Hamilton-connected {claw,net}-free graphs, II

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Abstract

In the first one in this series of two papers, we have proved that every 3-connected $\{K_{1,3}, N_{1,3,3}\}$-free graph is Hamilton-connected. In this paper, we continue in this direction by proving that every 3-connected $\{K_{1,3}, X\}$-free graph, where $X \in \{N_{1,1,5}, N_{2,2,3}\}$, is Hamilton-connected (where $N_{i,j,k}$ is the graph obtained by attaching endvertices of three paths of lengths i, j, k to a triangle). This together with a previous result of other authors completes the characterization of forbidden induced generalized nets implying Hamilton-connectedness of a 3-connected claw-free graph. We also discuss remaining open cases in a full characterization of connected graphs X such that every 3-connected $\{K_{1,3}, X\}$-free graph is Hamilton-connected.

Keywords: Hamilton-connected; closure; forbidden subgraph; claw-free; net-free

1 Introduction

In this paper, we generally follow the most common graph-theoretical notations and terminology as given e.g. in [3], and for problem-specific notations and terminology we refer to the first paper of this series, [12]. We recall here the special graphs in Fig. 1 that will be important for our results.

The following two results were proved in [13] and in [12], respectively.

Theorem A. *Let G be a 3-connected $\{K_{1,3}, X\}$-free graph, where*
(i) **[13]** $X = N_{1,2,4}$, *or*
(ii) **[12]** $X = N_{1,3,3}$.
Then G is Hamilton-connected.

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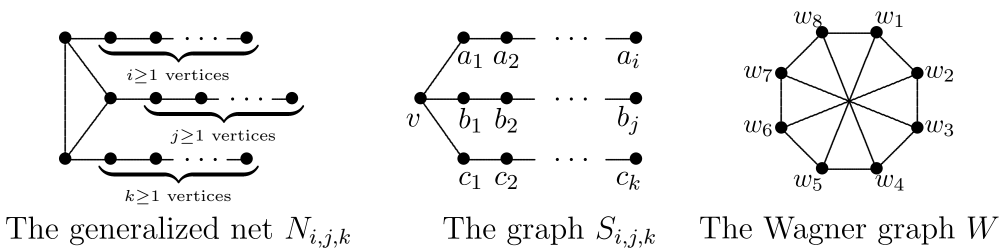


Figure 1: The graphs $N_{i,j,k}$, $S_{i,j,k}$ and the Wagner graph W

Recall that these results are sharp, as can be seen by considering the family of graphs $\mathcal{G} = \{L(H)|\ H \in \mathcal{W}\}$, where $\mathcal{W}$ is the family of graphs obtained by attaching at least one pendant edge to each of the vertices of the Wagner graph W (see Fig. 1). Then any $G \in \mathcal{G}$ is 3-connected, non-Hamilton-connected and $N_{i,j,k}$-free for $i + j + k \geq 8$. Hence the possible values that might imply a 3-connected $\{K_{1,3}, N_{i,j,k}\}$-free graph to be Hamilton-connected are those with $i + j + k \leq 7$. For $i + j + k = 7$, there are four possibilities, namely, $N_{1,1,5}$, $N_{1,2,4}$, $N_{1,3,3}$ and $N_{2,2,3}$. While the second and third possibilities show sharpness of Theorem A, the first and last ones show that the next result, which is the main result of this paper, is also sharp, and completes the characterization.

Theorem 1. *Let $X \in \{N_{1,1,5}, N_{2,2,3}\}$, and let G be a 3-connected $\{K_{1,3}, X\}$-free graph. Then G is Hamilton-connected.*

The proof of Theorem 1, which is a careful case analysis, is postponed to Section 3. In Section 2, we collect necessary known results and facts that allow to significantly reduce the number of cases to be considered. Finally, in Section 4, we discuss sharpness and remaining open cases.

2 Preliminaries

In this section, we summarize some known facts that will be needed in our proof of Theorem 1. All these fact and results are contained already in the first paper of this series, [12], and we include them here for the sake of completeness.

2.1 Line graphs of multigraphs and their preimages

The *line graph* of a multigraph H is the graph $G = L(H)$ with $V(G) = E(H)$, in which two vertices are adjacent if and only if the corresponding edges of H share at least one vertex. It is well-known that in line graphs of multigraphs, for a given line graph G, a multigraph H such that $G = L(H)$ is not uniquely determined. As shown in [16], this drawback can be overcome by an additional requirement that simplicial vertices correspond to pendant edges.

Theorem B [16]. *Let G be a connected line graph of a multigraph. Then there is, up to an isomorphism, a uniquely determined multigraph H such that $G = L(H)$ and a vertex $e \in V(G)$ is simplicial in G if and only if the corresponding edge $e \in E(H)$ is a pendant edge in H.*

The multigraph H with the properties given in Theorem B will be called the *preimage* of a line graph G and denoted $H = L^{-1}(G)$. We will also use the notation $a = L(e)$ and $e = L^{-1}(a)$ for an edge $e \in E(H)$ and the corresponding vertex $a \in V(G)$.

An edge-cut $R \subset E(H)$ of a multigraph H is *essential* if $H - R$ has at least two nontrivial components, and H is *essentially k-edge-connected* if every essential edge-cut of H is of size at least k. It is a well-known fact that a line graph G is k-connected if and only if $L^{-1}(G)$ is essentially k-edge-connected. It is also a well-known fact that if X is a line graph, then a line graph G is X-free if and only if $L^{-1}(G)$ does not contain as a subgraph (not necessarily induced) a graph F such that $L(F) = X$ (but not necessarily $F = L^{-1}(X)$). However, it is straightforward to verify that for the graph $N_{i,j,k}$ there is exactly one graph F such that $L(F) = N_{i,j,k}$, namely, the graph $L^{-1}(N_{i,j,k}) = S_{i+1,j+1,k+1}$ (see Fig. 1). Thus, we can conclude that a line graph G is $N_{i,j,k}$-free if and only if $L^{-1}(G)$ does not contain as a (not necessarily induced) subgraph the graph $L^{-1}(N_{i,j,k}) = S_{i+1,j+1,k+1}$. Recall that when listing vertices of an $S_{i,j,k}$ in a graph, we will write the list such that $i \le j \le k$, and we will use the notation $S_{i,j,k}(v; a_1a_2 \ldots a_i; b_1b_2 \ldots b_j; c_1c_2 \ldots c_k)$ (in the labeling of vertices as in Fig. 1).

Recall that a closed trail T is a *dominating closed trail* (abbreviated DCT) if T dominates all edges of G, and an (e, f)-trail is *an internally dominating (e, f)-trail* (abbreviated (e, f)-IDT) if $\mathrm{Int}(T)$ dominates all edges of G. The following result shows the relation between a hamiltonian cycle (hamiltonian (a_1, a_2)-path) in $G = L(H)$ and a DCT (an (e_1, e_2)-IDT) in H.

Theorem C. *Let H be a multigraph with $|E(H)| \ge 3$ and let $G = L(H)$.*

(i) **[8]** *The graph G is hamiltonian if and only if H has a DCT.*

(ii) **[11]** *For every $e_i \in E(H)$ and $a_i = L(e_i)$, $i = 1, 2$, G has a hamiltonian (a_1, a_2)-path if and only if H has an (e_1, e_2)-IDT.*

2.2 SM-closure

For $x \in V(G)$, the *local completion of G at x* is the graph $G^*_x = (V(G), E(G) \cup \{y_1y_2 |\ y_1, y_2 \in N_G(x)\})$ (i.e., G^*_x is obtained from G by adding all the missing edges with both vertices in $N_G(x)$). Obviously, if G is claw-free, then so is G^*_x. Note that in the special case when G is a line graph and $H = L^{-1}(G)$, we have $G^*_x = L(H|_e)$, where $e = L^{-1}(x)$, and $H|_e$ is obtained from H by contraction of e into a vertex and replacing the created loop(s) by pendant edge(s) (for more details on the contraction operation see Subsection 2.5). Finally, a vertex $x \in V(G)$ is *eligible* if $N_G(x)$ induces in G a connected noncomplete graph.

In [10], the concept of an SM-closure G^M of a claw-free graph G was defined by the following construction.

(i) If G is Hamilton-connected, we set $G^M = \mathrm{cl}(G)$.

(ii) If G is not Hamilton-connected, we recursively perform the local completion operation at such eligible vertices for which the resulting graph is still not Hamilton-connected, as long as this is possible. We obtain a sequence of graphs $G_1, \ldots, G_k$ such that

- $G_1 = G$,
- $G_{i+1} = (G_i)^*_{x_i}$ for some $x_i \in V_{EL}(G_i)$, $i = 1, \ldots, k-1$,
- G_k has no hamiltonian (a, b)-path for some $a, b \in V(G_k)$,
- for any $x \in V_{EL}(G_k)$, $(G_k)^*_x$ is Hamilton-connected,

and we set $G^M = G_k$.

A resulting G^M is called a *strong M-closure* (or briefly an *SM-closure*) of the graph G, and a graph G equal to its SM-closure is said to be *SM-closed*. Note that for a given graph G, its SM-closure is not uniquely determined.

As shown in [16] and [10], if G is SM-closed, then $G = L(H)$, where H does not contain any of the multigraphs shown in Fig. 2.

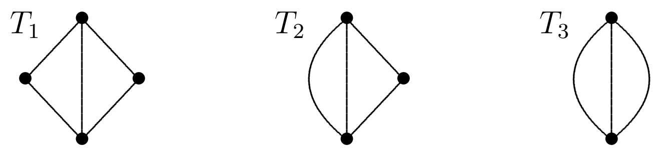


Figure 2: The diamond T_1, the multitriangle T_2 and the triple edge T_3

The following theorem summarizes basic properties of the SM-closure operation.

