A closure for Hamilton-connectedness in $\{K_{1,3}, \Gamma_3\}$-free graphs[1]

Adam Kabela Zdeněk Ryjáček Mária Skyvová Petr Vrána

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Abstract

We introduce a closure technique for Hamilton-connectedness of $\{K_{1,3}, \Gamma_3\}$-free graphs, where Γ_3 is the graph obtained by joining two vertex-disjoint triangles with a path of length 3. The closure turns a claw-free graph into a line graph of a multigraph while preserving its (non)-Hamilton-connectedness. The most technical parts of the proof are computer-assisted.

The main application of the closure is given in a subsequent paper showing that every 3-connected $\{K_{1,3}, \Gamma_3\}$-free graph is Hamilton-connected, thus resolving one of the two last open cases in the characterization of pairs of connected forbidden subgraphs implying Hamilton-connectedness.

Keywords: Hamilton-connected; closure; forbidden subgraph; claw-free; Γ_3-free

1 Terminology and notation

In this paper, we generally follow the most common graph-theoretical notation and terminology, and for notations and concepts not defined here we refer to [4]. Specifically, by a *graph* we always mean a simple finite undirected graph; whenever we admit multiple edges, we always speak about a *multigraph*.

We write $G_1 \subset G_2$ if G_1 is a sub(multi)graph of G_2, $G_1 \overset{\text{IND}}{\subset} G_2$ if G_1 is an induced sub(multi)graph of G_2, $G_1 \simeq G_2$ if the (multi)graphs G_1, G_2 are isomorphic, and $\langle M \rangle_G$ to denote the *induced sub(multi)graph* on a set $M \subset V(G)$. We use $d_G(x)$ to denote the *degree* of a vertex x in G (note that if G is a multigraph, then $d_G(x)$ equals the sum of multiplicities of the edges containing x), $N_G(x)$ denotes the *neighborhood* of a vertex x, and $N_G[x]$ the *closed neighborhood* of x, i.e., $N_G[x] = N_G(x) \cup \{x\}$. For $M \subset V(G)$, we denote $N_M(x) = N_G(x) \cap M$ and $N_G[M] = \cup_{x \in M} N_G[x]$. For $x, y \in V(G)$, $\text{dist}_G(x, y)$ denotes their *distance*, i.e., the length of a shortest (x, y)-path in G. More generally, if $F \subset G$ is connected and $x, y \in V(F)$, then $\text{dist}_F(x, y)$ denotes the length of a shortest (x, y)-path in F. By a *clique* in G we mean

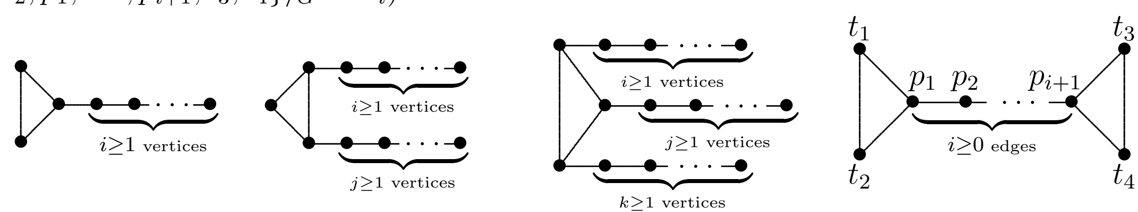
[1]All authors are affiliated with the Department of Mathematics; European Centre of Excellence NTIS - New Technologies for the Information Society, University of West Bohemia, Univerzitní 8, 301 00 Pilsen, Czech Republic. E-mails: {kabela,ryjacek,vranap}@kma.zcu.cz, mskyvova@ntis.zcu.cz. The research was supported by project GA20-09525S of the Czech Science Foundation.

a complete subgraph of G (not necessarily maximal), and $\alpha(G)$ denotes the *independence number* of G.

We say that a vertex $x \in V(G)$ is *simplicial* if $\langle N_G(x)\rangle_G$ is a clique, and we use $V_{SI}(G)$ to denote the set of all simplicial vertices of G, and $V_{NS}(G) = V(G) \setminus V_{SI}(G)$ the set of nonsimplicial vertices of G. For $k \geq 1$, we say that a vertex $x \in V(G)$ is *locally k-connected in G* if $\langle N_G(x)\rangle_G$ is a k-connected graph.

A graph is *Hamilton-connected* if, for any $u, v \in V(G)$, G has a hamiltonian (u,v)-path, i.e., an (u,v)-path P with $V(P) = V(G)$.

Finally, if $\mathcal{F}$ is a family of graphs, we say that G is *$\mathcal{F}$-free* if G does not contain an induced subgraph isomorphic to a member of $\mathcal{F}$, and the graphs in $\mathcal{F}$ are referred to in this context as *forbidden (induced) subgraphs.* If $\mathcal{F} = \{F\}$, we simply say that G is *F-free.* Here, the *claw* is the graph $K_{1,3}$, P_i denotes the path on i vertices, and Γ_i denotes the graph obtained by joining two triangles with a path of length i (see Fig. 1(d)). Several further graphs that will occur as forbidden subgraphs are shown in Fig. 1(a), (b), (c). Whenever we will list vertices of an induced claw $K_{1,3}$, we will always list its center as the first vertex of the list, and when listing vertices of an induced subgraph Γ_i, we always list first the vertices of degree 2 of one of the triangles, then the vertices of the path, and we finish with the vertices of degree 2 of the second triangle (i.e., in the labeling of vertices as in Fig. 1(d), we write $\langle\{t_1, t_2, p_1, \ldots, p_{i+1}, t_3, t_4\}\rangle_G \simeq \Gamma_i$).



(a) Z_i ($i \geq 1$) (b) $B_{i,j}$ ($i, j \geq 1$) (c) $N_{i,j,k}$ ($i, j, k \geq 1$) (d) Γ_i ($i \geq 0$)

Figure 1: The graphs Z_i, $B_{i,j}$, $N_{i,j,k}$ and Γ_i

2 Introduction

There are many results on forbidden induced subgraphs implying various Hamilton-type properties. While forbidden pairs of connected graphs for hamiltonicity in 2-connected graphs were completely characterized already in the early 90's [1, 8], the progress in forbidden pairs for Hamilton-connectedness is relatively slow.

Let W denote the Wagner graph and W^+ the graph obtained from W by attaching exactly one pendant edge to each of its vertices (see Fig. 2).

Figure 2: The Wagner graph W and the graph W^+

Theorem A below lists the best known results on pairs of forbidden subgraphs implying Hamilton-connectedness of a 3-connected graph.

Theorem A [3, 6, 12, 13, 14, 18, 19]. *Let G be a 3-connected $\{K_{1,3}, X\}$-free graph, where*
(*i*) **[6]** $X = \Gamma_1$, *or*
(*ii*) **[3]** $X = P_9$, *or*
(*iii*) **[18]** $X = Z_7$ *and* $G \not\simeq W^+$, *or*
(*iv*) **[19]** $X = B_{i,j}$ *for* $i + j \leq 7$, *or*
(*v*) **[12, 13, 14]** $X = N_{i,j,k}$ *for* $i + j + k \leq 7$.
Then G is Hamilton-connected.

Let $\mathcal{W}$ be the family of graphs obtained by attaching at least one pendant edge to each of the vertices of the Wagner graph W, and let $\mathcal{G} = \{L(H)|\ H \in \mathcal{W}\}$ be the family of their line graphs. Then any $G \in \mathcal{G}$ is 3-connected, non-Hamilton-connected, P_{10}-free, Z_8-free, $B_{i,j}$-free for $i + j = 8$ and $N_{i,j,k}$-free for $i + j + k = 8$. Thus, this example shows that parts (*ii*), (*iii*), (*iv*) and (*v*) of Theorem A are sharp.

According to the discussion in Section 6 of [19], there are two remaining connected graphs X that might imply Hamilton-connectedness of a 3-connected $\{K_{1,3}, X\}$-free graph, namely, the graph Γ_3, and the graph Γ_5 for $|V(G)| \geq 21$ (or possibly with a single exception of $G \simeq L(W^+)$). In this paper, we address the first of these graphs, the graph Γ_3. We develop the main tool, the closure operation, and in the subsequent paper [10], as an application of the main result of this paper, we prove the following.

Theorem B [10]. *Every 3-connected $\{K_{1,3}, \Gamma_3\}$-free graph is Hamilton-connected.*

In Section 3, we collect necessary known results and facts on line graphs and on closure operations, and then, in Section 4, we develop a closure technique that will be crucial for the proof of Theorem B. The most technical parts of the proofs (namely, Case 1 and Subcase 2.2 in the proof of Proposition 7) are computer-assisted. More details on the computation can be found in Section 5, and detailed results of the computation and source codes are available in the repository at link [22].

3 Preliminaries

In this section, we summarize known facts that will be needed in the proof of our main result, Theorem 2.

3.1 Line graphs of multigraphs and their preimages

The following characterization of line graphs of multigraphs was proved by Bermond and Meyer [2] (see also Zverovich [21]).

Theorem C [2]. *A graph G is a line graph* of a multigraph *if and only if G does not contain a copy of any of the graphs in Figure 3 as an induced subgraph.*

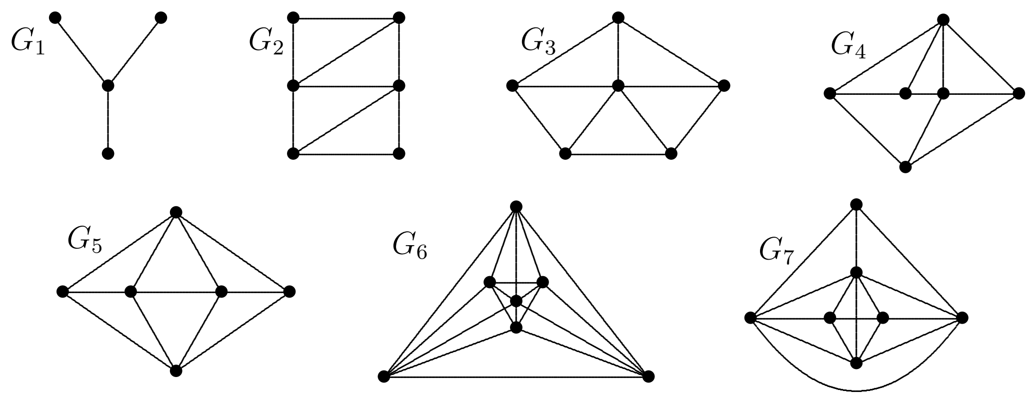


Figure 3: Forbidden subgraphs for line graphs of multigraphs

While in line graphs of graphs, for a connected line graph G, the graph H such that $G = L(H)$ is uniquely determined with a single exception of $G = K_3$, in line graphs of multigraphs this is not true: a simple example are the graphs $H_1 = Z_1$ and H_2 a double edge with one pendant edge attached to each vertex – while $H_1 \not\simeq H_2$, we have $L(H_1) \simeq L(H_2)$. Using a modification of an approach from [21], the following was proved in [17].

Theorem D [17]. *Let G be a connected line graph of a multigraph. Then there is, up to an isomorphism, a uniquely determined multigraph H such that a vertex $e \in V(G)$ is simplicial in G if and only if the corresponding edge $e \in E(H)$ is a pendant edge in H.*

The multigraph H with the properties given in Theorem D will be called the *preimage* of a line graph G and denoted $H = L^{-1}(G)$. We will also use the notation $a = L(e)$ and $e = L^{-1}(a)$ for an edge $e \in E(H)$ and the corresponding vertex $a \in V(G)$.

An edge-cut $R \subset E(H)$ of a multigraph H is *essential* if $H - R$ has at least two nontrivial components, and H is *essentially k-edge-connected* if every essential edge-cut of H is of size at least k. It is obvious that a set $M \subset V(G)$ of vertices of a line graph G is a vertex-cut of G if and only if the corresponding set $L^{-1}(M) \subset E(L^{-1}(G))$ is an essential edge-cut of $L^{-1}(G)$. Consequently, a noncomplete line graph G is k-connected if and only if $L^{-1}(G)$ is essentially k-edge-connected. It is also a well-known fact that if X is a line graph, then a line graph G is X-free if and only if $L^{-1}(G)$ does not contain as a subgraph (not necessarily induced) a graph F such that $L(F) = X$.

3.2 Closure operations

For $x \in V(G)$, the *local completion of G at x* is the graph $G^*_x = (V(G), E(G) \cup \{y_1y_2 |\ y_1, y_2 \in N_G(x)\})$ (i.e., G^*_x is obtained from G by adding all the missing edges with both vertices in $N_G(x)$). In this context, the edges in $E(G^*_x) \setminus E(G)$ will be refereed to as *new edges*, and the edges in $E(G)$ are *old*. Obviously, if G is claw-free, then so is G^*_x. Note that in the special case when G is a line graph and $H = L^{-1}(G)$, G^*_x is the line graph of the multigraph $H|_e$ obtained from H by contracting the edge $e = L^{-1}(x)$ into a vertex and replacing the created loop(s) by pendant edge(s) (Thus, if $G = L(H)$ and $x = L(e)$, then $G^*_x = L(H|_e)$).

Also note that clearly $x \in V_{SI}(G^*_x)$ for any $x \in V(G)$, and, more generally, $V_{SI}(G) \subset V_{SI}(G^*_x)$ for any $x \in V(G)$.

We say that a vertex $x \in V(G)$ is *eligible* if $\langle N_G(x)\rangle_G$ is a connected noncomplete graph, and we use $V_{EL}(G)$ to denote the set of all eligible vertices of G. Note that in the special case when G is a line graph and $H = L^{-1}(G)$, it is not difficult to observe that $x \in V(G)$ is eligible if and only if the edge $L^{-1}(x)$ is in a triangle or in a multiple edge of H. Based on the fact that if G is claw-free and $x \in V_{EL}(G)$, then G^*_x is hamiltonian if and only if G is hamiltonian, the *closure* $\mathrm{cl}(G)$ of a claw-free graph G was defined in [15] as the graph obtained from G by recursively performing the local completion operation at eligible vertices, as long as this is possible (more precisely: $\mathrm{cl}(G) = G_k$, where $G_1, \dots, G_k$ is a sequence of graphs such that $G_1 = G$, $G_{i+1} = (G_i)^*_{x_i}$ for some $x_i \in V_{EL}(G)$, $i = 1, \dots, k-1$, and $V_{EL}(G_k) = \emptyset$). The closure $\mathrm{cl}(G)$ of a claw-free graph G is uniquely determined, is a line graph of a triangle-free graph, and is hamiltonian if and only if so is G. However, as observed in [5], the closure operation does not preserve the (non-)Hamilton-connectedness of G.

In attempts to identify reasons of this problem, the following result on stability of hamiltonian path under local completion was proved in [5].

