its *endvertices*, and (any) its edge incident to an endvertex is a *terminal edge* (note that these definitions are equivalent with those in [6]). A *subtrail* of a trail is a subgraph which itself is a trail. For $x, y \in V(G)$, a path (trail) with endvertices $x, y$ is referred to as an $(x, y)$-*path* ($(x, y)$-*trail*), a trail with terminal edges $e, f \in E(G)$ is called an $(e, f)$-*trail*, and $\mathrm{Int}(T)$ denotes the set of interior vertices of a trail $T$. A set of vertices $M \subset V(G)$ *dominates* an edge $e$, if $e$ has at least one vertex in $M$, and a subgraph $F \subset G$ dominates $e$ if $V(F)$ dominates $e$. A closed trail $T$ is a *dominating closed trail* (abbreviated DCT) if $T$ dominates all edges of $G$, and an $(e, f)$-trail is an *internally dominating* $(e, f)$-*trail* (abbreviated $(e, f)$-IDT) if $\mathrm{Int}(T)$ dominates all edges of $G$. A graph is *Hamilton-connected* if, for any $u, v \in V(G)$, $G$ has a hamiltonian $(u, v)$-path, i.e., an $(u, v)$-path $P$ with $V(P) = V(G)$.

Finally, for a family of graphs $\mathcal{F}$, a graph $G$ is said to be $\mathcal{F}$-*free* if $G$ does not contain an induced subgraph isomorphic to a member of $\mathcal{F}$; the graphs in $\mathcal{F}$ are referred to in this context as *forbidden (induced) subgraphs*. If $\mathcal{F} = \{F\}$, we simply say that $G$ is $F$-*free*. Here, the *claw* is the graph $K_{1,3}$, $P_i$ denotes the path on $i$ vertices, and $\Gamma_i$ denotes the graph obtained by joining two triangles with a path of length $i$. Several further graphs that will be often used as forbidden subgraphs are shown in Fig. 1 (note that the graph $N_{i,j,k}$ in Fig. 1($c$) is often referred to as the *generalized net*). Whenever we will later on list vertices of an $S_{i,j,k}$ in a graph, we will always write the list such that $i \leq j \leq k$, and we will use the notation $S_{i,j,k}(v; a_1a_2 \dots a_i; b_1b_2 \dots b_j; c_1c_2 \dots c_k)$ (in the labeling of vertices as in Fig. 1($d$)).

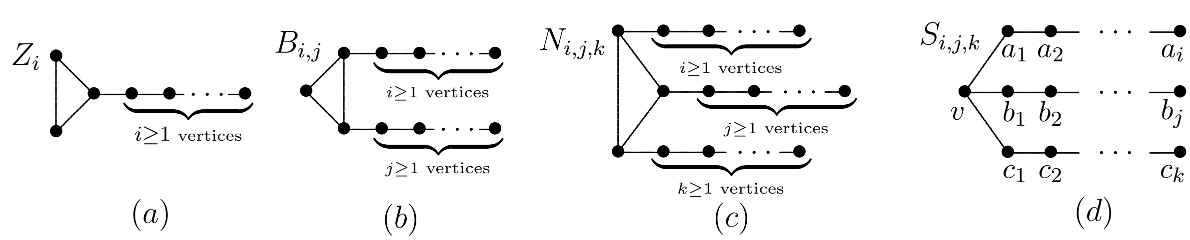


Figure 1: The graphs $Z_i$, $B_{i,j}$, $N_{i,j,k}$ and $S_{i,j,k}$

There are many results on forbidden induced subgraphs implying various Hamilton-type graph properties. While for hamiltonicity in 2-connected graphs (recall that 2-connectedness is the minimum connectivity level for the property), pairs of connected forbidden subgraphs are completely characterized [13], for Hamilton-connectedness in 3-connected graphs (where again, 3-connectedness is the minimum connectivity level for the property), the progress is relatively slow. Theorem A below reflects the history of consecutive improvements of sufficient conditions for Hamilton-connectedness in terms of pairs of connected forbidden subgraphs.

**Theorem A [29, 13, 11, 8, 12, 4, 17, 21].** *Let $G$ be a 3-connected $\{K_{1,3}, X\}$-free graph, where*

($i$) **[29]** $X = N_{1,1,1}$, *or*
($ii$) **[13]** $X = Z_2$, *or*
($iii$) **[11]** $X \in \{B_{1,2}, Z_3, P_6\}$, *or*
($iv$) **[8]** $X = \Gamma_1$, *or*
($v$) **[12]** $X \in \{N_{1,1,3}, N_{1,2,2}, P_8\}$, *or*
($vi$) **[4]** $X = P_9$, *or*
($vii$) **[17]** $X = N_{1,2,3}$, *or*
($viii$) **[21]** $X = N_{1,2,4}$.

*Then $G$ is Hamilton-connected.*

Note that Theorem A($viii$) immediately implies that every 3-connected $\{K_{1,3}, B_{2,4}\}$-free or $\{K_{1,3}, Z_4\}$-free graph is Hamilton-connected.

Let $\mathcal{W}$ be the family of graphs obtained by attaching at least one pendant edge to each of the vertices of the Wagner graph $W$ (see Fig. 2($b$)), and let $\mathcal{G} = \{L(H) | \ H \in \mathcal{W}\}$ be the family of their line graphs. Then any $G \in \mathcal{G}$ is 3-connected, non-Hamilton-connected, $P_{10}$-free and $N_{i,j,k}$-free for $i + j + k = 8$. Thus, this example shows that parts ($vi$) and ($viii$) of Theorem A are sharp, and, moreover, the largest generalized nets $N_{i,j,k}$ that might imply Hamilton-connectedness are those with $i + j + k = 7$. In view of part ($viii$) of Theorem A, we easily see that the only such remaining generalized nets are $N_{1,3,3}$, $N_{1,1,5}$ and $N_{2,2,3}$. In this series of two papers, we answer these questions in the affirmative. Specifically, the next result, which is one of the main results of this paper, is also sharp.

**Theorem 1.** *Every 3-connected $\{K_{1,3}, N_{1,3,3}\}$-free graph is Hamilton-connected.*

Theorem 1 immediately implies as a corollary that also every 3-connected $\{K_{1,3}, B_{3,3}\}$-free graph is Hamilton-connected.

The **proof** of Theorem 1, which is a careful case analysis, is postponed to Section 5. In Section 2, we collect necessary known results and facts, and in Sections 3 and 4, we develop techniques that allow to significantly reduce the number of cases to be considered.

In [20], the second one in this series of two papers, we will use the techniques developed in Sections 3 and 4 to prove an analogue of Theorem 1 for the remaining graphs $N_{i,j,k}$ with $i+j+k = 7$, namely, for $N_{1,1,5}$ and $N_{2,2,3}$. This will complete the characterization of generalized nets implying Hamilton-connectedness of a 3-connected claw-free graph. We will also include more details on sharpness and on remaining open cases.

# 2 Preliminaries

In this section, we summarize some known facts that will be needed in our proof of Theorem 1.

## 2.1 Line graphs of multigraphs and their preimages

The *line graph* of a multigraph $H$ is the graph $G = L(H)$ with $V(G) = E(H)$, in which two vertices are adjacent if and only if the corresponding edges of $H$ have at least one vertex in common. While in line graphs of graphs, for a line graph $G$, the graph $H$ such that $G = L(H)$ is uniquely determined with a single exception of $G = K_3$, in line graphs of multigraphs this is not true: a simple example are the graphs $H_1 = Z_1$ and $H_2$ a double edge with one pendant edge attached to each vertex – while $H_1 \not\simeq H_2$, we have $L(H_1) \simeq L(H_2) \simeq T_1$ (where $T_1$ is the diamond shown in Fig. 4). Using a modification of an approach from [31], the following was proved in [26].

**Theorem B [26].** *Let $G$ be a connected line graph of a multigraph. Then there is, up to an isomorphism, a uniquely determined multigraph $H$ such that a vertex $e \in V(G)$ is simplicial in $G$ if and only if the corresponding edge $e \in E(H)$ is a pendant edge in $H$.*

The multigraph $H$ with the properties given in Theorem B will be called the *preimage* of a line graph $G$ and denoted $H = L^{-1}(G)$. We will also use the notation $a = L(e)$ and $e = L^{-1}(a)$ for an edge $e \in E(H)$ and the corresponding vertex $a \in V(G)$.

An edge-cut $R \subset E(H)$ of a multigraph $H$ is *essential* if $H - R$ has at least two nontrivial components, and $H$ is *essentially $k$-edge-connected* if every essential edge-cut of $H$ is of size at least $k$. It is a well-known fact (see [28], Proposition 1.1.3), that a line graph $G$ is $k$-connected if and only if $L^{-1}(G)$ is essentially $k$-edge-connected. It is also a well-known fact that if $X$ is a line graph, then a line graph $G$ is $X$-free if and only if $L^{-1}(G)$ does not contain as a subgraph (not necessarily induced) a graph $F$ such that $L(F) = X$ (but not necessarily $F = L^{-1}(X)$). However, it is straightforward to verify that for the graph $N_{i,j,k}$ there is exactly one graph $F$ such that $L(F) = N_{i,j,k}$, namely, the graph $L^{-1}(N_{i,j,k}) = S_{i+1,j+1,k+1}$ (see Fig. 1($d$)). Thus, we can conclude that a line graph $G$ is $N_{i,j,k}$-free if and only if $L^{-1}(G)$ does not contain as a (not necessarily induced) subgraph the graph $L^{-1}(N_{i,j,k}) = S_{i+1,j+1,k+1}$.

Harary and Nash-Williams [15] established a correspondence between a DCT in $H$ and a hamiltonian cycle in $L(H)$ (the result was given in [15] for graphs, but it is easy to observe that it is true also for multigraphs). A similar result showing that $G = L(H)$ is Hamilton-connected if and only if $H$ has an $(e_1, e_2)$-IDT for any pair of edges $e_1, e_2 \in E(H)$, was given in [19]. Since the result was given without proof, and we need a slightly stronger statement, for the sake of completeness, we include the statement here with its (easy) proof.

**Theorem C [15, 19].** *Let $H$ be a multigraph with $|E(H)| \geq 3$ and let $G = L(H)$.*

($i$) **[15]** *The graph $G$ is hamiltonian if and only if $H$ has a DCT.*

($ii$) **[19]** *For every $e_i \in E(H)$ and $a_i = L(e_i)$, $i = 1, 2$, $G$ has a hamiltonian $(a_1, a_2)$-path if and only if $H$ has an $(e_1, e_2)$-IDT.*

**Proof.** $(ii)$ For $i = 1, 2$, subdivide $e_i$ with a vertex $v_i$ of degree 2 if $e_i$ is nonpendant, or let $v_i$ be the vertex of degree 1 of $e_i$ if $e_i$ is pendant; join $v_1, v_2$ with a path having at least two interior vertices, let $H'$ be the resulting graph, and set $G' = L(H')$. Then clearly $G$ has a hamiltonian $(a_1, a_2)$-path if and only if $G'$ is hamiltonian, and $H$ has an $(e_1, e_2)$-IDT if and only if $H'$ has a DCT. The rest follows from part $(i)$. ■

## 2.2 Strongly spanning trailable multigraphs

A multigraph $H$ is *strongly spanning trailable* if for any $e_1 = u_1v_1, e_2 = u_2v_2 \in E(H)$ (possibly $e_1 = e_2$), the multigraph $H(e_1, e_2)$, which is obtained from $H$ by replacing the edge $e_1$ by a path $u_1v_{e_1}v_1$ and the edge $e_2$ by a path $u_2v_{e_2}v_2$, has a spanning $(v_{e_1}, v_{e_2})$-trail.

We first recall two well-known graphs that will occur as exceptions in some of the results, namely, the Petersen graph $\Pi$ and the Wagner graph $W$ (see Fig. 2). It is a well-known fact that the Wagner graph can be obtained from the Petersen graph by removing an arbitrary edge and suppressing the two created vertices of degree 2. We will often refer to these graphs using the labeling of their vertices as indicated in Fig. 2.

