

Every 3-connected $\{K_{1,3}, \Gamma_3\}$ -free graph is Hamilton-connected¹

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Abstract

We show that every 3-connected $\{K_{1,3}, \Gamma_3\}$ -free graph is Hamilton-connected, where Γ_3 is the graph obtained by joining two vertex-disjoint triangles with a path of length 3. This resolves one of the two last open cases in the characterization of pairs of connected forbidden subgraphs implying Hamilton-connectedness. The proof is based on a new closure technique, developed in a previous paper, and on a structural analysis of small subgraphs, cycles and paths in line graphs of multigraphs. The most technical steps of the analysis are computer-assisted.

Keywords: Hamilton-connected; closure; forbidden subgraph; claw-free; Γ_3 -free

1 Terminology and notation

We generally follow the most common graph-theoretical notation and terminology, and for notations and concepts not defined here we refer the reader to [4]. Specifically, by a *graph* we always mean a simple finite undirected graph; whenever we admit multiple edges, we always speak about a *multigraph*. If G is a multigraph and $x_1x_2 \in E(G)$, we denote $\mu_G(x_1x_2)$ (or simply $\mu(x_1x_2)$) the *multiplicity* of the edge x_1x_2 in G (with $\mu(x_1x_2) = 0$ if x_1, x_2 are nonadjacent), and we use $E_S(G)$ ($E_M(G)$) to denote the set of all simple nonpendant (multiple) edges of G , respectively. If $x_1x_2 \in E_M(G)$ and we need to distinguish the individual edges joining x_1 and x_2 , we use the notation $(x_1x_2)^1, (x_1x_2)^2$ etc. We say that vertices $x_1, x_2 \in V(G)$ are *twins* in a multigraph G if $\mu_G(x_1u) = \mu_G(x_2u)$ for each $u \in V(G)$.

We say that S is an *induced subgraph* of a graph G if S can be obtained from G by removing some vertices, we denote this by $S \stackrel{\text{IND}}{\subset} G$. We say that a (multi)graph S is a *sub(multi)graph* of a multigraph G , denoted $S \subset G$, if $V(S) \subset V(G)$ and for every $x_1, x_2 \in V(S)$ we have $\mu_S(x_1x_2) \leq \mu_G(x_1x_2)$, and we say that S is a *flat subgraph* of G if S is a sub(multi)graph of G where for every $x_1, x_2 \in V(S)$, if $x_1x_2 \in E(G)$, then $\mu_S(x_1x_2) \in \{1, \mu_G(x_1x_2)\}$. We

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write $G_1 \simeq G_2$ if the (multi)graphs G_1, G_2 are isomorphic, and $\langle M \rangle_G$ to denote the *induced sub(multi)graph* on a set $M \subset V(G)$.

We use $d_G(x)$ to denote the *degree* of a vertex x in G (note that if G is a multigraph, then $d_G(x)$ equals the sum of multiplicities of the edges containing x). For $x \in V(G)$, $N_G(x)$ denotes the *neighborhood* of x in G , and for $M \subset V(G)$ we set $N_M(x) = N_G(x) \cap M$.

If $x \in V(G)$ is of degree 2 with $N_G(x) = \{y_1, y_2\}$, then the operation of replacing the path y_1xy_2 by the edge y_1y_2 is called *suppressing* the vertex x . The inverse operation is called *subdividing* the edge y_1y_2 with the vertex x . For $x, y \in V(G)$, $\text{dist}_G(x, y)$ denotes the *distance* of x and y in G , and if $F \subset G$ is a connected subgraph and $x, y \in V(F)$, then $\text{dist}_F(x, y)$ denotes the distance of x, y in F , i.e., the length of a shortest (x, y) -path in F . If C is a cycle in G , then an edge $xy \in E(G)$ such that $x, y \in V(C)$ and $\text{dist}_C(x, y) \geq 2$ is called a *chord* of C . If C has no chords, we say that C is *chordless* (note that some edges of a chordless cycle can still be multiple in H).

A triangle having a multiple edge is called a *multitriangle* (see Fig. 6(a)), and by a *diamond* we mean the graph $K_4 - e$ (see Fig. 6(b)). By a *clique* in G we mean a complete subgraph of G , not necessarily maximal.

We say that a vertex $x \in V(G)$ is *simplicial* if $\langle N_G(x) \rangle_G$ is a clique, and we use $V_{SI}(G)$ to denote the set of all simplicial vertices of G , and $V_{NS}(G) = V(G) \setminus V_{SI}(G)$ the set of nonsimplicial vertices of G . For $k \geq 1$, we say that a vertex $x \in V(G)$ is *locally k -connected* in G if $\langle N_G(x) \rangle_G$ is a k -connected graph.

A graph is *Hamilton-connected* if, for any $u, v \in V(G)$, G has a hamiltonian (u, v) -path, i.e., an (u, v) -path P with $V(P) = V(G)$.

Finally, if \mathcal{F} is a family of graphs, we say that G is \mathcal{F} -free if G does not contain an induced subgraph isomorphic to a member of \mathcal{F} , and the graphs in \mathcal{F} are referred to in this context as *forbidden (induced) subgraphs*. If $\mathcal{F} = \{F\}$, we simply say that G is F -free. Here, the *claw* is the graph $K_{1,3}$, P_i denotes the path on i vertices, and Γ_i denotes the graph obtained by joining two triangles with a path of length i (see Fig. 1(d)). Several further graphs that will occur as forbidden subgraphs are shown in Fig. 1(a), (b), (c). Whenever we will list vertices of an induced claw $K_{1,3}$, we will always list its center as the first vertex of the list, and when listing vertices of an induced subgraph Γ_i , we always list first the vertices of degree 2 of one of the triangles, then the vertices of the path, and we finish with the vertices of degree 2 of the second triangle (i.e., in the labeling of vertices as in Fig. 1(d), we write $\langle \{t_1, t_2, p_1, \dots, p_{i+1}, t_3, t_4\} \rangle_G \simeq \Gamma_i$).