Proposition E [5]. *Let x be an eligible vertex of a claw-free graph G, G^*_x the local completion of G at x, and a, b two distinct vertices of G. Then for every longest (a,b)-path $P'(a,b)$ in G^*_x there is a path P in G such that $V(P) = V(P')$ and P admits at least one of a, b as an endvertex. Moreover, there is an (a,b)-path $P(a,b)$ in G such that $V(P) = V(P')$ except perhaps in each of the following two situations (up to symmetry between a and b):*

- *(i) There is an induced subgraph $H \subset G$ isomorphic to the graph S in Figure 4 such that both a and x are vertices of degree 4 in H. In this case G contains a path P_b such that b is an endvertex of P and $V(P_b) = V(P')$. If, moreover, $b \in V(H)$, then G contains also a path P_a with endvertex a and with $V(P_a) = V(P')$.*
- *(ii) $x = a$ and $ab \in E(G)$. In this case there is always both a path P_a in G with endvertex a and with $V(P_a) = V(P')$ and a path P_b in G with endvertex b and with $V(P_b) = V(P')$.*

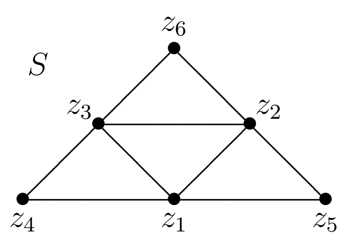


Figure 4: The graph S

The following consequence of Proposition E will be useful in our proof.

Corollary 1. *Let G be a claw-free graph that is not Hamilton-connected, and let $x_1, x_2 \in V_{EL}(G)$ be such that $x_1x_2 \notin E(G)$ and both $G^*_{x_1}$ and $G^*_{x_2}$ are Hamilton-connected. Then at least one of the vertices x_1, x_2 is a vertex of degree 4 in an induced subgraph $H \overset{IND}{\subset} G$ isomorphic to the graph S in Figure 4.*

Proof. If at least one of x_1, x_2 satisfies (i) of Proposition E, then we are done; thus, let both x_1 and x_2 satisfy (ii) of Proposition E. Thus, by (ii), there are vertices $b_1, b_2 \in V(G)$ such that $x_i b_i \in E(G)$ and G has no hamiltonian (x_i, b_i)-path, $i = 1, 2$. Since $x_1 x_2 \notin E(G)$, $b_1 \neq x_2$ and $b_2 \neq x_1$. Thus, again by (ii) of Proposition E, there is no hamiltonian (x_2, b_2)-path in $G^*_{x_1}$, implying that $G^*_{x_1}$ is not Hamilton-connected, a contradiction. ■

To handle the problem of unstability of Hamilton-connectedness, the closure concept was strengthened in [11] and [12] such that the closure operation (called SM-closure in [11], and its further strengthening, UM-closure in [12]), preserves the (non)-Hamilton-connectedness. However, these operations are not applicable to $\{K_{1,3}, \Gamma_3\}$-free graphs since a closure of a $\{K_{1,3}, \Gamma_3\}$-free graph is not necessarily Γ_3-free.

Before showing a way to overcome this problem, we first recall two classical results by Chvátal and Erdős [7] and by Fouquet [9] that will be needed in the proof of the main result.

Theorem F [7]. *Let G be an s-connected graph containing no independent set of s vertices. Then G is Hamilton-connected.*

Theorem G [9]. *Let G be a connected claw-free graph with independence number at least three. Then every vertex v satisfies exactly one of the following:*
(i) *$N(v)$ is covered by two cliques,*
(ii) *$\langle N(v) \rangle_G$ contains an induced C_5.*

4 Γ_3-closure

As already mentioned, the SM-closure and UM-closure operations (see [11] and [12]) preserve Hamilton-connectedness, but there is still a problem that the local completion G^*_x of a $\{K_{1,3}, \Gamma_3\}$-free graph G is not necessarily Γ_3-free. To handle this problem, we define the concept of a Γ_3-closure G^{Γ_3} of a $\{K_{1,3}, \Gamma_3\}$-free graph G. For a set $M = \{x_1, x_2, \ldots, x_k\} \subset V(G)$, we set $G^*_M = ((G^*_{x_1})^*_{x_2} \ldots)^*_{x_k}$. It is implicit in the proof of uniqueness of $\mathrm{cl}(G)$ in [15] (and easy to see) that, for a given set $M = \{x_1, x_2, \ldots, x_k\} \subset V(G)$, G^*_M is uniquely determined (i.e., does not depend on the order of the vertices $x_1, x_2, \ldots, x_k$ used during the construction).

If G is not Hamilton-connected, then a vertex $x \in V_{NS}(G)$, for which the graph G^*_x is still not Hamilton-connected, is said to be *feasible in* G. A set of vertices $M \subset V(G)$ is said to be *feasible in* G if the vertices in M can be ordered in a sequence $x_1, \ldots, x_k$ such that x_1 is feasible in $G_0 = G$, and x_{i+1} is feasible in $G_i = (G_{i-1})^*_{x_i}$, $i = 1, \ldots, k-1$. Thus, if $M \subset V(G)$ is feasible, then $M \subset V_{SI}(G^*_M)$, but G^*_M is still not Hamilton-conected.

Note that it is possible that some two vertices x, y of a graph G are feasible in G, but x is not feasible in G^*_y (for example, if H is obtained from the Petersen graph by adding a pendant edge to each vertex, subdividing a nonpendant edge $x_1 x_2$ with a vertex w, replacing each of the edges $x_i w$ with a double edge, and if $G = L(H)$ and $x'_i, x''_i \in V(G)$ correspond to the two edges joining x_i and w in H, $i = 1, 2$, then G is not Hamilton-connected, each of the vertices x'_i, x''_i is feasible in G, $i = 1, 2$, but e.g. x'_1 and x''_1 are not feasible in $G^*_{x'_2} \simeq G^*_{x''_2}$). Thus, the recursive form of the definition is essential for verifying feasibility of a set $M \subset V(G)$ (although the resulting graph G^*_M does not depend on their order).

Now, for a $\{K_{1,3}, \Gamma_3\}$-free graph G, we define its Γ_3-closure G^{Γ_3} by the following construction.

(i) If G is Hamilton-connected, we define G^{Γ_3} as the complete graph.

(ii) If G is not Hamilton-connected, we recursively perform the local completion operation at such feasible sets of vertices for which the resulting graph is still Γ_3-free, as long as this is possible. We obtain a sequence of graphs $G_1, \ldots, G_k$ such that

- $G_1 = G$,
- $G_{i+1} = (G_i)^*_{M_i}$ for some set $M_i \subset V(G_i)$, $i = 1, \ldots, k-1$,
- G_k has no hamiltonian (a, b)-path for some $a, b \in V(G_k)$,
- for any feasible set $M \subset V_{NS}(G_k)$, $(G_k)^*_M$ contains an induced subgraph isomorphic to Γ_3,

and we set $G^{\Gamma_3} = G_k$.

A resulting graph G^{Γ_3} is called a Γ_3-*closure* of the graph G, and a graph G equal to (some) its Γ_3-closure is said to be Γ_3-*closed*. Note that for a given graph G, its Γ_3-closure is not uniquely determined.

The following two results from [16] and [17] will be useful to identify feasible vertices.

Theorem H [16]. *Let G be a claw-free graph and let $x \in V(G)$ be locally 2-connected in G. Then G is Hamilton-connected if and only if G^*_x is Hamilton-connected.*

Thus, in our terminology, Theorem H says that a locally 2-connected vertex is feasible.

Lemma I [17]. *Let G be a claw-free graph, $x \in V(G)$, and let $H \overset{IND}{\subset} \langle N_G(x)\rangle_G$ be a 2-connected graph containing two disjoint pairs of independent vertices. Then x is locally 2-connected in G.*

Note that if a vertex $x \in V(G)$ is feasible by virtue of Theorem H (i.e., x is locally 2-connected in G), then, for any $y \in V(G)$ $y \neq x$, x is locally 2-connected also in G^*_y, but $\langle N_{G^*_y}(x)\rangle_{G^*_y}$ can be complete (if $N_G(x) \subset N_G(y)$). Thus, for any $y \in V(G)$, x is feasible or simplicial in G^*_y.

We thus define more generally: a set $M \subset V(G)$ is *weakly feasible in* G if the vertices in M can be ordered in a sequence $x_1, \ldots, x_k$ such that x_1 is feasible in $G_0 = G$, and x_{i+1} is feasible or simplicial in $G_i = (G_{i-1})^*_{x_i}$, $i = 1, \ldots, k-1$. Thus, similarly, if G is not Hamilton-connected and $M \subset V(G)$ is weakly feasible in G, then G^*_M is still not Hamilton-conected and all vertices of M are simplicial in G^*_M.

The following theorem is the main result of this paper.

Theorem 2. *Let G be a 3-connected $\{K_{1,3}, \Gamma_3\}$-free graph and let G^{Γ_3} be its Γ_3-closure. Then there is a multigraph H such that $G^{\Gamma_3} = L(H)$.*

Proof. Let G be a $\{K_{1,3}, \Gamma_3\}$-free graph, and let $\bar{G}$ be (some) its Γ_3-closure. To show that $\bar{G}$ is a line graph of a multigraph, by Theorem C, we show that $\bar{G}$ does not contain as an induced subgraph any of the graphs $G_1, \ldots, G_7$ of Figure 3. If G is Hamilton-connected, then $\bar{G}$ is complete and the statement is trivial. So, assume that G (and hence also $\bar{G}$) is not Hamilton-connected.

Since $G_1 \simeq K_{1,3}$ and $\bar{G}$ was obtained from G by a series of local completions, obviously $\bar{G}$ is G_1-free. We prove the following three facts (for the graphs W_5, W_4, P_6^2 and P_6^{2+} see Fig. 5):

- $\bar{G}$ is W_5-free,
- $\bar{G}$ is W_4-free,
- $\bar{G}$ is $\{P_6^2, P_6^{2+}\}$-free.

This will establish the result since $G_3 \simeq W_5$, each of the graphs G_5, G_6, G_7 contains an induced W_4, $G_2 \simeq P_6^2$, and $G_4 \simeq P_6^{2+}$.

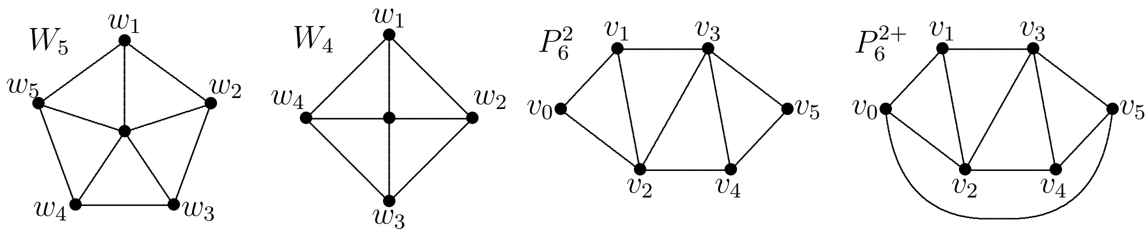


Figure 5: The 5-wheel W_5, the 4-wheel W_4, and the graphs P_6^2 and P_6^{2+}

This will be done in the following Propositions 3, 7 and 10.

Throughout the proof, when listing vertices of an induced W_5 or W_4, we will always list the center first, and we will list vertices of a P_6^2 or a P_6^{2+} in the order indicated by the indices in Fig. 5.

Proposition 3. *Let G be a $\{K_{1,3}, \Gamma_3\}$-free graph and let $\bar{G}$ be its Γ_3-closure. Then $\bar{G}$ is W_5-free.*

Proof. If x is a center of an induced W_5 in $\bar{G}$, then x is locally 2-connected in $\bar{G}$ by Lemma I, hence x is feasible in $\bar{G}$ by Theorem H. Thus, by the definition of the Γ_3-closure, $\bar{G}_x^*$ contains an induced Γ_3. To reach a contradiction, we show that this is not possible.

We first prove several lemmas.

Lemma 4. *Let G be a claw-free graph and $F \overset{IND}{\subset} G$, $F \simeq W_5$, with center x and cycle C. Then*

(*i*) *every vertex $y \in N_G(x) \setminus V(F)$ has at least three consecutive neighbors on C,*

(*ii*) *for every $y_1, y_2 \in N_G(x)$, $dist_{\langle N_G(x)\rangle_G}(y_1, y_2) \leq 2$,*

(*iii*) *for every $y \in N_G(x) \setminus V(F)$ that is not in an induced C_5 in $\langle N_G(x)\rangle_G$, every induced C_5 in $\langle N_G(x)\rangle_G$ contains at least 4 neighbors of y.*

Proof. Let $F = \langle\{w, w_1, w_2, w_3, w_4, w_5\}\rangle_G \simeq W_5$.

(*i*) If $N_G(y) \cap V(C) = \emptyset$, then x is a center of a claw; thus, say, $w_1 y \in E(G)$. Since $\langle\{w_1, y, w_2, w_5\}\rangle_G \not\simeq K_{1,3}$, by symmetry, $w_2 y \in E(G)$. If y is adjacent to neither w_3 nor w_5, then $\langle\{x, w_3, w_5, y\}\rangle_G \simeq K_{1,3}$; thus, by symmetry, say, $y w_3 \in E(G)$.

(*ii*) By (*i*), any $y_1, y_2 \in N_G(x)$ are adjacent or have a common neighbor in $N_G(x)$.

(*iii*) Let $C' = w_1' w_2' w_3' w_4' w_5' w_1'$ be an induced C_5 in $N_G(x)$. By (*i*), y has at least 3 consecutive neighbors on C'. If $|N_G(y) \cap V(C')| \geq 4$, we are done, thus, let, say, $N_G(y) \cap V(C') = \{w_1', w_2', w_3'\}$. But then $\langle\{w_1', y, w_3', w_4', w_5'\}\rangle_G \simeq C_5$, contradicting the assumption. ■

Lemma 5. *Let $i \geq 1$, let G be a $\{K_{1,3}, \Gamma_i\}$-free graph, and let $x \in V_{EL}(G)$. Let $F \overset{IND}{\subset} G^*_x$ be such that $F \simeq \Gamma_i$ and a triangle of F contains a new edge y_1y_2. Then $dist_{\langle N_G(x)\rangle_G}(y_1, y_2) = 3$.*

Proof. Let $F \overset{\text{IND}}{\subset} G^*_x$, $F \simeq \Gamma_i$, and let y_1y_2 be a new edge in some of its triangles. We use the labeling of the vertices of F as in Fig. 1(d). Since $\langle N_G(x)\rangle_{G^*_x}$ is a clique, all new edges in F are in one of its triangles, say, $t_1t_2p_1t_1$. If t_1t_2 is the only new edge, then $\langle\{p_1, p_2, t_1, t_2\}\rangle_G \simeq K_{1,3}$, and if all three edges are new, then $\langle\{x, t_1, t_2, p_1\}\rangle_G \simeq K_{1,3}$. Thus, by symmetry, we can choose the notation such that the edge t_1p_1 (and possibly also one of the edges t_2p_1, t_1t_2) is new, i.e., $y_1 = t_1$ and $y_2 = p_1$.