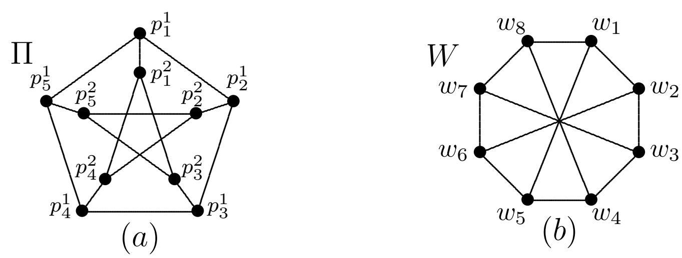


Figure 2: The Petersen graph $\Pi$ and the Wagner graph $W$

We will need the following two results on "small" strongly spanning trailable multigraphs from [21]. Here, $\mathbb{W}$ is the set of multigraphs that are obtained from the Wagner graph $W$ by subdividing one of its edges and adding at least one edge between the new vertex and exactly one of its neighbors.

**Theorem D [21].**

$(i)$ *Every 2-connected 3-edge-connected multigraph $H$ with circumference $c(H) \leq 8$ other than the Wagner graph $W$ is strongly spanning trailable.*

$(ii)$ *Every 3-edge-connected multigraph $H$ with $|V(H)| \leq 9$ such that $H \notin \{W\} \cup \mathbb{W}$ is strongly spanning trailable.*

## 2.3 $A$-contractible multigraphs

We will also use the following operation introduced in [24]. The concept was defined in [24] for graphs, but it is easy to observe that it remains true also for multigraphs. For a multigraph $H$ and $F \subset H$, $H|_F$ denotes the multigraph obtained from $H$ by identifying the vertices of $F$ as a (new) vertex $v_F$, and by replacing the created loops by pendant edges. Specifically, if $E(F) = \{e\}$, we simply write $G|_e$. If $H$ is a multigraph, $X \subset V(H)$, and $\mathcal{A}$ is a partition of $X$ into subsets, then $E(\mathcal{A})$ denotes the set of all edges $a_1a_2$ (not necessarily in $H$) such that $a_1$, $a_2$ are in the same element of $\mathcal{A}$. Further $H^{\mathcal{A}}$ denotes the multigraph with vertex set

$V(H^{\mathcal{A}}) = V(H)$ and edge set $E(H^{\mathcal{A}}) = E(H) \cup E(\mathcal{A})$ (where $E(H)$ and $E(\mathcal{A})$ are considered to be disjoint, i.e., if $e_1 = a_1a_2 \in E(H)$ and $e_2 = a_1a_2 \in E(\mathcal{A})$, then $e_1$, $e_2$ are parallel edges in $H^{\mathcal{A}}$).

Let $F$ be a multigraph and let $A \subset V(F)$. Then $F$ is said to be $A$-*contractible*, if for every even subset $X \subset A$ and for every partition $\mathcal{A}$ of $X$ into two-element subsets, the graph $F^{\mathcal{A}}$ has a DCT containing all vertices of $A$ and all edges of $E(\mathcal{A})$. Note that, in this definition, we admit $X$ to be empty, in which case $F^{\mathcal{A}} = F$. Also, if $F$ is $A$-contractible, then $F$ is $A'$-contractible for any $A' \subset A$ (since every subset $X$ of $A'$ is a subset of $A$). The following important property of the contractibility concept follows from the results in [24].

**Theorem E [24].** *Let $H$ be a multigraph and let $F \subset H$ be an $A_H(F)$-contractible submultigraph of $H$. Then $H$ has a DCT if and only if $H|_F$ has a DCT.*

Note that if $F$ is collapsible in the sense of Catlin [10], then $F$ is $V(F)$-contractible, and, similarly, the $A$-contractibility concept also generalizes $X$-collapsibility by Veldman [30]. For more details, we refer to [24].

In Fig. 3, we give several examples of $A$-contractible graphs (in the figure, the vertices in the set $A$ are double-circled). Note that detailed proofs of $A$-contractibility are for $F_2$ and $F_3$ given in [24].

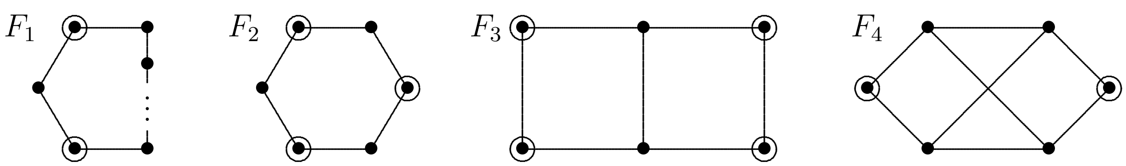


Figure 3: Examples of $A$-contractible graphs

# 3 Closure operations for Hamilton-connectedness

## 3.1 M-closure and SM-closure

A vertex $x \in V(G)$ is said to be *locally connected* if $\langle N(x)\rangle_G$ is a connected subgraph of $G$, and $x$ is *eligible* if $x$ is locally connected and $\langle N(x)\rangle_G$ is noncomplete. We will use $V_{EL}(G)$ to denote the set of all eligible vertices in $G$. It is easy to observe that in the special case when $G$ is a line graph and $H = L^{-1}(G)$, a nonsimplicial vertex $x \in V(G)$ is locally connected if and only if the corresponding edge $e = L_G^{-1}(x)$ is in a triangle or in a multiedge in $H$.

For $x \in V(G)$, the *local completion of $G$ at $x$* is the graph $G_x^* = (V(G), E(G) \cup \{y_1y_2 |\ y_1, y_2 \in N_G(x))$ (i.e., $G_x^*$ is obtained from $G$ by adding to $\langle N(x)\rangle_G$ all missing edges). Obviously, if $G$ is claw-free, then so is $G_x^*$. Note that in the special case when $G$ is a line graph, $H = L^{-1}(G)$ and $e = L^{-1}(x)$, we have $G_x^* = L(H|_e)$.

In [23], it was shown that $G$ is claw-free and $x \in V_{EL}(G)$, then $G_x^*$ is hamiltonian if and only if $G$ is hamiltonian, and the *closure* $\mathrm{cl}(G)$ of a claw-free graph $G$ was defined as the graph obtained from $G$ by recursively performing the local completion operation at eligible vertices, as long as this is possible (more precisely: $\mathrm{cl}(G) = G_k$, where $G_1, \ldots, G_k$ is a sequence of graphs such that $G_1 = G$, $G_{i+1} = (G_i)^*_{x_i}$ for some $x_i \in V_{EL}(G_i)$, $i = 1, \ldots, k-1$, and $V_{EL}(G_k) = \emptyset$). We say that $G$ is *closed* if $G = \mathrm{cl}(G)$.

For a claw-free graph $G$, the closure $\mathrm{cl}(G)$ is uniquely determined, is the line graph of a triangle-free graph, and is hamiltonian if and only if $G$ is hamiltonian. Note that this allows us to decide questions about hamiltonicity in a claw-free graph by looking at the corresponding question in the line graph preimage of the closure. However, as observed in [7], the closure operation does not preserve (non-)Hamilton-connectedness of $G$. This motivated the concept of $k$-closure as introduced in [5]: for an integer $k \geq 1$, a vertex $x$ is *$k$-eligible* if $\langle N(x) \rangle_G$ is $k$-connected noncomplete, and the $k$-closure $\mathrm{cl}_k(G)$ is defined analogously as the graph obtained by recursively performing the local completion operation at $k$-eligible vertices, as long as this is possible. The resulting graph is again unique (see [5]). The following facts were conjectured in [5] and proved in [25].

**Theorem F [25].** *Let $G$ be a claw-free graph.*
- (*i*) *If $x \in V(G)$ is 2-eligible, then $G$ is Hamilton-connected if and only if $G^*_x$ is Hamilton-connected,*
- (*ii*) *$G$ is Hamilton-connected if and only if $\mathrm{cl}_2(G)$ is Hamilton-connected.*

It is easy to observe that, in general, $\mathrm{cl}_2(G)$ is not a line graph, and even not a line graph of a multigraph. To avoid this disadvantage, the second and third authors developed in [26] the concept of the *multigraph closure* (or briefly *M-closure*) $\mathrm{cl}^M(G)$ of a claw-free graph $G$ as the graph $\mathrm{cl}^M(G)$ obtained from $\mathrm{cl}_2(G)$ by a sequence of local completions at some (but not all) eligible vertices, where the eligible vertices are chosen in a special way such that the resulting graph is a line graph of a multigraph, and the operation still preserves the (non-)Hamilton-connectedness of $G$. We do not give technical details of the construction since we will not need them in our proofs. We only note here that $\mathrm{cl}^M(G)$ can be constructed in polynomial time, and we refer for more details to [25], [26].

The following result summarizes basic properties of $\mathrm{cl}^M(G)$.

**Theorem G [26].** *Let $G$ be a claw-free graph and let $\mathrm{cl}^M(G)$ be its M-closure. Then*
- (*i*) *$\mathrm{cl}^M(G)$ is uniquely determined,*
- (*ii*) *there is a multigraph $H$ such that $\mathrm{cl}^M(G) = L(H)$,*
- (*iii*) *$\mathrm{cl}^M(G)$ is Hamilton-connected if and only if $G$ is Hamilton-connected.*

We say that $G$ is *M-closed* if $G = \mathrm{cl}^M(G)$. Consider the multigraphs $T_1, T_2, T_3$ in Fig. 4.

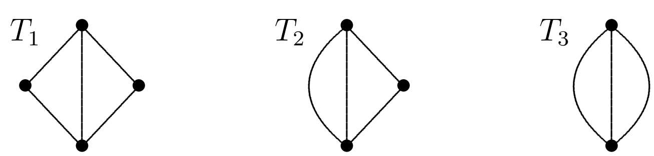


Figure 4: The diamond $T_1$, the multitriangle $T_2$ and the triple edge $T_3$

**Theorem H [26].** *Let $G$ be a claw-free graph and let $T_1, T_2, T_3$ be the multigraphs shown in Fig. 4. Then $G$ is M-closed if and only if $G$ is a line graph of a multigraph and $L^{-1}(G)$*

*does not contain a subgraph (not necessarily induced) isomorphic to any of the multigraphs* $T_1$*,* $T_2$ *or* $T_3$*.*

The M-closure operation was further strengthened in [18] in such a way that a closure of a claw-free graph is the line graph of a multigraph with either at most two triangles and no multiedge, or with at most one double edge and no triangle.

For a given claw-free graph $G$, a graph $G^M$ is defined in [18] by the following construction.

($i$) If $G$ is Hamilton-connected, we set $G^M = \mathrm{cl}(G)$.

($ii$) If $G$ is not Hamilton-connected, we recursively perform the local completion operation at such eligible vertices for which the resulting graph is still not Hamilton-connected, as long as this is possible. We obtain a sequence of graphs $G_1, \ldots, G_k$ such that

- $G_1 = G$,
- $G_{i+1} = (G_i)^*_{x_i}$ for some $x_i \in V_{EL}(G_i)$, $i = 1, \ldots, k-1$,
- $G_k$ has no hamiltonian $(a, b)$-path for some $a, b \in V(G_k)$,
- for any $x \in V_{EL}(G_k)$, $(G_k)^*_x$ is Hamilton-connected,

and we set $G^M = G_k$.

A resulting $G^M$ is called a *strong M-closure* (or briefly an *SM-closure*) of the graph $G$, and a graph $G$ equal to its SM-closure is said to be *SM-closed*. Note that, for a given graph $G$, its SM-closure $G^M$ is not uniquely determined.

It is straightforward to see that if $G$ is SM-closed, then $G$ is also M-closed, implying $G = L(H)$, where $H$ does not contain any of the multigraphs shown in Fig. 4. The following theorem summarizes basic properties of the SM-closure operation.