Theorem D [10]. *Let G be a claw-free graph and let G^M be one of its SM-closures. Then G^M has the following properties:*

(i) *$V(G) = V(G^M)$ and $E(G) \subset E(G^M)$,*
(ii) *G^M is obtained from G by a sequence of local completions at eligible vertices,*
(iii) *G is Hamilton-connected if and only if G^M is Hamilton-connected,*
(iv) *if G is Hamilton-connected, then $G^M = \mathrm{cl}(G)$,*
(v) *if G is not Hamilton-connected, then either*
  (α) *$V_{EL}(G^M) = \emptyset$ and $G^M = \mathrm{cl}(G)$, or*
  (β) *$V_{EL}(G^M) \neq \emptyset$ and $(G^M)^*_x$ is Hamilton-connected for any $x \in V_{EL}(G^M)$,*
(vi) *$G^M = L(H)$, where H contains either*
  (α) *at most 2 triangles and no multiedge, or*
  (β) *no triangle, at most one double edge and no other multiedge,*
(vii) *if G^M contains no hamiltonian (a,b)-path for some $a, b \in V(G^M)$ and*
  (α) *X is a triangle in H, then $E(X) \cap \{L^{-1}_{G^M}(a), L^{-1}_{G^M}(b)\} \neq \emptyset$,*
  (β) *X is a multiedge in H, then $E(X) = \{L^{-1}_{G^M}(a), L^{-1}_{G^M}(b)\}$.*

We will also need the following lemma on SM-closed graphs proved in [17].

Lemma E [17]. *Let G be an SM-closed graph, let $H = L^{-1}(G)$ and let F be the graph with $V(F) = \{v_1, v_2, v_3, v_4, v_5, z\}$ and $E(F) = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_1, v_3v_5, zv_1, zv_2\}$ (see Fig. 3). Then H does not contain a subgraph $\bar{H}$ isomorphic to the graph F such that $N_H(\{v_1, v_2, v_3, v_5\}) \subset V(\bar{H})$.*

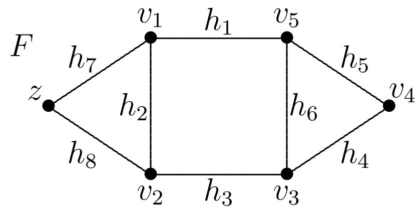


Figure 3: The graph F

2.3 The core of the preimage of an SM-closed graph

The definition of the core is slightly problematic for multigraphs, therefore we restrict our observations to the case that we need, i.e., to preimages of 3-connected SM-closed graphs. The difficulties then do not occur since such a multigraph cannot have pendant multiedges by Theorem B, and cannot have pendant multitriangles (since there are no multitriangles at all).

Thus, let G be a 3-connected SM-closed graph and let $H = L^{-1}(G)$. The *core of* H is the multigraph $\mathrm{co}(H)$ obtained from H by removing all pendant edges and suppressing all vertices of degree 2.

Shao [20] proved the following properties of the core of a multigraph.

Theorem F [20]. *Let H be an essentially 3-edge-connected multigraph. Then*

(i) $\mathrm{co}(H)$ *is uniquely determined,*

(ii) $\mathrm{co}(H)$ *is 3-edge-connected,*

(iii) $V(\mathrm{co}(H))$ *dominates all edges of* H*,*

(iv) *if* $\mathrm{co}(H)$ *has a spanning closed trail, then* H *has a DCT.*

2.4 UM-closure

As shown in [12], the concept of SM-closure can be further strengthened by omitting the eligibility assumption in the local completion operation. Specifically, for a given claw-free graph G, we construct a graph G^U by the following construction.

(i) If G is Hamilton-connected, we set $G^U = K_{|V(G)|}$.

(ii) If G is not Hamilton-connected, we recursively perform the local completion operation at such vertices for which the resulting graph is still not Hamilton-connected, as long as this is possible. We obtain a sequence of graphs $G_1, \ldots, G_k$ such that

- $G_1 = G$,
- $G_{i+1} = (G_i)^*_{x_i}$ for some $x_i \in V(G_i)$, $i = 1, \ldots, k-1$,
- G_k has no hamiltonian (a,b)-path for some $a, b \in V(G_k)$,
- for any $x \in V(G_k)$, $(G_k)^*_x$ is Hamilton-connected,

and we set $G^U = G_k$.

A graph G^U obtained by the above construction is called an *ultimate M-closure* (or briefly a *UM-closure*) of the graph G, and a graph G equal to its UM-closure is said to be *UM-closed*.

Obviously, by the definition, if G is UM-closed, then G is also SM-closed, implying that G is a line graph and $H = L^{-1}(G)$ has special structure (contains no diamond etc. – see Fig. 2 and Theorem D $(vi), (vii)$). The next theorem shows that for UM-closed graphs, not only H, but also $\mathrm{co}(H)$ has these strong structural properties.

Theorem G [12]. *Let G be a claw-free graph and let G^U be one of its UM-closures. Then G^U has the following properties:*

(i) $V(G) = V(G^U)$ *and* $E(G) \subset E(G^U)$*,*

(ii) G^U *is obtained from* G *by a sequence of local completions at vertices,*

(iii) G *is Hamilton-connected if and only if* G^U *is Hamilton-connected,*

(iv) *if* G *is Hamilton-connected, then* $G^U = K_{|V(G)|}$*,*

(v) *if* G *is not Hamilton-connected, then* $(G^U)^*_x$ *is Hamilton-connected for any* $x \in V(G^U)$*,* (*vi*) $G^U = L(H)$, *where* $\mathrm{co}(H)$ *contains no diamond, no mutitriangle and no triple edge, and either*

(α) *at most 2 triangles and no multiedge, or*

(β) *no triangle, at most one double edge and no other multiedge, and if* $\mathrm{co}(H)$ *contains a double edge, then this double edge is also in* H,

(*vii*) *if* G^U *contains no hamiltonian* (a, b)*-path for some* $a, b \in V(G^U)$ *and*

(α) X *is a triangle in* $\mathrm{co}(H)$, *then* $E(X) \cap \{L^{-1}_{G^U}(a), L^{-1}_{G^U}(b)\} \neq \emptyset$,

(β) X *is a multiedge in* $\mathrm{co}(H)$, *then* $E(X) = \{L^{-1}_{G^U}(a), L^{-1}_{G^U}(b)\}$.

The following result was first established in [5], and later on reconsidered in [14] in a more general setting.

Theorem H [14]. *Let* G *be a* $\{K_{1,3}, N_{i,j,k}\}$*-free graph,* $i, j, k \geq 1$, *and let* $x \in V(G)$. *Then the graph* G^*_x *is* $\{K_{1,3}, N_{i,j,k}\}$*-free.*

Specifically, Theorem H implies that a UM-closure of a $\{K_{1,3}, N_{i,j,k}\}$-free graph is also $\{K_{1,3}, N_{i,j,k}\}$-free.

2.5 A-contractible multigraphs

For a multigraph H and $F \subset H$, $H|_F$ denotes the multigraph obtained from H by identifying the vertices of F as a (new) vertex v_F, and by replacing the created loops by pendant edges. Specifically, if $E(F) = \{e\}$, we simply write $G|_e$. If H is a multigraph, $X \subset V(H)$, and $\mathcal{A}$ is a partition of X into subsets, then $E(\mathcal{A})$ denotes the set of all edges a_1a_2 (not necessarily in H) such that a_1, a_2 are in the same element of $\mathcal{A}$. Further $H^{\mathcal{A}}$ denotes the multigraph with vertex set $V(H^{\mathcal{A}}) = V(H)$ and edge set $E(H^{\mathcal{A}}) = E(H) \cup E(\mathcal{A})$ (where $E(H)$ and $E(\mathcal{A})$ are considered to be disjoint, i.e., if $e_1 = a_1a_2 \in E(H)$ and $e_2 = a_1a_2 \in E(\mathcal{A})$, then e_1, e_2 are parallel edges in $H^{\mathcal{A}}$).

Let F be a multigraph and let $A \subset V(F)$. We say that F is A*-contractible*, if for every even subset $X \subset A$ and for every partition $\mathcal{A}$ of X into two-element subsets, the graph $F^{\mathcal{A}}$ has a DCT containing all vertices of A and all edges of $E(\mathcal{A})$. Note that this definition allows X to be empty, in which case $F^{\mathcal{A}} = F$. Also, if F is A-contractible, then F is A'-contractible for any $A' \subset A$ (since every subset X of A' is a subset of A). As shown in [15], if F is an $A_H(F)$-contractible submultigraph of a multigraph H, then H has a DCT if and only if $H|_F$ has a DCT (note that the concept was defined in [15] for graphs, but it is easy to observe that it remains true also for multigraphs).

Several examples of A-contractible graphs are shown in Fig. 4 (where the vertices in the set A are double-circled). Note that detailed proofs of A-contractibility are for F_2 and F_3 given in [15].

In our proof, we will need the following lemma from [12].

Lemma I [12]. *Let* H *be a multigraph,* F *an* $A_H(F)$*-contractible submultigraph of* H, *and let* $e_1, e_2 \in E(H) \setminus E(F)$. *Then* H *has an* (e_1, e_2)*-IDT if and only if* $H|_F$ *has an* (e_1, e_2)*-IDT.*

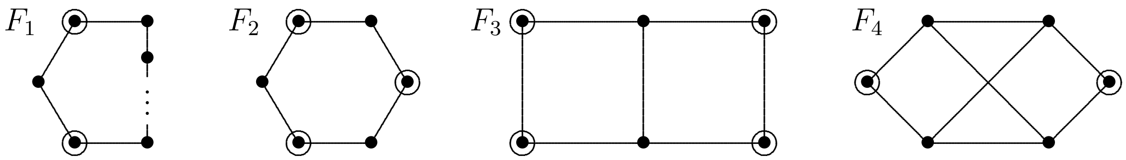


Figure 4: Examples of A-contractible graphs

2.6 A special version of the "Nine-point-theorem"

The well-known "Nine point theorem" by Holton et al. [9] states that a 3-connected cubic graph contains a cycle passing through any 9 prescribed vertices. For our proof, we will need its special version, based on a stronger version by Bau and Holton [1], and developed in [12].

Theorem J [12]. *Let $X \in \{N_{1,1,5}, N_{1,3,3}, N_{2,2,3}\}$, and let G be a 3-connected UM-closed $\{K_{1,3}, X\}$-free graph such that $\mathrm{co}(H)$, where $H = L^{-1}(G)$, is 2-connected. Let $e_1, e_2 \in E(H)$ be such that there is no (e_1, e_2)-IDT in H. Then for every set $A \subset V(\mathrm{co}(H))$, $|A| = 8$, there is an (e_1, e_2)-trail T in H such that $A \subset \mathrm{Int}(T)$.*

The following lemma will be crucial in our proof. Recall that W denotes the Wagner graph (see Fig. 1).