Note that $2 \leq \text{dist}_{\langle N_G(x)\rangle_G}(t_1, p_1) \leq 3$ since t_1p_1 is a new edge and x is not the center of an induced claw. Suppose, to the contrary, that $\text{dist}_{N_G(x)}(t_1, p_1) = 2$. If $xt_2, p_1t_2 \in E(G)$, then $\langle\{x, t_2, p_1, p_2, \ldots, p_{i+1}, t_3, t_4\}\rangle_G \simeq \Gamma_i$, a contradiction. Thus, by the assumption that $\text{dist}_{\langle N_G(x)\rangle_G}(t_1, p_1) = 2$, there is a vertex $z \in N_G(x)$ such that $zt_1, zp_1 \in E(G)$ (note that possibly $x = t_2$). Since $\langle\{z, x, p_1, p_2, \ldots, p_{i+1}, t_3, t_4\}\rangle_G \not\simeq \Gamma_i$, $zw \in E(G)$ for some $w \in \{p_2, \ldots, p_{i+1}, t_3, t_4\}$. However, if $w \in \{p_3, \ldots, p_{i+1}, t_3, t_4\}$, then $\langle\{z, t_1, p_1, w\}\rangle_G \simeq K_{1,3}$. Hence $zw \notin E(G)$ for $w \in \{p_3, \ldots, p_{i+1}, t_3, t_4\}$, implying $zp_2 \in E(G)$, and then $\langle\{t_1, x, z, p_2, \ldots, p_{i+1}, t_3, t_4\}\rangle_G \simeq \Gamma_i$, a contradiction. ∎

Lemma 6. *Let $i \geq 2$ and $1 \leq j \leq i$. Let G be a $\{K_{1,3}, \Gamma_i\}$-free graph and $x \in V_{NS}(G)$ such that G^*_x contains an induced subgraph $F \simeq \Gamma_i$ with a new edge p_jp_{j+1} in the path of F. For $k = 1, 2$, let $z_k \in V(G)$ be such that $\{p_j, p_{j+1}, x\} \subset N_G(z_k)$. Then $z_1z_2 \in E(G)$.*

Proof. Let, to the contrary, $z_1z_2 \notin E(G)$. By symmetry, we can choose the notation such that $1 \leq j \leq i - 1$ (i.e., p_{j+1} is not the last vertex of the path of F). Since $\langle\{p_{j+1}, z_1, z_2, p_{j+2}\}\rangle_G \not\simeq K_{1,3}$, we have, up to a symmetry, $z_2p_{j+2} \in E(G)$. Since each of the edges z_2w, $w \in \{t_1, t_2, p_1, \ldots, p_{j-1}, p_{j+3}, \ldots, p_{i+1}, t_3, t_4\}$ yields an induced $K_{1,3}$ in G with center at z_2, we have $\langle\{t_1, t_2, p_1, \ldots, p_j, z_2, p_{j+2}, \ldots, p_{i+1}, t_3, t_4\}\rangle_G \simeq \Gamma_i$, a contradiction. ∎

Suppose now, to the contrary, that $\bar{G}$ contains an induced subgraph $W \simeq W_5$, set $W = \langle\{x, w_1, w_2, w_3, w_4, w_5\}\rangle_{\bar{G}}$, and let $F \overset{\text{IND}}{\subset} \bar{G}^*_x$ be such that $F \simeq \Gamma_3$. Set $F = \langle\{t_1, t_2, p_1, p_2, p_3, p_4, t_3, t_4\}\rangle_{\bar{G}}$. Since $\bar{G}$ is Γ_3-free, at least one edge of F is new.

If there is a new edge y_1y_2 in a triangle of F, then we have $\text{dist}_{\bar{G}}(y_1, y_2) = 2$ by Lemma 4(ii), and $\text{dist}_{\bar{G}}(y_1, y_2) = 3$ by Lemma 5, a contradiction. Thus, a new edge is on the path of F, and, moreover, F has exactly one new edge since it is induced and x is simplicial in $\bar{G}^*_x$.

Choose $W \simeq W_5$ so that $|V(W) \cap V(F)|$ is maximized. Clearly, $|V(W) \cap V(F)| \leq 2$, and we can choose the notation such that the new edge is p_1p_2 or p_2p_3.

<u>**Case 1:**</u> $|V(W) \cap V(F)| = 2$.

<u>**Subcase 1.1:**</u> *The edge p_1p_2 is new.*

We can choose the notation such that $p_1 = w_2$ and $p_2 = w_5$. First observe that since $\langle\{w_2, w_3, x, w_5, p_3, p_4, t_3, t_4\}\rangle_{\bar{G}} \not\simeq \Gamma_3$, $w_3z \in E(\bar{G})$ for some $z \in \{p_3, p_4, t_3, t_4\}$. Since $\langle\{w_2, t_1, w_1, w_3\}\rangle_{\bar{G}} \not\simeq K_{1,3}$, $t_1w_1 \in E(\bar{G})$ or $t_1w_3 \in E(\bar{G})$. But if $t_1w_3 \in E(\bar{G})$, then $\langle\{w_3, t_1, x, z\}\rangle_{\bar{G}} \simeq K_{1,3}$, a contradiction. Hence $t_1w_1 \in E(\bar{G})$. Since $\langle\{t_1, w_2, w_1, w_5, p_3, p_4, t_3, t_4\}\rangle_{\bar{G}} \not\simeq \Gamma_3$, we have $w_1z' \in E(\bar{G})$ for some $z' \in \{p_3, p_4, t_3, t_4\}$, but for each of these possibilities, $\langle\{w_1, t_1, x, z'\}\rangle_{\bar{G}} \simeq K_{1,3}$, a contradiction.

Subcase 1.2: *The edge p_2p_3 is new.*
Choose the notation such that $p_2 = w_2$ and $p_3 = w_5$. Considering $\langle\{w_2, w_1, w_3, p_1\}\rangle_{\bar{G}} \not\simeq K_{1,3}$, we have $p_1w_1 \in E(\bar{G})$ or $p_1w_3 \in E(\bar{G})$. Let first $p_1w_1 \in E(\bar{G})$. Since $\langle\{t_1, t_2, p_1, w_1, w_3, p_4, t_3, t_4\}\rangle_{\bar{G}} \not\simeq \Gamma_3$, $w_1z \in E(\bar{G})$ for some $z \in \{t_1, t_2\} \cup \{p_4, t_3, t_4\}$; however, if $z \in \{t_1, t_2\}$, then $\langle\{w_1, z, w_2, w_5\}\rangle_{\bar{G}} \simeq K_{1,3}$, and if $z \in \{p_4, t_3, t_4\}$, then $\langle\{w_1, z, x, p_1\}\rangle_{\bar{G}} \simeq K_{1,3}$. Hence $p_1w_3 \in E(\bar{G})$. Symmetrically, $p_4w_4 \in E(\bar{G})$. Since $\langle\{t_1, t_2, p_1, w_3, w_4, p_4, t_3, t_4\}\rangle_{\bar{G}} \not\simeq \Gamma_3$, w_3 or w_4 has a neighbor among the vertices t_1, t_2, t_3, t_4. However, if $w_3z \in E(\bar{G})$ for $z \in \{t_1, t_2\}$, then $\langle\{w_3, z, w_2, w_4\}\rangle_{\bar{G}} \simeq K_{1,3}$, and if $w_3z \in E(\bar{G})$ for $z \in \{t_3, t_4\}$, then $\langle\{w_3, p_1, x, z\}\rangle_{\bar{G}} \simeq K_{1,3}$; the situation for w_4 is symmetric. Thus, $\langle\{t_1, t_2, p_1, w_3, w_4, p_4, t_3, t_4\}\rangle_{\bar{G}} \simeq \Gamma_3$, a contradiction.

Case 2: $|V(W) \cap V(F)| = 1$.
Let the new edge be p_ip_{i+1}, $i \in \{1, 2\}$, and, by symmetry, choose the notation such that $p_{i+1} = w_1$ (note that our proof of this case does not use the rest of F and hence it is symmetric also for $i = 1$). Since $\langle\{x, p_i, w_2, w_5\}\rangle_{\bar{G}} \not\simeq K_{1,3}$, by symmetry, we have $p_iw_5 \in E(\bar{G})$, and, by Lemma 6, $p_iw_2 \notin E(\bar{G})$. Since $\langle\{x, p_i, w_1, w_3\}\rangle_{\bar{G}} \not\simeq K_{1,3}$, $p_iw_3 \in E(\bar{G})$. Then the subgraph $W' = \langle\{x, w_1, w_2, w_3, p_i, w_5\}\rangle_{\bar{G}} \simeq W_5$ is an induced W_5 in $\bar{G}$ with $|V(W') \cap V(F)| = 2$, contradicting the choice of W.

Case 3: $V(W) \cap V(F) = \emptyset$.
Let again p_ip_{i+1}, $i \in \{1, 2\}$, be the new edge in F. By Lemma 4(iii), each of p_i, p_{i+1} has at least 4 neighbors on the 5-cycle $C = W - x$. By the pigeonhole principle, p_i and p_{i+1} have at least 3 common neighbors on C. Since C is induced, at least two of these common neighbors are nonadjacent, contradicting Lemma 6. ■

Note that now we know that $\bar{G}$ is $\{K_{1,3}, W_5\}$-free. Since $\bar{G}$ is 3-connected and not Hamilton-connected, we have $\alpha(\bar{G}) \geq 3$ by Theorem F, hence for every $v \in V(G)$, $N_{\bar{G}}(v)$ can be covered by two cliques. This fact will be useful in the next proof.

Proposition 7. *Let G be a 3-connected $\{K_{1,3}, \Gamma_3\}$-free graph and let $\bar{G}$ be its Γ_3-closure. Then $\bar{G}$ is W_4-free.*

Proof. Similarly as before, if $x \in V(G)$ is a center of an induced W_4 in $\bar{G}$, then x is locally 2-connected in $\bar{G}$ by Lemma I, hence x is feasible in $\bar{G}$ by Theorem H, and $\bar{G}^*_x$ contains an induced Γ_3. We again show that this is not possible.

Lemma 8. *Let G be a claw-free graph, let W be an induced subgraph of G such that $W = \langle\{x, w_1, w_2, w_3, w_4\}\rangle_G \simeq W_4$, let $z_1, z_2 \in N_G(x)$ be such that $\mathrm{dist}_{\langle N_G(x)\rangle_G}(z_1, z_2) = 3$, and let R be the graph shown in Fig. 6. Then G contains R as an induced subgraph.*

Proof. Let $C = W - x$ be the 4-cycle $C = w_1w_2w_3w_4$ of the 4-wheel W. First observe that $\mathrm{dist}_{\langle N_G(x)\rangle_G}(z_i, C) \leq 1$ since otherwise e.g. $\langle\{x, z_i, w_1, w_3\}\rangle_G \simeq K_{1,3}$, $i = 1, 2$. On the other hand, if some z_i is on C, say, $z_1 = w_1$, then $w_2z_2, w_4z_2 \notin E(G)$ by the assumption that $\mathrm{dist}_{\langle N_G(x)\rangle_G}(z_1, z_2) = 3$, and then $\langle\{x, w_2, w_4, z_2\}\rangle_G \simeq K_{1,3}$. Hence $\mathrm{dist}_{\langle N_G(x)\rangle_G}(z_i, C) = 1$, $i = 1, 2$.

Up to a symmetry, let $z_1w_1 \in E(G)$. Since $\langle\{x, w_2, w_4, z_2\}\rangle_G \not\simeq K_{1,3}$, up to a symmetry, $w_4z_2 \in E(G)$. Since $\langle\{x, z_1, w_2, w_4\}\rangle_G \not\simeq K_{1,3}$ and $z_1w_4 \notin E(G)$ by the assumption

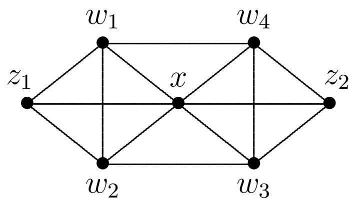


Figure 6: The graph R used in Lemma 8.

$\text{dist}_{\langle N_G(x)\rangle_G}(z_1, z_2) = 3$, we have $z_1w_2 \in E(G)$. Symmetrically, $w_3z_2 \in E(G)$. Thus, we have the graph R. Finally, the subgraph is induced since any additional edge would contradict either the assumption $\text{dist}_{\langle N_G(x)\rangle_G}(z_1, z_2) = 3$, or the fact that $\langle\{x, w_1, w_2, w_3, w_4\}\rangle_G \simeq W_4$. ■

Let now, to the contrary, $W = \langle\{x, w_1, w_2, w_3, w_4\}\rangle_{\bar{G}} \simeq W_4$ be an induced subgraph of $\bar{G}$, and let $F \overset{\text{IND}}{\subset} \bar{G}^*_x$ be such that $F \simeq \Gamma_3$. Set again $F = \langle\{t_1, t_2, p_1, p_2, p_3, p_4, t_3, t_4\}\rangle_{\bar{G}}$. Since $\bar{G}$ is Γ_3-free, at least one edge of F is new. Since $x \in V_{SI}(\bar{G}^*_x)$, either all new edges are in one of the triangles of F, or the only new edge of F is some edge on the path.

Case 1: *New edges are in a triangle of F.*
We can choose the notation such that the new edges are in the triangle $p_1t_1t_2p_1$. As before, if t_1t_2 is the only new edge in F, then $\langle\{p_1, t_1, t_2, p_2\}\rangle_{\bar{G}} \simeq K_{1,3}$; hence we can choose the notation such that t_1p_1 (and possibly also one of t_1t_2, t_2p_1) is new. By Lemma 5, $\text{dist}_{\langle N_{\bar{G}}(x)\rangle_{\bar{G}}}(t_1, p_1) = 3$. By Lemma 8, p_1, t_1 and W are in the induced subgraph R of Fig. 6. Thus, in all the cases (independently of whether the edges t_1t_2, t_2p_1 are present in $\bar{G}$ or not), $\bar{G}$ contains the subgraph F_0 shown in Fig. 7.

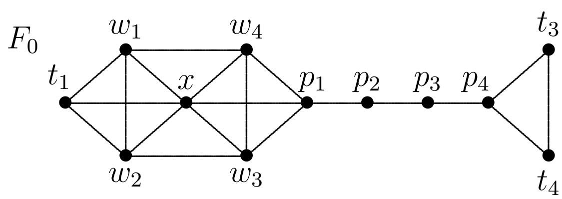


Figure 7: The subgraph F_0

Since e.g. $\langle\{x, w_3, p_1, p_2, p_3, p_4, t_3, t_4\}\rangle_{F_0} \not\simeq \Gamma_3$, F_0 is not an induced subgraph of $\bar{G}$. Checking by computer all possible sets of additional edges such that the resulting graph is still $\{K_{1,3}, \Gamma_3, W_5\}$-free, we obtain 10 exceptional graphs $F_1, \ldots, F_{10}$ (see [22]). For each of them, $V(F_i) = V(F_0)$, $i = 1, \ldots, 10$, and their edge sets are:
$E(F_1) = E(F_0) \cup \{w_1t_3, w_2t_4, w_3t_4, w_4t_3\}$,
$E(F_2) = E(F_0) \cup \{w_1t_3, w_2p_2, w_2p_3, w_3p_2, w_3p_3, w_4t_3\}$,
$E(F_3) = E(F_0) \cup \{w_1t_3, w_2p_3, w_2p_4, w_3p_3, w_3p_4, w_4t_3\}$,
$E(F_4) = E(F_0) \cup \{w_1p_2, w_1p_3, w_2p_3, w_2p_4, w_3p_3, w_3p_4, w_4p_2, w_4p_3\}$,
$E(F_5) = E(F_0) \cup \{w_1p_2, w_1p_3, w_2t_3, w_2t_4, w_3t_3, w_3t_4, w_4p_2, w_4p_3\}$,
$E(F_6) = E(F_0) \cup \{w_1p_3, w_1p_4, w_2t_3, w_2t_4, w_3t_3, w_3t_4, w_4p_3, w_4p_4\}$,
$E(F_7) = E(F_0) \cup \{w_1p_4, w_1t_3, w_1t_4, w_2t_4, w_3t_4, w_4p_4, w_4t_3, w_4t_4\}$,
$E(F_8) = E(F_0) \cup \{w_1p_2, w_1p_3, w_2p_4, w_2t_3, w_2t_4, w_3p_4, w_3t_3, w_3t_4, w_4p_2, w_4p_3\}$,
$E(F_9) = E(F_0) \cup \{w_1p_3, w_1p_4, w_2p_4, w_2t_3, w_2t_4, w_3p_4, w_3t_3, w_3t_4, w_4p_3, w_4p_4\}$,
$E(F_{10}) = E(F_0) \cup \{w_1p_4, w_1t_3, w_1t_4, w_2t_3, w_2t_4, w_3t_3, w_3t_4, w_4p_4, w_4t_3, w_4t_4\}$.