**Theorem I [18].** *Let* $G$ *be a claw-free graph and let* $G^M$ *be one of its SM-closures. Then* $G^M$ *has the following properties:*

($i$) $V(G) = V(G^M)$ *and* $E(G) \subset E(G^M)$*,*

($ii$) $G^M$ *is obtained from* $G$ *by a sequence of local completions at eligible vertices,*

($iii$) $G$ *is Hamilton-connected if and only if* $G^M$ *is Hamilton-connected,*

($iv$) *if* $G$ *is Hamilton-connected, then* $G^M = \mathrm{cl}(G)$*,*

($v$) *if* $G$ *is not Hamilton-connected, then either*

  ($\alpha$) $V_{EL}(G^M) = \emptyset$ *and* $G^M = \mathrm{cl}(G)$*, or*

  ($\beta$) $V_{EL}(G^M) \neq \emptyset$ *and* $(G^M)^*_x$ *is Hamilton-connected for any* $x \in V_{EL}(G^M)$*,*

($vi$) $G^M = L(H)$*, where* $H$ *contains either*

  ($\alpha$) *at most 2 triangles and no multiedge, or*

  ($\beta$) *no triangle, at most one double edge and no other multiedge,*

($vii$) *if* $G^M$ *contains no hamiltonian* $(a, b)$*-path for some* $a, b \in V(G^M)$ *and*

  ($\alpha$) $X$ *is a triangle in* $H$*, then* $E(X) \cap \{L^{-1}_{G^M}(a), L^{-1}_{G^M}(b)\} \neq \emptyset$*,*

  ($\beta$) $X$ *is a multiedge in* $H$*, then* $E(X) = \{L^{-1}_{G^M}(a), L^{-1}_{G^M}(b)\}$*.*

We will also need the following lemma on SM-closed graphs proved in [27].

**Lemma J [27].** *Let* $G$ *be an SM-closed graph and let* $H = L^{-1}(G)$*. Then* $H$ *does not contain a triangle with a vertex of degree 2 in* $H$*.*

### 3.2 The core of the preimage of an SM-closed graph

The concept of the core of a graph is an important tool for studying hamiltonian properties of line graphs. As the definition is slightly problematic for multigraphs, we restrict our observations to the case that we need, i.e., to preimages of 3-connected SM-closed graphs. The difficulties then do not occur since such a multigraph cannot have pendant multiedges by Theorem B, and cannot have pendant multitriangles by Theorem H.

Thus, let $G$ be a 3-connected SM-closed graph and let $H = L^{-1}(G)$. The *core of* $H$ is the multigraph $\mathrm{co}(H)$ obtained from $H$ by removing all pendant edges and suppressing all vertices of degree 2.

Shao [28] proved the following properties of the core of a multigraph.

**Theorem K [28].** *Let $H$ be an essentially 3-edge-connected multigraph. Then*
($i$) $\mathrm{co}(H)$ *is uniquely determined,*
($ii$) $\mathrm{co}(H)$ *is 3-edge-connected,*
($iii$) $V(\mathrm{co}(H))$ *dominates all edges of* $H$*,*
($iv$) *if* $\mathrm{co}(H)$ *has a spanning closed trail, then* $H$ *has a DCT,*
($v$) *if* $\mathrm{co}(H)$ *is strongly spanning trailable, then* $L(H)$ *is Hamilton-connected.*

### 3.3 UM-closure

In this subsection we show that the concept of SM-closure can be further strengthened by omitting the eligibility assumption in the local completion operation. Specifically, for a given claw-free graph $G$, we construct a graph $G^U$ by the following construction.

($i$) If $G$ is Hamilton-connected, we set $G^U = K_{|V(G)|}$.
($ii$) If $G$ is not Hamilton-connected, we recursively perform the local completion operation at such vertices for which the resulting graph is still not Hamilton-connected, as long as this is possible. We obtain a sequence of graphs $G_1, \ldots, G_k$ such that
- $G_1 = G$,
- $G_{i+1} = (G_i)^*_{x_i}$ for some $x_i \in V(G_i)$, $i = 1, \ldots, k-1$,
- $G_k$ has no hamiltonian $(a, b)$-path for some $a, b \in V(G_k)$,
- for any $x \in V(G_k)$, $(G_k)^*_x$ is Hamilton-connected,

and we set $G^U = G_k$.

A graph $G^U$ obtained by the above construction will be called an *ultimate M-closure* (or briefly an *UM-closure*) of the graph $G$, and a graph $G$ equal to its UM-closure will be said to be *UM-closed*. Note that since the construction of a UM-closure requires deciding Hamilton-connectedness, there is not much hope to construct a UM-closure of a claw-free graph in polynomial time.

Obviously, by the definition, if $G$ is UM-closed, then $G$ is also SM-closed, implying that $G$ is a line graph and $H = L^{-1}(G)$ has special structure (contains no diamond etc. - see Theorems H and I ($vi$), ($vii$)). In the next theorem, summarizing basic properties of the UM-closure operation, we will see that for UM-closed graphs, not only $H$, but also $\mathrm{co}(H)$ has these strong structural properties.

**Theorem 2.** *Let $G$ be a claw-free graph and let $G^U$ be one of its UM-closures. Then $G^U$ has the following properties:*

$(i)$ $V(G) = V(G^U)$ and $E(G) \subset E(G^U)$,
$(ii)$ $G^U$ is obtained from $G$ by a sequence of local completions at vertices,
$(iii)$ $G$ is Hamilton-connected if and only if $G^U$ is Hamilton-connected,
$(iv)$ if $G$ is Hamilton-connected, then $G^U = K_{|V(G)|}$,
$(v)$ if $G$ is not Hamilton-connected, then $(G^U)^*_x$ is Hamilton-connected for any $x \in V(G^U)$,
$(vi)$ $G^U = L(H)$, where $\mathrm{co}(H)$ contains no diamond, no mutitriangle and no triple edge, and either
  $(\alpha)$ at most 2 triangles and no multiedge, or
  $(\beta)$ no triangle, at most one double edge and no other multiedge, and if $\mathrm{co}(H)$ contains a double edge, then this double edge is also in $H$,
$(vii)$ if $G^U$ contains no hamiltonian $(a, b)$-path for some $a, b \in V(G^U)$ and
  $(\alpha)$ $X$ is a triangle in $\mathrm{co}(H)$, then $E(X) \cap \{L^{-1}_{G^U}(a), L^{-1}_{G^U}(b)\} \neq \emptyset$,
  $(\beta)$ $X$ is a multiedge in $\mathrm{co}(H)$, then $E(X) = \{L^{-1}_{G^U}(a), L^{-1}_{G^U}(b)\}$.

For the proof of Theorem 2, we will need three lemmas.

**Lemma 3.** *Let $H$ be a multigraph, $F$ an $A_H(F)$-contractible submultigraph of $H$, and let $e_1, e_2 \in E(H) \setminus E(F)$. Then $H$ has an $(e_1, e_2)$-IDT if and only if $H|_F$ has an $(e_1, e_2)$-IDT.*

**Proof.** Let $H_1$ be the multigraph obtained from $H$ by subdividing the edges $e_1, e_2$ with new vertices $a_1, a_2$, and connecting $a_1, a_2$ by a path with new inner vertices $b_1, b_2$. Then clearly $H$ has an $(e_1.e_2)$-IDT if and only if $H_1$ has a DCT. By Theorem E, $H_1$ has a DCT if and only if the multiograph $H_2 = H_1|_F$ has a DCT. Finally, $H_2$ has a DCT if and only if the multigraph $H_3$, obtained from $H_2$ by removing $b_1, b_2$ and suppressing $a_1, a_2$ has an $(e_1, e_2)$-IDT. However, $H_3 = H|_F$. ■

Let $H$ be a multigraph, $u \in V(H)$ a vertex of degree 2, and let $v_1, v_2$ be the neighbors of $u$. Then $H|_{(u)}$ denotes the multigraph obtained from $H$ by suppressing the vertex $u$ and by adding two pendant edges $f_1$ and $f_2$ such that $f_1$ is incident with $v_1$ and $f_2$ is incident with $v_2$. The following lemma was proved in [18].

**Lemma L [18].** *Let $H$ be a multigraph, $u \in V(H)$ a vertex of degree 2, and let $v_1, v_2$ be the neighbors of $u$. Set $H' = H|_{(u)}$, $h = v_1v_2 \in E(H')$, and let $f_1, f_2 \in E(H') \setminus E(H)$ be the two pendant edges attached to $v_1$ and $v_2$, respectively.*
$(i)$ *If $L(H)$ is Hamilton-connected, then $L(H')$ has a hamiltonian $(x, y)$-path for every $x, y \in V(L(H'))$ for which either $L(h) \notin \{x, y\}$, or $L(h) \in \{x, y\}$ and $\{x, y\} \cap \{L(f_1), L(f_2)\} \neq \emptyset$.*
$(ii)$ *If $L(H')$ is Hamilton-connected, then $L(H)$ has a hamiltonian $(x, y)$-path for every $x, y \in V(L(H))$ for which $\{x, y\} \neq \{L(uv_1), L(uv_2)\}$.*

**Lemma 4.** *Let $H$ be a multigraph, and let $T = x_1x_2x_3$ be a triangle in $\mathrm{co}(H)$ such that the edge $x_1x_2$ is subdivided in $H$ by a vertex $x_{12}$ of degree 2, and the edges $x_1x_3$, $x_2x_3$ are possibly (but not necessarily) subdivided in $H$ by vertices $x_{13}$, $x_{23}$ of degree 2. Let $T'$ be the subgraph of $H$ corresponding to $T$. Set $e_i = x_ix_{12}$, and set $f_i = x_ix_{i3}$ if $x_ix_3$ is subdivided or $f_i = x_ix_3$ otherwise, $i = 1, 2$. If $H$ contains an $(f_1, f_2)$-IDT, then $H$ contains an $(e_1, e_2)$-IDT.*

**Proof.** Let $Q$ be an $(f_1, f_2)$-IDT in $H$. We will consider nontrivial components of $Q - E(T')$. Note that each such component is an $(x_i, x_j)$-subtrail of $Q$ for some $i, j \in \{1, 2, 3\}$ (possibly $i = j$) with all edges in $E(H) \setminus E(T')$. First observe that since $\{f_1, f_2\}$ is an edge-cut of $T'$, separating $x_3$ from $x_1$, $x_{12}$ and $x_2$, there is no nontrivial $(x_3, x_3)$-subtrail of $Q$ among the components of $Q - E(T')$. Thus, if $x_3$ is in a nontrivial subtrail, then it is in an $(x_3, x_i)$-subtrail for some $i = 1, 2$. Since two distinct subtrails must have distinct endvertices (otherwise they form one component of $Q - E(T')$), there are at most two such subtrails. Up to a symmetry, we have the following possibilities.

| Number of subtrails | Endvertices of subtrails | $(e_1, e_2)$-IDT in $H$ |
|---|---|---|
| 1 | $\{x_1, x_3\}$ | $x_{12}x_1Qx_3(x_{23})x_2x_{12}$ |
| 1 | $\{x_1, x_2\}$ | $x_{12}x_1Qx_2x_{12}$ |
| 1 | $\{x_1, x_1\}$ | $x_{12}x_1Qx_1(x_{13})x_3(x_{23})x_2x_{12}$ |
| 2 | $\{x_1, x_1\}, \{x_2, x_2\}$ | $x_{12}x_1Qx_1(x_{13})x_3(x_{23})x_2Qx_2x_{12}$ |
| 2 | $\{x_1, x_3\}, \{x_2, x_2\}$ | $x_{12}x_1Qx_3(x_{23})x_2Qx_2x_{12}$ |

(For the last case, see Fig. 5). In each of the possible cases, we have obtained an $(e_1, e_2)$-IDT in $H$. ■

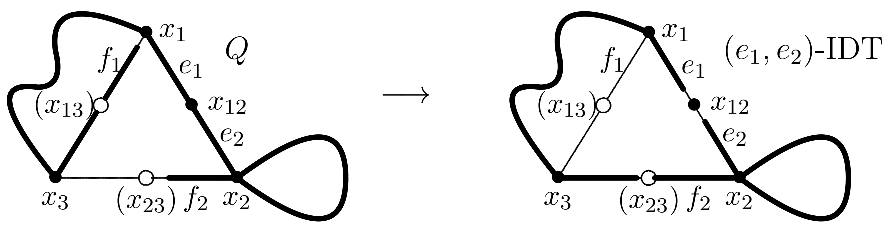


Figure 5: Transformation of the trail $Q$ into an $(e_1, e_2)$-IDT

**Proof of Theorem 2.** Let $G$ be a claw-free graph and let $G^U$ be its UM-closure. The properties $(i)$ – $(v)$ follow immediately by the construction of $G^U$. Set $H = L^{-1}(G^U)$. If $G^U$ is Hamilton-connected, then $H$ is a star and $(vi)$, $(vii)$ are trivially satisfied. So, suppose that $G^U$ is not Hamilton-connected.