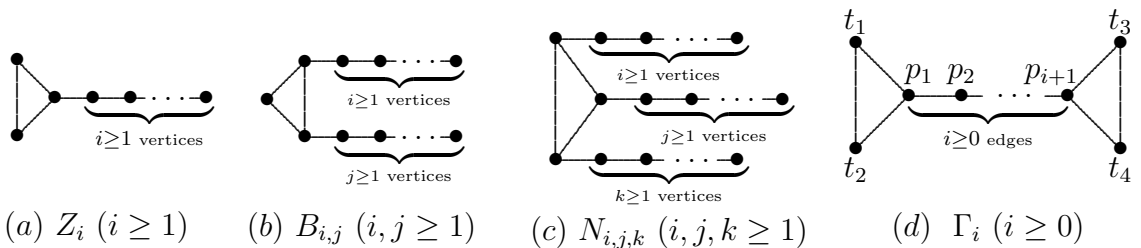


Figure 1: The graphs $Z_i, B_{i,j}, N_{i,j,k}$ and Γ_i

2 Introduction and main result

There are many results on forbidden induced subgraphs implying various Hamilton-type properties. While forbidden pairs of connected graphs for hamiltonicity in 2-connected graphs were completely characterized already in the early 90's [1, 8], the progress in forbidden pairs for Hamilton-connectedness is relatively slow. For forbidden pairs of connected graphs, a list of potential candidates is known: one of them has to be the claw $K_{1,3}$, and the second one belongs to a list that will be mentioned in Section 6.

Let W denote the Wagner graph and W^+ the graph obtained from W by attaching exactly one pendant edge to each of its vertices (see Fig. 2).



Figure 2: The Wagner graph W and the graph W^+

Theorem A below lists the best known results on pairs of forbidden subgraphs implying Hamilton-connectedness of a 3-connected graph.

Theorem A [3, 6, 12, 13, 14, 19, 20]. *Let G be a 3-connected $\{K_{1,3}, X\}$ -free graph, where*

- (i) [6] $X = \Gamma_1$, or
- (ii) [3] $X = P_9$, or
- (iii) [19] $X = Z_7$ and $G \not\cong L(W^+)$, or
- (iv) [20] $X = B_{i,j}$ for $i + j \leq 7$, or
- (v) [12, 13, 14] $X = N_{i,j,k}$ for $i + j + k \leq 7$.

Then G is Hamilton-connected.

Let \mathcal{W} be the family of graphs obtained by attaching at least one pendant edge to each of the vertices of the Wagner graph W , and let $\mathcal{G} = \{L(H) \mid H \in \mathcal{W}\}$ be the family of their line graphs. Then any $G \in \mathcal{G}$ is 3-connected, non-Hamilton-connected, P_{10} -free, Z_8 -free, $B_{i,j}$ -free for $i + j = 8$ and $N_{i,j,k}$ -free for $i + j + k = 8$. Thus, this example shows that parts (ii), (iii), (iv) and (v) of Theorem A are sharp.

The following theorem is our main result.

Theorem 1. *Every 3-connected $\{K_{1,3}, \Gamma_3\}$ -free graph is Hamilton-connected.*

Proof of Theorem 1 is postponed to Section 5.

In Section 3, we collect necessary known results and facts on line graphs and on closure operations, and then, in Section 4, we present a closure technique, introduced in the previous paper [10], that will be crucial for the proof of the main result. Section 5 contains the proof of the main result, in which the most technical parts (namely, the proof of Lemma 4, and the introductory part and Case 1 of the proof of Theorem 1) are computer-assisted. More details

on the computation can be found in Section 6, and detailed results of the computation and source codes are available at [24] and [25]. Finally, in Section 6, we briefly update the discussion of remaining open cases in the characterization of forbidden pairs for Hamilton-connectedness from [13] and [20].

3 Preliminaries

In Subsections 3.1 – 3.4, we summarize some known facts that will be needed in the proof of Theorem 1.

3.1 Line graphs of multigraphs and their preimages

The following characterization of line graphs of multigraphs was proved by Bermond and Meyer [2] (see also Zverovich [23]).

Theorem B [2]. *A graph G is a line graph of a multigraph if and only if G does not contain a copy of any of the graphs in Figure 3 as an induced subgraph.*

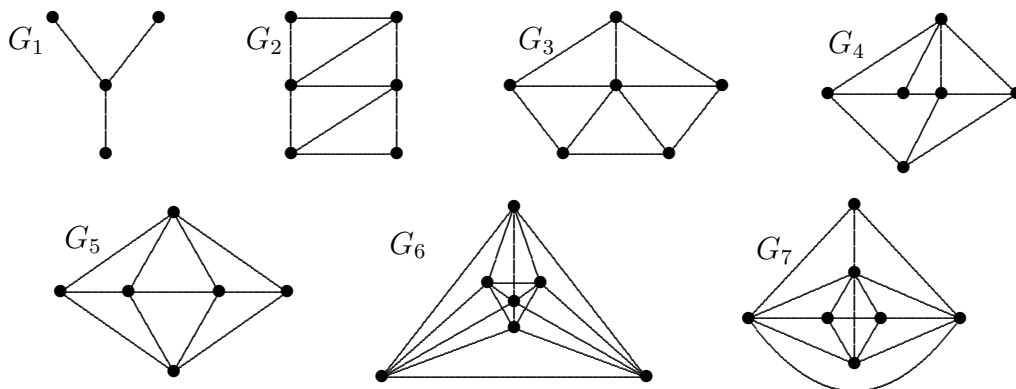


Figure 3: Forbidden subgraphs for line graphs of multigraphs

While in line graphs of graphs, for a connected line graph G , the graph H such that $G = L(H)$ is uniquely determined with a single exception of $G = K_3$, in line graphs of multigraphs this is not true: a simple example is the graphs $H_1 = Z_1$ and H_2 a double edge with one pendant edge attached to each vertex – while $H_1 \not\cong H_2$, we have $L(H_1) \cong L(H_2)$. Using a modification of an approach from [23], the following was proved in [17].