The graphs $F_1, \ldots, F_{10}$ are also shown in Fig. 8 (where the double-circled vertices indicate the weakly feasible sets given by Claim 3).

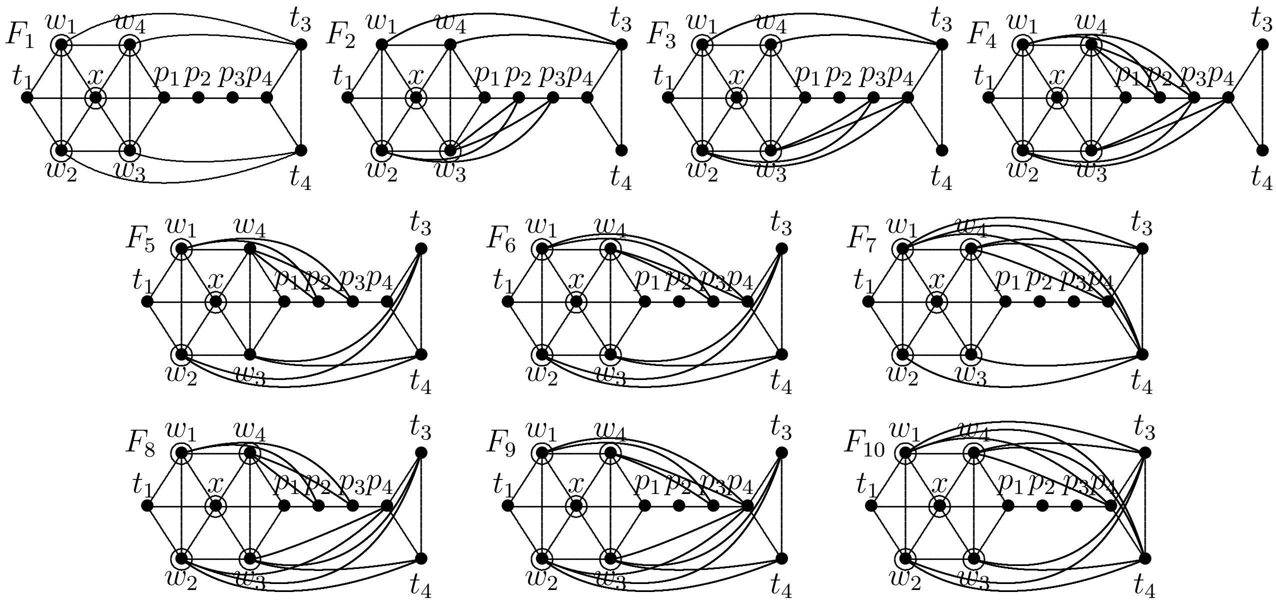


Figure 8: The 10 exceptional graphs

Each of the graphs $F_1, \ldots, F_{10}$ is $\{K_{1,3}, \Gamma_3\}$-free, we have $F_i \overset{\text{IND}}{\subset} \bar{G}$ for some $i \in \{1, \ldots, 10\}$, however, in $(F_i)^*_x$ we have $\langle\{x, t_1, p_1, p_2, p_3, p_4, t_3, t_4\}\rangle_{(F_i)^*_x} \simeq \Gamma_3$. By the definition of the Γ_3-closure, in each of these possibilities not only $(F_i)^*_x$, but even G^*_M for any (weakly) feasible set $M \subset V(\bar{G})$, contains an induced Γ_3. To reach a contradiction, in each of the subgraphs $F_1, \ldots, F_{10}$, we identify a weakly feasible set $M \subset V(\bar{G})$ with $|M| > 1$ and $x \in M$, for which this is not possible.

First observe that, in all the graphs $F_1, \ldots, F_{10}$, x is locally is 2-connected in $\bar{G}$, hence x is feasible in $\bar{G}$ by Theorem H. Set $G_1 = (\bar{G})^*_x$.

Claim 1. *Let $v \in \{w_1, w_2, w_3, w_4\}$, and let S be the graph in Fig. 4. Then either v is locally 2-connected in G_1, or v is not a vertex of degree 4 in an induced subgraph $F \overset{\text{IND}}{\subset} G_1$ such that $F \simeq S$.*

Proof. Let, say, $v = w_1$. Then w_1 and w_4 have, in all the graphs $F_1, \ldots, F_{10}$, a common neighbor $u \in \{p_3, p_4, t_3, t_4\}$. Since $\bar{G}$ is 3-connected and not Hamilton-connected, by Theorem F, $\alpha(\bar{G}) \geq 3$. By Proposition 3, $\bar{G}$ is W_5-free, thus, by Theorem G, $N_{\bar{G}}(v)$, hence also $N_{\bar{G}}[v]$, can be covered by two cliques, say, K_1 and K_2. Choose the notation such that K_1 contains the triangle $w_1w_4t_1$, and K_2 contains the triangle w_1w_4u. Then, in G_1, K_1 extends to a clique K'_1 containing the vertices $w_1, w_2, w_3, w_4, t_1, x, p_1$. Thus, $\{w_1, w_4\} \subset K'_1 \cap K_2$.

If $|K'_1 \cap K_2| \geq 3$, then w_1 is locally 2-connected in G_1 and we are done by Theorem H; thus, let $K'_1 \cap K_2 = \{w_1, w_4\}$. Then, if w_1 is a vertex of degree 4 in $F \overset{\text{IND}}{\subset} G_1$ with $F \simeq S$, there are vertices $z_i \in K_i \setminus \{w_1, w_4\}$, $i = 1, 2$, such that $z_1z_2 \in E(G_1)$, but then w_1 is locally 2-connected in G_1.

The proof for $v \in \{w_2, w_3, w_4\}$ is symmetric (since the argument in our proof never used the vertex p_2). □

Claim 2. *At least one of the vertices w_1, w_2, w_3, w_4 is feasible in G_1.*

Proof. If, say, w_1 is not feasible in G_1, then, by Claim 1 and by Proposition E(ii), $(G_1)^*_{w_1}$ has a hamiltonian (w_1, w)-path for some $w \in N_{G_1}(w_1)$. Hence there is no hamiltonian (w_1, w)-path in G_1, and, by Proposition E(ii), there is still no hamiltonian (w_1, w)-path in $(G_1)^*_{w'}$ for any $w' \in \{w_2, w_3, w_4\} \setminus \{w\}$. Thus, w' is feasible in G_1. □

Claim 3. *Let $F_i \overset{IND}{\subset} \bar{G}$ for some i, $1 \leq i \leq 10$, and set $M_i = \{x, w_1, w_2, w_3, w_4\}$ if $i \in \{1, 3, 4, 6, 7, 8, 9, 10\}$, or $M_2 = \{x, w_2, w_3\}$, or $M_5 = \{x, w_1, w_4\}$. Then the set M_i is weakly feasible in $\bar{G}$.*

Proof. Throughout this proof, we will use $V_{\text{L2C}}(G)$ to denote the set of all locally 2-connected vertices in a graph G.

First of all, observe that $x \in V_{\text{L2C}}(\bar{G})$ in all cases $i = 1, \ldots, 10$ by Lemma I (the independent sets are $\{w_1, w_3\}$ and $\{w_2, w_4\}$), hence x is feasible in $\bar{G}$ by Theorem H, and feasible or simplicial in any graph obtained from $\bar{G}$ by a series of local completions.

Let $F_j \overset{\text{IND}}{\subset} \bar{G}$, and let, by Claim 2, $v_j \in \{w_1, w_2, w_3, w_4\} \cap V_{\text{L2C}}(G_1)$.

Case Cl 3-1: *$F_1 \overset{IND}{\subset} \bar{G}$.*

By symmetry, we can assume that $v_1 = w_4$. Then, by Lemma I, $w_3 \in V_{\text{L2C}}(\bar{G}^*_{w_4})$ (the independent sets are $\{w_2, w_4\}$ and $\{w_1, t_4\}$), and also $w_1 \in V_{\text{L2C}}(\bar{G}^*_{w_4})$ (the independent sets are $\{w_2, w_4\}$ and $\{t_1, t_3\}$). By symmetry, $w_2 \in V_{\text{L2C}}(\bar{G}^*_{w_1})$. Hence also $w_3 \in V_{\text{L2C}}((\bar{G}^*_{w_4})^*_x) = V_{\text{L2C}}((\bar{G}^*_x)^*_{w_4})$, i.e., w_3 is feasible or simplicial in $(\bar{G}^*_x)^*_{w_4}$. Similarly with w_1 and w_2.

Thus, if $F_1 \overset{\text{IND}}{\subset} \bar{G}$, then the set $M_1 = \{x, w_1, w_2, w_3, w_4\}$ is weakly feasible in $\bar{G}$.

We summarize the above discussion in the following table (where "L2C" stands for "locally 2-connected").

v_1	L2C vertex	Argument
w_4	$w_3 \in V_{\text{L2C}}(\bar{G}^*_{w_4})$	Lemma I, indep. sets $\{w_2, w_4\}$, $\{x, t_4\}$
	$w_1 \in V_{\text{L2C}}(\bar{G}^*_{w_4})$	Lemma I, indep. sets $\{w_2, w_4\}$, $\{t_1, t_3\}$
	$w_2 \in V_{\text{L2C}}(\bar{G}^*_{w_1})$	Symmetric to $w_3 \in V_{\text{L2C}}(\bar{G}^*_{w_4})$

Case Cl 3-2: *$F_2 \overset{IND}{\subset} \bar{G}$.*

Immediately $w_3 \in V_{\text{L2C}}(\bar{G})$ (independent sets $\{w_2, w_4\}$ and $\{x, p_3\}$), and then $w_2 \in V_{\text{L2C}}(\bar{G}^*_{w_3})$ (independent sets $\{w_1, w_3\}$ and $\{t_1, p_3\}$).

Thus, if $F_2 \overset{\text{IND}}{\subset} \bar{G}$, then the set $M_2 = \{x, w_2, w_3\}$ is weakly feasible in $\bar{G}$.

Case Cl 3-3: *$F_3 \overset{IND}{\subset} \bar{G}$.*

By symmetry, we can assume that $v_3 \in \{w_3, w_4\}$. We summarize the possibilities in the following table.

v_3	L2C vertex	Argument
w_4	$w_3 \in V_{\mathrm{L2C}}(\bar{G}^*_{w_4})$	Lemma I, indep. sets $\{w_2, w_4\}$, $\{x, p_4\}$
	$w_1 \in V_{\mathrm{L2C}}(\bar{G}^*_{w_4})$	Lemma I, indep. sets $\{w_2, w_4\}$, $\{t_1, t_3\}$
	$w_2 \in V_{\mathrm{L2C}}(\bar{G}^*_{w_1})$	Symmetric to $w_3 \in V_{\mathrm{L2C}}(\bar{G}^*_{w_4})$
w_3	$w_4 \in V_{\mathrm{L2C}}(\bar{G}^*_{w_3})$	Lemma I, indep. sets $\{w_1, w_3\}$, $\{x, t_3\}$
	$w_2 \in V_{\mathrm{L2C}}(\bar{G}^*_{w_3})$	Lemma I, indep. sets $\{w_1, w_3\}$, $\{t_1, p_4\}$
	$w_1 \in V_{\mathrm{L2C}}(\bar{G}^*_{w_2})$	Symmetric to $w_4 \in V_{\mathrm{L2C}}(\bar{G}^*_{w_3})$

Thus, if $F_3 \overset{\text{IND}}{\subset} \bar{G}$, then the set $M_3 = \{x, w_1, w_2, w_3, w_4\}$ is weakly feasible in $\bar{G}$.

<u>Case Cl 3-4:</u> $F_4 \overset{\textit{IND}}{\subset} \bar{G}$.
In this case, already $w_i \in V_{\mathrm{L2C}}(\bar{G})$, $i = 1, 2, 3, 4$:

L2C vertex	Argument
$w_1 \in V_{\mathrm{L2C}}(\bar{G})$	Lemma I, indep. sets $\{w_2, w_4\}$, $\{x, p_3\}$
$w_2 \in V_{\mathrm{L2C}}(\bar{G})$	Lemma I, indep. sets $\{w_1, w_3\}$, $\{x, p_3\}$
$w_3 \in V_{\mathrm{L2C}}(\bar{G})$	Lemma I, indep. sets $\{w_2, w_4\}$, $\{x, p_3\}$
$w_4 \in V_{\mathrm{L2C}}(\bar{G})$	Lemma I, indep. sets $\{w_1, w_3\}$, $\{x, p_3\}$

Thus, if $F_4 \overset{\text{IND}}{\subset} \bar{G}$, then the set $M_4 = \{x, w_1, w_2, w_3, w_4\}$ is weakly feasible in $\bar{G}$.

<u>Case Cl 3-5:</u> $F_5 \overset{\textit{IND}}{\subset} \bar{G}$.
In this case, $w_4 \in V_{\mathrm{L2C}}(\bar{G})$ (Lemma I with sets $\{w_1, w_3\}$, $\{x, p_3\}$), and then $w_1 \in V_{\mathrm{L2C}}(\bar{G}^*_{w_4})$ (Lemma I with sets $\{w_2, w_4\}$, $\{t_1, p_3\}$).

Thus, if $F_5 \overset{\text{IND}}{\subset} \bar{G}$, then the set $M_5 = \{x, w_1, w_4\}$ is weakly feasible in $\bar{G}$.