Lemma K [12]. *Let G be a 3-connected non-Hamilton-connected UM-closed claw-free graph. Then G has an induced subgraph $\tilde{G}$ (possibly $\tilde{G} = G$) such that $\tilde{G}$ is 3-connected, non-Hamilton-connected and UM-closed, and, moreover, $\tilde{H}_0 = \mathrm{co}(L^{-1}(\tilde{G}))$ is 2-connected, and either $c(\tilde{H}_0) \geq 9$ and $|V(\tilde{H}_0)| \geq 10$, or $\tilde{H}_0 \simeq W$.*

3 Proof of Theorem 1

Let G be a 3-connected $\{K_{1,3}, N_{1,1,5}\}$-free or $\{K_{1,3}, N_{2,2,3}\}$-free graph and suppose, to the contrary, that G is not Hamilton-connected. By Theorem G and Theorem H, we can suppose that G is UM-closed. Let thus $H = L^{-1}(G)$, and set $H_0 = \mathrm{co}(H)$. By Theorem F(ii), H_0 is 3-edge-connected. By Lemma K, we can suppose that H_0 is 2-connected and $c(H_0) \geq 9$ and $|V(H_0)| \geq 10$, unless $H_0 \simeq W$. Then, by Theorems J and C(ii), we have the following claim.

Claim 1. *Let $A \subset V(H_0)$ be such that $|A| = 8$. Then A does not dominate all edges of H.*

Proof. Since G is not Hamilton-connected, by Theorem C(ii), there are edges $e_1, e_2 \in E(H)$ such that there is no (e_1, e_2)-IDT in H. Then, by Theorem J, there is an (e_1, e_2)-trail T in H such that $A \subset \mathrm{Int}(T)$. But if A dominates all the edges in H, then T would be an (e_1, e_2)-IDT in H. □

Now, if $H_0 \simeq W$, then $|V(H_0)| = 8$ and $V(H_0)$ dominates all edges of H, contradicting Claim 1. Thus, we have $c(H_0) \geq 9$ and $|V(H_0)| \geq 10$. We consider separate cases for possible values of $c(H_0)$ and $|V(H_0)|$.

Throughout the proof, in each of the cases, $C = x_1x_2 \dots x_{c(H_0)}$ always denotes a longest cycle in H_0, $R = V(H) \setminus V(C)$, $N = \{y \in V(H_0) | \ N_R(y) = \emptyset\}$, $R_0 = R \cap V(H_0)$, and if $R_0 \neq \emptyset$, we set $R_0 = \{y_1, \dots, y_{|R_0|}\}$ and we choose the notation such that $y_1x_1 \in E(H_0)$. An edge x_ix_j with $x_i, x_j \in V(C)$, $1 \leq i, j \leq |V(C)|$, will be called a *chord* of C, and we say that x_ix_j is a *k-chord* if the shorter one of the two subpaths of C determined by x_i and x_j has k interior vertices. Similarly, for a vertex $y \in R_0$ with $N_C(y) \neq \emptyset$, the subpaths of C determined by the neighbors of y will be called *segments* of C, and a segment with k interior vertices will be referred to as a *k-segment.*

Claim 2. *If $E(\langle R \rangle_H) = \emptyset$, then H_0 has no double edge.*

Proof. Suppose that H_0 has a double edge $\{e, f\}$. Then, since $V(C)$ dominates all edges of H, it follows that H has an (e, f)-IDT. But then, by Theorem G$(vii)(\beta)$, G has an (a, b)-path for every pair $a, b \in V(G)$, a contradiction. $\square$

In the proof, we will often list vertices of a subgraph $S_{i,j,k}$. There are two general comments to all these situations.

- When some edge $e = x_ix_j$ of the $S_{i,j,k}$ is in $E(H_0)$, it can always happen that e is subdivided in H, i.e., formally, $e \notin E(H)$. However, it is immediate to see that if this happens, then the corresponding subgraph of H, which instead of $e = x_ix_j$ contains a path x_izx_j with $z \in V_2(H)$, also contains $S_{i,j,k}$ as a subgraph.
- When a vertex $x_i \in V(C)$ has a (potential) neighbor $z \in R$ and the vertex z occurs as the last vertex of a branch of the $S_{i,j,k}$, then such a vertex z can be an endvertex of a pendant edge attached to x_i, or can be $z \in V_2(H)$ and z subdivides some of the edges incident to x_i. It should be noted that in the second case, the vertices x_i and z can occur in reverse order in the list (i.e., x_i being the last vertex of the branch).

Throughout the proof, we always implicitly understand that there are also these possibilities.

Case 1: *G is $\{K_{1,3}, N_{1,1,5}\}$-free.*

Then H does not contain as a subgraph the graph $S_{2,2,6}$.

Subcase 1.1: *$c(H_0) = 9$ and $|V(H_0)| \geq 10$.*

First observe that $E(\langle R \rangle_H) = \emptyset$, since if e.g. $y_1z \in E(H)$ for some $z \in R$, then H contains the subgraph $S_{2,2,6}(x_1; y_1z; x_2x_3; x_9x_8x_7x_6x_5x_4)$, a contradiction. Consequently, no vertex in R_0 is connected to C by a double edge by Claim 2. Thus, y_1 has at least three distinct neighbors on C since $d_{H_0}(y_1) \geq 3$.

Suppose that $y_1x_3 \in E(H_0)$. Then $x_2 \in N$, for otherwise, if $x_2z \in E(H)$ for some $z \in R$, either there is $S_{2,2,6}(x_1; x_2z; y_1x_3; x_9x_8x_7x_6x_5x_4)$ in H if $z \neq y_1$, or the cycle $C' = x_1y_1x_2 \dots x_9x_1$ is longer than C if $z = y_1$, a contradiction. Hence $\{x_2, y_1\} \subset N$ and, by the same argument for $y_2, \dots, y_{|R_0|}$ and since $E(\langle R \rangle_H) = \emptyset$, we have $\{x_2, y_1, \dots, y_{|R_0|}\} \subset N$. This implies that the set $A = \{x_1, x_3, x_4, x_5, x_6, x_7, x_8, x_9\}$ with $|A| = 8$ dominates all edges in H, contradicting Claim 1. Thus, $y_1x_3 \notin E(H_0)$.

Since $d_C(y_1) \geq 3$ and $|V(C)| = 9$, the only possibility is, up to a symmetry, that $N_C(y_1) = \{x_1, x_4, x_7\}$. Then again $x_2 \in N$, for otherwise, for a neighbor z of x_2 in R, we have $S_{2,2,6}(x_1; x_2z; x_9x_8; y_1x_7x_6x_5x_4x_3)$ in H if $z \neq y_1$, or a longer cycle if $z = y_1$, a contradiction. By symmetry, we have $V(C) \setminus \{x_1, x_4, x_7\} \subset N$, implying that, if $|V(H_0)| > 10$, all y_i, $i = 2, \dots, |R_0|$, are adjacent to x_1, x_4 and x_7, and to no other vertices of C. Then the set $A = V(C) \setminus \{x_9\}$ with $|A| = 8$ dominates all edges in H, contradicting Claim 1.

Subcase 1.2: $c(H_0) = |V(H_0)| = 10$.

By Theorem G(vi), H_0 has at most two triangles. Hence, since $\delta(H_0) \geq 3$, it follows that C has a k-chord for some $k \geq 2$. We consider possible cases.

Subcase 1.2.1: $x_1x_4 \in E(H_0)$.

We show that $\{x_2, x_5, x_7\} \subset N$. Thus, let $z \in R$ and consider the following possibilities.

Edge	$S_{2,2,6}$ in H
$x_2z \in E(H)$	$S_{2,2,6}(x_1; x_2z; x_4x_3; x_{10}x_9x_8x_7x_6x_5)$
$x_5z \in E(H)$	$S_{2,2,6}(x_4; x_3x_2; x_5z; x_1x_{10}x_9x_8x_7x_6)$
$x_7z \in E(H)$	$S_{2,2,6}(x_4; x_3x_2; x_5x_6; x_1x_{10}x_9x_8x_7z)$

Hence we have $\{x_2, x_5, x_7\} \subset N$ and, by symmetry, also $\{x_3, x_8, x_{10}\} \subset N$. Now, the set $A = V(C) \setminus \{x_3, x_5\}$ with $|A| = 8$ dominates all edges of H, contradicting Claim 1, unless $x_3x_5 \in E(H_0)$. Hence $x_3x_5 \in E(H_0)$. Symmetrically, $x_2x_{10} \in E(H_0)$. Similarly, considering the set $A = V(C) \setminus \{x_5, x_7\}$ with $|A| = 8$, we have $x_5x_7 \in E(H_0)$. But then x_3x_5, x_2x_{10} and x_5x_7 are three 1-chords in C, determining three triangles in H_0, which contradicts Theorem G(vi). Thus, C has no 2-chord.

Subcase 1.2.2: $x_1x_5 \in E(H_0)$.

Then $x_6 \in N$, for otherwise we have a subgraph $S_{2,2,6}(x_1; x_2x_3; x_5x_4; x_{10}x_9x_8x_7x_6z)$ in H for $z \in R$ with $x_6z \in E(H)$. Hence $x_6 \in N$ and, symmetrically, $x_{10} \in N$. If $x_6x_{10} \notin E(H)$, then $A = V(C) \setminus \{x_6, x_{10}\}$ is a set of 8 vertices in H_0 that dominates all edges of H, contradicting Claim 1. Thus, $x_6x_{10} \in E(H_0)$. By symmetry, $\{x_1, x_5, x_6, x_{10}\} \subset N$. Then, considering the set $A = V(C) \setminus \{x_1, x_6\}$, Claim 1 implies $x_1x_6 \in E(H_0)$. Symmetrically, $x_5x_{10} \in E(H_0)$. But then $\langle\{x_1, x_5, x_6, x_{10}\}\rangle_H \simeq K_4$, contradicting Theorem G(vi). Thus, C has no 3-chord.

By Subcases 1.2.1 and 1.2.2 and since $|V(C)| = 10$, C can have only 4-chords, plus at most two 1-chords. Thus, at most 4 vertices of C can be in a 1-chord, implying that C has at least three 4-chords. Since $|V(C)| = 10$, there always is a pair of 4-chords such that their endvertices are on C at distance 2. Thus, let, say, $x_1x_6 \in E(H_0)$ and $x_3x_8 \in E(H_0)$. Then $x_2 \in N$, since otherwise, for a $z \in R$ with $x_2z \in E(H)$, we have $S_{2,2,6}(x_1; x_2z; x_{10}x_9; x_6x_7x_8x_3x_4x_5)$ in H. Symmetrically also $x_7 \in N$. Considering the set $A = V(C) \setminus \{x_2, x_7\}$ with $|A| = 8$, by Claim 1 we have $x_2x_7 \in E(H_0)$. Now, $x_1 \in N$, since otherwise, for a $z \in R$ with $x_1z \in E(H)$ we have $S_{2,2,6}(x_6; x_1z; x_7x_2; x_5x_4x_3x_8x_9x_{10})$ in H. Thus, we have $\{x_1, x_2, x_7\} \subset N$. Considering the set $A = V(C) \setminus \{x_1, x_7\}$ with $|A| = 8$, we have $x_1x_7 \in E(H_0)$ by Claim 1. However, x_1x_7 is a 3-chord, and, by symmetry, we are back in Subcase 1.2.2.