Theorem C [17]. *Let G be a connected line graph of a multigraph. Then there is, up to an isomorphism, a uniquely determined multigraph H such that $G = L(H)$ and a vertex $e \in V(G)$ is simplicial in G if and only if the corresponding edge $e \in E(H)$ is a pendant edge in H .*

The multigraph H with the properties given in Theorem C will be called the *preimage* of a line graph G and denoted $H = L^{-1}(G)$. We will also use the notation $a = L(e)$ and $e = L^{-1}(a)$ for an edge $e \in E(H)$ and the corresponding vertex $a \in V(G)$.

An edge-cut $R \subset E(H)$ of a multigraph H is *essential* if $H - R$ has at least two nontrivial components, and H is *essentially k -edge-connected* if every essential edge-cut of H is of size at least k . It is a well-known fact that a line graph G is k -connected if and only if $L^{-1}(G)$ is essentially k -edge-connected. It is also a well-known fact that if X is a line graph, then a line graph G is X -free if and only if $L^{-1}(G)$ does not contain as a subgraph (not necessarily induced) a graph F such that $L(F) = X$.

Note that in the special case of the graph Γ_3 , there are three nonisomorphic multigraphs F_1, F_2, F_3 such that $L(F_i) = \Gamma_3$, $i = 1, 2, 3$, see Fig. 4. It is straightforward to verify that if $G = \Gamma_3$, then $L^{-1}(G) = F_1$, however, if G contains $F \simeq \Gamma_3$ as a proper induced subgraph, then the corresponding subgraph $L^{-1}(F)$ can be any of F_1, F_2, F_3 (for example, if G is obtained from Γ_3 by adding to $F = \Gamma_3$ new vertices u_1, u_2 and edges u_1u_2, u_2t_1, u_2t_2 , then in $L^{-1}(G)$, $L^{-1}(F)$ is the multigraph F_2).

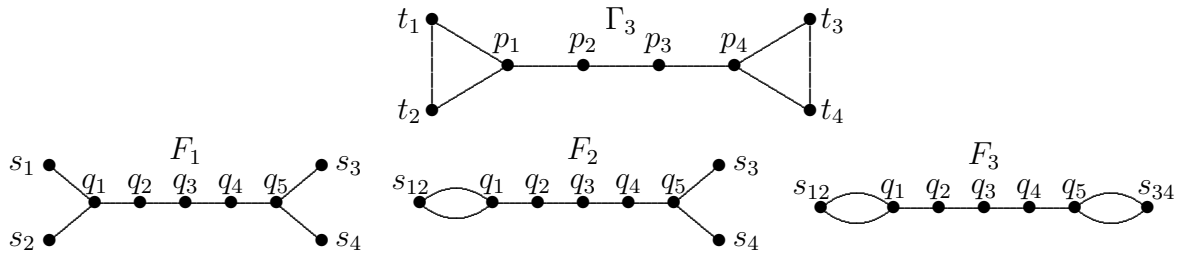


Figure 4: The graph Γ_3 and its three preimages

Recall that a closed trail T is a *dominating closed trail* (abbreviated DCT) if T dominates all edges of G , and an (e, f) -trail is an *internally dominating (e, f) -trail* (abbreviated (e, f) -IDT) if $\text{Int}(T)$ dominates all edges of G . Harary and Nash-Williams [9] established a correspondence between a DCT in H and a hamiltonian cycle in $L(H)$. A similar result showing that $G = L(H)$ is Hamilton-connected if and only if H has an (e_1, e_2) -IDT for any pair of edges $e_1, e_2 \in E(H)$, was given in [11] (in fact, part (ii) of the following theorem is slightly stronger than the result from [11], and its easy proof is given in [12]). Note that these results were proved for line graphs of graphs but it is easy to verify that they remain true also for line graphs of multigraphs.

Theorem D [9, 11]. *Let H be a multigraph with $|E(H)| \geq 3$ and let $G = L(H)$.*

- (i) [9] *The graph G is hamiltonian if and only if H has a DCT.*
- (ii) [11] *For every $e_i \in E(H)$ and $a_i = L(e_i)$, $i = 1, 2$, G has a hamiltonian (a_1, a_2) -path if and only if H has an (e_1, e_2) -IDT.*

3.2 Strongly spanning trailable multigraphs

A multigraph H is *strongly spanning trailable* if for any $e_1 = u_1v_1, e_2 = u_2v_2 \in E(H)$ (possibly $e_1 = e_2$), the multigraph $H(e_1, e_2)$, which is obtained from H by replacing the edge e_1 by a path $u_1v_{e_1}v_1$ and the edge e_2 by a path $u_2v_{e_2}v_2$, has a spanning (v_{e_1}, v_{e_2}) -trail.

By Theorem D(ii), it is straightforward to see that if H is strongly spanning trailable, then $G = L(H)$ is Hamilton-connected.

We will need the following two results on “small” strongly spanning trailable multigraphs from [14]. Here, \mathbb{W} is the set of multigraphs that are obtained from the Wagner graph W by subdividing one of its edges and adding at least one edge between the new vertex and exactly one of its neighbors.

Theorem E [14].

- (i) Every 2-connected 3-edge-connected multigraph H with circumference $c(H) \leq 8$ other than the Wagner graph W is strongly spanning trailable.
- (ii) Every 3-edge-connected multigraph H with $|V(H)| \leq 9$ such that $H \notin \{W\} \cup \mathbb{W}$ is strongly spanning trailable.

3.3 The core of the preimage of a 3-connected line graph

To avoid difficulties that can occur with the core of a multigraph, we define the core only for the case we need, i.e., for the preimage of a 3-connected line graph (then e.g. vertices of degree 2 are independent by the connectivity assumption, and pendant multiedges cannot occur by Theorem C).