$(vi)$ We first show that $\mathrm{co}(H)$ contains no diamond, no multitriangle and no triple edge. In what follows, the common edge of the two triangles of a diamond $D$ will be called the *middle edge* of $D$.

Claim 1. *If $D$ is a diamond in $\mathrm{co}(H)$, then the middle edge of $D$ is subdivided in $H$.*

Proof. Let, to the contrary, $D$ be a diamond in $\mathrm{co}(H)$ such that its middle edge is also an edge in $H$. Then in $H$, the subgraph $D'$ corresponding to $D$ has at least one subdivided edge (since $G$ is SM-closed, implying that $H$ is diamond-free). Let thus $e = x_1x_2 \in E(D)$ be such an edge that is subdivided in $D'$ by a vertex $y$ of degree 2, and let $H' = H|_{(y)}$.

Suppose that $L(H')$ is Hamilton-connected. Then, by Lemma L$(ii)$ and by the fact that $L(H)$ is not Hamilton-connected, $H$ has an $(e, f)$-IDT if and only if $\{e, f\} \neq \{x_1y, x_2y\}$, but then, if $T$ is the triangle of $D$ not containing $e$ and $T'$ is the corresponding subgraph of $D'$, the multigraph $H|_{T'}$ has no $(x_1y, x_2y)$-IDT by Lemma 3 (since $T'$ is $A_H(T')$-contractible - see the graph $F_2$ in Fig. 3). However, $L(H|_{T'})$ can be alternatively obtained from $L(H)$ by a series of contractions of edges of $H$, i.e., of local completions at vertices of $L(H)$, contradicting the fact that $G^U = L(H)$ is UM-closed. Thus, $L(H')$ is not Hamilton-connected.

By induction, we conclude that the line graph of the multigraph $H_1$, obtained from $H$ by suppressing all vertices of degree 2 in $D'$ and adding a pendant edge to each of the vertices of the new edges, is not Hamilton-connected (note that the resulting multigraph of the inductive construction can have different number of pendant edges since e.g. in the second step we apply the construction to $H'$). Moreover, if $e$ is the middle edge of $D$, then $v = L(e)$ is 2-eligible in $L(H_1)$, hence $(L(H_1))^*_v = L(H_1|_e)$ (see an example in Fig. 6($a$)), is not Hamilton-connected by Theorem F.

However, the same multigraph as $H_1|_e$, with only possibly different number of pendant edges at vertices, can be alternatively obtained from $H$ by a series of contractions of edges (i.e., local completions at vertices of $G^U$, see an example in Fig. 6($b$)), implying that $L(H_1|_e)$ is Hamilton-connected by Theorem 2($v$), a contradiction. $\square$

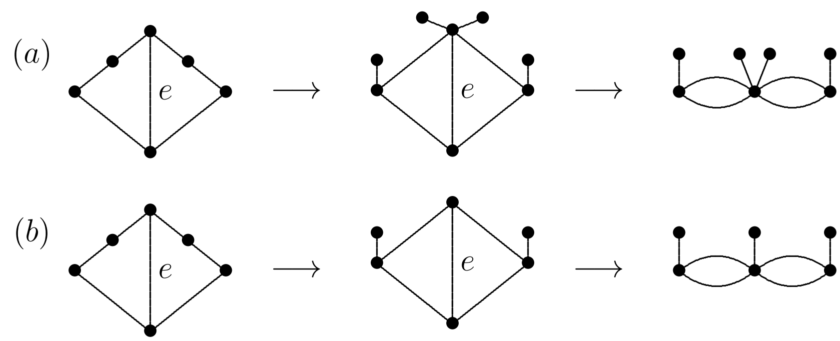


Figure 6: Two alternative constructions of a part of $H_1|_e$

Claim 2. *Let $D$ be a diamond in* co$(H)$, *$e = v_1v_2$ the middle edge of $D$, $u \in V(H)$ the vertex with $d_H(u) = 2$ and $uv_1, uv_2 \in V(H)$, and set $y_i = L(uv_i)$, $i = 1, 2$. Then, for any $x_1, x_2 \in V(G^U)$, there is a hamiltonian $x_1, x_2$-path in $G^U$ if and only if $\{x_1, x_2\} \neq \{y_1, y_2\}$.*

Proof. Suppose, to the contrary, that co$(H)$ contains a diamond $D$ not satisfying the statement of Claim 2. Then either $G^U$ has no hamiltonian $(x_1, x_2)$-path for some $x_1, x_2 \in V(G^U)$ such that $\{x_1, x_2\} \neq \{y_1, y_2\}$, or $G^U$ has a hamiltonian $(y_1, y_2)$-path.

In the first case, we again construct $H_1$ from $H$ by suppressing all vertices of degree 2 in $D$ and adding a pendant edge to each of the vertices of the new edges, observe that the vertex of $L(H_1)$ corresponding to the middle edge of $D$ is 2-eligible, and obtain a contradiction in the same way as in the proof of Claim 1. Thus, $G^U$ has a hamiltonian $(y_1, y_2)$-path. But $G^U$ is not Hamilton-connected, hence there is no hamiltonian $(z_1, z_2)$-path in $G^U$ for some other two vertices $z_1, z_2$ with $\{z_1, z_2\} \neq \{y_1, y_2\}$, and we are back in the first case. $\square$

Claim 3. *If $X$ is a double edge in* co$(H)$, *then neither of the edges of $X$ is subdivided in $H$.*

Proof. Set $V(X) = \{x_1, x_2\}$. If $X$ is not a double edge in $H$, then, by Lemma J, both edges of $X$ are subdivided in $H$. Thus, let $y_1, y_2 \in V(H)$ be of degree 2 in $H$ such that $x_1y_i, x_2y_i \in E(G)$, $i = 1, 2$. Set $H_1 = H|_{(y_2)}$.

If $L(H_1)$ is Hamilton-connected, then, by Lemma L($ii$), $H$ has an $(e, f)$-IDT for all pairs of edges $\{e, f\}$ except $\{x_1y_2, x_2y_2\}$, hence also an $\{x_1y_1, x_2y_1\}$-IDT, but then we easily also have an $\{x_1y_2, x_2y_2\}$-IDT in $H$, a contradiction. Hence $L(H_1)$ is not Hamilton-connected.

The multigraph $H_1$ contains the triangle $x_1x_2y_1$ with $d_{H_1}(y_1) = 2$, thus, by Lemma J, $L(H_1)$ is not SM-closed, hence also not UM-closed. Let $\bar{G}$ be a UM-closure of $L(H_1)$, and set $\bar{H} = L^{-1}(\bar{G})$. By Lemma J, $\bar{H}$ cannot contain the triangle $x_1x_2y_1$, implying that, in $\bar{H}$, some of its edges is contracted, and the line graph of the resulting multigraph is still not Hamilton-connected. However, it is easy to see that each of the resulting multigraphs can be (up to possibly different number of pendant edges at vertices) alternatively obtained from $H$ by a series of contraction of edges, contradicting the fact that $G^U = L(H)$ is UM-closed. $\square$

Now, let $D$ be a diamond in co($H$), and set $V(D) = \{v_1, v_2, v_3, v_4\}$, where $v_1v_2$ is the middle edge of $D$. By Claim 1, the edge $v_1v_2$ is subdivided in $H$ by a vertex, say, $u$, with $d_H(u) = 2$, and by Claim 2, $H$ has an $(e, f)$-IDT if and only if $\{e, f\} \neq \{v_1u, v_2u\}$. Specifically, $H$ has an $(f_1, f_2)$-IDT, where $f_i = v_iv_3$ if $v_iv_3$ is not subdivided in $H$, or $f_i = v_iu_i$ if $v_iv_3$ is subdivided in $H$ by a vertex $u_i$, $i = 1, 2$. But then, by Lemma 4, $H$ has also a $(v_1u, v_2u)$-IDT, a contradiction. Hence co($H$) contains no diamond.

Let next $F$ be a multitriangle in co($H$) and set $V(F) = \{v_1, v_2, v_3\}$, where $\langle\{v_1, v_2\}\rangle_F$ is a double edge. Then at least one of the edges $v_1v_3, v_2v_3$ is subdivided in $H$. Let thus $u_1 \in V(H)$ be subdividing $v_1v_3$, and possibly also $u_2 \in V(H)$ be subdividing $v_2v_3$. Set $H_1 = (H|_{(u_1)})|_{(u_2)}$ if $v_2v_3$ is subdivided, or $H_1 = H|_{(u_1)}$ otherwise. By Lemma L($ii$), Claim 3 and Theorem I($vii$)($\beta$), $L(H_1)$ is not Hamilton-connected. But in $L(H_1)$ the vertex $L(h)$, where $h$ is either of the two edges joining $v_1, v_2$, is 2-eligible, hence the graph $L(H_2)$, where $H_2 = H_1|_h$, is also not Hamilton-connected by Theorem F($i$). However, the same multigraph, with only possibly different number of pendant edges at vertices, can be obtained from $H$ by a series of edge contractions, contradicting Theorem I($v$). Hence co($H$) contains no multitriangle.

Finally, co($H$) contains no triple edge immediately by Claim 3 and by Theorem I($vi$).

($vi$)($\alpha$), ($\beta$). If co($H$) contains three triangles, then these triangles are edge-disjoint, for otherwise we have a diamond in co($H$). Then one of the corresponding subgraphs of $H$ contains neither of the edges $e, f$ for which there is no $(e, f)$-IDT and can be contracted by Lemma 3, contradicting the fact that $G^U = L(H)$ is UM-closed. Hence co($H$) contains at most two triangles.

Now, if $T$ is a triangle in co($H$) and $T'$ the corresponding subgraph of $H$, then co($H$) cannot contain a double edge, for otherwise similarly, by Claim 3 and by Theorem I($vii$)($\beta$), $T'$ can be contracted by Lemma 3, a contradiction. Hence co($H$) contains either at most two triangles and no multiedge, or a double edge and no triangle.

The proof of ($vii$)($\alpha$) follows analogously by Lemma 3, and ($vii$)($\beta$) follows by Claim 3 and by Theorem I($vii$)($\beta$). ■

The following result was first established in [9], and later on reconsidered in [22] in a more general setting implying its validity without the eligibility assumption.

**Theorem M [22].** *Let $G$ be a $\{K_{1,3}, N_{i,j,k}\}$-free graph, $i, j, k \geq 1$, and let $x \in V(G)$. Then the graph $G^*_x$ is $\{K_{1,3}, N_{i,j,k}\}$-free.*

Specifically, Theorem M implies that a UM-closure of a $\{K_{1,3}, N_{i,j,k}\}$-free graph is also $\{K_{1,3}, N_{i,j,k}\}$-free.

# 4 Variants of the "Nine-point-theorem"

The well-known "Nine-point-theorem" by Holton et al. [16] states that a 3-connected cubic graph contains a cycle passing through any 9 prescribed vertices. The nine-point-theorem was strengthened by Bau and Holton [3] to cycles through 12 vertices (with the help of a computer). For our purposes, we will use another stronger version by Bau and Holton [2] that deals with a set of vertices and an edge (proved without computer). For this, we will need some more terminology from [1].

Let $G$ be a multigraph, $R \subset G$ a spanning subgraph of $G$, and let $\mathcal{R}$ be the set of components of $R$. Then $G/R$ is the multigraph with $V(G/R) = \mathcal{R}$, in which, for each edge in $E(G)$ between two components of $R$, there is an edge in $E(G/R)$ joining the corresponding vertices of $G/R$ (note that this means that $G/R$ can have multiple edges even if $G$ is a graph). The (multi-)graph $G/R$ is said to be a *contraction* of $G$. (Roughly, in $G/R$, components of $R$ are contracted to single vertices while keeping the adjacencies between them). Clearly, if $R$ is connected, then $G/R = K_1$, and if $R$ is edgeless, then $G/R = G$; these two contractions are called *trivial*.

The contraction operation maps $V(G)$ onto $V(G/R)$ (where vertices of a component of $R$ are mapped on a vertex of $G/R$). If $G/R \simeq F$, then this defines a function $\alpha : G \to F$ which is called a *contraction of $G$ on $F$*.