Subcase 1.3: $c(H_0) = 10$ *and* $|V(H_0)| \geq 11$.

Then $E(\langle R\rangle_H) = \emptyset$, for if there is a $z \in R$ with e.g. $zy_1 \in E(H)$, then H contains $S_{2,2,6}(x_1; y_1z; x_2x_3; x_{10}x_9x_8x_7x_6x_5)$. Hence $R \subset N$, and no vertex in R_0 is attached to C by a double edge by Claim 2. Since $\delta(H_0) \geq 3$, every vertex in R_0 has at least three distinct neighbors on C. We consider the possible cases.

Subcase 1.3.1: $y_1x_4 \in E(H_0)$.

Then we have $S_{2,2,6}(x_1; y_1x_4; x_2x_3; x_{10}x_9x_8x_7x_6x_5)$ in H_0, a contradiction.

Subcase 1.3.2: $y_1x_5 \in E(H_0)$.

Then $x_6 \in N$ since otherwise H contains $S_{2,2,6}(x_1; y_1x_5; x_2x_3; x_{10}x_9x_8x_7x_6z)$ for some $z \in R$ with $zx_6 \in E(H)$. Symmetrically also $x_{10} \in N$. If $x_6x_{10} \in E(H_0)$, then the set $A = V(C) \setminus \{x_6, x_{10}\}$ with $|A| = 8$ dominates all edges of H (recall that $R \subset N$). Hence $x_6x_{10} \in E(H_0)$ by Claim 1. But then H contains $S_{2,2,6}(x_6; x_7x_8; x_{10}x_9; x_5y_1x_1x_2x_3x_4)$, a contradiction.

By Subcases 1.3.1 and 1.3.2, by symmetry and since C is longest, the neighbors of y_1 on C can be at distance (along C) either 2, or at least 5. However, it is straightforward to verify that this is not possible since $|V(C)| = 10$.

Subcase 1.4: $c(H_0) = |V(H_0)| = 11$.

Since $\delta(H_0) \geq 3$ and $R_0 = \emptyset$, every vertex of C is in a chord. We consider the possible cases.

Subcase 1.4.1: $x_1x_4 \in E(H_0)$.

Then we have $S_{2,2,6}(x_1; x_2x_3; x_4x_5; x_{11}x_{10}x_9x_8x_7x_6)$ in H_0, a contradiction. Thus, by symmetry, there is no 2-chord.

Subcase 1.4.2: $x_1x_5 \in E(H_0)$.

Then we have $S_{2,2,6}(x_1; x_2x_3; x_5x_4; x_{11}x_{10}x_9x_8x_7x_6)$ in H_0, a contradiction. Thus, by symmetry, there is no 3-chord.

We observe the following fact.

Claim 3. *If $e = x_ix_{i+5}$ is a 4-chord of C, then $\{x_{i-1}, x_{i+6}\} \subset N$ (indices modulo 11).*

Proof. Choose the notation such that $i = 1$. If x_7 is adjacent to a $z \in R$, then we have $S_{2,2,6}(x_1; x_2x_3; x_6x_5; x_{11}x_{10}x_9x_8x_7z)$ in H, hence $x_7 \in N$. Symmetrically, $x_{11} \in N$. □

We show that C has no 1-chord. Thus, let, to the contrary, $x_1x_3 \in E(H_0)$. By Theorem G(vi), H_0 contains no diamond, and hence $x_4, x_{11} \notin N_{H_0}(x_2)$. Since $\delta(H_0) \geq 3$ and C has neither 2-chords nor 3-chords, we may assume, by symmetry, that $x_2x_7 \in E(H_0)$. By Claim 3, we have $\{x_1, x_8\} \subset N$. Moreover, $x_2 \in N$, for otherwise we have $S_{2,2,6}(x_1; x_2z; x_3x_4; x_{11}x_{10}x_9x_8x_7x_6)$ in H for some neighbor z of x_2 in R, and also $x_{10} \in N$ since for a $z \in R$ with $zx_{10} \in E(H)$ we have $S_{2,2,6}(x_7; x_8x_9; x_6x_5; x_2x_3x_1x_{11}x_{10}z)$ in H. Thus, we have $\{x_1, x_2, x_8, x_{10}\} \subset N$.

Considering the set $A = V(C) \setminus \{x_2, x_8, x_{10}\}$ with $|A| = 8$ and since $x_2x_{10} \notin E(H_0)$ by Subcase 2.4.1, either $x_2x_8 \in E(H_0)$ or $x_8x_{10} \in E(H_0)$ by Claim 1. Similarly, considering the set $A = V(C) \setminus \{x_1, x_8, x_{10}\}$ with $|A| = 8$ and since $x_1x_8 \notin E(H_0)$ by Subcase 2.4.2, either $x_1x_{10} \in E(H_0)$ or $x_8x_{10} \in E(H_0)$. Thus, there is at least one of the edges x_2x_8, x_8x_{10}, and also at least one of the edges x_1x_{10}, x_8x_{10}. Since each of these edges creates a triangle, and there already is one triangle in H_0 (created by the 1-chord x_1x_3), the only possibility is that $x_8x_{10} \in E(H_0)$. Now, since $d_{H_0}(x_{11}) \geq 3$, by the previous subcases and since H_0 does not contain a diamond, we have $x_{11}x_5 \in E(H_0)$ or $x_{11}x_6 \in E(H_0)$ (recall that we also have $\{x_1, x_2, x_8, x_{10}\} \subset N$).

Let first $x_{11}x_5 \in E(H_0)$. Then $x_6 \in N$ by Claim 3, and considering the set $A = V(C) \setminus \{x_2, x_6, x_{10}\}$ with $|A| = 8$, Claim 1 implies that H_0 contains at least one of the edges x_2x_6, x_6x_{10}, x_2x_{10}. However, each of these edges contradicts Subcase 1.4.1 or 1.4.2. Thus, we have $x_{11}x_6 \in E(H_0)$. Then similarly $x_5 \in N$ by Claim 3, and we have an analogous contradiction by Claim 1 considering the set $A = V(C) \setminus \{x_1, x_5, x_8\}$. Thus, C has no 1-chord.

By the previous considerations, C has no k-chord for $k = 1, 2, 3$. Since $\delta(H_0) \geq 3$, every vertex of C is in a 4-chord. By symmetry, let $x_1x_6 \in E(H_0)$. Then x_9 is in a 4-chord and, again by symmetry, we can suppose that $x_3x_9 \in E(H_0)$. Then H_0 contains $S_{2,2,6}(x_9; x_8x_7; x_{10}x_{11}; x_3x_2x_1x_6x_5x_4)$, a contradiction.

<u>Subcase 1.5:</u> *$c(H_0) \geq 11$ and $|V(H_0)| > c(H_0)$.*

First observe that, as in Subcase 1.3, $E(\langle R \rangle_H) = \emptyset$ (otherwise there is an $S_{2,2,6}$ in H), implying $R \subset N$, and that no vertex in R_0 is attached to C by a double edge by Claim 2. Since $\delta(H_0) \geq 3$, every vertex in R_0 has at least three distinct neighbors on C, and we consider possible cases. Set $|V(C)| = t$.

<u>Subcase 1.5.1:</u> $y_1x_4 \in E(H_0)$.

Then we have $S_{2,2,6}(x_1; x_2x_3; y_1x_4; x_tx_{t-1}x_{t-2}x_{t-3}x_{t-4}x_{t-5})$. Thus, there is no 2-segment on C.

<u>Subcase 1.5.2:</u> $y_1x_5 \in E(H_0)$.

Then we similarly have $S_{2,2,6}(x_1; x_2x_3; y_1x_5; x_tx_{t-1}x_{t-2}x_{t-3}x_{t-4}x_{t-5})$. Thus, there is no 3-segment on C.

By Subcases 1.5.1 and 1.5.2, y_1 determines on C only 1-segments and k-segments for $k \geq 4$, and there are at least three segments determined by y_1. We distinguish possible cases, where, in each of the cases, we always consider k_1-segments and k_2-segments such that $k_1 \leq k_2$ and $k_1 + k_2$ is smallest possible.

<u>Subcase 1.5.3:</u> $k_1 = k_2 = 1$.

By symmetry, we can choose the notation such that $\{x_1, x_3, x_5\} \subset N_C(y_1)$, and then we have $S_{2,2,6}(y_1; x_1x_2; x_3x_4; x_5x_6x_7x_8x_9x_{10})$ in H.

<u>Subcase 1.5.4:</u> $k_1 = 1$, $k_2 \geq 4$.

Then, by symmetry, $\{x_1, x_3, x_{k_2+4}\} \subset N_C(y_1)$ and, by the choice of k_1 and k_2, $t \geq 2k_2 + 4 \geq 12$. Then we have $S_{2,2,6}(x_3; x_2x_1; x_4x_5; y_1x_{k_2+4}x_{k_2+5}x_{k_2+6}x_{k_2+7}x_{k_2+8})$ in H.

<u>Subcase 1.5.5:</u> $4 \leq k_1 \leq k_2$.

Then $\{x_1, x_{k_1+2}, x_{k_1+k_2+3}\} \subset N_C(y_1)$ and $t \geq k_1 + 2k_2 + 3 \geq 15$. Then we have $S_{2,2,6}(x_1; x_2x_3; x_tx_{t-1}; y_1x_{k_1+2}x_{k_1+3}x_{k_1+4}x_{k_1+5}x_{k_1+6})$.

<u>Subcase 1.6:</u> $c(H_0) = |V(H_0)| \geq 12$.

Since $\delta(H_0) \geq 3$ and $R_0 = \emptyset$, every vertex of C is in a chord, and we consider the possible cases. Set $|V(C)| = t$.

Subcase 1.6.1: $x_1x_4 \in E(H_0)$.

Then we have $S_{2,2,6}(x_1; x_2x_3; x_4x_5; x_tx_{t-1}x_{t-2}x_{t-3}x_{t-4}x_{t-5})$ in H_0, a contradiction. By symmetry, there is no 2-chord.