Thus, let G be a 3-connected line graph and let $H = L^{-1}(G)$. The *core* of H is the multigraph $\text{co}(H)$ obtained from H by removing all pendant edges and suppressing all vertices of degree 2.

Shao [22] proved the following properties of the core of a multigraph.

Theorem F [22]. *Let H be an essentially 3-edge-connected multigraph. Then*

- (i) $\text{co}(H)$ is uniquely determined,
- (ii) $\text{co}(H)$ is 3-edge-connected,
- (iii) $V(\text{co}(H))$ dominates all edges of H ,
- (iv) if $\text{co}(H)$ has a spanning closed trail, then H has a DCT,
- (v) if $\text{co}(H)$ is strongly spanning trailable, then $L(H)$ is Hamilton-connected.

3.4 Closure operations

For $x \in V(G)$, the *local completion* of G at x is the graph $G_x^* = (V(G), E(G) \cup \{y_1y_2 \mid y_1, y_2 \in N_G(x)\})$ (i.e., G_x^* is obtained from G by adding all the missing edges with both vertices in $N_G(x)$). In this context, the edges in $E(G_x^*) \setminus E(G)$ will be referred to *new edges*, and the edges in $E(G)$ are *old*. Obviously, if G is claw-free, then so is G_x^* . Note that in the special case when G is a line graph and $H = L^{-1}(G)$, G_x^* is the line graph of the multigraph $H|_e$ obtained from H by contracting the edge $e = L^{-1}(x)$ into a vertex and replacing the created loop(s) by pendant edge(s) (Thus, if $G = L(H)$ and $x = L(e)$, then $G_x^* = L(H|_e)$).

Also note that clearly $x \in V_{SI}(G_x^*)$ for any $x \in V(G)$, and, more generally, $V_{SI}(G) \subset V_{SI}(G_x^*)$ for any $x \in V(G)$.

We say that a vertex $x \in V(G)$ is *eligible* if $\langle N_G(x) \rangle_G$ is a connected noncomplete graph, and we use $V_{EL}(G)$ to denote the set of all eligible vertices of G . Note that in the special case when G is a line graph and $H = L^{-1}(G)$, it is not difficult to observe that $x \in V(G)$ is eligible if and only if the edge $L^{-1}(x)$ is in a triangle or in a multiple edge of H . Based on the fact

that if G is claw-free and $x \in V_{EL}(G)$, then G_x^* is hamiltonian if and only if G is hamiltonian, the *closure* $\text{cl}(G)$ of a claw-free graph G was defined in [15] as the graph obtained from G by recursively performing the local completion operation at eligible vertices, as long as this is possible (more precisely: $\text{cl}(G) = G_k$, where G_1, \dots, G_k is a sequence of graphs such that $G_1 = G$, $G_{i+1} = (G_i)_{x_i}^*$ for some $x_i \in V_{EL}(G)$, $i = 1, \dots, k-1$, and $V_{EL}(G_k) = \emptyset$). The closure $\text{cl}(G)$ of a claw-free graph G is uniquely determined, is a line graph of a triangle-free graph, and is hamiltonian if and only if so is G . However, as observed in [5], the closure operation does not preserve the (non-)Hamilton-connectedness of G .

To handle this problem, the closure concept was strengthened in [12] by omitting the eligibility assumption for the application of the local completion operation. Specifically, for a given claw-free graph G , we construct a graph G^U by the following construction.

- (i) If G is Hamilton-connected, we set $G^U = K_{|V(G)|}$.
- (ii) If G is not Hamilton-connected, we recursively perform the local completion operation at such vertices for which the resulting graph is still not Hamilton-connected, as long as this is possible. We obtain a sequence of graphs G_1, \dots, G_k such that
 - $G_1 = G$,
 - $G_{i+1} = (G_i)_{x_i}^*$ for some $x_i \in V(G_i)$, $i = 1, \dots, k-1$,
 - G_k has no hamiltonian (a, b) -path for some $a, b \in V(G_k)$,
 - for any $x \in V(G_k)$, $(G_k)_x^*$ is Hamilton-connected,
and we set $G^U = G_k$.

A graph G^U obtained by the above construction is called an *ultimate M -closure* (or briefly a *UM-closure*) of the graph G , and a graph G equal to its UM-closure is said to be *UM-closed*.

The following theorem summarizes basic properties of the UM-closure operation.

Theorem G [12]. *Let G be a claw-free graph and let G^U be one of its UM-closures. Then G^U has the following properties:*

- (i) $V(G) = V(G^U)$ and $E(G) \subset E(G^U)$,
- (ii) G^U is obtained from G by a sequence of local completions at vertices,
- (iii) G is Hamilton-connected if and only if G^U is Hamilton-connected,
- (iv) if G is Hamilton-connected, then $G^U = K_{|V(G)|}$,
- (v) if G is not Hamilton-connected, then $(G^U)_x^*$ is Hamilton-connected for any $x \in V(G^U)$,
- (vi) $G^U = L(H)$, where $\text{co}(H)$ contains no diamond, no multitriangle and no triple edge, and either
 - (α) at most 2 triangles and no multiedge, or
 - (β) no triangle, at most one double edge and no other multiedge, and if $\text{co}(H)$ contains a double edge, then this double edge is also in H ,
- (vii) if G^U contains no hamiltonian (a, b) -path for some $a, b \in V(G^U)$ and
 - (α) X is a triangle in $\text{co}(H)$, then $E(X) \cap \{L_{G^U}^{-1}(a), L_{G^U}^{-1}(b)\} \neq \emptyset$,
 - (β) X is a multiedge in $\text{co}(H)$, then $E(X) = \{L_{G^U}^{-1}(a), L_{G^U}^{-1}(b)\}$.

We will also need the following lemma from [18].