<u>Case Cl 3-6:</u> $F_6 \overset{\textit{IND}}{\subset} \bar{G}$.
By symmetry, we can assume that $v_6 \in \{w_3, w_4\}$. We then have the following possibilities.

v_6	L2C vertex	Argument
w_3	$w_2 \in V_{\mathrm{L2C}}(\bar{G}^*_{w_3})$	Lemma I, indep. sets $\{w_1, w_3\}$, $\{t_1, t_3\}$
	$w_4 \in V_{\mathrm{L2C}}(\bar{G}^*_{w_3})$	Lemma I, indep. sets $\{w_1, w_3\}$, $\{x, p_3\}$
	$w_1 \in V_{\mathrm{L2C}}(\bar{G}^*_{w_2})$	Symmetric to $w_4 \in V_{\mathrm{L2C}}(\bar{G}^*_{w_3})$
w_4	$w_1 \in V_{\mathrm{L2C}}(\bar{G}^*_{w_4})$	Lemma I, indep. sets $\{w_2, w_4\}$, $\{t_1, p_4\}$
	$w_3 \in V_{\mathrm{L2C}}(\bar{G}^*_{w_4})$	Lemma I, indep. sets $\{w_2, w_4\}$, $\{x, t_3\}$
	$w_2 \in V_{\mathrm{L2C}}(\bar{G}^*_{w_1})$	Symmetric to $w_3 \in V_{\mathrm{L2C}}(\bar{G}^*_{w_4})$

Thus, if $F_6 \overset{\text{IND}}{\subset} \bar{G}$, then the set $M_6 = \{x, w_1, w_2, w_3, w_4\}$ is weakly feasible in $\bar{G}$.

<u>Case Cl 3-7:</u> $F_7 \overset{\textit{IND}}{\subset} \bar{G}$.
By symmetry, it is sufficient to verify w_3 and w_4, however, $w_3, w_4 \in V_{\mathrm{L2C}}(\bar{G})$ by Lemma I: w_3 with sets $\{w_2, w_4\}$, $\{x, t_4\}$, w_4 with sets $\{w_1, w_3\}$, $\{x, p_4\}$.

Thus, if $F_7 \overset{\text{IND}}{\subset} \bar{G}$, then the set $M_7 = \{x, w_1, w_2, w_3, w_4\}$ is weakly feasible in $\bar{G}$.

<u>Case Cl 3-8:</u> $F_8 \overset{\textit{IND}}{\subset} \bar{G}$.
Then $w_4 \in V_{\mathrm{L2C}}(\bar{G})$ (Lemma I with sets $\{w_2, w_4\}$, $\{x, p_3\}$), and for the remaining vertices we have the following possibilities.

L2C vertex	Argument
$w_1 \in V_{\mathrm{L2C}}(\bar{G}^*_{w_4})$	Lemma I, indep. sets $\{w_2, w_4\}$, $\{t_1, p_3\}$
$w_3 \in V_{\mathrm{L2C}}(\bar{G}^*_{w_4})$	Lemma I, indep. sets $\{w_2, w_4\}$, $\{x, t_3\}$
$w_2 \in V_{\mathrm{L2C}}(\bar{G}^*_{w_1})$	Symmetric to $w_3 \in V_{\mathrm{L2C}}(\bar{G}^*_{w_4})$

Thus, if $F_8 \overset{\mathrm{IND}}{\subset} \bar{G}$, then the set $M_8 = \{x, w_1, w_2, w_3, w_4\}$ is weakly feasible in $\bar{G}$.

<u>Case Cl 3-9:</u> $F_9 \overset{IND}{\subset} \bar{G}$.
In this case, $\{w_1, w_2, w_3, w_4\} \subset V_{\mathrm{L2C}}(\bar{G})$: w_1 and w_2 by Lemma I with sets $\{w_2, w_4\}$, $\{t_1, p_3\}$) for w_1, and $\{w_1, w_3\}$, $\{x, p_4\}$) for w_2; w_3 and w_4 follow by symmetry.

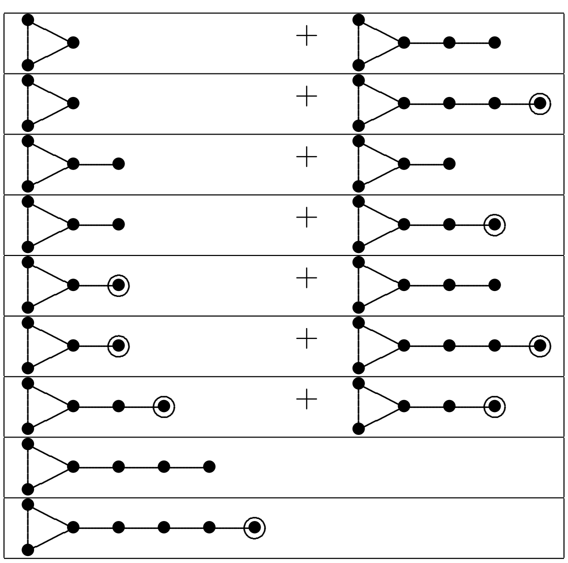
Thus, if $F_9 \overset{\mathrm{IND}}{\subset} \bar{G}$, then the set $M_9 = \{x, w_1, w_2, w_3, w_4\}$ is weakly feasible in $\bar{G}$.

<u>Case Cl 3-10:</u> $F_{10} \overset{IND}{\subset} \bar{G}$.
In this case similarly $\{w_1, w_2, w_3, w_4\} \subset V_{\mathrm{L2C}}(\bar{G})$: w_1 and w_2 by Lemma I with sets $\{w_2, w_4\}$, $\{x, p_4\}$) for w_1, and $\{w_1, w_3\}$, $\{x, t_3\}$) for w_2; w_3 and w_4 follow by symmetry.

Thus, if $F_{10} \overset{\mathrm{IND}}{\subset} \bar{G}$, then the set $M_{10} = \{x, w_1, w_2, w_3, w_4\}$ is weakly feasible in $\bar{G}$. □

Now, since each of the sets M_i is weakly feasible in $\bar{G}$, by the definition of the Γ_3-closure, each $\bar{G}^*_{M_i}$ contains an induced subgraph $F' \simeq \Gamma_3$, $i = 1, \ldots, 10$. Since each $N_{\bar{G}}[M_i]$ induces in $\bar{G}^*_{M_i}$ a clique, the clique $\langle N_{\bar{G}}[M_i] \rangle_{\bar{G}^*_{M_i}}$ contains either a triangle of F', or one of the edges of the path of F'. The rest of F' outside $\langle N_{\bar{G}}[M_i] \rangle_{\bar{G}^*_{M_i}}$ consists of a triangle and a path of appropriate length in the first case, or of two triangles with a path of appropriate length in the second case. Let us call these parts of F' outside $\langle N_{\bar{G}}[M_i] \rangle_{\bar{G}^*_{M_i}}$ "tails". Then each tail consists of a triangle with a path, and the length of the path is 3 in the first case, or the lengths of the paths sum up to 2 in the second case. Moreover, since each vertex of M_i is simplicial in $\bar{G}^*_{M_i}$, it cannot be an end of a tail directly, but a tail can be attached to some its neighbor, which corresponds to the situation with a tail one longer.

Considering all possible combinations of such tails, we obtain the possibilities that are listed in the following table.



Here, the double-circled endvertices of tails can be identified with a vertex in M_i, and the non-double-circled endvertices of tails can be identified with a vertex in $N_{\bar{G}}[M_i] \setminus M_i$.

We have generated by computer all possible ways of extending the graph F_i, $i = 1, \ldots, 10$, by joining some of the combinations of the tails in the table to vertices in $N_{\bar{G}}[M_i]$. To the resulting graphs, we have added all possible sets of edges between the new vertices and F_i. Finally, we have tested whether the resulting graphs are $\{\Gamma_3, W_5, K_{1,3}\}$-free. In each of the possible cases, we have obtained a contradiction (see [22]).

Case 2: *The new edge is on the path of F.*
If $|V(W) \cap V(F)| = 2$, then, choosing the notation such that $w_1, w_3 \in V(F)$, Lemma 6 implies $w_2w_4 \in E(G)$, a contradiction. Hence $|V(W) \cap V(F)| \leq 1$.

Subcase 2.1: $|V(W) \cap V(F)| = 1$.
The proof will depend on the position of the new edge on the path of F.

Subcase 2.1.1: *The edge p_1p_2 is new.*
We distinguish two subcases.

Subcase 2.1.1.1: $p_1 \in V(W)$.
Choose the notation such that $p_1 = w_1$. Since $\langle\{x, w_1, p_2, w_3\}\rangle_{\bar{G}} \not\simeq K_{1,3}$, we have $p_2w_3 \in E(\bar{G})$, and from $\langle\{x, w_2, w_4, p_2\}\rangle_{\bar{G}} \not\simeq K_{1,3}$, up to a symmetry, $p_2w_4 \in E(\bar{G})$. By Lemma 6, $w_2p_2 \notin E(\bar{G})$. Since $\langle\{w_1, w_2, x, p_2, p_3, p_4, t_3, t_4\}\rangle_{\bar{G}} \not\simeq \Gamma_3$, we have $w_2z \in E(\bar{G})$ for some $z \in \{p_3, p_4, t_3, t_4\}$. Then $w_2t_i \notin E(\bar{G})$ for otherwise $\langle\{w_2, t_i, x, z\}\rangle_{\bar{G}} \simeq K_{1,3}$, $i = 1, 2$, and from $\langle\{w_1, t_i, w_2, w_4\}\rangle_{\bar{G}} \not\simeq K_{1,3}$ we have $t_iw_4 \in E(\bar{G})$, $i = 3, 4$. Since $\langle\{t_1, t_2, w_4, p_2, p_3, p_4, t_3, t_4\}\rangle_{\bar{G}} \not\simeq \Gamma_3$, we have $w_4z' \in E(\bar{G})$ for some $z' \in \{p_3, p_4, t_3, t_4\}$, but in each of these cases, $\langle\{w_4, t_1, x, z'\}\rangle_{\bar{G}} \simeq K_{1,3}$, a contradiction.

Subcase 2.1.1.2: $p_2 \in V(W)$.
Set $p_2 = w_1$. Then similarly $\langle\{x, p_1, w_1, w_3\}\rangle_{\bar{G}} \not\simeq K_{1,3}$ implies $p_1w_3 \in E(\bar{G})$, $\langle\{x, p_1, w_2, w_4\}\rangle_{\bar{G}} \not\simeq K_{1,3}$, implies, up to a symmetry, $p_1w_4 \in E(\bar{G})$, and Lemma 6 implies $w_2p_1 \notin E(\bar{G})$. From $\langle\{w_1, w_2, w_4, p_3\}\rangle_{\bar{G}} \not\simeq K_{1,3}$ we now have $w_4p_3 \in E(\bar{G})$ or $w_2p_3 \in E(\bar{G})$. However, if $w_4p_3 \in E(\bar{G})$, then immediately $\langle\{t_1, t_2, p_1, w_4, p_3, p_4, t_3, t_4\}\rangle_{\bar{G}} \simeq \Gamma_3$ since each of the edges w_4z, $z \in \{t_1, t_2, p_4, t_3, t_4\}$, yields an induced $K_{1,3}$ with center at w_4. Thus, $w_4p_3 \notin E(\bar{G})$ and $w_2p_3 \in E(\bar{G})$.

Now $w_2t_i \notin E(\bar{G})$, for otherwise $\langle\{w_2, t_i, x, p_3\}\rangle_{\bar{G}} \simeq K_{1,3}$, $i = 1, 2$. Considering $\langle\{p_1, w_4, x, w_2, p_3, p_4, t_3, t_4\}\rangle_{\bar{G}} \not\simeq \Gamma_3$, we have $w_4z \in E(\bar{G})$ for some $z \in \{p_4, t_3, t_4\}$, or $w_2z' \in E(\bar{G})$ for some $z' \in \{p_4, t_3, t_4\}$. However, in the first case $\langle\{w_4, z, w_1, p_1\}\rangle_{\bar{G}} \simeq K_{1,3}$, and in the second case $\langle\{w_2, x, p_3, z'\}\rangle_{\bar{G}} \simeq K_{1,3}$ for $z' \in \{t_3, t_4\}$. Thus, $z' = p_4$, i.e., $w_2p_4 \in E(\bar{G})$. But then $\langle\{t_1, t_2, p_1, x, w_2, p_4, t_3, t_4\}\rangle_{\bar{G}} \simeq \Gamma_3$, a contradiction.

Subcase 2.1.2: *The edge p_2p_3 is new.*
By symmetry, we can set $w_1 = p_2$. As before, from $\langle\{x, w_1, p_3, w_3\}\rangle_{\bar{G}} \not\simeq K_{1,3}$ we have $w_1w_3 \in E(\bar{G})$, from $\langle\{x, w_2, w_4, p_3\}\rangle_{\bar{G}} \not\simeq K_{1,3}$, up to a symmetry, $p_3w_4 \in E(\bar{G})$, and, by Lemma 6, $w_2p_3 \notin E(\bar{G})$.

Claim 4. *If $y \in V(\bar{G})$ is such that $\{x, w_1, p_3\} \subset N_{\bar{G}}(y)$, then $N_{\bar{G}}(y) \cap \{t_1, t_2, p_1, p_4, t_3, t_4\} = \emptyset$.*

Proof. If $yz \in E(\bar{G})$ for some $z \in \{t_1, t_2, t_3, t_4\}$, then $\langle\{y, z, w_1, p_3\}\rangle_{\bar{G}} \simeq K_{1,3}$. If both $yp_1 \in E(\bar{G})$ and $yp_4 \in E(\bar{G})$, then $\langle\{y, p_1, p_4, x\}\rangle_{\bar{G}} \simeq K_{1,3}$, and if y is adjacent to one of p_1, p_4, say, $yp_4 \in E(\bar{G})$, then $\langle\{t_1, t_2, p_1, w_1, y, p_4, t_3, t_4\}\rangle_{\bar{G}} \simeq \Gamma_3$. □

Now, since $w_4p_1 \notin E(\bar{G})$ by Claim 4, from $\langle\{w_1, p_1, w_2, w_4\}\rangle_{\bar{G}} \not\simeq K_{1,3}$ we have $p_1w_2 \in E(\bar{G})$. Since $\langle\{p_1, w_2, w_1, w_4, p_3, p_4, t_3, t_4\}\rangle_{\bar{G}} \not\simeq \Gamma_3$, by Claim 4, $w_2z \in E(\bar{G})$ for some $z \in \{p_4, t_3, t_4\}$, but in each of these cases, $\langle\{w_2, p_1, x, z\}\rangle_{\bar{G}} \simeq K_{1,3}$, a contradiction.

Subcase 2.2: $V(W) \cap V(F) = \emptyset$.
In this case, the vertex x must be adjacent in G to the two vertices of the new edge (and to no other vertex of F since F is induced in G^*_x). Up to a symmetry, there are two possible subcases, namely, that the edge p_2p_3 is new, see Fig 9(a), and that the edge p_1p_2 is new, see Fig 9(b).

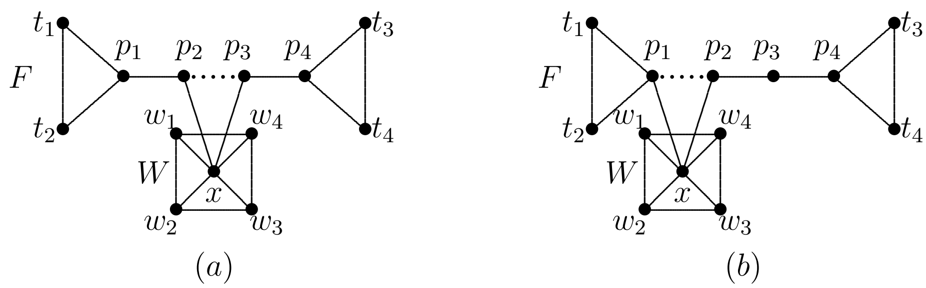


Figure 9: The two possibilities in Subcase 2.2 (dotted lines indicate new edges).