Note that there is a difference between $G/R$ and the contraction $G|_F$, as defined in Section 2: in $G/R$, the components of $R$ are contracted to single vertices while removing the created loops, while in $G|_F$, the subgraph $F$ is contracted to a single vertex and the created loops are replaced by pendant edges.

Throughout the rest of this section, $\Pi$ denotes the Petersen graph.

**Theorem N [2].** *Let $G$ be a 3-connected cubic graph, $A \subset V(G)$, $|A| = 8$, and let $e \in E(G)$. Then there is a cycle in $G$ which contains $A \cup \{e\}$, unless there is a contraction $\alpha : G \to \Pi$ such that $\alpha(e) = xy \in E(\Pi)$ and $\alpha(A) = V(\Pi) \setminus \{x, y\}$.*

**Corollary O [2].** *Let $G$ be a 3-connected cubic graph, $A \subset V(G)$, $|A| = 7$, and let $e \in E(G)$. Then there is a cycle in $G$ which contains $A \cup \{e\}$.*

We will also need the following two easy consequences of Corollary O.

**Lemma 5.** *Let $G$ be a 3-connected cubic graph and let $A \subset V(G)$, $|A| = 7$. Then for any $e, f \in E(G)$, $G$ has an $(e, f)$-trail $T$ such that $A \subset \mathrm{Int}(T)$.*

**Proof.** Let $G'$ be obtained from $G$ by subdividing the edges $e, f$ with new vertices $v_1, v_2$ and adding the edge $h = v_1v_2$. By Corollary O, $G'$ has a cycle $C$ containing $A \cup \{h\}$, and removing $h$ and suppressing $v_1, v_2$, we obtain the requested $(e, f)$-trail $T$ with $A \subset \mathrm{Int}(T)$. ∎

**Lemma 6.** *Let $H$ be a graph such that $\mathrm{co}(H) = W$. If there is a vertex $x \in V(\mathrm{co}(H))$ such that $N_H(x) = N_{\mathrm{co}(H)}(x)$, then $L(H)$ is Hamilton-connected.*

**Proof.** Let $x \in V(\mathrm{co}(H))$ be such that $N_H(x) = N_{\mathrm{co}(H)}(x)$, i.e., $x$ is incident in $H$ to neither a pendant edge nor a subdivided edge, and set $A = V(\mathrm{co}(H)) \setminus \{x\}$. By Lemma 5, for any $e, f \in E(H)$, $H$ has an $(e, f)$-trail $T$ with $A \subset \mathrm{Int}(T)$. However, $T$ is an $(e, f)$-IDT in $H$ since all edges incident to $v$ have a vertex on $T$. Thus, $L(H)$ is Hamilton-connected by Theorem C($ii$). ■

To apply Theorem N to our proof, we need to transform the line graph preimage to a cubic graph. This was first done in [14] by an inflation operation which replaces a vertex of high degree by a cycle. To avoid possible difficulties with edge-connectivity of the created graph, we modify the inflation operation as follows.

For $k \geq 4$, the *$k$-prism* $\Pi_k$ is the Cartesian product $C_k \Box K_2$, and $\Pi'_k$ is the graph obtained from $\Pi_k$ by subdividing each edge of one of the two $k$-cycles in $\Pi_k$ by a vertex of degree 2 (for $k = 4$, see Fig. 7($a$)). If $z \in V(H)$ is of degree $t \geq 4$ and $v_1, \ldots, v_t$ are the neighbors of $z$ (where we allow repetition in case of multiple edges), then the *inflation of $H$ at $z$* is the graph $H^I_z$ obtained from $H - z$ and $\Pi'_t$ with $V_2(\Pi'_t) = \{w_1, \ldots, w_t\}$ by adding the edges $v_i w_i$, $i = 1, \ldots, t$ (for $t = 4$, see Fig. 7($b$)).

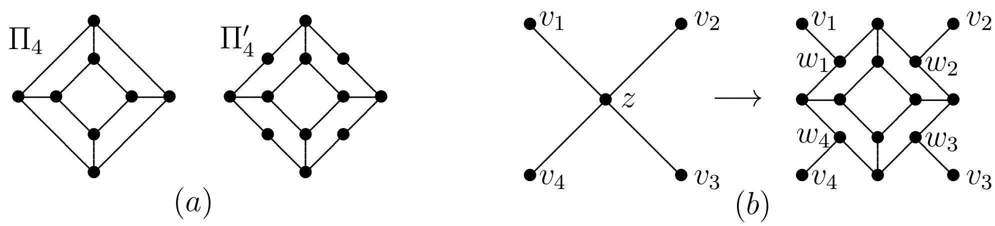


Figure 7: The 4-prism and the inflation operation

**Lemma 7.** *Let $H$ be a 3-edge-connected multigraph, let $z \in V(H)$ be of degree $d_H(z) \geq 4$, and let $F \simeq \Pi'_k$ be the subgraph of $H^I_z$ replacing the vertex $z$ in $H^I_z$. Then, for every essential 3-edge-cut $R$ of $H^I_z$, $R \cap E(F) = \emptyset$.*

**Proof.** Let $R \subset E(H^I_z)$ be an essential 3-edge-cut of $H^I_z$ and suppose that $R$ contains at least one edge of $F$. Then $R_F = R \cap E(F)$ is an edge-cut of $F$.

Suppose that $R_F$ is not essential, i.e., $R_F$ separates some vertex $u \in V(F)$ from the rest of $F$. If $d_F(u) = 3$, then $R$ is not essential in $H^I_z$, a contradiction. Hence $d_F(u) = 2$, and $u$ has a neighbor $u' \in V(H^I_z) \setminus V(F)$. Set $f = uu'$. Then $f \notin R$ since $R$ is essential, and since $d_F(u) = 2$, we have $|R \cap E(F)| = 2$, implying that $R$ contains one edge in $V(H^I_z) \setminus V(F)$, say, $f'$. Then $\{f, f'\}$ is an edge-cut of $H$, contradicting the assumption that $H$ is 3-edge-connected. Hence $R_F$ is essential. Since $F$ is essentially 3-edge-connected, we have $R = R_F$.

It is straightforward to observe that every essential 3-edge-cut of $F$ separates one edge of $F$ incident to a vertex of degree 2 in $F$ from the rest of $F$. Thus, let $e = uv$, $d_F(u) = 2$, be the edge of $F$ separated by $R$ from the rest of $F$, and let $w$ be the neighbor of $u$ in $V(H^I_z) \setminus V(F)$. Since $H$ is 3-edge-connected, there is a path in $H - F$ joining $w$ with some vertex in $V_2(F) \setminus \{u\}$. But this implies that $R$ is not an edge-cut of $H$, a contradiction. ■

If $H$ is a multigraph with $\delta(H) \geq 3$, then, by successively performing the inflation operation at every vertex of degree greater than 3, we obtain a cubic graph $H^I$ called a *cubic inflation* of $H$. It is straightforward to observe that if $H$ is 3-edge-connected, then $H^I$ is a 3-connected cubic graph. We will use $I$ to denote the inflation operation that assigns $H^I$ to $H$, and, for a vertex $z \in V(H)$ with $d_H(z) = t$, we use $I(z)$ to denote the copy of $\Pi'_t$ replacing $z$ in $H^I$ if $t > 3$, or the vertex $z$ itself (which we consider to be also in $V(H^I)$) if $t = 3$, respectively.

**Theorem 8.** *Let $H$ be a 3-edge-connected multigraph, $A \subset V(H)$, $|A| = 8$, and let $e \in E(H)$. Then either*

($i$) *$H$ contains a closed trail $T$ such that $A \subset V(T)$ and $e \in E(T)$, or*

($ii$) *there is a contraction $\alpha : H \to \Pi$ such that $\alpha(e) = xy \in E(\Pi)$ and $\alpha(A) = V(\Pi)\setminus\{x, y\}$.*

**Proof.** If ($ii$) is true, then ($i$) cannot be true since otherwise $\alpha(T)$ is a hamiltonian cycle in $\Pi$, a contradiction. So, suppose that ($ii$) is not true, i.e., there is no contraction $\alpha : H \to \Pi$ with the requested properties.

Let $H^I$ be a cubic inflation of $H$. Set $A = \{a_1, \ldots, a_8\}$ and $e = b_1b_2$, choose arbitrary vertices $a_i^I \in I(a_i)$, $i = 1, \ldots, 8$, set $A^I = \{a_1^I, \ldots, a_8^I\}$, and choose two vertices $b_1^I \in I(b_1)$ and $b_2^I \in I(b_2)$ such that $e^I = b_1^I b_2^I \in E(H^I)$. We want to show that there is no contraction $\alpha_I : H^I \to \Pi$ such that $\alpha_I(e^I) = x^I y^I \in E(\Pi)$ and $\alpha_I(A^I) = V(\Pi) \setminus \{x^I, y^I\}$.

Suppose, to the contrary, that there is such an $\alpha_I$. Let $z \in V(H)$ and $w \in V(\alpha_I(H^I))$. We observe that if $\alpha_I^{-1}(w)$ is nontrivial, then cannot be $V(I(z)) \setminus V(\alpha_I^{-1}(w)) \neq \emptyset$ and $V(I(z)) \cap V(\alpha_I^{-1}(w)) \neq \emptyset$ (specifically, $\alpha_I^{-1}(w)$ cannot be a proper subgraph of $I(z)$), since $\alpha_I^{-1}(w)$ is separated from the rest of $H^I$ by an essential 3-edge-cut and we would have a contradiction with Lemma 7. Thus, for every $z \in V(H)$, we have two possibilities: either

($a$) $V(I(z)) \subset V(\alpha_I^{-1}(w))$ (including possible equality if both are trivial) for some $w \in V(\alpha_I(H^I))$, or

($b$) every vertex of $I(z)$ is a trivial $\alpha_I^{-1}(w)$.

Recall that, by the definition of $\alpha_I$, every vertex of $H^I$ is in some $\alpha_I^{-1}(w)$.

However, the case ($b$) provides an embedding of the prism $\Pi'_t$ into the Petersen graph $\Pi$, which is not possible (e.g. since $t \geq 4$, implying $|V(\Pi'_t)| = 3t > 10 = |V(\Pi)|$). Hence we have the case ($a$), i.e., every $I(z)$ is contained in $\alpha_I^{-1}(w)$ for some vertex $w$ of $\alpha_I(H^I) \simeq \Pi$. But then the mapping $\alpha : H \to \Pi$, defined by $\alpha(H) = \alpha_I(H^I)$, is a contraction of $H$ on $\Pi$ satisfying ($ii$), a contradiction.

Thus, there is no contraction $\alpha_I : H^I \to \Pi$ such that $\alpha_I(e^I) = x^I y^I \in E(\Pi)$ and $\alpha_I(A^I) = V(\Pi) \setminus \{x^I, y^I\}$. By Theorem N, there is a cycle $C$ in $H^I$ containing $A^I \cup \{e^I\}$. Contracting back all the inflated prisms $I(z)$, we have a closed trail $T$ in $H$ with the required properties. ■

**Theorem 9.** *Let $X \in \{N_{1,1,5}, N_{1,3,3}, N_{2,2,3}\}$, and let $G$ be a 3-connected UM-closed $\{K_{1,3}, X\}$-free graph such that $\mathrm{co}(H)$, where $H = L^{-1}(G)$, is 2-connected. Let $e_1, e_2 \in E(H)$ be such that there is no $(e_1, e_2)$-IDT in $H$. Then for every set $A \subset V(\mathrm{co}(H))$, $|A| = 8$, there is an $(e_1, e_2)$-trail $T$ in $H$ such that $A \subset \mathrm{Int}(T)$.*

**Proof.** Let $H'$ be the graph obtained from $H$ by the following construction:

($i$) if $e_1, e_2$ share a vertex of degree 2, say, $e_i = v_iv$, $i = 1, 2$ with $v \in V_2(H)$, we suppress $v$ and set $h = v_1v_2$,

$(ii)$ otherwise, we subdivide $e_i$ (or some edge in $\mathrm{co}(H)$ sharing a vertex with $e_i$ if $e_i$ is pendant) with a vertex $v_i$, $i = 1, 2$, and add a new edge $h = v_1v_2$.