Lemma H [18]. *Let G be an SM-closed graph and let $H = L^{-1}(G)$. Then H does not contain a triangle with a vertex of degree 2 in H .*

Note that Lemma H was proved in [18] for SM-closed graphs (which we do not define here), but since every UM-closed graph is also SM-closed (see e.g. [12]), it is true also for UM-closed graphs.

4 Γ_3 -closure

The UM-closure operation preserves Hamilton-connectedness, but there is still a problem that the local completion G_x^* of a $\{K_{1,3}, \Gamma_3\}$ -free graph G is not necessarily Γ_3 -free. To handle this problem, we define the concept of a Γ_3 -closure G^{Γ_3} of a $\{K_{1,3}, \Gamma_3\}$ -free graph G . For a set $M = \{x_1, x_2, \dots, x_k\} \subset V(G)$, we set $G_M^* = ((G_{x_1}^*)_{x_2}^* \dots)_{x_k}^*$. It is implicit in the proof of uniqueness of $\text{cl}(G)$ in [15] (and easy to see) that, for a given set $M = \{x_1, x_2, \dots, x_k\} \subset V(G)$, G_M^* is uniquely determined (i.e., does not depend on the order of the vertices x_1, x_2, \dots, x_k used during the construction).

If G is not Hamilton-connected, then a vertex $x \in V_{NS}(G)$, for which the graph G_x^* is still not Hamilton-connected, is said to be *feasible in G* . A set of vertices $M \subset V(G)$ is said to be *feasible in G* if the vertices in M can be ordered in a sequence x_1, \dots, x_k such that x_1 is feasible in $G_0 = G$, and x_{i+1} is feasible in $G_i = (G_{i-1})_{x_i}^*$, $i = 1, \dots, k-1$. Thus, if $M \subset V(G)$ is feasible, then $M \subset V_{SI}(G_M^*)$, but G_M^* is still not Hamilton-connected.

Note that it is possible that some two vertices x, y of a graph G are feasible in G , but x is not feasible in G_y^* (for example, if H is obtained from the Petersen graph by adding a pendant edge to each vertex, subdividing a nonpendant edge x_1x_2 with a vertex w , replacing each of the edges x_iw with a double edge, and if $G = L(H)$ and $x'_i, x''_i \in V(G)$ correspond to the two edges joining x_i and w in H , $i = 1, 2$, then G is not Hamilton-connected, each of the vertices x'_i, x''_i is feasible in G , $i = 1, 2$, but e.g. x'_1 and x''_1 are not feasible in $G_{x'_2}^* \simeq G_{x''_2}^*$). Thus, the recursive form of the definition is essential for verifying feasibility of a set $M \subset V(G)$ (although the resulting graph G_M^* does not depend on their order).

Recall that in the special case when $G = L(H)$, a local completion at a vertex $x \in V(G)$ corresponds to the contraction of the corresponding edge $e = L^{-1}(x)$. In this case, when $x = L(e)$ is feasible in $G = L(H)$, we also say that the edge $e \in E(H)$ is *contractible in H* , and, similarly, if a set $M \subset V(G)$ is feasible, then the corresponding set of edges $L^{-1}(M) \subset E(H)$ is said to be contractible in H .

Now, for a $\{K_{1,3}, \Gamma_3\}$ -free graph G , we define its Γ_3 -closure G^{Γ_3} by the following construction.

- (i) If G is Hamilton-connected, we define G^{Γ_3} as the complete graph.
- (ii) If G is not Hamilton-connected, we recursively perform the local completion operation at such feasible sets of vertices for which the resulting graph is still Γ_3 -free, as long as this is possible. We obtain a sequence of graphs G_1, \dots, G_k such that
 - $G_1 = G$,
 - $G_{i+1} = (G_i)_{M_i}^*$ for some set $M_i \subset V(G_i)$, $i = 1, \dots, k-1$,
 - G_k has no hamiltonian (a, b) -path for some $a, b \in V(G_k)$,
 - for any feasible set $M \subset V_{NS}(G_k)$, $(G_k)_M^*$ contains an induced subgraph isomorphic to Γ_3 ,

and we set $G^{\Gamma_3} = G_k$.

A resulting graph G^{Γ_3} is called a Γ_3 -closure of the graph G , and a graph G equal to (some) its Γ_3 -closure is said to be Γ_3 -closed. Note that for a given graph G , its Γ_3 -closure is not uniquely determined.

The following two theorems give basic properties of the Γ_3 -closure operation, proved in [10] (for the graphs W_5 , W_4 , P_6^2 and P_6^{2+} , see Fig. 5).

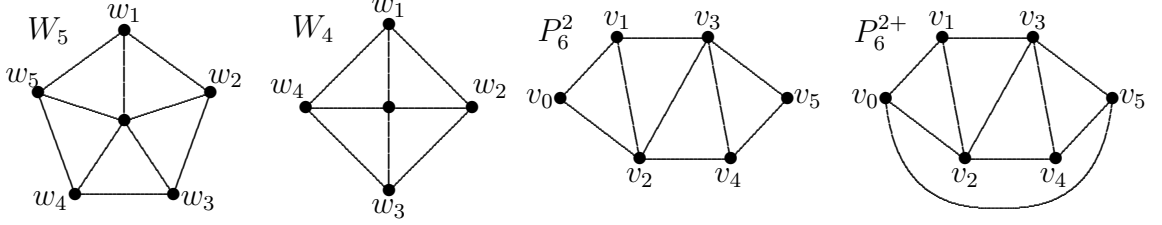


Figure 5: The 5-wheel W_5 , the 4-wheel W_4 , and the graphs P_6^2 and P_6^{2+}

Theorem I [10]. *Let G be a $\{K_{1,3}, \Gamma_3\}$ -free graph and let G^{Γ_3} be its Γ_3 -closure. Then G^{Γ_3} is $\{K_{1,3}, W_5, W_4, P_6^2, P_6^{2+}\}$ -free.*

To show that G^{Γ_3} is a line graph of a multigraph, by Theorem B, it is sufficient to show that G^{Γ_3} does not contain as an induced subgraph any of the graphs G_1, \dots, G_7 of Figure 3. Since $G_1 \simeq K_{1,3}$, $G_3 \simeq W_5$, each of the graphs G_5, G_6, G_7 contains an induced W_4 , $G_2 \simeq P_6^2$, and $G_4 \simeq P_6^{2+}$, Theorem I immediately implies the following crucial fact.