Subcase 1.6.2: $x_1x_i \in E(H_0), i \in \{5, 6\}$.

Then we have $S_{2,2,6}(x_1; x_2x_3; x_ix_{i-1}; x_tx_{t-1}x_{t-2}x_{t-3}x_{t-4}x_{t-5})$ in H_0, a contradiction. Thus, by symmetry, there is no 3-chord and no 4-chord.

Subcase 1.6.3: $x_1x_7 \in E(H_0)$.

If $t \geq 13$, we similarly have $S_{2,2,6}(x_1; x_2x_3; x_7x_6; x_tx_{t-1}x_{t-2}x_{t-3}x_{t-4}x_{t-5})$. Thus, suppose that $t = 12$. By the previous subcases, there are only 1-chords and 5-chords. Since H_0 has at most two triangles, there are at most two 1-chords, and since $t = 12$ and every vertex of C is in a chord, there are three vertices that are consecutive on C and each of them is in a 5-chord. Choose the notation such that $x_1x_7 \in E(H_0)$, $x_2x_8 \in E(H_0)$ and $x_3x_9 \in E(H_0)$. Then we have $S_{2,2,6}(x_1; x_2x_8; x_{12}x_{11}; x_7x_6x_5x_4x_3x_9)$ in H_0.

Subcase 1.6.4: $x_1x_i \in E(H_0), i \geq 8$.

Then, by the definition of a chord (the shorter one of the two paths), $t \geq 2i - 2 \geq 14$, and we have $S_{2,2,6}(x_1; x_ix_{i+1}; x_tx_{t-1}; x_2x_3x_4x_5x_6x_7)$ in H_0.

Case 2: *G is $\{K_{1,3}, N_{2,2,3}\}$-free.*

Then H does not contain as a subgraph the graph $S_{3,3,4}$. In this case, we make an additional choice of the cycle C; namely, we choose C such that

(i) C is a longest cycle in H_0, and

(ii) subject to (i), C dominates the maximum number of edges of H.

Subcase 2.1: *$c(H_0) = 9$ and $|V(H_0)| \geq 10$.*

Subcase 2.1.1: $E(\langle R\rangle_H) \neq \emptyset$.

Let $e \in E(\langle R\rangle_H)$. By Theorem F($iii$), we can choose the notation such that $e = y_1z_1$, where $y_1 \in R_0$ and $z_1 \in R$, and, if possible, we choose e and y_1 such that $|N_C(y_1)| \geq 3$.

Subcase 2.1.1.1: $|N_C(y_1)| \geq 3$.

We first show that $x_5, x_6 \notin N_{H_0}(y_1)$. Let, say, e.g. $y_1x_5 \in E(H_0)$. Clearly $x_2, x_4, x_6, x_9 \notin N_C(y_1)$ since C is longest. By the assumption of the subcase and by symmetry, $y_1x_3 \in E(H_0)$ or $y_1x_7 \in E(H_0)$. If $y_1x_3 \in E(H_0)$, then $N_R(x_4) = \emptyset$, since otherwise, for a $z \in N_R(x_4)$ we have $S_{3,3,4}(x_1; y_1x_5x_6; x_9x_8x_7; x_2x_3x_4z)$ in H, but then the cycle $C' = x_1x_2x_3y_1x_5 \dots x_9x_1$ dominates in H more edges than C, contradicting choice (ii) of C. Similarly, if $y_1x_7 \in E(H_0)$, then $N_R(x_6) = \emptyset$ (otherwise we have $S_{3,3,4}(x_1; x_2x_3x_4; x_9x_8x_7; y_1x_5x_6z)$ in H for a $z \in N_R(x_6)$), and then the cycle $C' = x_1x_2x_3x_4x_5y_1x_7x_8x_9x_1$ dominates in H more edges than C, contradicting choice (ii) of C. Thus, by symmetry, $x_5, x_6 \notin N_{H_0}(y_1)$.

Since $|N_C(y_1)| \geq 3$, y_1 determines on C three segments, say, k_i-segments, $i = 1, 2, 3$, and since $|V(C) \setminus N_C(y_1)| \leq 6$, the possible distributions of interior vertices of the segments are $(k_1, k_2, k_3) = (1, 1, 4)$, $(k_1, k_2, k_3) = (1, 2, 3)$ or $(k_1, k_2, k_3) = (2, 2, 2)$

(where we admit that the 4-segment in the case $(1, 1, 4)$ or the 3-segment in the case $(1, 2, 3)$ can be further subdivided by another neighbor of y_1 if $|N_C(y_1)| > 3$).

However, in the first case we have $y_1x_5 \in E(H_0)$ and in the second case we have $y_1x_6 \in E(H_0)$ (or a symmetric situation), a contradiction. Thus, we have $(k_1, k_2, k_3) = (2, 2, 2)$, implying $N_C(y_1) = \{x_1, x_4, x_7\}$.

We observe that $N_R(x_i) = \emptyset$ for $i = 2, 3, 5, 6, 8, 9$ since if, say, $y_2z \in E(H)$ for some $z \in R$, we have $S_{3,3,4}(x_4; x_3x_2z; x_5x_6x_7; y_1x_1x_9x_8)$ in H; other cases are symmetric. Since H_0 is 3-edge-connected, each of the vertices $x_2, x_3, x_5, x_6, x_8, x_9$ is either in a double edge or in a chord. By Theorem G(vi) and by symmetry, we can choose the notation such the segment $x_1x_2x_3x_4$ contains neither a double edge nor a 1-chord. Considering possible chords containing x_2, we have the following possibilities.

Chord	Contradiction
x_2x_5	$C' = x_1y_1x_4x_3x_2x_5x_6x_7x_8x_9x_1$ is longer than C
x_2x_6	$S_{3,3,4}(x_2; x_3x_4x_5; x_1y_1z_1; x_6x_7x_8x_9)$ in H
x_2x_7	$S_{3,3,4}(x_2; x_7y_1z_1; x_1x_9x_8; x_3x_4x_5x_6)$ in H
x_2x_8	$C' = x_1y_1x_7x_6x_5x_4x_3x_2x_8x_9x_1$ is longer than C
x_2x_9	$S_{3,3,4}(x_2; x_1y_1z_1; x_3x_4x_5; x_9x_8x_7x_6)$ in H

In each of the cases, we have reached a contradiction.

<u>Subcase 2.1.1.2:</u> $|N_C(y_1)| = 2$.

By the connectivity, one of the two connections of y_1 to C is a double edge and hence, by Theorem G$(vi)(\beta)$, there is no other double edge in H_0. Hence y_1 is the only vertex in H_0 that has a neighbor in R. This also implies that H_0 is triangle-free. Obviously, $x_2, x_9 \notin N_C(y_1)$ since C is longest. Thus, by symmetry, we have $y_1x_3 \in E(H_0)$, $y_1x_4 \in E(H_0)$ or $y_1x_5 \in E(H_0)$. In each of these cases, each of the vertices in $V(C) \setminus N_C(y_1)$ (specifically, x_2 in the first two cases and x_3 in the third case) must be connected to some other vertex of C by a chord or by a path P of length two with an interior vertex in R_0 since H_0 is 3-edge-connected.

Below we list (up to a symmetry) all possible cases. Here, if we give an $S_{3,3,4}$ for some chord, say, x_2x_j, it is always implicitly understood that x_2x_j can also be a path of length two, in which case the listed graph contains an $S_{3,3,4}$ as a proper subgraph. Also note that although there is no 1-chord since H_0 is triangle-free, x_2 can be still connected to x_4 or to x_9 by a path of length two with interior vertex, say, $y_2 \in R_0$.

Case	Chord or path	Contradiction: $S_{3,3,4}$ in H
$y_1x_3 \in E(H_0)$	$P = x_2y_2x_4$	$S_{3,3,4}(x_2; y_2x_4x_5; x_3y_1z_1; x_1x_9x_8x_7)$
	$x_2x_5 \in E(H_0)$	$S_{3,3,4}(x_5; x_2x_1x_9; x_6x_7x_8; x_4x_3y_1z_1)$
	$x_2x_6 \in E(H_0)$	$S_{3,3,4}(x_2; x_3x_4x_5; x_1y_1z_1; x_6x_7x_8x_9)$
$y_1x_4 \in E(H_0)$	$P = x_2y_2x_4$	$S_{3,3,4}(x_4; y_2x_2x_3; y_1x_1x_9; x_5x_6x_7x_8)$
	$x_2x_5 \in E(H_0)$	$S_{3,3,4}(x_2; x_5x_6x_7; x_1x_9x_8; x_3x_4y_1z_1)$
	$x_2x_6 \in E(H_0)$	$S_{3,3,4}(x_2; x_3x_4x_5; x_1y_1z_1; x_6x_7x_8x_9)$
	$x_2x_7 \in E(H_0)$	$S_{3,3,4}(x_2; x_7x_8x_9; x_1y_1z_1; x_3x_4x_5x_6)$
	$x_2x_8 \in E(H_0)$	$S_{3,3,4}(x_8; x_2x_3x_4; x_7x_6x_5; x_9x_1y_1z_1)$
	$P = x_2y_2x_9$	$S_{3,3,4}(x_2; x_3x_4x_5; x_1y_1z_1; y_2x_9x_8x_7)$
$y_1x_5 \in E(H_0)$	$P = x_3y_2x_5$	$S_{3,3,4}(x_5; y_2x_3x_4; y_1x_1x_2; x_6x_7x_8x_9)$
	$x_3x_6 \in E(H_0)$	$S_{3,3,4}(x_3; x_6x_7x_8; x_2x_1x_9; x_4x_5y_1z_1)$
	$x_3x_7 \in E(H_0)$	$S_{3,3,4}(x_3; x_4x_5x_6; x_7x_8x_9; x_2x_1y_1z_1)$

In each of the possible cases, we have reached a contradiction.

Subcase 2.1.2: $E(\langle R\rangle_H) = \emptyset$.