Since the vertex x, the vertices of the new edge, and any of the vertices w_i, $i = 1, 2, 3, 4$, cannot induce a claw in G, there must be some more additional edges in G between $\{w_1, w_2, w_3, w_4\}$ and F. Checking by computer all possible sets of additional edges uv such that $u \in \{w_1, w_2, w_3, w_4\}$ and $v \in V(F)$, we conclude that, in the first case (when p_2p_3 is new), the resulting graph always contains at least one of the graphs $K_{1,3}$, Γ_3 or W_5 as an induced subgraph, a contradiction (see [22]); while in the second case (when p_1p_2 is new), the computing gives one possible graph shown in Fig. 10.

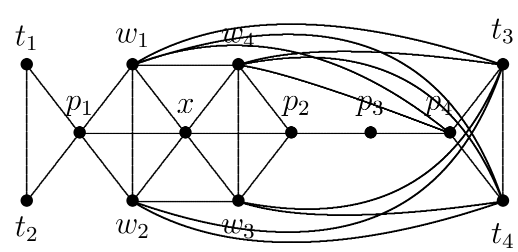


Figure 10: The only possibility in the second case of Subcase 2.2

By the connectivity assumption, we have $d_G(p_3) \geq 3$, hence the vertex p_3 has another neighbor $v \in V(G) \setminus (V(F) \cup V(W))$. Since $\langle\{p_3, p_2, p_4, v\}\rangle_G \not\simeq K_{1,3}$, there must be some more edges. Checking by computer all possibilities (see [22]), we again conclude that the resulting graph always contains at least one of the graphs $K_{1,3}$, Γ_3 or W_5 as an induced subgraph, a contradiction. ■

We now know that $\bar{G}$ is $\{K_{1,3}, W_4, W_5\}$-free and we will repeatedly use this property. For the sake of brevity, we introduce the following notion. Given a graph G and its vertex x, we say that $N_G(x)$ contains an *endgame* if $N_G(x)$ contains vertices $x_1, \ldots, x_k$ satisfying at least one of the following conditions (see also Fig. 11):

(*i*) $k = 3$ and $\{x_1x_2, x_2x_3, x_3x_1\} \cap E(G) = \emptyset$,

(*ii*) $k = 4$, $\{x_1x_2, x_2x_3, x_3x_4, x_4x_1\} \subset E(G)$ and $\{x_1x_3, x_2x_4\} \cap E(G) = \emptyset$,

(*iii*) $k = 5$ and $\{x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_1\} \cap E(G) = \emptyset$,

(*iv*) $k = 5$, $\{x_1x_2, x_2x_3, x_3x_4, x_4x_5\} \cap E(G) = \emptyset$ and $\{x_1x_4, x_2x_5\} \subset E(G)$,

(*v*) $k = 5$, $\{x_2x_3, x_3x_4, x_1x_5\} \cap E(G) = \emptyset$ and $\{x_2x_5, x_5x_3, x_3x_1, x_1x_4\} \subset E(G)$.

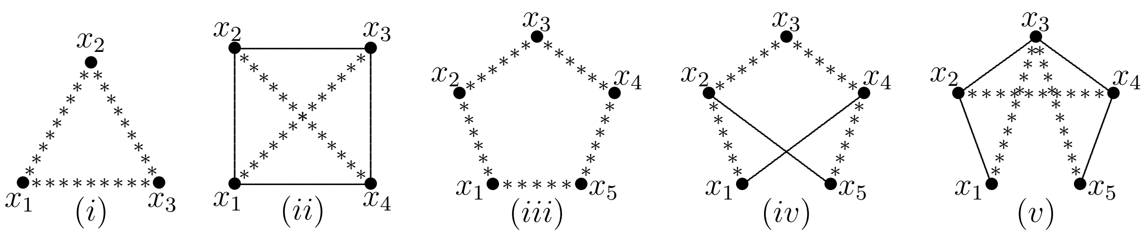


Figure 11: The endgames used in Proposition 9. The star-lines indicate pairs of nonadjacent vertices (i.e., edges in the complement of G).

Proposition 9. *Let G be a graph and let x be its vertex such that $N_G(x)$ contains an endgame. Then G contains an induced $K_{1,3}$, W_4 or W_5 with center at x.*

Proof. (*i*) $\langle\{x, x_1, x_2, x_3\}\rangle_G \simeq K_{1,3}$.

(*ii*) $\langle\{x, x_1, x_2, x_3, x_4\}\rangle_G \simeq W_4$.

(*iii*) If, say, $x_1x_3 \notin E(G)$, then the vertices x_1, x_2, x_3 satisfy (*i*). Thus, by symmetry, $\{x_1x_3, x_3x_5, x_5x_2, x_2x_4, x_4x_1\} \subset E(G)$, and then $\langle\{x, x_1, x_3, x_5, x_2, x_4\}\rangle_G \simeq W_5$.

(*iv*) If $x_1x_5 \notin E(G)$, then $x_1, \ldots, x_5$ satisfy (*iii*), hence $x_1x_5 \in E(G)$. Now, if $x_2x_4 \notin E(G)$, then x_2, x_3, x_4 satisfy (*i*), and if $x_2x_4 \in E(G)$, then x_1, x_4, x_2, x_5 satisfy (*ii*).

(*v*) If $x_2x_5 \notin E(G)$, then x_1, x_3, x_5, x_2, x_4 satisfy (*iv*), and if $x_2x_5 \in E(G)$, then x_2, x_3, x_4, x_5 satisfy (*ii*). ■

Proposition 10. *Let G be a 3-connected $\{K_{1,3}, \Gamma_3\}$-free graph and let $\bar{G}$ be its Γ_3-closure. Then $\bar{G}$ is $\{P_6^2, P_6^{2+}\}$-free.*

Proof. Let, to the contrary, $Q = \langle\{v_0, \ldots, v_5\}\rangle_{\bar{G}} \overset{\text{IND}}{\subset} \bar{G}$ be such that $Q \simeq P_6^2$ or $Q \simeq P_6^{2+}$ (with the labeling of vertices as in Fig. 5). We will prove the statement for both P_6^2 and P_6^{2+} at the same time since our proof does not depend on whether the edge v_0v_5 is in Q or not. We show that either none of the vertices v_1, v_4 is a vertex of degree 4 in an induced subgraph $F \simeq S$, or for at least one $v_i \in \{v_1, v_4\}$, $\langle\{N_{\bar{G}}(v_i)\rangle_{\bar{G}}$ is 2-connected. By Corollary 1 and by

Theorem H, this will imply that at least one $v_i \in \{v_1, v_4\}$ is feasible, and hence $G^*_{v_i}$ contains an induced Γ_3. Showing that this is not possible, we obtain the requested contradiction.

For the vertices of the induced subgraph $F \simeq S$, we will use the labeling as in Fig. 4, and we will use C to denote its central triangle $C = z_1z_2z_3$. By symmetry, we suppose that $v_1 \in V(C)$, and we distinguish several cases according to the mutual position of C and Q.

Case 1: $\{v_1, v_3\} \subset V(C)$.
Up to a symmetry, set $v_1 = z_1$ and $v_3 = z_2$. Then there is a vertex $w_1 \in V(F) \setminus V(Q)$ such that $w_1v_1, w_1v_3 \in E(\bar{G})$: if $z_3 \notin V(F)$, we simply set $w_1 = z_3$; otherwise necessarily $z_3 = v_2$, and we set $w_1 = z_5$.

Subcase 1.1: $w_1v_5 \in E(\bar{G})$.
If $w_1v_4 \notin E(\bar{G})$, then $w_1v_4v_1v_5v_2$ is endgame (iv) in $N_{\bar{G}}(v_3)$ (note that $v_4v_1, v_1v_5, v_5v_2 \notin E(\bar{G})$ since $Q \overset{\text{IND}}{\subset} \bar{G}$; similarly in the following), hence $w_1v_4 \in E(\bar{G})$. If $w_1v_2 \notin E(\bar{G})$, then $w_1v_4v_2v_1$ is endgame (ii) in $N_{\bar{G}}(v_3)$, hence $w_1v_2 \in E(\bar{G})$. This implies that $v_2 \notin V(F)$ (since $F \simeq S$ is induced). Thus, there is a vertex $w_2 \in V(F) \setminus V(Q)$ such that $w_1v_1, w_2v_3 \in E(\bar{G})$. If $w_2v_5 \in E(\bar{G})$, then $w_2v_5w_1v_1$ is endgame (ii) in $N_{\bar{G}}(v_3)$, hence $w_2v_5 \notin E(\bar{G})$. Since $\langle\{v_3, v_2, w_2, v_5\}\rangle_{\bar{G}} \not\simeq K_{1,3}$, $v_2w_2 \in E(\bar{G})$. Now we have $v_0w_1 \notin E(\bar{G})$ and $v_0w_2 \notin E(\bar{G})$, for otherwise $\langle N_{\bar{G}}(v_1)\rangle_{\bar{G}}$ is 2-connected by Lemma I and v_1 is feasible. But then $\langle\{v_1, w_1, v_0, w_2\}\rangle_{\bar{G}} \simeq K_{1,3}$, a contradiction.

Subcase 1.2: $w_1v_5 \notin E(\bar{G})$.
Since $\langle\{v_3, w_1, v_5, v_2\}\rangle_{\bar{G}} \not\simeq K_{1,3}$, we have $w_1v_2 \in E(\bar{G})$. This implies that $v_2 \notin V(F)$, hence there is again a vertex $w_2 \in V(F) \setminus V(Q)$ such that $w_2v_1, w_2v_3 \in E(\bar{G})$. Note that $w_1w_2 \notin E(\bar{G})$ since F is induced. Then $w_2v_5 \in E(\bar{G})$ since otherwise $\langle\{v_3, w_1, w_2, v_5\}\rangle_{\bar{G}} \simeq K_{1,3}$. If $v_0w_2 \in E(\bar{G})$, then $\langle N_{\bar{G}}(v_1)\rangle_{\bar{G}}$ is 2-connected by Lemma I, hence $v_0w_2 \notin E(\bar{G})$, and from $\langle\{v_1, v_0, w_1, w_2\}\rangle_{\bar{G}} \not\simeq K_{1,3}$ we have $v_0w_1 \in E(\bar{G})$. Now, $v_2w_2 \notin E(\bar{G})$ for otherwise $\langle N_{\bar{G}}(v_1)\rangle_{\bar{G}}$ is 2-connected by Lemma I. If $w_2v_4 \notin E(\bar{G})$, then $w_2v_4v_1v_5w_1$ is endgame (iii) in $N_{\bar{G}}(v_3)$, hence $w_2v_4 \in E(\bar{G})$. However, then $w_2v_4v_2v_1$ is endgame (ii) in $N_{\bar{G}}(v_3)$, a contradiction.

Case 2: $\{v_1, v_2\} \subset V(C)$.

Subcase 2.1: $v_0 \in V(C)$.
By the structure of F, there is a vertex $w_1 \in V(F) \setminus V(Q)$ such that $w_1v_0, w_1v_2 \in E(F)$. Since $w_1v_1 \notin E(\bar{G})$ (F is induced), $v_1v_4 \notin E(\bar{G})$ (Q is induced), and $\langle\{v_2, w_1, v_1, v_4\}\rangle_{\bar{G}} \not\simeq K_{1,3}$, we have $w_1v_4 \in E(\bar{G})$. Then $w_1v_1v_4v_0v_3$ is endgame (iv) in $N_{\bar{G}}(v_2)$.

Subcase 2.2: $v_0 \notin V(C)$.
Note that also $v_3 \notin V(C)$ since otherwise we are back in Case 1. Then there is a vertex $w_1 \in V(C) \setminus V(Q)$ such that $w_1v_1, w_1v_2 \in E(C)$.

Suppose that $w_1v_3 \notin E(\bar{G})$. Then necessarily $w_1v_4 \in E(\bar{G})$, for otherwise the subgraph $Q' = \langle\{w_1, v_1, v_2, v_3, v_4, v_5\}\rangle_{\bar{G}}$ is isomorphic to P_6^2 or to P_6^{2+} and, for Q', we are back in Case 1; however, then $w_1v_1v_3v_4$ is endgame (ii) in $N_{\bar{G}}(v_2)$, a contradiction.

Thus, $w_1v_3 \in E(\bar{G})$, implying that $v_3 \notin V(F)$. Hence there is a vertex $w_2 \in V(F) \setminus V(Q)$ such that $w_2w_1, w_2v_1 \in E(F)$. If $w_2v_0 \in E(\bar{G})$, then $\langle N_{\bar{G}}(v_1)\rangle_{\bar{G}}$ is 2-connected by

Lemma I, hence $w_2v_0 \notin E(\bar{G})$. Since $\langle\{v_1, v_0, w_2, v_3\}\rangle_{\bar{G}} \not\simeq K_{1,3}$, we have $w_2v_3 \in E(\bar{G})$. Then $w_2v_2v_5v_1v_4$ is endgame (iv) in $N_{\bar{G}}(v_3)$, a contradiction.

Case 3: $\{v_0, v_1\} \subset V(C)$, $v_2 \notin V(C)$.
Then there is a vertex $w_1 \in V(C) \setminus V(Q)$ such that $w_1v_0, w_1v_1 \in E(C)$.

Subcase 3.1: $w_1v_2 \notin E(\bar{G})$.
If $w_1v_3 \in E(\bar{G})$, then $w_1v_0v_2v_3$ is endgame (ii) in $N_{\bar{G}}(v_1)$, hence $w_1v_3 \notin E(\bar{G})$. Then v_3 cannot have two neighbors in $V(C)$, implying $v_3 \notin V(F)$. Hence there is a vertex $w_2 \in V(F) \setminus V(Q)$ such that $w_2v_1, w_2w_1 \in E(F)$. Now, $w_2v_3 \notin E(\bar{G})$ for otherwise $\langle N_{\bar{G}}(v_1)\rangle_{\bar{G}}$ is 2-connected by Lemma I, and then $\langle\{v_1, w_2, v_0, v_3\}\rangle_{\bar{G}} \simeq K_{1,3}$, a contradiction.