Theorem J [10]. *Let G be a $\{K_{1,3}, \Gamma_3\}$ -free graph and let G^{Γ_3} be its Γ_3 -closure. Then there is a multigraph H such that $G^{\Gamma_3} = L(H)$.*

Further structural properties of a Γ_3 -closure of a graph and of its preimage will be shown in Section 5 (Lemma 2 and Lemma 3).

The following results will be useful to identify feasible vertices.

Theorem K [16]. *Let G be a claw-free graph and let $x \in V(G)$ be locally 2-connected in G . Then G is Hamilton-connected if and only if G_x^* is Hamilton-connected.*

Thus, in our terminology, Theorem K says that a locally 2-connected vertex is feasible.

Lemma L [17]. *Let G be a claw-free graph, $x \in V(G)$, and let $H \stackrel{IND}{\subset} \langle N_G(x) \rangle_G$ be a 2-connected graph containing two disjoint pairs of independent vertices. Then x is locally 2-connected in G .*

Note that if a vertex $x \in V(G)$ is feasible by virtue of Theorem K (i.e., x is locally 2-connected in G), then, for any $y \in V(G)$ $y \neq x$, x is locally 2-connected also in G_y^* , but $\langle N_{G_y^*}(x) \rangle_{G_y^*}$ can be complete (if $N_G(x) \subset N_G(y)$). Thus, for any $y \in V(G)$, x is feasible or simplicial in G_y^* .

We thus define more generally: a set $M \subset V(G)$ is *weakly feasible in G* if the vertices in M can be ordered in a sequence x_1, \dots, x_k such that x_1 is feasible in $G_0 = G$, and x_{i+1} is feasible or simplicial in $G_i = (G_{i-1})_{x_i}^*$, $i = 1, \dots, k-1$. Thus, similarly, if G is not Hamilton-connected and $M \subset V(G)$ is weakly feasible in G , then G_M^* is still not Hamilton-connected and all vertices of M are simplicial in G_M^* .

5 Proof of Theorem 1

In the proof, we will need the following three lemmas. Let D be the diamond (see Fig. 6(b)), and D^1 and D^2 the diamond in which one or two of the edges ac_i , $i = 1, 2$, are subdivided, respectively (see Fig. 6(c), (d)). We will use the labeling of the vertices of D , D^1 and D^2 as in Fig. 6.

Lemma 2. *Let G be a 3-connected $\{K_{1,3}, \Gamma_3\}$ -free graph that is not Hamilton-connected, let \bar{G} be its Γ_3 -closure, and let $H = L^{-1}(\bar{G})$. Then*

- (i) H does not contain as a subgraph the diamond D ,
- (ii) for any triangle $T \subset H$, every vertex $x \in V(T)$ has a neighbor in $V(H) \setminus V(T)$,
- (iii) H contains as a subgraph neither the graph D^1 such that $N_H(d_1) = \{a, c_1\}$, nor the graph D^2 such that $N_H(d_i) = \{a, c_i\}$, $i = 1, 2$.

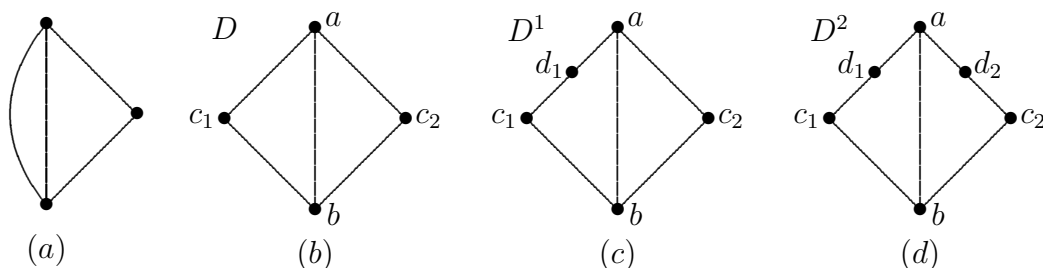


Figure 6: The multitrangle, the diamond D , and the graphs D^1 and D^2 .

Proof. (i) If H contains a subgraph $F \simeq D$, then, since $L(D) = W_4$, \bar{G} is not W_4 -free, contradicting Proposition I.

(ii) Let $T = u_1u_2u_3$ be a triangle in H , denote $e_1 = u_1u_2$, $e_2 = u_2u_3$, $e_3 = u_3u_1$, $v_i = L(e_i)$, $i = 1, 2, 3$, and suppose, to the contrary, that $N_H(u_1) = \{u_2, u_3\}$. By Theorem G(vi) and Lemma H, \bar{G} is not UM-closed, hence some of the vertices v_1, v_2, v_3 is feasible in \bar{G} , implying that some of the edges e_1, e_2, e_3 is contractible in H . By the definition of the Γ_3 -closure, some $\bar{G}_{v_i}^*$, $i \in \{1, 2, 3\}$, contains an induced Γ_3 , i.e., the corresponding subgraph $H|_{e_i}$ contains an $L^{-1}(\Gamma_3)$. We will use the labeling of the vertices of Γ_3 and $L^{-1}(\Gamma_3)$ as in Fig. 4.

By symmetry, it is sufficient to consider the cases when v_1 or v_2 is feasible.

Case 1: v_1 is feasible in \bar{G} .

Then the edge $e_1 = u_1u_2$ is contractible in H , and $H|_{e_1}$ contains a subgraph F such that $L(F) = \Gamma_3$, i.e., F is one of the three graphs in Fig. 4.