If there is no contraction $\alpha' : H' \to \Pi$ such that $\alpha'(h) = x_1x_2 \in E(\Pi)$ and $\alpha'(A) = V(\Pi) \setminus \{x_1, x_2\}$, then, by Theorem 8, there is a closed trail $T'$ in $H'$ such that $A \subset V(T')$ and $h \in E(T')$. Returning to $H$, i.e., subdividing $h$ in case $(i)$, or removing $h$ and suppressing $v_1, v_2$ (and extending the trail to $e_i$ if $e_i$ is pendant) in case $(ii)$, we obtain an $(e_1, e_2)$-trail $T$ in $H$ with $A \subset \mathrm{Int}(T)$.

Thus, we suppose that there is a contraction $\alpha' : H' \to \Pi$ such that $\alpha'(h) = x_1x_2 \in E(\Pi)$ and $\alpha'(A) = V(\Pi) \setminus \{x_1, x_2\}$. In case $(i)$, $H$ contains a subgraph that can be contracted to a graph isomorphic to the Petersen graph with at least one subdivided edge; however, this graph contains each of the graphs $S_{2,2,6}$, $S_{2,4,4}$ and $S_{3,3,4}$: in the labeling of vertices as in Fig. 2$(a)$, if, say, the edge $p_1^1p_2^1$ is subdivided with a vertex $q$, we have $S_{2,2,6}(p_1^1; qp_2^1; p_5^1p_4^1; p_1^2p_4^2p_2^2p_5^2p_3^2p_3^1)$, $S_{2,4,4}(p_1^1; qp_2^1; p_5^1p_4^1p_3^1p_3^2; p_1^2p_4^2p_2^2p_5^2)$ and $S_{3,3,4}(p_1^1; qp_2^1p_3^1; p_5^1p_4^1p_4^2; p_1^2p_3^2p_5^2p_2^2)$ as subgraphs of $H$, a contradiction. Thus, for the rest of the proof, we suppose that $H'$ is obtained by construction $(ii)$.

Set $H_0 = \mathrm{co}(H)$, and recall that $H_0$ is 3-edge-connected (since $H$ is essentially 3-edge-connected). Let $R'$ be the spanning subgraph of $H'$ that defines $\alpha'$, and suppose that, say, the component $R_1 = (\alpha')^{-1}(x_1)$ of $R'$ is nontrivial. Since $x_1 \in V(\Pi)$, the subgraph $R_1$ is separated from the rest of $H'$ by a 3-edge-cut containing the edge $h$, implying that in $H_0$, the subgraph $R_1 - v_1$ is separated from the rest of $H_0$ by a 2-edge-cut, contradicting the fact that $H_0$ is 3-edge-connected. Hence $(\alpha')^{-1}(x_1)$, and symmetrically also $(\alpha')^{-1}(x_2)$, are trivial, i.e., $V((\alpha')^{-1}(x_i)) = \{v_i\}$, $i = 1, 2$. Removing from $H'$ the edge $h$ and suppressing $v_1$ and $v_2$, we obtain from $R'$ the corresponding spanning subgraph $R$ of $H$, and from $R$, in a standard way a spanning subgraph $R_0$ of $H_0$. Note that clearly every component of $R'$ except $\{v_1\}$ and $\{v_2\}$ corresponds to a nonempty component of $R_0$ since $\alpha'$ maps $H'$ on a cubic graph and hence every component of $R'$ must contain a vertex of degree more than 2. Then the components of $R_0$ define a contraction $\alpha : H_0 \to W$, where $W$ is the Wagner graph (see Fig. 2$(b)$; recall that $W$ can be obtained from $\Pi$ by removing an edge and suppressing the created vertices of degree 2).

**<u>Case 1:</u>** $\alpha^{-1}(w)$ *is trivial for any* $w \in V(W)$.

Then we have $H_0 \simeq W$. By Lemma 6, every vertex of $H_0$ is incident in $H$ to a pendant edge or to a subdivided edge.

We claim that each of the edges $w_1w_5$, $w_2w_6$, $w_3w_7$, $w_4w_8$ (in the labeling of vertices as in Fig. 2$(b)$) is subdivided in $H$. Let, to the contrary, say, $w_1w_5 \in E(H)$. If both $w_1$ and $w_5$ are incident in $H$ to a pendant edge, say, $w_1w_1', w_2w_2' \in E(H)$ with $w_1', w_2' \in V_1(H)$, we consider the subgraphs $T_1 \simeq S_{1,2,6}(w_2; w_6; w_1w_1'; w_3w_7w_8w_4w_5w_5')$, $T_2 \simeq S_{2,3,4}(w_2; w_1w_1'; w_6w_7w_8; w_3w_4w_5w_5')$ and $T_3 \simeq S_{2,3,4}(w_3; w_2w_6; w_4w_5w_5'; w_7w_8w_1w_1')$. We have $d_{T_1}(w_6) = d_{T_2}(w_8) = d_{T_3}(w_6) = 1$ and, by Lemma 6, each of these vertices must be incident in $H$ to a pendant edge or to a subdivided edge. However, it is straightforward to see that in each of the possible cases, each of the subpaths $w_2w_6$ of $T_1$, $w_2w_6w_7w_8$ of $T_2$ and $w_3w_2w_6$ of $T_3$ can be extended in $H$ by one edge, which yields in $H$ from $T_1$ a subgraph $T_1' \simeq S_{2,2,6}$, from $T_2$ a subgraph $T_2' \simeq S_{2,4,4}$, and from $T_3$ a subgraph $T_3' \simeq S_{3,3,4}$, a contradiction. Thus, by symmetry, we can suppose that $w_1$ is not incident in $H$ to a pendant edge.

By Lemma 6, by the assumption that $w_1w_5 \in E(H)$ and by symmetry, we can suppose that

$w_1w_2$ is subdivided in $H$ with a vertex, say, $w' \in V_2(H)$. Now we consider the subgraphs $T_1 \simeq S_{1,2,5}(w_1; w_8; w'w_2; w_5w_4w_3w_7w_6)$, $T_2 \simeq S_{2,3,3}(w_1; w'w_2; w_5w_4w_3; w_8w_7w_6)$ and $T_3 \simeq S_{2,3,3}(w_1; w_5w_4; w'w_2w_3; w_8w_7w_6)$. Then similarly $d_{T_1}(w_6) = d_{T_1}(w_8) = d_{T_2}(w_3) = d_{T_2}(w_6) = d_{T_3}(w_4) = d_{T_3}(w_6) = 1$. Moreover, we observe that each of the sets $\{w_6, w_8\} \subset V(T_1)$, $\{w_3, w_6\} \subset V(T_2)$ and $\{w_4, w_6\} \subset V(T_3)$ is independent in $H$. Since each of these vertices must be incident in $H$ to a pendant edge or to a subdivided edge by Lemma 6, we again easily observe that in each of the possible cases, each of the subpaths $w_1w_8$, $w_1w_5w_4w_3w_7w_6$ of $T_1$, $w_1w_5w_4w_3$, $w_1w_8w_7w_6$ of $T_2$, and $w_1w_5w_4$, $w_1w_8w_7w_6$ of $T_3$ can be extended in $H$ by one edge, which again yields in $H$ from $T_1$ a subgraph $T_1' \simeq S_{2,2,6}$, from $T_2$ a subgraph $T_2' \simeq S_{2,4,4}$, and from $T_3$ a subgraph $T_3' \simeq S_{3,3,4}$, a contradiction. Consequently, we conclude that $w_1w_5 \notin E(H)$, and, by symmetry, $w_iw_{i+4} \notin E(H)$, $i = 1, 2, 3, 4$.

Let $w_i' \in V_2(H)$ be the vertex subdividing the edge $w_iw_{i+4}$ in $H$, $i = 1, 2, 3, 4$. Then we have $S_{2,2,6}(w_1; w_1'w_5; w_8w_4'; w_2w_2'w_6w_7w_3'w_3)$, $S_{2,4,4}(w_1; w_1'w_5; w_8w_4'w_4w_3; w_2w_2'w_6w_7)$ and $S_{3,3,4}(w_1; w_2w_3w_4; w_8w_7w_3'; w_1'w_5w_6w_2')$ as subgraphs of $H$, a contradiction.

**<u>Case 2:</u>** *$\alpha^{-1}(w)$ is nontrivial for some $w \in V(W)$.*

Let $R_1, \ldots, R_8$ be the components of the graph $R$ that defines $\alpha$, and choose the notation such that $R_i = \alpha^{-1}(w_i)$, $i = 1, \ldots, 8$, and such that $R_1 = \alpha^{-1}(w_1)$ is nontrivial. Recall that $\cup_{i=1}^{8}(V(R_i)) = V(R) = V(H_0)$.

We observe that $e_1, e_2 \in E(H_0) \setminus E(R)$ since, by the construction of $H'$, $\alpha^{-1}(x_i) = v_i$ are trivial and after deleting the edge $h$ and suppressing the vertices $v_1, v_2$, each of the edges $e_1, e_2$ has its vertices in different components of $R$. By Theorem 2$(vi)$,$(vii)$, this implies that each $R_i$ is a triangle-free (simple) graph. Moreover, each $R_i$ is 2-edge-connected since $R_i = \alpha^{-1}(w_i)$ is separated from the rest of $H_0$ by a 3-edge-cut and a cut-edge in $R_i$ would create a 2-edge-cut in $H_0$.

We introduce the following notation. For any edge $w_iw_j \in E(W)$, we set $f_{ij} = \alpha^{-1}(w_iw_j)$ (i.e., $f_{ij}$ joins $R_i$ and $R_j$), and we denote $b_j^i$ its vertex in $R_i$ and $b_i^j$ its vertex in $R_j$. Thus, we e.g. have $A_{H_0}(R_1) = \{b_2^1, b_5^1, b_8^1\}$, where $2 \le |\{b_2^1, b_5^1, b_8^1\}| \le 3$, and $\{f_{12}, f_{15}, f_{18}\}$ is the 3-edge-cut that separates $R_1$ from the rest of $H_0$.

<u>Claim 1.</u> *Let $R_i$ be a component of $R$, $1 \le i \le 8$, and let $A_{H_0}(R_i) = \{b_{j_1}^i, b_{j_2}^i, b_{j_3}^i\}$. Then there is a vertex $d^i \in V(R_i)$ and three internally vertex-disjoint (possibly trivial) $(d^i, b_{j_k}^i)$-paths $P_{j_k}^i$, $k = 1, 2, 3$.*

<u>Proof.</u> Let $P$ be an arbitrary (possibly trivial) $(b_{j_1}^i, b_{j_2}^i)$-path in $R_i$, and let $P_{j_3}^i$ be a shortest $(d^i, b_{j_3}^i)$-path with $d^i \in V(P)$. Then the vertex $d^i$ and the paths $P_{j_1}^i = d^iPb_{j_1}^i$ $P_{j_2}^i = d^iPb_{j_2}^i$ and $P_{j_3}^i$ have the required properties. □

<u>Claim 2.</u> *The component $R_1$ contains a cycle $C$ of length at least 4, vertices $c_2, c_5, c_8 \in V(C)$ and paths $Q_2^1, Q_5^1, Q_8^1$ (possibly trivial) such that*

$(i)$ $2 \le |\{c_2, c_5, c_8\}| \le 3$,

$(ii)$ *$Q_2^1$ is a $(c_2, b_2^1)$-path, $Q_5^1$ is a $(c_5, b_5^1)$-path and $Q_8^1$ is a $(c_8, b_8^1)$-path,*

$(iii)$ *the paths $Q_2^1, Q_5^1, Q_8^1$ are internally vertex-disjoint.*

Proof.

Let $d^1$ and $P_2^1$, $P_5^1$, $P_8^1$ be the vertex and paths in $R_1$ given by Claim 1. Since $R_1$ is nontrivial, at least one of $P_2^1, P_5^1, P_8^1$ is nontrivial. Suppose that, say, $P_5^1$ is nontrivial. We consider a $(b_2^1, b_8^1)$-path $P$ and choose two edge-disjoint paths $P_5'$, $P_5''$ such that

- $P_5'$ is a $(b_5^1, c_2)$-path and $P_5''$ is a $(b_5^1, c_8)$-path for some $c_2, c_8 \in V(P)$,
- if $c_2 \neq c_8$, then $c_2$ is on $P$ between $c_8$ and $b_2^1$, and
- $c_2$, $c_8$, $P_5'$ and $P_5''$ are chosen such that $|E(P_5')| + |E(P_5'')|$ is smallest possible.