Subcase 3.2: $w_1v_2 \in E(\bar{G})$.
Then $|N_{\bar{G}}(v_2) \cap V(C)| = 3$, implying $v_2 \notin V(F)$. Hence there is a vertex $w_2 \in V(F) \setminus V(Q)$ with $w_2v_0, w_2v_1 \in E(F)$. If $w_2v_3 \in E(\bar{G})$, then $\langle N_{\bar{G}}(v_1)\rangle_{\bar{G}}$ is 2-connected by Lemma I, hence $w_2v_3 \notin E(\bar{G})$, and from $\langle\{v_1, w_1, w_2, v_3\}\rangle_{\bar{G}} \not\simeq K_{1,3}$ we have $w_1v_3 \in E(\bar{G})$. Now, $v_5 \in V(F)$ would imply $v_5w_1, v_5v_0 \in E(\bar{G})$, and then $v_0v_2v_3v_5$ is endgame (ii) in $N_{\bar{G}}(w_1)$; hence $v_5 \notin V(F)$. Then there is a vertex $w_3 \in V(F) \setminus V(Q)$ with $w_3w_1, w_3v_0 \in E(F)$ (clearly $v_2, v_3, v_4 \notin V(F)$). Now $w_2v_2 \notin E(\bar{G})$ for otherwise $\langle N_{\bar{G}}(v_1)\rangle_{\bar{G}}$ is 2-connected by Lemma I, from $\langle\{v_0, w_3, v_2, w_2\}\rangle_{\bar{G}} \not\simeq K_{1,3}$ we have $w_3v_2 \in E(\bar{G})$, and from $\langle\{v_2, w_3, v_1, v_4\}\rangle_{\bar{G}} \not\simeq K_{1,3}$ we have $w_3v_4 \in E(\bar{G})$.

If $w_3v_3 \notin E(\bar{G})$, then $v_0v_1v_3v_4w_3$ is endgame (v) in $N_{\bar{G}}(v_2)$, hence $w_3v_3 \in E(\bar{G})$. This implies that $v_3 \notin V(F)$, hence there is a vertex $w_4 \in V(F) \setminus V(Q)$ such that $w_1v_1, w_4w_1 \in E(F)$. Since $\langle\{v_1, v_0, w_4, v_3\}\rangle_{\bar{G}} \not\simeq K_{1,3}$, $w_4v_3 \in E(\bar{G})$. Since $\langle\{v_1, w_4, v_2, w_2\}\rangle_{\bar{G}} \not\simeq K_{1,3}$, $w_4v_2 \in E(\bar{G})$, and since $\langle\{v_2, v_0, w_4, v_4\}\rangle_{\bar{G}} \not\simeq K_{1,3}$, $w_4v_4 \in E(\bar{G})$. Now, if $w_4v_5 \in E(\bar{G})$ or $w_3v_5 \in E(\bar{G})$, then $\langle N_{\bar{G}}(v_4)\rangle_{\bar{G}}$ is 2-connected by Lemma I (the independent sets are $\{v_2, v_5\}$ and $\{w_3, v_4\}$), hence $w_4v_5 \notin E(\bar{G})$ and $w_3v_5 \notin E(\bar{G})$. However, then we have $\langle\{v_3, w_4, w_3, v_5\}\rangle_{\bar{G}} \simeq K_{1,3}$, a contradiction.

Case 4: $\{v_0, v_3\} \cap V(C) = \emptyset$.
Then $C = v_1w_1w_2$ for some $w_1, w_2 \in V(F) \setminus V(Q)$. If $|V(F) \cap \{v_0, v_2, v_3\}| \geq 2$, then $v_0 \in V(F)$ and $v_3 \in V(F)$ (since vertices of degree 2 in F are independent), and $\langle N_{\bar{G}}(v_1)\rangle_{\bar{G}}$ is 2-connected by Lemma I. Thus, at most one of the vertices v_0, v_2, v_3 is in $V(F)$. This implies that there is a vertex $w_3 \in V(F) \setminus (V(Q) \cup \{w_1, w_2\})$ such that $w_3v_1, w_3w_2 \in E(F)$. Since $\langle\{v_1, w_3, w_1, v_0\}\rangle_{\bar{G}} \not\simeq K_{1,3}$, up to a symmetry, $w_3v_0 \in E(\bar{G})$. Then $w_1v_3 \notin E(\bar{G})$, for otherwise $\langle N_{\bar{G}}(v_1)\rangle_{\bar{G}}$ is 2-connected by Lemma I. Since $\langle\{v_1, w_3, w_1, v_3\}\rangle_{\bar{G}} \not\simeq K_{1,3}$, we have $w_3v_3 \in E(\bar{G})$, and since $\langle\{v_1, v_0, w_1, v_3\}\rangle_{\bar{G}} \not\simeq K_{1,3}$, we have $v_0w_1 \in E(\bar{G})$. But then $\langle N_{\bar{G}}(v_1)\rangle_{\bar{G}}$ is 2-connected by Lemma I.

Thus, we know that at least one of the vertices v_1, v_4 is feasible. We choose the notation such that v_1 is feasible. By the definition of the Γ_3-closure, $\bar{G}^*_{v_1}$ contains an induced Γ_3. To reach a contradiction, we show that this is not possible.

Set $x = v_1$, let $F = \langle\{t_1, t_2, p_1, p_2, p_3, p_4, t_3, t_4\}\rangle_{\bar{G}^*_x} \simeq \Gamma_3$, and let $y_1y_2 \in E(F)$ be a new edge. Since t_1t_2 cannot be the only new edge (then would be $\langle\{p_1, t_1, t_2, p_2\}\rangle_{\bar{G}} \simeq K_{1,3}$), by symmetry, we can assume that $y_1y_2 \in \{t_2p_1, p_1p_2, p_2p_3\}$.

<u>**Case 1:**</u> $dist_{\bar{G}}(y_1, y_2) = 2$.
Then, by Lemma 5, $y_1y_2 \in \{p_1p_2, p_2p_3\}$. Let $y_1z_1y_2$ be a shortest (y_1, y_2)-path in $\langle N_{\bar{G}}(x)\rangle_{\bar{G}}$.

<u>Claim 1.</u> $N_{\bar{G}}(z_1) \cap V(F) = \{y_1, y_2\}$.

<u>Proof.</u> Let first $y_1y_2 = p_1p_2$, and let $z_1z_2 \in E(\bar{G})$ for some $z_2 \in V(F) \setminus \{p_1, p_2\}$. If $z_2 \in \{p_3, t_3, t_4\}$, then $\langle\{z_1, p_1, p_2, z_2\}\rangle_{\bar{G}} \simeq K_{1,3}$, and if both $z_1p_3 \in E(\bar{G})$ and $z_2t_i \in E(\bar{G})$ for some $i \in \{1, 2\}$, then $\langle\{z_1, t_i, x, p_3\}\rangle_{\bar{G}} \simeq K_{1,3}$. Hence, by symmetry, either $z_1p_3 \in E(\bar{G})$, or, say, $z_1t_2 \in E(\bar{G})$, but in the first case $\langle\{t_1, t_2, p_1, z_1, p_3, p_4, t_3, t_4\}\rangle_{\bar{G}} \simeq \Gamma_3$, and in the second case $\langle\{t_2, p_1, z_1, p_2, p_3, p_4, t_3, t_4\}\rangle_{\bar{G}} \simeq \Gamma_3$.

Let next $y_1y_2 = p_2p_3$, and let $z_1z_2 \in E(\bar{G})$ for some $z_2 \in V(F) \setminus \{p_2, p_3\}$. If $z_2 \in \{t_1, t_2, t_3, t_4\}$, then $\langle\{z_1, z_2, p_2, p_3\}\rangle_{\bar{G}} \simeq K_{1,3}$, and if both $z_1p_1 \in E(\bar{G})$ and $z_1p_4 \in E(\bar{G})$, then $\langle\{z_1, p_1, x, p_4\}\rangle_{\bar{G}} \simeq K_{1,3}$. Thus, by symmetry, we can assume that $z_1p_1 \in E(\bar{G})$ and $z_1p_4 \notin E(\bar{G})$, but then $\langle\{t_1, t_2, p_1, z_1, p_3, p_4, t_3, t_4\}\rangle_{\bar{G}} \simeq \Gamma_3$, a contradiction. □

<u>Claim 2.</u> $N_{\bar{G}}[x] = N_{\bar{G}}[z_1]$.

<u>Proof.</u> By Claim 1, $N_{\bar{G}}(x) \cap V(F) = N_{\bar{G}}(z_1) \cap V(F)$.

<u>Case Cl-2-1:</u> $y_1y_2 = p_1p_2$.
We first show that $N_{\bar{G}}[x] \subset N_{\bar{G}}[z_1]$. Let thus, to the contrary, $z_2 \in N_{\bar{G}}[x] \setminus N_{\bar{G}}[z_1]$. Since $\langle\{x, p_1, p_2, z_2\}\rangle_{\bar{G}} \not\simeq K_{1,3}$, $p_1z_2 \in E(\bar{G})$ or $p_2z_2 \in E(\bar{G})$. On the other hand, if both $p_1z_2 \in E(\bar{G})$ and $p_2z_2 \in E(\bar{G})$, then we have a contradiction with Claim 1. Hence z_2 is adjacent to exactly one of p_1, p_2. Let first $p_1z_2 \in E(\bar{G})$ and $p_2z_2 \notin E(\bar{G})$. By Claim 1, $\langle\{p_1, t_1, z_1, z_2\}\rangle_{\bar{G}} \not\simeq K_{1,3}$ implies $t_1z_2 \in E(\bar{G})$. From $\langle\{p_1, z_2, x, p_2, p_3, p_4, t_3, t_4\}\rangle_{\bar{G}} \not\simeq \Gamma_3$ we have $z_2z_3 \in E(\bar{G})$ for some $z_3 \in \{p_3, p_4, t_3, t_4\}$, but then $\langle\{z_2, t_1, x, z_3\}\rangle_{\bar{G}} \simeq K_{1,3}$, a contradiction. Thus, we have $p_2z_2 \in E(\bar{G})$ and $p_1z_2 \notin E(\bar{G})$. By Claim 1, $\langle\{p_2, z_1, z_2, p_3\}\rangle_{\bar{G}} \not\simeq K_{1,3}$ implies $p_3z_2 \in E(\bar{G})$. From $\langle\{p_1, z_1, x, z_2, p_3, p_4, t_3, t_4\}\rangle_{\bar{G}} \not\simeq \Gamma_3$ we have $z_2z_3 \in E(\bar{G})$ for some $z_3 \in \{p_4, t_3, t_4\}$. However, if $z_3 \in \{t_3, t_4\}$, then $\langle\{z_2, z_3, p_3, x\}\rangle_{\bar{G}} \simeq K_{1,3}$, hence $z_2p_4 \in E(\bar{G})$, but then $\langle\{t_1, t_2, p_1, x, z_2, p_4, t_3, t_4\}\rangle_{\bar{G}} \simeq \Gamma_3$, a contradiction. Thus, $N_{\bar{G}}[x] \subset N_{\bar{G}}[z_1]$.

Next we show that also $N_{\bar{G}}[z_1] \subset N_{\bar{G}}[x]$. Let, to the contrary, $z_2 \in N_{\bar{G}}[z_1] \setminus N_{\bar{G}}[x]$. Let first $p_1z_2 \in E(\bar{G})$. From $\langle\{p_1, t_1, x, z_2\}\rangle_{\bar{G}} \not\simeq K_{1,3}$ then $t_1z_2 \in E(\bar{G})$. Since $\langle\{p_1, z_2, z_1, p_2, p_3, p_4, t_3, t_4\}\rangle_{\bar{G}} \not\simeq \Gamma_3$, we have $z_2z_3 \in E(\bar{G})$ for some $z_3 \in \{p_2, p_3, p_4, t_3, t_4\}$; however, $z_3 \in \{p_3, p_4, t_3, t_4\}$ implies $\langle\{z_2, z_3, t_1, z_1\}\rangle_{\bar{G}} \simeq K_{1,3}$, and $z_3 = p_2$ implies $\langle\{p_2, p_3, z_2, x\}\rangle_{\bar{G}} \simeq K_{1,3}$. Thus, $p_1z_2 \notin E(\bar{G})$, and from $\langle\{z_1, p_1, p_2, z_2\}\rangle_{\bar{G}} \not\simeq K_{1,3}$ we have $z_2p_2 \in E(\bar{G})$. Since $\langle\{p_2, x, z_2, p_3\}\rangle_{\bar{G}} \not\simeq K_{1,3}$, also $z_2p_3 \in E(\bar{G})$. Now, since $\langle\{p_1, x, z_1, z_2, p_3, p_4, t_3, t_4\}\rangle_{\bar{G}} \not\simeq \Gamma_3$, by Claim 1, we have $z_2z_3 \in E(\bar{G})$ for some $z_3 \in \{p_4, t_3, t_4\}$. However, $z_3 \in \{t_3, t_4\}$ implies $\langle\{z_2, z_1, p_3, z_3\}\rangle_{\bar{G}} \simeq K_{1,3}$, and for $z_3 = p_4$ we have $\langle\{t_1, t_2, p_1, z_1, z_2, p_4, t_3, t_4\}\rangle_{\bar{G}} \simeq \Gamma_3$, a contradiction.

<u>Case Cl-2-2:</u> $y_1y_2 = p_2p_3$.
We again first show that $N_{\bar{G}}[x] \subset N_{\bar{G}}[z_1]$. Let, to the contrary, $z_2 \in N_{\bar{G}}[x] \setminus N_{\bar{G}}[z_1]$. Since $\langle\{x, p_1, p_2, z_2\}\rangle_{\bar{G}} \not\simeq K_{1,3}$, up to a symmetry, $p_2z_2 \in E(\bar{G})$, and since $\langle\{p_2, p_1, z_1, z_2\}\rangle_{\bar{G}} \not\simeq$

$K_{1,3}$, by Claim 1, $p_1z_2 \in E(\bar{G})$. Since $\langle\{p_1, z_2, p_2, z_1, p_3, p_4, t_3, t_4\}\rangle_{\bar{G}} \not\simeq \Gamma_3$ and by Claim 1, $z_2z_3 \in E(\bar{G})$ for some $z_3 \in \{p_3, p_4, t_3, t_4\}$. However, for $z_3 \in \{p_4, t_3, t_4\}$, $\langle\{z_2, p_1, x, z_3\}\rangle_{\bar{G}} \simeq K_{1,3}$, hence $z_3 = p_3$, but then $\langle\{t_1, t_2, p_1, z_2, p_3, p_4, t_3, t_4\}\rangle_{\bar{G}} \simeq \Gamma_3$, a contradiction. Thus, $N_{\bar{G}}[x] \subset N_{\bar{G}}[z_1]$.