By Claim 2, H_0 has no double edge, implying that every vertex in R_0 has three distinct neighbors on C. If, say, $y_1x_5 \in E(H_0)$, then x_4 has no neighbor in R (otherwise, for a $z \in N_R(x_4)$, we have $S_{3,3,4}(x_1; y_1x_5x_6; x_9x_8x_7; x_2x_3x_4z)$ in H), and then, considering the set $A = V(C) \setminus \{x_4\}$ with $|A| = 8$, we have a contradiction by Claim 1. Hence, by symmetry, y_1 adjacent to neither x_5 nor x_6. This implies that, similarly as in Subcase 2.1.1.1, for y_1, $(k_1, k_2, k_3) = (2, 2, 2)$ is the only possible distribution of segments on C. Thus, we have $N_C(y_1) = \{x_1, x_4, x_7\}$. If there is a $z \in N_R(x_3)$, we have $S_{3,3,4}(x_1; x_2x_3z; y_1x_4x_5; x_9x_8x_7x_6)$ in H, hence $N_R(x_3) = \emptyset$. But then again, considering the set $A = V(C) \setminus \{x_3\}$ with $|A| = 8$, we have a contradiction by Claim 1.

Subcase 2.2: $c(H_0) = |V(H_0)| = 10$.

Subcase 2.2.1: *C has a 3-chord.*

Let $x_1x_5 \in E(H_0)$ be a 3-chord of C, and let $z \in R$ be adjacent (in H) to a vertex on C. We consider the following cases.

Case	Contrdiction: $S_{3,3,4}$ in H
$zx_2 \in E(H)$	$S_{3,3,4}(x_5; x_6x_7x_8; x_1x_{10}x_9; x_4x_3x_2z)$
$zx_6 \in E(H)$	$S_{3,3,4}(x_1; x_2x_3x_4; x_5x_6z; x_{10}x_9x_8x_7)$
$zx_7 \in E(H)$	$S_{3,3,4}(x_5; x_4x_3x_2; x_6x_7z; x_1x_{10}x_9x_8)$

Thus, we have $N_R(\{x_2, x_6, x_7\}) = \emptyset$, and, by symmetry, also $N_R(\{x_4, x_9, x_{10}\}) = \emptyset$. Now, considering the set $A_1 = V(C) \setminus \{x_2, x_4\}$ with $|A_1| = 8$, we have $x_2x_4 \in E(H_0)$ by Claim 1. Similarly, considering the set $A_2 = V(C) \setminus \{x_7, x_9\}$ with $|A_2| = 8$ and the set $A_3 = V(C) \setminus \{x_2, x_{10}\}$ with $|A_3| = 8$, we have $x_7x_9 \in E(H_0)$ and $x_2x_{10} \in E(H_0)$ by Claim 1. But then $T_1 = x_2x_3x_4x_2$, $T_2 = x_7x_8x_9x_7$ and $T_3 = x_2x_1x_{10}x_2$ are three triangles in H_0, contradicting Theorem G(v). Thus, C has no 3-chord.

Subcase 2.2.2: *C has a 4-chord.*

Let $x_1x_6 \in E(H_0)$ be a 4-chord of C. Then $N_R(x_3) = \emptyset$ (otherwise, for a $z \in N_R(x_3)$, we have $S_{3,3,4}(x_1; x_2x_3z; x_6x_5x_4; x_{10}x_9x_8x_7)$ in H), and, symmetrically, $N_R(x_9) = \emptyset$.

Moreover, $x_3x_9 \notin E(H_0)$ by Subcase 2.2.1. But then, considering the set $A = V(C) \setminus \{x_3, x_9\}$ with $|A| = 8$, we have a contradiction by Claim 1. Thus, C has no 4-chord.

By Subcases 2.2.1 and 2.2.2., every vertex of C is in a 1-chord or in a 2-chord. Since there are at most two 1-chords by Theorem G(v), we can choose the notation such that x_1 is in a 2-chord, i.e., $x_1x_4 \in E(H_0)$.

Claim 4. *If x_ix_{i+3} is a 2-chord of C, then $N_R(\{x_{i+1}, x_{i+2}\}) = \emptyset$ (indices modulo 10).*

Proof. By symmetry, set $i = 1$. If $z \in N_R(x_3)$, then $S_{3,3,4}(x_1; x_2x_3z; x_4x_5x_6; x_{10}x_9x_8x_7)$ is a subgraph of H, a contradiction. Hence $N_R(x_3) = \emptyset$. Symmetrically, $N_R(x_2) = \emptyset$. □

Thus, by Claim 4, we have $N_R(\{x_2, x_3\}) = \emptyset$. Suppose that x_2 is in a 2-chord. Then either $x_2x_5 \in E(H_0)$, or $x_2x_9 \in E(H_0)$.

Let first $x_2x_5 \in E(H_0)$. Then we have $N_R(x_4) = \emptyset$ by Claim 4, and, considering the set $A = V(C) \setminus \{x_2, x_4\}$ with $|A| = 8$, we have $x_2x_4 \in E(H_0)$ by Claim 1. But then $\langle\{x_1, x_2, x_4, x_5\}\rangle_{H_0}$ is a diamond in H_0, a contradiction. Hence $x_2x_5 \notin E(H_0)$.

Secondly, let $x_2x_9 \in E(H_0)$. Then, again by Claim 4, we have $N_R(\{x_1, x_{10}\}) = \emptyset$. Considering the set $A = V(C) \setminus \{x_2, x_{10}\}$ with $|A| = 8$, we have $x_2x_{10} \in E(H_0)$ by Claim 1, but then $\langle\{x_1, x_2, x_9, x_{10}\}\rangle_{H_0}$ is a diamond or a K_4 in H_0, a contradiction again. Hence x_2 is in a 1-chord. Symmetrically, x_3 is also in a 1-chord, and since H_0 is diamond-free, we have $x_2x_{10} \in E(H_0)$ and $x_3x_5 \in E(H_0)$. Then $N_R(x_4) = \emptyset$, since otherwise we have $S_{3,3,4}(x_2; x_1x_4z; x_3x_5x_6; x_{10}x_9x_8x_7)$ in H for a $z \in N_R(x_4)$. Recall that also $N_R(x_2) = \emptyset$ by Claim 4. But then, considering the set $A = V(C) \setminus \{x_2, x_4\}$ with $|A| = 8$, we have $x_2x_4 \in E(H_0)$, implying that $\langle\{x_1, x_2, x_3, x_4\}\rangle_{H_0}$ is a diamond or a K_4 in H_0, a contradiction.

Subcase 2.3: *$c(H_0) \geq 10$ and $|V(H_0)| > c(H_0)$.*

Set $|V(C)| = t$. If y_1 determines on C a 3-segment, i.e., $y_1x_5 \in E(H_0)$, we have $S_{3,3,4}(x_1; x_2x_3x_4; y_1x_5x_6; x_tx_{t-1}x_{t-2}x_{t-3})$ in H_0, if y_1 determines on C a 4-segment, i.e., $y_1x_6 \in E(H_0)$, we have $S_{3,3,4}(x_1; y_1x_6x_7; x_tx_{t-1}x_{t-2}; x_2x_3x_4x_5)$ in H_0, and if y_1 determines on C a k-segment with $k \geq 5$, i.e., $y_1x_{k+2} \in E(H_0)$, then, by the definition of a segment (the shorter one of the two subpaths of C), necessarily $t \geq 2k + 2$, and then we have $S_{3,3,4}(x_1; y_1x_{k+2}x_{k+3}; x_tx_{t-1}x_{t-2}; x_2x_3x_4x_5)$ in H_0. Thus, by symmetry, every vertex in R_0 determines on C only 1-segments and 2-segments.

Now, if a vertex in R_0, say, y_1, has three distinct neighbors on C, then necessarily y_1 is adjacent to some of x_5, x_6 or x_7, and we are in some of the previous cases. Thus, y_1 has two neighbors on C, and one of the two connections of y_1 to C is a double edge. This implies that $R_0 = \{y_1\}$, $|V(H_0)| = t + 1$, and H_0 is triangle-free. Choose the notation such that $N_C(y_1) = \{x_1, x_3\}$ or $N_C(y_1) = \{x_1, x_4\}$. Since $\delta(H_0) \geq 3$, the vertex x_2 has, besides x_1 and x_3, another neighbor in H_0. Clearly $N_{R_0}(x_2) = \emptyset$ since $R_0 = \{y_1\}$ and C is longest, and also $x_2x_4, x_2x_t \notin E(H_0)$ since H_0 is triangle-free.

Let first $y_1x_3 \in E(H_0)$. Up to a symmetry, we have the following possibilities (note that if $x_2x_k \in E(H_0)$ for $k \geq 7$, then $t \geq 2k - 4$ by symmetry).

Case	Contradiction: $S_{3,3,4}$ in H_0
$x_2x_5 \in E(H_0)$	$S_{3,3,4}(x_1; y_1x_3x_4; x_2x_5x_6; x_tx_{t-1}x_{t-2}x_{t-3})$
$x_2x_6 \in E(H_0)$	$S_{3,3,4}(x_1; y_1x_3x_4; x_2x_6x_5; x_tx_{t-1}x_{t-2}x_{t-3})$
$x_2x_k \in E(H_0)$, $k \geq 7$	$S_{3,3,4}(x_1; y_1x_3x_4; x_tx_{t-1}x_{t-2}; x_2x_kx_{k-1}x_{k-2})$

Thus, $y_1x_4 \in E(H_0)$. Then we have the following possibilities.

Case	Contradiction: $S_{3,3,4}$ in H_0
$x_2x_5 \in E(H_0)$	$S_{3,3,4}(x_1; y_1x_4x_3; x_2x_5x_6; x_tx_{t-1}x_{t-2}x_{t-3})$
$x_2x_6 \in E(H_0)$	$S_{3,3,4}(x_1; y_1x_4x_3; x_2x_6x_5; x_tx_{t-1}x_{t-2}x_{t-3})$
$x_2x_k \in E(H_0)$, $7 \leq k \leq t-3$	$S_{3,3,4}(x_1; y_1x_4x_3; x_tx_{t-1}x_{t-2}; x_2x_kx_{k-1}x_{k-2})$
$x_2x_{t-2} \in E(H_0)$	$S_{3,3,4}(x_2; x_3x_4y_1; x_1x_tx_{t-1}; x_{t-2}x_{t-3}x_{t-4}x_{t-5})$
$x_2x_{t-1} \in E(H_0)$	$S_{3,3,4}(x_4; y_1x_1x_t; x_3x_2x_{t-1}; x_5x_6x_7x_8)$

Subcase 2.4: $c(H_0) = |V(H_0)| \geq 11$.

Set $|V(C)| = t$. We recall that H_0 has no double edge by Claim 2. Since $\delta(H_0) \geq 3$, every vertex of C is in at least one chord.