If $c_2 \neq c_8$, we choose $c_5$ as the last common vertex of $P_5'$ and $P_5''$, and we set $C = c_2Pc_8P_5''c_5P_5'c_2$, $Q_2^1 = c_2Pb_2^1$, $Q_8^1 = c_8Pb_8^1$, and, say, $Q_5^1 = c_5P_5'b_5^1$. If $c_2 = c_8$, we choose $c_5$ as the last common vertex of $P_5'$ and $P_5''$ distinct from the vertex $c_2 = c_8$ (possibly $c_5 = b_5^1$), and set $C = c_2P_5'c_5P_5''c_2$, $Q_2^1 = c_2Pb_2^1$, $Q_8^1 = c_8Pb_8^1$, and, say, $Q_5^1 = c_5P_5'b_5^1$.

If $P_2$ or $P_8$ is nontrivial, we get $C$, $Q_2^1$, $Q_5^1$ and $Q_8^1$ in the same way with the only difference that possibly $c_5 = c_8$ or $c_2 = c_5$. $\square$

By Claim 2, we have, up to a symmetry, the following possibilities (note that $W$ has two types of symmetries – rotations and reflections, but is not edge-transitive): $|\{c_2, c_5, c_8\}| = 3$; $|\{c_2, c_5, c_8\}| = 2$ and $c_2 = c_8$; $|\{c_2, c_5, c_8\}| = 2$ and $c_2 = c_5$. For each of the requested graphs $S_{2,2,6}$, $S_{2,4,4}$ and $S_{3,3,4}$, we describe a subgraph of $H_0$ in which it is contained, in all three possible cases. Here, for integers $i_0, j_0, k_0$, $1 \leq i_0 \leq j_0 \leq k_0$, we use $S_{\geq i_0, \geq j_0, \geq k_0}$ to denote a graph containing an $S_{i_0,j_0,k_0}$ as a subgraph. If a component $R_i$ contains the vertex of degree 3 of the $S_{\geq i_0, \geq j_0, \geq k_0}$, then it is located in the vertex $d^i$ and uses the paths $P_{j_k}^i$, $k = 1, 2, 3$, given by Claim 1, and for any other component $R_i$, $2 \leq i \leq 8$, and $b_j^i, b_k^i \in A_{H_0}(R_i)$, we use $Q_{j,k}^i$ to denote an arbitrarily chosen $(b_j^i, b_k^i)$-path in $R_i$ (of course, if $R_i$ is trivial, all these paths collapse to a single vertex). Finally, we relabel the vertices of the cycle $C$ given by Claim 2 such that $C = u_1u_2 \ldots u_{|V(C)|}$ with $u_1 = c_5$ (and also $u_1 = c_5 = c_2$ in the third case). Then, for each of the graphs $S_{2,2,6}$, $S_{2,4,4}$ and $S_{3,3,4}$, the requested subgraphs can be (for all three cases) described as follows.

| Subgraph | Subgraph of $H_0$ in which is contained |
|---|---|
| $S_{2,2,6}$ | $S_{\geq 2,\geq 2,\geq 6}(d^3; P_2^3Q_{3,6}^2b_2^6; P_7^3Q_{3,8}^7b_7^8; P_4^3Q_{3,5}^4Q_{4,1}^5Q_5^1u_1u_2u_3u_4)$ |
| $S_{2,4,4}$ | $S_{\geq 2,\geq 4,\geq 4}(d^5; P_6^5Q_{5,2}^6b_6^2; P_4^5Q_{5,3}^4Q_{4,7}^3Q_{3,8}^7b_7^8; P_1^5Q_5^1u_1u_2u_3u_4)$ |
| $S_{3,3,4}$ | $S_{\geq 3,\geq 3,\geq 4}(d^5; P_6^5Q_{5,7}^6Q_{6,8}^7b_7^8; P_4^5Q_{5,3}^4Q_{4,2}^3b_3^2; P_1^5Q_5^1u_1u_2u_3u_4)$ |

In each of the possible cases, we have obtained a contradiction. ■

# 5 Proof of Theorem 1

The following lemma, combining techniques developed in the previous sections, will be crucial in our proof.

**Lemma 10.** *Let $G$ be a 3-connected non-Hamilton-connected UM-closed claw-free graph. Then $G$ has an induced subgraph $\tilde{G}$ (possibly $\tilde{G} = G$) such that $\tilde{G}$ is 3-connected, non-Hamilton-connected and UM-closed, and, moreover, $\tilde{H}_0 = \mathrm{co}(L^{-1}(\tilde{G}))$ is 2-connected, and either $c(\tilde{H}_0) \geq 9$ and $|V(\tilde{H}_0)| \geq 10$, or $\tilde{H}_0 \simeq W$.*

**Proof.** Let $H = L^{-1}(G)$, and set $H_0 = \mathrm{co}(H)$. By Theorem K($ii$), $H_0$ is 3-edge-connected.

Suppose first that $H_0$ is not 2-connected, let $B_1^0, \ldots, B_b^0$ be blocks of $H_0$, let $B_1, \ldots, B_b$ be the corresponding subgraphs of $H$ (i.e., $B_i^0 = \mathrm{co}(B_i)$, $i = 1, \ldots, b$), and let $B_i'$ be obtained from $B_i$ by attaching a pendant edge to every its vertex which is a cutvertex of $H_0$, $i = 1, \ldots, b$. Then obviously $\mathrm{co}(B_i') = \mathrm{co}(B_i) = B_i^0$, and $B_i^0$ is 2-connected, $i = 1, \ldots, b$. If every $B_i'$ has an $(f_1, f_2)$-IDT for any $f_1, f_2 \in E(B_i')$, then an easy induction shows that $G = L(H)$ is Hamilton-connected, a contradiction. Hence there is a $B_{i_0}'$ having no $(f_1, f_2)$-IDT for some $f_1, f_2 \in E(B_{i_0}')$.

Set $\tilde{H} = B_{i_0}'$ and $\tilde{G} = L(\tilde{H})$. Then $\tilde{G}$ is an induced subgraph of $G$ (since $\tilde{H}$ is a subgraph of $H$), is 3-connected (since $\tilde{H}$ is essentially 3-edge-connected), non-Hamilton-connected (since $\tilde{H} = B_{i_0}'$ has no $(f_1, f_2)$-IDT) and UM-closed (since a local completion in $\tilde{G}$ is a local completion in $G$), and, by the construction, $\tilde{H}_0 = \mathrm{co}(\tilde{H}) = B_{i_0}^0$ is 2-connected. By Theorem K($v$), $\tilde{H}_0$ is not strongly spanning trailable, implying that, by Theorem D, $c(\tilde{H}_0) \geq 9$ and $|V(\tilde{H}_0)| \geq 10$, unless $\tilde{H}_0 \simeq W$ or $\tilde{H}_0 \in \mathbb{W}$.

Suppose that $\tilde{H}_0 \in \mathbb{W}$. By Theorem 2($vi$), $\tilde{H}_0$ contains only one double edge. Let $\{e_1, e_2\}$ be the double edge in $\tilde{H}_0$. By the definition of $\mathbb{W}$ and by symmetry, there are two possibilities: $V(e_1) = V(e_2) = \{w_1 z\}$ with $zw_2 \in E(\tilde{H}_0)$, or $V(e_1) = V(e_2) = \{w_1 z\}$ with $zw_5 \in E(\tilde{H}_0)$, however, in the first case $e_1 z w_2 w_3 w_4 w_5 w_6 w_7 w_8 w_1 e_2$, and in the second case $e_1 w_1 w_2 w_3 w_4 w_8 w_7 w_6 w_5 z e_2$ is an $(e_1, e_2)$-IDT in $\tilde{H}$, contradicting Theorem 2($vii$)($\beta$). Hence $\tilde{H}_0 \notin \mathbb{W}$. Thus, we conclude that $c(\tilde{H}_0) \geq 9$ and $|V(\tilde{H}_0)| \geq 10$, unless $\tilde{H}_0 \simeq W$. ■

**Proof of Theorem 1**. Let $G$ be a 3-connected $\{K_{1,3}, N_{1,3,3}\}$-free graph and suppose, to the contrary, that $G$ is not Hamilton-connected. By Theorem 2 and Theorem M, we can suppose that $G$ is UM-closed. Let thus $H = L^{-1}(G)$, and set $H_0 = \mathrm{co}(H)$. By Theorem K($ii$), $H_0$ is 3-edge-connected. By Lemma 10, we can suppose that $H_0$ is 2-connected and $c(H_0) \geq 9$ and $|V(H_0)| \geq 10$, unless $H_0 \simeq W$. Then, by Theorems 9 and C($ii$), we have the following claim.

<u>Claim 1.</u> *Let $A \subset V(H_0)$ be such that $|A| = 8$. Then $A$ does not dominate all edges of $H$.*

<u>Proof.</u> Since $G$ is not Hamilton-connected, by Theorem C($ii$), there are edges $e_1, e_2 \in E(H)$ such that there is no $(e_1, e_2)$-IDT in $H$. Then, by Theorem 9, there is an $(e_1, e_2)$-trail $T$ in $H$ such that $A \subset \mathrm{Int}(T)$. But if $A$ dominates all the edges in $H$, then $T$ would be an $(e_1, e_2)$-IDT in $H$. □

Now, if $H_0 \simeq W$, then $|V(H_0)| = 8$ and $V(H_0)$ dominates all edges of $H$, contradicting Claim 1. Thus, we have $c(H_0) \geq 9$ and $|V(H_0)| \geq 10$. Moreover, $H$ does not contain as a subgraph the graph $S_{2,4,4}$. We consider the possible cases separately.

Throughout the proof, in each of the cases, $C = x_1 x_2 \ldots x_{c(H_0)}$ always denotes a longest cycle in $H_0$, $R = V(H) \setminus V(C)$, $N = \{y \in V(H_0) | \; N_R(y) = \emptyset\}$, $R_0 = R \cap V(H_0)$, and if $R_0 \neq \emptyset$, we set $R_0 = \{y_1, \ldots, y_{|R_0|}\}$ and we choose the notation such that $y_1 x_1 \in E(H_0)$.

<u>Claim 2.</u> *If $E(\langle R \rangle_H) = \emptyset$, then $H_0$ has no double edge.*

<u>Proof.</u> Suppose that $H_0$ has a double edge $\{e, f\}$. Then, since $V(C)$ dominates all edges of $H$, it follows that $H$ has an $(e, f)$-IDT. But then, by Theorem 2($vii$)($\beta$), $G$ has an $(a, b)$-path for every pair $a, b \in V(G)$, a contradiction. □

In the proof, we will often list vertices of a subgraph $S_{i,j,k}$. There are two general comments to all these situations.

- When some edge $e = x_i x_j$ of the $S_{i,j,k}$ is in $E(H_0)$, it can always happen that $e$ is subdivided in $H$, i.e., formally, $e \notin E(H)$. However, it is immediate to see that if this happens, then the corresponding subgraph of $H$, which instead of $e = x_i x_j$ contains a path $x_i z x_j$ with $z \in V_2(H)$, also contains $S_{i,j,k}$ as a subgraph.
- When a vertex $x_i \in V(C)$ has a (potential) neighbor $z \in R$ and the vertex $z$ occurs as the last vertex of a branch of the $S_{i,j,k}$, then such a vertex $z$ can be an endvertex of a pendant edge attached to $x_i$, or can be $z \in V_2(H)$ and $z$ subdivides some of the edges incident to $x_i$. It should be noted that in the second case, the vertices $x_i$ and $z$ can occur in reverse order in the list (i.e., $x_i$ being the last vertex of the branch).

Throughout the proof, we always implicitly understand that there are also these possibilities.

**<u>Case 1:</u>** $c(H_0) = 9$ *and* $|V(H_0)| \geq 10$.

If $E(\langle R \rangle_H) \neq \emptyset$, we can choose an edge $e \in E(\langle R \rangle_H)$ and the notation such that $e = y_1 z$ for some $z \in R$, and then $H$ contains $S_{2,4,4}(x_1; y_1 z; x_2 x_3 x_4 x_5; x_9 x_8 x_7 x_6)$, a contradiction. Thus, $R$ is an independent set in $H$, i.e., $R \subset N$. Specifically, $R_0 \subset N$. By Claim 2, $H_0$ has no double edge. Since $H_0$ is 3-edge-conncted, $y_1$ has three distinct neighbors on $C$. Clearly, no two neighbors of $y_1$ can be consecutive on $C$ since $C$ is longest. Since $|V(C)| = 9$, we can choose the notation such that either $y_1 x_3 \in E(H_0)$, or $y_1 x_4 \in E(H_0)$ (see Fig. 8).