Let $u_{12} \in V(H|_{e_1})$ be the vertex obtained by identifying u_1 and u_2 . Clearly, $u_{12} \in V(F)$ (otherwise also $F \subset H$), and at least one of the edges $u_{12}u_3$ is in $E(F)$ (since $N_H(u_1) = \{u_2, u_3\}$). If $u_{12}u_3 = q_iq_{i+1}$ for some $i \in \{1, 2, 3, 4\}$, then also $F \subset H$, a contradiction. Hence, up to a symmetry, $u_{12}u_3 = s_{12}q_1$. Then, replacing in F the edges $(u_{12}u_3)^1, (u_{12}u_3)^2$ by u_1u_3, u_2u_3 (if $q_1 = u_3$), or by u_1u_2, u_1u_3 (if $q_1 = u_{12}$), we have an $L^{-1}(\Gamma_3)$ in H , a contradiction.

Case 2: v_2 is feasible in \bar{G} .

Then the edge $e_2 = u_2u_3$ is contractible in H and $H|_{e_2}$ contains a subgraph $F \simeq L^{-1}(\Gamma_3)$. Since $N_{H|_{e_1}}(u_1) = \{u_{23}\}$, by symmetry, $u_{23} \in \{q_1, q_2, q_3\}$.

If $u_{23} = q_1$, then it is straightforward to obtain an $L^{-1}(\Gamma_3)$ in H .

Let $u_{23} = q_2$. Then, replacing in F the edges q_1q_2 and q_1s_1, q_1s_2 (or $(q_1s_{12})^1, (q_1s_{12})^2$) by the edges u_2u_3, u_2u_1, u_2q_1 (if $u_3 = q_2$), or u_3u_2, u_3u_1, u_3q_1 (if $u_2 = q_2$), we have an $L^{-1}(\Gamma_3)$ in H .

Finally, let $u_{23} = q_3$. By symmetry, choose the notation such that, in H , $q_2u_2 \in E(H)$ and $q_4u_3 \in E(H)$. Since $N_H(u_1) = \{u_2, u_3\}$ and $\{q_2u_2, u_3q_4\}$ cannot be an essential edge-cut in H , u_2 or u_3 has a neighbor $w \in V(H) \setminus (V(F) \cup \{u_1, u_2, u_3\})$. By symmetry, let $u_3w \in E(H)$. Then the edges $(s_{12}q_1)^1, (s_{12}q_1)^2$ (or s_1q_1, s_2q_1) and $q_1q_2, q_2u_2, u_2u_1, u_1u_3, u_3q_4, u_3w$ determine an $L^{-1}(\Gamma_3)$ in H , a contradiction.

(iii) Suppose that H contains a subgraph $F \in \{D^1, D^2\}$ such that $N_H(d_1) = \{a, c_1\}$, and if $F = D^2$, then also $N_H(d_2) = \{a, c_2\}$.

We first show that, in each of the cases, the subgraph F is contractible to a double edge with some pendant edges (i.e., the subgraph $F \subset H$ contains some contractible subgraphs such that their contraction turns F into a double edge plus some pendant edges). This will imply that $L(H)$ is not UM-closed, implying that $L(\tilde{H})$ (where \tilde{H} is the multigraph obtained from H by the contractions) is still not Hamilton-connected. Consequently, $L(\tilde{H})$ must contain an induced Γ_3 , and we show that this is not possible.

Case 1: the edges ad_1, c_1d_1 , and ad_2, c_2d_2 if $F = D^2$, are simple.

Then $\text{co}(H)$ contains a diamond, thus, by Theorem G(vi), $L(H)$ is not UM-closed, implying that some edge $f \in E(F)$ is contractible.

Subcase 1.1: $F = D^1$.

First, if $f \in \{ad_1, c_1d_1, c_1b\}$, then $F_1 = F|_f$ is a diamond, hence $L(ab)$ has a 2-connected neighborhood in $H_1 = H|_f$, implying that ab is contractible in H_1 . Then, in $H_2 = H_1|_{ab}$, F contracts to two double edges. By Theorem G(vi)(β), one of the double edges is contractible, and its contraction yields a double edge (plus some pendant edges).

Secondly, if $f = ab$, then $F|_f$ is a triangle with a vertex of degree 2 plus a double edge. By Theorem G, some of the edges of the triangle is contractible and the contraction yields a double edge (plus some pendant edges).

Finally, if $f \in \{ac_2, bc_2\}$, then $F_1 = F|_f$ is the graph with edges ad_1, c_1d_1, c_1b and a double edge ab , and since d_1 is of degree 2, F_1 corresponds to a multitriangle in $\text{co}(H|_f)$. Thus, by Theorem G(vi), some edge $f_1 \in E(F_1)$ is contractible in $H_1 = H|_f$. If $f_1 \in \{ad_1, c_1d_1, c_1b\}$, then $F_2 = F_1|_{f_1}$ is a multitriangle, ab is contractible in $H_2 = H_1|_{f_1}$ (since it has 2-connected neighborhood), and $F_2|_{ab}$ yields a double edge. If $f_1 = ab$, then $F_2 = F_1|_{ab}$ is a triangle with a vertex of degree 2, and a contraction of any of its edges (by Theorem K) yields again a double edge (plus pendant edges).

Subcase 1.2: $F = D^2$.

If $f \in E(F) \setminus \{ab\}$, then $F_1 = F|_f \simeq D^1$ and we are in some of the previous cases. Thus,

let $f = ab$. Then $F_1 = F|_f$ consists of two triangles, each with a vertex of degree 2, which by Theorem G and Lemma H yields again a double edge.

Case 2: *some of the edges ad_1, c_1d_1 , and ad_2, c_2d_2 if $F = D^2$, is multiple in H .*

In this case, we have the following fact.

Claim. *Each of the edges ad_1, c_1d_1 , and ad_2, c_2d_2 if $F = D^2$, that is multiple in H , is contractible.*

Proof. Suppose that some $f \in \{ad_1, c_1d_1\}$ is multiple but not contractible.