Finally, we show that also $N_{\bar{G}}[z_1] \subset N_{\bar{G}}[x]$. Let thus again, to the contrary, $z_2 \in N_{\bar{G}}[z_1] \setminus N_{\bar{G}}[x]$. Since $\langle\{z_1, p_2, p_3, z_2\}\rangle_{\bar{G}} \not\simeq K_{1,3}$, up to a symmetry, $z_2p_2 \in E(\bar{G})$, and since $\langle\{p_2, p_1, z_2, x\}\rangle_{\bar{G}} \not\simeq K_{1,3}$, also $z_2p_1 \in E(\bar{G})$. Since $\langle\{p_1, z_2, p_2, x, p_3, p_4, t_3, t_4\}\rangle_{\bar{G}} \not\simeq \Gamma_3$, we have $z_2z_3 \in E(\bar{G})$ for some $z_3 \in \{p_3, p_4, t_3, t_4\}$; however, if $z_3 \in \{p_4, t_3, t_4\}$, then $\langle\{z_2, p_1, z_1, z_3\}\rangle_{\bar{G}} \simeq K_{1,3}$ by Claim 1, and if $z_3 = p_3$, then $\langle\{p_3, z_2, x, p_4\}\rangle_{\bar{G}} \simeq K_{1,3}$, a contradiction. $\square$

By the assumption, set $Q = \langle\{v_0, v_1, v_2, v_3, v_4, v_5\}\rangle_{\bar{G}}$, where $Q \simeq P_6^2$ or $Q \simeq P_6^{2+}$ and $x = v_1$. Since $\bar{G}$ is 3-connected and not Hamilton-connected, by Theorem F, $\alpha(\bar{G}) \geq 3$. Thus, by Theorem G and by Proposition 3, $N_{\bar{G}}(x)$, hence also $N_{\bar{G}}[x]$, can be covered by two cliques, say, K_1 and K_2. Since $y_1, y_2 \in N_{\bar{G}}(x)$ and $y_1y_2 \notin E(\bar{G})$ (where $y_1 = p_1$ and $y_2 = p_2$, or $y_1 = p_2$ and $y_2 = p_3$, depending on the case), we can choose the notation such that $y_1 \in K_1 \setminus K_2$ and $y_2 \in K_2 \setminus K_1$. Clearly $v_1 = x \in K_1 \cap K_2$, and, by Claim 2, also $z_1 \in K_1 \cap K_2$. On the other hand, since $v_4 \in N_{\bar{G}}(v_2) \setminus N_{\bar{G}}(v_1)$, by Claim 2, $v_2 \notin K_1 \cap K_2$. Hence either $v_2 \in K_1 \setminus K_2$ and, since $v_0, v_3 \in N_{\bar{G}}(v_1)$ but $v_0v_3 \notin E(\bar{G})$, one of v_0, v_3 is in $K_2 \setminus K_1$, or, symmetrically, $v_2 \in K_2 \setminus K_1$ and one of v_0, v_3 is in $K_1 \setminus K_2$. Thus, we conclude that there are vertices $w_1 \in K_1 \setminus K_2$ and $w_2 \in K_2 \setminus K_1$ such that $w_1, w_2 \in V(Q)$ and $w_1w_2 \in E(Q)$. By Claim 2, $w_1y_2 \notin E(\bar{G})$ and $w_2y_1 \notin E(\bar{G})$.

Subcase 1.1: $y_1y_2 = p_1p_2$.
Since $\langle\{p_1, w_1, x, p_2, p_3, p_4, t_3, t_4\}\rangle_{\bar{G}} \not\simeq \Gamma_3$, $w_1z_2 \in E(\bar{G})$ for some $z_2 \in \{p_3, p_4, t_3, t_4\}$. Since $\langle\{w_1, p_1, w_2, z_2\}\rangle_{\bar{G}} \not\simeq K_{1,3}$, also $w_2z_2 \in E(\bar{G})$. If $z_3 = p_3$, then $w_1p_3p_2x$ is endgame *(ii)* in $N_{\bar{G}}(w_2)$, hence $z_2 \in \{p_4, t_3, t_4\}$. Recall that $w_1, w_2 \in V(Q)$ and either $w_1 = v_2$ and $w_2 \in \{v_0, v_3\}$, or $w_1 \in \{v_0, v_3\}$ and $w_2 = v_2$. Then, if $w_1 = v_2$ and $w_2 = v_0$, $z_2v_1v_4v_0v_3$ is endgame *(iv)* in $N_{\bar{G}}(v_2)$, and if $w_1 = v_2$ and $w_2 = v_3$, then $p_2v_2v_5v_1v_4$ is endgame *(iv)* in $N_{\bar{G}}(v_3)$. The remaining two cases $w_2 = v_2$ are symmetric (since our argument did not use the vertices $t_1, t_2, p_3, p_4, t_3, t_4$).

Subcase 1.2: $y_1y_2 = p_2p_3$.
Since $w_1, w_2 \in V(Q)$, by symmetry, we can assume that $w_1 = v_2$ and $w_2 \in \{v_0, v_3\}$. However, if $w_2 = v_3$, then $p_3v_2v_5v_1v_4$ is endgame *(iv)* in $N_{\bar{G}}(v_3)$. Hence $w_2 = v_0$.

Next observe that a vertex $t \in \{t_1, t_2, t_3, t_4\}$ is adjacent to either both the vertices v_0, v_2, or to none of them: if, say, $t_1v_2 \in E(\bar{G})$, then $\langle\{v_2, t_1, p_2, v_0\}\rangle_{\bar{G}} \not\simeq K_{1,3}$ implies $t_1v_0 \in E(\bar{G})$, and, conversely, if $t_1v_0 \in E(\bar{G})$, then $\langle\{v_0, t_1, p_3, v_2\}\rangle_{\bar{G}} \not\simeq K_{1,3}$ implies $t_1v_2 \in E(\bar{G})$. However, if $tv_0, tv_2 \in E(\bar{G})$ for some $t \in \{t_1, t_2, t_3, t_4\}$, then $tv_1v_4v_0v_3$ is endgame *(iv)* in $N_{\bar{G}}(v_2)$. Thus, there are no edges between $\{v_0, v_2\}$ and $\{t_1, t_2, t_3, t_4\}$.

Now, if $v_0p_1 \in E(\bar{G})$, then $\langle\{p_1, t_1, p_2, v_0\}\rangle_{\bar{G}} \simeq K_{1,3}$, hence $v_0p_1 \notin E(\bar{G})$, and, symmetrically, $v_2p_4 \notin E(\bar{G})$. Thus, the only possible edges between $\{v_0, v_2\}$ and $V(F) \setminus \{p_2, p_3\}$ are the edges v_2p_1 and v_0p_4. If both are present, then $\langle\{t_1, t_2, p_1, v_2, v_0, p_4, t_3, t_4\}\rangle_{\bar{G}} \simeq \Gamma_3$, and if, say, $v_2p_1 \in E(\bar{G})$ but $v_0p_4 \notin E(\bar{G})$, then $\langle\{p_1, p_2, v_2, v_0, p_3, p_4, t_3, t_4\}\rangle_{\bar{G}} \simeq \Gamma_3$. Thus, there are no edges between $\{v_0, v_2\}$ and $V(F) \setminus \{p_2, p_3\}$.

If $v_3 = p_2$, then, by the above observations, $v_4 \neq p_1$, and from $\langle\{v_3, p_1, v_4, v_1\}\rangle_{\bar{G}} \not\simeq K_{1,3}$ we have $p_1v_4 \in E(\bar{G})$. Since $\langle\{p_1, v_4, v_3, v_1, p_3, p_4, t_3, t_4\}\rangle_{\bar{G}} \not\simeq \Gamma_3$, we have $v_4z \in E(\bar{G})$ for some $z \in \{p_3, p_4, t_3, t_4\}$, but in each of these cases, $\langle\{v_4, p_1, v_2, z\}\rangle_{\bar{G}} \simeq K_{1,3}$, a contradiction. Thus, $v_3 \neq p_2$.

Since $\langle\{v_2, v_4, p_2, v_0\}\rangle_{\bar{G}} \not\simeq K_{1,3}$, we have $p_2v_4 \in E(\bar{G})$, and from $\langle\{p_2, p_1, v_4, v_1\}\rangle_{\bar{G}} \not\simeq K_{1,3}$ also $p_1v_4 \in E(\bar{G})$. Since $\langle\{p_1, v_4, p_2, v_1, p_3, p_4, t_3, t_4\}\rangle_{\bar{G}} \not\simeq \Gamma_3$, we have $v_4z \in E(\bar{G})$ for some $z \in \{p_3, p_4, t_3, t_4\}$, but in each of these cases, $\langle\{v_4, p_1, v_2, z\}\rangle_{\bar{G}} \simeq K_{1,3}$, a contradiction.

Case 2: $dist_{\bar{G}}(y_1, y_2) = 3$.
Let K_1, K_2 be the two cliques that cover $N_{\bar{G}}[x]$, chosen such that $y_1 \in K_1$ and $y_2 \in K_2$. By the assumption of the case, $K_1 \cap K_2 = \{x\}$. Since $x = v_1$, by the structure of Q, there are vertices $w_1 \in K_1 \setminus K_2$ and $w_2 \in K_2 \setminus K_1$ such that $w_1, w_2 \in V(Q)$, $w_1w_2 \in E(Q)$, and either $w_1 = v_2$ and $w_2 \in \{v_0, v_3\}$, or $w_1 \in \{v_0, v_3\}$ and $w_2 = v_2$. Note that also $y_iw_i \in E(\bar{G})$, $i = 1, 2$, and $y_1w_2, y_2w_1 \notin E(\bar{G})$.

Now, if, say, $w_1 = v_2$ and $w_2 = v_3$, then $y_2v_2v_5v_1v_4$ is endgame (iv) in $N_{\bar{G}}(v_3)$. Since the case $w_1 = v_3$ and $w_2 = v_2$ is symmetric, we conclude that either $w_1 = v_2$ and $w_2 = v_0$, or $w_1 = v_0$ and $w_2 = v_2$.

Subcase 2.1: $y_1y_2 = t_2p_1$.
We show that w_1, w_2 have a common neighbor $z \in \{p_3, p_4, t_3, t_4\}$.

Since $\langle\{v_1, w_2, p_1, p_2, p_3, p_4, t_3, t_4\}\rangle_{\bar{G}} \not\simeq \Gamma_3$, $w_2z \in E(\bar{G})$ for some $z \in \{p_2, p_3, p_4, t_3, t_4\}$. If $z \in \{p_3, p_4, t_3, t_4\}$, then $\langle\{w_2, w_1, p_1, z\}\rangle_{\bar{G}} \not\simeq K_{1,3}$ implies that also $w_1z \in E(\bar{G})$; thus, let $z = p_2$. Then from $\langle\{w_1, v_1, w_2, p_2, p_3, p_4, t_3, t_4\}\rangle_{\bar{G}} \not\simeq \Gamma_3$ we have $w_1z' \in E(\bar{G})$ for some $z' \in \{p_3, p_4, t_3, t_4\}$, but then $\langle\{w_1, t_2, w_2, z'\}\rangle_{\bar{G}} \not\simeq K_{1,3}$ implies that also $w_2z' \in E(\bar{G})$. Thus, in all cases, there is $z \in \{p_3, p_4, t_3, t_4\}$ such that $w_1z, w_2z \in E(\bar{G})$. Since $\{w_1, w_2\} = \{v_0, v_2\}$, necessarily $z \notin \{v_3, v_4\}$, and then $zv_1v_4v_0v_3$ is endgame (iv) in $N_{\bar{G}}(v_2)$.

Subcase 2.2: $y_1y_2 = p_1p_2$.
We similarly show that w_1 and w_2 have a common neighbor $z \in \{p_4, t_3, t_4\}$. Since $\langle\{p_1, w_1, v_1, p_2, p_3, p_4, t_3, t_4\}\rangle_{\bar{G}} \not\simeq \Gamma_3$, we have $w_1z \in E(\bar{G})$ for some $z \in \{p_3, p_4, t_3, t_4\}$.

If $z \in \{p_4, t_3, t_4\}$, then $\langle\{w_1, p_1, w_2, z\}\rangle_{\bar{G}} \not\simeq K_{1,3}$ implies that also $w_2z \in E(\bar{G})$; thus, let $z = p_3$. Then $\langle\{p_3, p_2, p_4, w_1\}\rangle_{\bar{G}} \not\simeq K_{1,3}$ implies that also $w_1p_4 \in E(\bar{G})$, and we are in the previous case. Thus, we have $w_1z, w_2z \in E(\bar{G})$ for some $z \in \{p_4, t_3, t_4\}$. Again, $z \notin \{v_3, v_4\}$ since $\{w_1, w_2\} = \{v_0, v_2\}$, and then $zv_1v_4v_0v_3$ is endgame (iv) in $N_{\bar{G}}(v_2)$.

Subcase 2.3: $y_1y_2 = p_2p_3$.
We proceed in a similar way as in Subcase 1.2. If, say, $w_1t_i \in E(\bar{G})$ for some $i \in \{1, 2, 3, 4\}$, then $\langle\{w_1, w_2, p_2, t_i\}\rangle_{\bar{G}} \not\simeq K_{1,3}$ implies that also $w_2t_i \in E(\bar{G})$ and $t_iv_1v_4v_0v_3$ is endgame (iv) in $N_{\bar{G}}(v_2)$; and if, say, $w_2p_4 \in E(\bar{G})$, then since $\langle\{p_4, t_4, p_3, w_1\}\rangle_{\bar{G}} \not\simeq K_{1,3}$, $w_1t_4 \in E(\bar{G})$, and we are in the previous case. Hence w_1p_1 and w_2p_4 are the only possible edges between $\{w_1, w_2\}$ and $V(F) \setminus \{p_2, p_3\}$; however, if both are present, then $\langle\{t_1, t_2, p_1, w_1, w_2, p_4, t_3, t_4\}\rangle_{\bar{G}} \simeq \Gamma_3$, and if, say, $w_1p_1 \in E(\bar{G})$ but $w_2p_4 \notin E(\bar{G})$, then $\langle\{p_1, p_2, w_1, w_2, p_3, p_4, t_3, t_4\}\rangle_{\bar{G}} \simeq \Gamma_3$. Thus, there are no edges between $\{w_1, w_2\}$ and $V(F) \setminus \{p_2, p_3\}$. By symmetry, set $w_1 = v_2$ and $w_2 = v_0$.

If $v_3 = p_2$, then $v_4 \neq p_1$, $\langle\{p_2, p_1, v_4, v_1\}\rangle_{\bar{G}} \not\simeq K_{1,3}$ implies $p_1v_4 \in E(\bar{G})$, and since $\langle\{p_1, v_4, p_2, v_1, p_3, p_4, t_3, t_4\}\rangle_{\bar{G}} \not\simeq \Gamma_3$, we have $v_4z \in E(\bar{G})$ with $z \in \{p_3, p_4, t_3, t_4\}$, implying $\langle\{v_4,p_1,v_2,z\}\rangle_{\bar{G}} \simeq K_{1,3}$. Hence $v_3 \neq p_2$. Then $\langle\{v_2,v_4,p_2,v_0\}\rangle_{\bar{G}} \not\simeq K_{1,3}$ implies $p_2v_4 \in E(\bar{G})$, $\langle\{p_2,p_1,v_4,v_1\}\rangle_{\bar{G}} \not\simeq K_{1,3}$ implies $p_1v_4 \in E(\bar{G})$, and from $\langle\{p_1,v_4,p_2,v_1,p_3,p_4,t_3,t_4\}\rangle_{\bar{G}} \not\simeq \Gamma_3$ we have $v_4z \in E(\bar{G})$ with $z \in \{p_3,p_4,t_3,t_4\}$, implying $\langle\{v_4,p_1,v_2,z\}\rangle_{\bar{G}} \simeq K_{1,3}$, a contradiction. ■

5 Concluding remarks

1. A Γ_3-closure of a graph G, as defined in Section 4, is not unique in general. However, in view of Theorem B, it is unique in 3-connected $\{K_{1,3},\Gamma_3\}$-free graphs since each such graph is Hamilton-connected, hence has complete closure.

2. The source codes of our proof-assisting programs are available at [22]. The codes are written in SageMath 9.6 and use some of its functions. We thank the SageMath community [20] for developing a valuable open-source mathematical software.

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Computing source codes

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