If C has a 3-chord, say, $x_1x_5 \in E(H_0)$, we have $S_{3,3,4}(x_1; x_2x_3x_4; x_5x_6x_7; x_tx_{t-1}x_{t-2}x_{t-3})$ in H_0, and, similarly, if C has a k-chord for $k \geq 4$, then $t \geq \max\{2k+2, 11\}$ by the definition of a chord, and we have $S_{3,3,4}(x_1; x_{k+2}x_{k+3}x_{k+4}; x_tx_{t-1}x_{t-2}; x_2x_3x_4x_5)$ in H_0. Thus, the only possible chords in C are 1-chords and 2-chords.

We show that there is no 1-chord. Let, to the contrary, $x_1x_3 \in E(H_0)$ be a 1-chord of C. Since x_2 must be in a chord and a 1-chord at x_2 would create a diamond, by symmetry, we have $x_2x_5 \in E(H_0)$. We consider possible chords at x_4. Since both x_1x_4 and x_2x_4 create a diamond, necessarily $x_4x_6 \in E(H_0)$ or $x_4x_7 \in E(H_0)$. However, if $x_4x_7 \in E(H_0)$, we have $S_{3,3,4}(x_1; x_2x_5x_6; x_3x_4x_7; x_tx_{t-1}x_{t-2}x_{t-3})$ in H_0, hence $x_4x_6 \in E(H_0)$. Now, if x_5 has another neighbor on C, then the only possibility is $x_5x_8 \in E(H_0)$ (all other chords at x_5 create a diamond), but then we have $S_{3,3,4}(x_1; x_3x_4x_6; x_tx_{t-1}x_{t-2}; x_2x_5x_8x_7)$ in H_0. Thus, $x_5x_8 \notin E(H_0)$, and, symmetrically, $x_2x_{10} \notin E(H_0)$.

We summarize that the subgraph $F = \langle\{x_1, x_2, x_3, x_4, x_5, x_6\}\rangle_{H_0}$ is isomorphic to the graph F in Fig. 3, and the vertices x_2, x_3, x_4, x_5 have no other neighbors on C outside F. Moreover, if, say, x_5 has a neighbor $z \in R$, we have $S_{3,3,4}(x_1; x_2x_5z; x_3x_4x_6; x_tx_{t-1}x_{t-2}x_{t-3})$ in H; thus, by symmetry, none of the vertices x_2, x_3, x_4, x_5 has a neighbor in R. We conclude that F is a subgraph of H, and $N_H(\{x_2, x_3, x_4, x_5\}) \subset V(F)$, which contradicts Lemma E. Thus, C has no 1-chord, implying that the only possible chords are 2-chords.

Claim 5. *If $x_ix_{i+3} \in E(H_0)$ is a 2-chord of C for some i, $1 \leq i \leq t$, then $d_{H_0}(x_{i+1}) = d_{H_0}(x_{i+2}) = 3$ (indices modulo t).*

Proof. Clearly $3 \leq d_{H_0}(x) \leq 4$ for $x \in V(C)$. Let, say, $x_3x_6 \in E(H_0)$ be a 2-chord of C, and suppose that $d_{H_0}(x_4) = 4$. Then $x_1x_4, x_4x_7 \in E(H_0)$, and we have $S_{3,3,4}(x_1; x_4x_7x_8; x_tx_{t-1}x_{t-2}; x_2x_3x_6x_5)$ in H_0. Hence $d_{H_0}(x_4) = 3$, and, symmetrically, $d_{H_0}(x_5) = 3$. $\square$

Claim 6. *Let x_ix_{i+3}, $x_{i+1}x_{i+4}$, $x_{i+2}x_{i+5}$ be 2-chords of C for some i, $1 \leq i \leq t$ (indices modulo t), and let $X = \langle\{x_i, x_{i+1}, x_{i+2}, x_{i+3}, x_{i+4}, x_{i+5}\}\rangle_{H_0}$. Then*

(i) X is isomorphic to the graph F_4 in Fig. 4,

(ii) $N_H(\{x_{i+1}, x_{i+2}, x_{i+3}, x_{i+4}\}) \subset V(X)$,

(iii) $A_{H_0}(X) = \{x_i, x_{i+5}\}$.

Proof. (i) Part (i) is obvious since any further edge in X would be a k-chord of C for $k \neq 2$.

(ii) Let, say, $x_{i+2}y \in E(H)$ for some $y \notin V(X)$. If $y \in V(C)$, then $d_{H_0}(x_{i+2}) = 4$, contradicting Claim 5, and if $y \in R$, we have $S_{3,3,4}(x_i; x_{i+1}x_{i+2}y; x_{i+3}x_{i+4}x_{i+5}; x_{i-1}x_{i-2}x_{i-3}x_{i-4})$ in H. Thus, $N_H(x_{i+2}) \subset V(X)$, and, by symmetry, $N_H(\{x_{i+1}, x_{i+2}, x_{i+3}, x_{i+4}\}) \subset V(X)$.

(iii) By (ii), x_i and x_{i+5} are the only vertices of X having a neighbor outside X. $\square$

We show that H_0 has no vertices of degree 4. Let, to the contrary, say, $d_{H_0}(x_6) = 4$. Then x_6 is in two 2-chords, i.e., $x_3x_6, x_6x_9 \in E(H_0)$. By Claim 5, $d_{H_0}(x_4) = d_{H_0}(x_5) = 3$, implying $|N_{H_0}(x_4) \cap \{x_1, x_7\}| = |N_{H_0}(x_5) \cap \{x_2, x_8\}| = 1$. If $x_4x_7 \in E(H_0)$, then H_0 contains $S_{3,3,4}(x_6; x_3x_2x_1; x_9x_{10}x_{11}; x_5x_4x_7x_8)$; hence $x_1x_4 \in E(H_0)$. By symmetry, $x_5x_8 \notin E(H_0)$, implying $x_2x_5 \in E(H_0)$. Symmetrically, $x_7x_{10}, x_8x_{11} \in E(H_0)$. Set $X_1 = \langle\{x_1, x_2, x_3, x_4, x_5, x_6\}\rangle_{H_0}$ and $X_2 = \langle\{x_6, x_7, x_8, x_9, x_{10}, x_{11}\}\rangle_{H_0}$. By Claim 6, we have $X_1 \simeq X_2 \simeq F_4$ (where F_4 is the graph in Fig. 4), $A_{H_0}(X_1) = \{x_1, x_6\}$, $A_{H_0}(X_2) = \{x_6, x_{11}\}$, and X_i is $A_{H_0}(X_i)$-contractible, $i = 1, 2$.

Let first $t = 11$. The set $\{x_3, x_5, x_9\}$ is independent in H_0 and, by Claim 6(ii), we have $N_R(\{x_3, x_5, x_9\}) = \emptyset$. Considering the set $A = V(C) \setminus \{x_3, x_5, x_9\}$ with $|A| = 8$, we have a contradiction by Claim 1. Thus, $t \geq 12$.

Then necessarily $d_{H_0}(x_1) = 4$ or $d_{H_0}(x_{11}) = 4$, for otherwise $\{x_1x_t, x_{11}x_{12}\}$ is an edge-cut of H_0, a contradiction. By symmetry, let $d_{H_0}(x_{11}) = 4$. Then x_{11} is, besides the chord x_8x_{11}, in another 2-chord of C, implying $t \geq 14$ and $x_{11}x_{14} \in E(H_0)$. However, since x_{12} and x_{13} must have a chord, by Claim 5, we have $t \geq 16$ and $x_{12}x_{15}, x_{13}x_{16} \in E(H_0)$. Set $X_3 = \langle\{x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}\}\rangle_{H_0}$. By Claim 6, we have $X_3 \simeq F_4$, $N_H(\{x_{12}, x_{13}, x_{14}, x_{15}\}) \subset V(X_3)$, $A_{H_0}(X_3) = \{x_{11}, x_{16}\}$, and X_3 is $A_{H_0}(X_3)$-contractible. But then X_1, X_2 and X_3 are three contractible subgraphs of H, contradicting Lemma I. Thus, there is no vertex of degree 4, i.e., H_0 is cubic, implying that t is even and $t \geq 12$.

Let now x_1x_4 be a 2-chord of C. Since $d_{H_0}(x_2) = d_{H_0}(x_3) = 3$, up to a symmetry, either $x_2x_5, x_3x_6 \in E(H_0)$, or $x_2x_{t-1}, x_3x_6 \in E(H_0)$ (or $x_2x_5, x_3x_t \in E(H_0)$, which is symmetric with the first possibility). If $x_2x_5, x_3x_6 \in E(H_0)$, then, by Claim 6, again $X_1 = \langle\{x_1, x_2, x_3, x_4, x_5, x_6\}\rangle_{H_0} \simeq F_4$, $N_H(\{x_2, x_3, x_4, x_5\}) \subset V(X_1)$ and $A_{H_0}(X_1) = \{x_1, x_6\}$, but then $\{x_1x_t, x_6x_7\}$ is an edge-cut of H_0, a contradiction. Thus, $x_2x_{t-1}, x_3x_6 \in E(H_0)$. Now, $x_2x_5 \notin E(H_0)$ by Claim 5, and since $d_{H_0}(x_5) = 3$, $x_5x_8 \in E(H_0)$. But then we have $S_{3,3,4}(x_1; x_4x_5x_8; x_tx_{t-1}x_{t-2}; x_2x_3x_6x_7)$ in H_0, a contradiction. ■

4 Concluding remarks

1. Theorems A and 1 can be slightly extended as follows. For $s \geq 0$, a graph G is *s-Hamilton-connected* if the graph $G - M$ is Hamilton-connected for any set $M \subset V(G)$ with $|M| \leq s$. Obviously, an s-Hamilton-connected graph must be $(s+3)$-connected. Since an induced subgraph of a $\{K_{1,3}, N_{i,j,k}\}$-free graph is also $\{K_{1,3}, N_{i,j,k}\}$-free, we immediately have the following fact, showing that the obvious necessary condition is also sufficient in $\{K_{1,3}, N_{i,j,k}\}$-free graphs.

Corollary 2. *Let s, i, j, k be integers such that $s \geq 0$, $i, j, k \geq 1$ and $i + j + k \leq 7$, and let G be a $\{K_{1,3}, N_{i,j,k}\}$-free graph. Then G is s-Hamilton-connected if and only if G is $(s+3)$-connected.*

2. We will now discuss sharpness of the known results and the remaining open cases in the characterization of all pairs of connected graphs X, Y that might imply a 3-connected $\{X, Y\}$-free graph to be Hamilton-connected. To avoid trivial cases, we restrict the observations to $X, Y \not\simeq P_3$. Starting from the negative side, we recall the following result that appeared in [4].