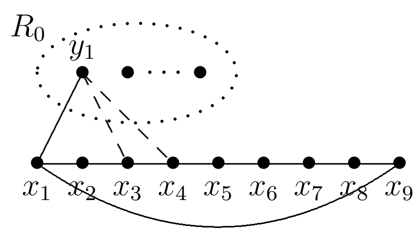


Figure 8: The situation in Case 1

**<u>Subcase 1.1:</u>** $y_1 x_3 \in E(H_0)$.

If $x_2 z \in E(H)$ for some $z \in R$, then $H$ contains $S_{2,4,4}(x_1; x_2 z; y_1 x_3 x_4 x_5; x_9 x_8 x_7 x_6)$, a contradiction. Hence $x_2 \in N$. We set $A = \{x_1, x_3, x_4, x_5, x_6, x_7, x_8, x_9\}$. Then $|A| = 8$ and $A$ dominates all edges of $H$, contradicting Claim 1.

**<u>Subcase 1.2:</u>** $y_1 x_4 \in E(H_0)$.

Similarly, if $x_9 z \in E(H)$ for some $z \in R$, we have $S_{2,4,4}(x_4; x_3 x_2; x_5 x_6 x_7 x_8; y_1 x_1 x_9 z)$ in $H$, a contradiction; hence $R_0 \cup \{x_9\} \subset N$. We set $A = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$. Then again $|A| = 8$ and $A$ dominates all edges of $H$, contradicting Claim 1.

**<u>Case 2:</u>** $c(H_0) = |V(H_0)| = 10$.

Since $H_0$ is 3-edge-connected, we have $d_{H_0}(x_1) \geq 3$, and since $c(H_0) = |V(H_0)|$, $x_1$ has, besides $x_2$ and $x_{10}$, another neighbor on $C$. By Theorem 2($vi$), we can suppose that $x_1$ is not in a triangle. Thus, by symmetry, $x_1$ is adjacent to $x_4$, $x_5$ or $x_6$.

**<u>Subcase 2.1:</u>** $x_1 x_4 \in E(H_0)$.

We show that $\{x_6, x_7, x_8, x_9\} \subset N$. Let thus $z \in R$. If $x_6z \in E(H)$, then we have $S_{2,4,4}(x_1; x_2x_3; x_4x_5x_6z; x_{10}x_9x_8x_7)$ in $H$, and if $x_7z \in E(H)$, we have in $H$ the subgraph $S_{2,4,4}(x_4; x_3x_2; x_5x_6x_7z; x_1x_{10}x_9x_8)$; the cases $x_8z \in E(H)$ and $x_9z \in E(H)$ are symmetric. Thus, $\{x_6, x_7, x_8, x_9\} \subset N$. Now, if $x_6x_8 \notin E(H_0)$, then the set $A = \{x_1, x_2, x_3, x_4, x_5, x_7, x_9, x_{10}\}$ dominates all edges of $H$ and $|A| = 8$, contradicting Claim 1. Hence $x_6x_8 \in E(H_0)$, and, symmetrically, $x_7x_9 \in E(H_0)$. But then $\langle\{x_6, x_7, x_8, x_9\}\rangle_H$ is a diamond or a $K_4$ in $H_0$, contradicting Theorem 2.

**Subcase 2.2:** $x_1x_6 \in E(H_0)$.

Then $x_7 \in N$ for otherwise, for some $z \in R$ with $zx_7 \in E(H)$, $H$ contains the subgraph $S_{2,4,4}(x_6; x_7z; x_5x_4x_3x_2; x_1x_{10}x_9x_8)$. Symmetrically, $x_{10} \in N$. Then the set $A = V(C) \setminus \{x_7, x_{10}\}$ dominates all edges in $H$, unless $x_7x_{10} \in E(H_0)$. Since $|A| = 8$, we have $x_7x_{10} \in E(H_0)$ by Claim 1, but then, for the vertex $x_7$, we are back in Subcase 2.1.

**Subcase 2.3:** $x_1x_5 \in E(H_0)$.

Then $x_4 \in N$ (otherwise we have $S_{2,4,4}(x_1; x_{10}x_9; x_2x_3x_4z; x_5x_6x_7x_8)$ in $H$ for some $z \in R$ with $zx_4 \in E(H)$), and, symmetrically, $x_2 \in N$. Considering the set $A = V(C) \setminus \{x_2, x_4\}$ with $|A| = 8$, we have $x_2x_4 \in E(H_0)$ by Claim 1. Then $x_3 \in N$, for otherwise we have $S_{2,4,4}(x_2; x_3z; x_4x_5x_6x_7; x_1x_{10}x_9x_8)$ in $H$ for some $z \in R$, $zx_3 \in E(H)$. Since $d_{H_0}(x_3) \geq 3$ and $x_3 \in N$, $x_3$ has, besides $x_2$ and $x_4$, another neighbor on $C$. Then, by the previous cases, since $H_0$ does not contain a diamond and by symmetry, the only possibility is that $x_3x_7 \in E(H_0)$. Then $x_6 \in N$, for otherwise we have $S_{2,4,4}(x_3; x_2x_1; x_4x_5x_6z; x_7x_8x_9x_{10})$ in $H$ for some $z \in R$, $zx_6 \in E(H)$. Thus, we have $x_3, x_6 \in N$, and considering the set $A = V(C) \setminus \{x_3, x_6\}$ with $|A| = 8$, we have $x_3x_6 \in E(H_0)$ by Claim 1. But then, for the vertex $x_3$, we are back in Subcase 2.1.

**Case 3:** $c(H_0) = 10$ *and* $|V(H_0)| \geq 11$.

First observe that $E(\langle R\rangle_H) = \emptyset$: if, say, $y_1z \in E(H)$ for some $z \in R$, then we have $S_{2,4,4}(x_1; y_1z; x_2x_3x_4x_5; x_{10}x_9x_8x_7)$ in $H$, a contradiction. Hence $R$ is an independent set, implying $R \subset N$. By Claim 2, $H_0$ has no double edge, hence $y_1$ has three distinct neighbors on $C$. If $y_1x_4 \in E(H_0)$, we have $S_{2,4,4}(x_1; x_2x_3; y_1x_4x_5x_6; x_{10}x_9x_8x_7)$ in $H$, and if $y_1x_6 \in E(H_0)$, we have $S_{2,4,4}(x_1; x_2x_3; y_1x_6x_5x_4; x_{10}x_9x_8x_7)$ in $H$. Hence each of the subpaths of $C$ determined by any two neighbors of $y_1$ on $C$ has odd number of interior vertices. This implies that, up to a symmetry, either $\{x_1, x_3, x_7\} \subset N_C(y_1)$, or $\{x_1, x_3, x_5\} \subset N_C(y_1)$. But if $\{x_1, x_3, x_7\} \subset N_C(y_1)$, we have $S_{2,4,4}(y_1; x_1x_2; x_3x_4x_5x_6; x_7x_8x_9x_{10})$ in $H$, hence $\{x_1, x_3, x_5\} \subset N_C(y_1)$.

Then $x_2 \in N$ (otherwise we have $S_{2,4,4}(x_1; x_2z; y_1x_5x_4x_3; x_{10}x_9x_8x_7)$ in $H$ for some $z \in R$, $zx_2 \in E(H_0)$), and, symmetrically, $x_4 \in N$. Since also $y_1 \in N$, considering the set $A = V(C) \setminus \{x_2, x_4\}$ with $|A| = 8$, by Claim 1, $H_0$ contains some of the edges $x_2y_1$, $x_4y_1$, $x_2x_4$. However, each of these edges can be used to extend $C$ through $y_1$, contradicting the fact that $C$ is a longest cycle.

**Case 4:** $c(H_0) = |V(H_0)| \geq 11$.

Set $t = c(H_0) = |V(H_0)|$. Since $d_{H_0}(x_1) \geq 3$, $x_1$ has in $H_0$, besides $x_2$ and $x_t$, another neighbor on $C$. Since $H_0$ has at most two triangles and $t \geq 11$, we can choose the notation such that $x_1$ is not in a triangle, i.e., $x_1x_3 \notin E(H_0)$. Then we have, up to a symmetry, the following possibilities:

| Case | Possible values of $t$, $i$ | $S_{2,4,4}$ in $H$ |
|---|---|---|
| $x_1x_4 \in E(H_0)$ | $t \geq 11$ | $S_{2,4,4}(x_1; x_2x_3; x_4x_5x_6x_7; x_tx_{t-1}x_{t-2}x_{t-3})$ |
| $x_1x_5 \in E(H_0)$ | $t \geq 12$ | $S_{2,4,4}(x_1; x_2x_3; x_5x_6x_7x_8; x_tx_{t-1}x_{t-2}x_{t-3})$ |
| $x_1x_i \in E(H_0)$ | $t \geq 11$, $6 \leq i \leq \lfloor \frac{t}{2} + 1 \rfloor$ | $S_{2,4,4}(x_1; x_ix_{i+1}; x_2x_3x_4x_5; x_tx_{t-1}x_{t-2}x_{t-3})$ |

Thus, the only remaining case is $x_1x_5 \in E(H_0)$ for $t = 11$. By the same argument for $x_3$ we have, up to a symmetry, $x_3x_5 \in E(H_0)$ or $x_3x_7 \in E(H_0)$. If $x_3x_5 \in E(H_0)$, then for $x_4$ we have, by symmetry and since $H_0$ does not contain a diamond, the only possibility $x_4x_8 \in E(H_0)$, and then we have $S_{2,4,4}(x_8; x_7x_6; x_4x_5x_3x_2; x_9x_{10}x_{11}x_1)$ in $H$ (note that the argument does not use the edge $x_1x_5$), and if $x_3x_7 \in E(H_0)$, we immediately have $S_{2,4,4}(x_1; x_5x_4; x_2x_3x_7x_6; x_{11}x_{10}x_9x_8)$ in $H$.

**Case 5:** *$c(H_0) \geq 11$ and $|V(H_0)| > c(H_0)$.*
Set $c(H_0) = t$. Immediately $E(\langle R\rangle_H) = \emptyset$, since if e.g. $y_1z \in E(H)$ for some $z \in R$, we have $S_{2,4,4}(x_1; y_1z; x_2x_3x_4x_5; x_tx_{t-1}x_{t-2}x_{t-3})$ in $H$. Thus, $R \subset N$ and $C$ is dominating in $H$, implying that $y_1$ cannot be connected to $C$ by a double edge by Claim 2. Since $d_{H_0}(y_1) \geq 3$, $y_1$ has in $H_0$, besides $x_1$, another neighbor on $C$. We consider the following possibilities:

| Case | Possible values of $t$, $i$ | $S_{2,4,4}$ in $H$ |
|---|---|---|
| $y_1x_4 \in E(H_0)$ | $t \geq 11$ | $S_{2,4,4}(x_1; x_2x_3; y_1x_4x_5x_6; x_tx_{t-1}x_{t-2}x_{t-3})$ |
| $y_1x_5 \in E(H_0)$ | $t \geq 11$ | $S_{2,4,4}(x_1; x_2x_3; y_1x_5x_6x_7; x_tx_{t-1}x_{t-2}x_{t-3})$ |
| $y_1x_i \in E(H_0)$ | $t \geq 11$, $6 \leq i \leq \lfloor \frac{t}{2} + 1 \rfloor$ | $S_{2,4,4}(x_1; y_1x_i; x_2x_3x_4x_5; x_tx_{t-1}x_{t-2}x_{t-3})$ |

Thus, the only remaining case is $y_1x_3 \in E(H_0)$. Since $d_{H_0}(y_1) \geq 3$, $y_1$ has in $H_0$, besides $x_1$ and $x_3$, another neighbor $x_i$ on $C$, $i \neq 1, 3$, and then, for some two of the three vertices $x_1$, $x_3$, $x_i$, we are back in some of the previous cases.

■

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