Theorem L [4]. *If X, Y is a pair of connected graphs such that $X, Y \not\simeq P_3$ and every 3-connected $\{X, Y\}$-free graph is Hamilton connected, then, up to a symmetry, $X = K_{1,3}$ and Y satisfies each of the following conditions:*

- (a) $\Delta(Y) \leq 3$,
- (b) *any longest induced path in Y has at most 9 vertices,*
- (c) *Y contains no cycles of length at least 4,*
- (d) *the distance between two distinct triangles in Y is either 1 or at least 3,*
- (e) *There are at most two triangles in Y,*
- (f) *Y is claw-free.*

Moreover, item (d) was reduced in [7] to read

- (d') *the distance between two distinct triangles in Y is either 1 or 3.*

The reduction in [7] consists in two steps: excluding even lengths, and showing that length 5 is not possible (knowing that length more than 5 is not possible by (b)). The graph used in the first step can be easily turned into an infinite family by attaching in the preimage arbitrarily many pendant edges to vertices of degree 3; however, the graph used in the second step has 20 vertices and any additional pendant edge or subdivision in the preimage makes it contain an induced Γ_5. Thus, distance 5 might be still possible for $n = |V(G)| \geq 21$. We therefore replace (d') with the following more precise statement:

- (d'') *the distance between two distinct triangles in Y is 1 or 3, or possibly also 5 for $n = |V(G)| \geq 21$.*

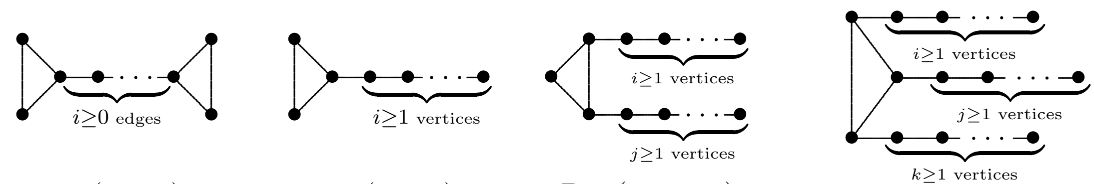
This implies that, as noted in [2], the only possibilities for the graph Y are (see Fig. 5 for the graphs Z_i, $B_{i,j}$, $N_{i,j,k}$ and Γ_i):

- (i) *the path P_i with $4 \leq i \leq 9$,*
- (ii) *the graph Z_i,*
- (iii) *the generalized bull $B_{i,j}$,*
- (iv) *the generalized net $N_{i,j,k}$,*
- (v) *the generalized hourglass Γ_1 or Γ_3, or also Γ_5 for $n = |V(G)| \geq 21$,*

(*vi*) *the generalized hourglass* Γ_1 *or* Γ_3, *or also* Γ_5 *for* $n = |V(G)| \geq 21$, *with paths possibly attached to either of the two triangles.*

In [2], this list was further reduced by excluding the possibility (*vi*).

Note that the reduction in [2], and also all observations in the proof of Theorem L in [4], are based on infinite families of graphs. Thus, except for the Γ_5, there is no hope to get a corresponding result for "sufficiently large" graphs.



Γ_i $(i \geq 0)$ Z_i $(i \geq 1)$ $B_{i,j}$ $(i, j \geq 1)$ $N_{i,j,k}$ $(i, j, k \geq 1)$

Figure 5: The graphs Γ_i, Z_i, $B_{i,j}$ and $N_{i,j,k}$

To obtain upper bounds on the possible values of the parameters i, j, k, consider the family of graphs $\mathcal{G} = \{L(H)|\ H \in \mathcal{W}\}$, introduced in Section 1. The graphs in $\mathcal{G}$ are 3-connected, non-Hamilton-connected, P_{10}-free, Z_i-free for $i \geq 7$ and with a single exception of the smallest graph in $\mathcal{G}$ (having 20 vertices) even Z_i-free for $i \geq 8$, $B_{i,j}$-free for $i + j \geq 8$, and $N_{i,j,k}$-free for $i+j+k \geq 8$. Hence the possible graphs Y that might imply a 3-connected $\{K_{1,3}, Y\}$-free graph to be Hamilton-connected are Γ_1, Γ_3, Γ_5 for $n \geq 21$, P_i for $i \leq 9$, Z_i for $i \leq 6$ and Z_7 for $n \geq 21$, $B_{i,j}$ for $i + j \leq 7$, and $N_{i,j,k}$ for $i + j + k \leq 7$. Among these, Γ_1 was proved in [4], P_9 was proved in [2], and the proof for $N_{i,j,k}$ with $i + j + k = 7$ was completed in this series of two papers. The best known explicit result for Z_i is Z_3 [6]; however, the proof for $N_{1,1,5}$ in this paper gives implicitly Z_5, leaving open Z_i for $6 \leq i \leq 7$. Similarly, the best known explicit result for $B_{i,j}$ is $B_{1,2}$ (also in [6]); however, the results for $N_{i,j,k}$ with $i+j+k = 7$ give implicitly all possible $B_{i,j}$ with $i + j \leq 6$, leaving open $B_{i,j}$ for $i + j = 7$. Finally, Γ_3, and Γ_5 for $n \geq 21$ remain open.

We summarize this discussion of possible graphs Y implying a 3-connected $\{K_{1,3}, Y\}$-free graph to be Hamilton-connected in the following table.

The graph Y	Possible	Known	Open
Γ_i	Γ_1, Γ_3, Γ_5 for $n \geq 21$	Γ_1	Γ_3; Γ_5 for $n \geq 21$
P_i	$4 \leq i \leq 9$	P_9	—
Z_i	$i \leq 7$	Z_5	Z_6, Z_7 for $n \geq 21$
$B_{i,j}$	$i + j \leq 7$	$i + j \leq 6$	$i + j = 7$
$N_{i,j,k}$	$i + j + k \leq 7$	$i + j + k \leq 7$	—

3. Notice. During the refereeing process of this paper, sharp results for Z_i and $B_{i,j}$ were proved in [18] and [19]. Namely, the following was shown (here W^+ denotes the graph obtained from the Wagner graph W by attaching exactly one pendant edge to each of its vertices).

- [18] *If* G *is a 3-connected* $\{K_{1,3}, Z_7\}$*-free graph that is not isomorphic to the graph* $L(W^+)$, *then* G *is Hamilton-connected. Specifically, every 3-connected* $\{K_{1,3}, Z_6\}$*-free graph is Hamilton-connected.*

- [19] *If* G *is a 3-connected* $\{K_{1,3}, B_{i,j}\}$*-free graph with* $i + j \leq 7$, *then* G *is Hamilton-connected.*

Thus, the only remaining open cases are the graphs Γ_3 (for all graphs), and Γ_5 for $|V(G)| \geq 21$.

References

[1] S. Bau, D.A. Holton: On cycles containing eight vertices and an edge in 3-connected cubic graphs. Ars Comb. 26A (1988), 21-34.

[2] Q. Bian, R.J. Gould, P. Horn, S. Janiszewski, S. Fleur, P. Wrayno: 3-connected $\{K_{1,3}, P_9\}$-free graphs are hamiltonian-connected. Graphs Combin. 30 (2014), 1099-1122.

[3] J.A. Bondy, U.S.R. Murty: Graph Theory. Springer, London, 2008.

[4] H. Broersma, R.J. Faudree, A. Huck, H. Trommel, H.J. Veldman: Forbidden subgraphs that imply Hamiltonian-connectedness. J. Graph Theory 40 (2002), 104-119.

[5] J. Brousek, O. Favaron, Z. Ryjáček: Forbidden subgraphs, hamiltonicity and closure in claw-free graphs. Discrete Math. 196 (1999), 29-50.

[6] G. Chen, R.J. Gould: Hamiltonian connected graphs involving forbidden subgraphs. Bull. Inst. Combin. Appl. 29 (2000), 25-32.

[7] J.R. Faudree, R.J. Faudree, Z. Ryjáček , P. Vrána: On forbidden pairs implying Hamilton-connectedness. J. Graph Theory 72 (2012), 247-365.

[8] F. Harary, C.St.J.A. Nash-Williams: On eulerian and hamiltonian graphs and line graphs. Canad. Math. Bull. 8 (1965), 701-710.

[9] D.A Holton, B.D. McKay, M.D. Plummer, C. Thomassen: A nine point theorem for 3-connected graphs. Combinatorica 2 (1982), 57-62.

[10] R. Kužel, Z. Ryjáček, J. Teska, P. Vrána: Closure, clique covering and degree conditions for Hamilton-connectedness in claw-free graphs. Discrete Math. 312 (2012), 2177-2189.

[11] D. Li, H.-J. Lai, M. Zhan: Eulerian subgraphs and Hamilton-connected line graphs. Discrete Appl. Math. 145 (2005), 422-428.

[12] X. Liu, Z.Ryjáček, P. Vrána, L. Xiong, X. Yang: Hamilton-connected {claw,net}-free graphs I. J. Graph Theory (2022), 1–26. https://doi.org/10.1002/jgt.22863.

[13] X. Liu, L. Xiong, H.-J. Lai: Strongly spanning trailable graphs with small circumference and Hamilton-connected claw-free graphs. Graphs Combin. 37 (2021), 65–85.

[14] M. Miller, J. Ryan, Z. Ryjáček, J. Teska, P. Vrána: Stability of hereditary graph classes under closure operations. J. Graph Theory 74 (2013), 67-80.

[15] Z. Ryjáček, R. H. Schelp: Contractibility techniques as a closure concept. J. Graph Theory 43 (2003), 37-48.

[16] Z. Ryjáček, P. Vrána: Line graphs of multigraphs and Hamilton-connectedness of claw-free graphs. J. Graph Theory 66 (2011), 152-173.

[17] Z. Ryjáček, P. Vrána: A closure for 1-Hamilton-connectedness in claw-free graphs. J. Graph Theory 75 (2014), 358–376.

[18] Z. Ryjáček, P. Vrána: Every 3-connected $\{K_{1,3}, Z_7\}$-free graph of order at least 21 is Hamilton-connected. Discrete Math. 344 (2021), 112350.

[19] Z. Ryjáček, P. Vrána: Hamilton-connected {claw,bull}-free graphs. J. Graph Theory (2022), 1–26. https://doi.org/10.1002/jgt.22861.

[20] Y. Shao: Claw-free graphs and line graphs. Ph.D Thesis, West Virginia University, 2005.