First observe that the edge ab is not contractible, since otherwise, in $F|_{ab}$, the edges ad_1, c_1d_1 and c_1b create a multitriangle in which f is in a multiple edge, hence $L(f)$ has 2-connected neighborhood and f is contractible, a contradiction. Thus, ab is not contractible, specifically, ab is simple.

If both ad_1 and c_1d_1 are multiple, then one of them is contractible (by Theorem G(vi)), and its contraction yields again a multitriangle with f in multiple edge, implying contractibility of f , a contradiction. Thus, exactly one of the edges ad_1, c_1d_1 is multiple.

Now, if $F = D^1$, then abc_2 is a triangle, and since f is a noncontractible multiedge, some of the edges ac_2, bc_2 is contractible, implying contractibility of ab , a contradiction. Thus, $F = D^2$.

If none of the edges ad_2, c_2d_2 is multiple, then abc_2 is a triangle in $\text{co}(H)$. Since f is a noncontractible multiedge, some of the edges ad_2, c_2d_2 is contractible, implying again contractibility of ab , a contradiction. Thus, some edge $f_1 \in \{ad_2, c_2d_2\}$ is multiple. But then again, by Theorem G(vi), f_1 is contractible, implying, as before, contractibility of ab , a contradiction. \square

We summarize that some edges of F are multiple, and each multiple edge of F is contractible. But now, contracting some of the multiple edges of F , in each of the cases, we are in some of the previous cases. Thus, we conclude that F can be contracted to a double edge plus some pendant edges, and for the resulting multigraph \tilde{H} , $L(\tilde{H})$ is still not Hamilton-connected.

By the definition of the Γ_3 -closure, \tilde{H} contains a subgraph $\tilde{F} \simeq L^{-1}(\Gamma_3)$. Moreover, observe that, in each of the cases, the contraction of F to a double edge contracts some of the two “subdivided triangles” of F to a vertex (plus some pendant edges). More specifically, either the subgraph F^1 with edges ad_1, d_1c_1, c_1b and ab , or the subgraph F^2 with edges ad_2, d_2c_2 (or ac_2 if $F = D^1$), c_2b and ab , is contracted to a vertex plus pendant edges, and this turns F into a double edge (plus some pendant edges). Thus, at most one of the edges of F is an edge of \tilde{F} . Denote the vertices of \tilde{F} as in Fig. 4.

We will consider the vertex of \tilde{H} resulting from the contraction of F^1 or F^2 as a “new” vertex, and will distinguish cases according to which of the vertices of \tilde{F} is new, and, subject to this, which of the edges q_iq_{i+1} , $i = 1, 2, 3, 4$, is an edge of F . To reach a contradiction, in each of the cases, we will list edges of an $L^{-1}(\Gamma_3)$ in H . We will also list which part of F (i.e., F^1 or F^2) corresponds to the new vertex, and its vertices that are used in the $L^{-1}(\Gamma_3)$ in H . The cases and subcases are distinguished up to a symmetry, and, in symmetric situations, we will always list the possibility that F^1 is contracted (note that if $F = D^1$, it is possible that

the $L^{-1}(\Gamma_3)$ in H uses the edge ad_1 or c_1d_1 when F^1 is contracted, while it uses the edge ab if F^2 is contracted, and we consider these situations also symmetric). In all cases, when the edges q_1s_1 and q_1s_2 , or q_5s_3 and q_5s_4 are used, it is always implicitly understood that there is also a possibility of a double edge q_1s_{12} or q_5s_{34} .

Case 1: q_1 is new.

Then possibly $E(F) \cap E(\tilde{F}) = \emptyset$ or $q_1q_2 \in E(F)$, and (up to a symmetry) $q_1 \in \{c_1, b\}$ if $q_1q_2 \notin E(F)$, and $q_1 = b$ if $q_1q_2 \in E(F)$. We thus have the following possibilities.

$E(F) \cap E(\tilde{F})$	q_1	edges of an $L^{-1}(\Gamma_3)$ in H
\emptyset	$F^1; c_1$	$c_1d_1, c_1b, c_1q_2, q_2q_3, q_3q_4, q_4q_5, q_5s_3, q_5s_4$
\emptyset	$F^1; b$	$bc_1, ba, bq_2, q_2q_3, q_3q_4, q_4q_5, q_5s_3, q_5s_4$
$q_1q_2 = bc_2$	$F^1; b$	$bc_1, ba, bc_2, c_2q_3, q_3q_4, q_4q_5, q_5s_3, q_5s_4$

Case 2: q_2 is new.

In this case, possibly $E(F) \cap E(\tilde{F}) = \emptyset$, $q_1q_2 \in E(F)$, or $q_2q_3 \in E(F)$, and we have the following possibilities.

$E(F) \cap E(\tilde{F})$	q_2	edges of an $L^{-1}(\Gamma_3)$ in H
\emptyset	$F^1; a, b$	$bc_1, bq_1, ba, aq_3, q_3q_4, q_4q_5, q_5s_3, q_5s_4$
$q_1q_2 = bc_2$	$F^1; b, c_1$	$bc_2, ba, bc_1, c_1q_3, q_3q_4, q_4q_5, q_5s_3, q_5s_4$
$q_1q_2 = bc_2$	$F^1; b$	$ad_1, ad_2(ac_2), ab, bq_3, q_3q_4, q_4q_5, q_5s_3, q_5s_4$
$q_2q_3 = bc_2$	$F^1; b, c_1$	$c_1d_1, c_1q_1, c_1b, bc_2, c_2q_4, q_4q_5, q_5s_3, q_5s_4$
$q_2q_3 = bc_1$	$F^2; a, b$	$aq_1, ab, ad_1, d_1c_1, c_1q_4, q_4q_5, q_5s_3, q_5s_4$

Case 3: q_3 is new.

In this case, possibly $E(F) \cap E(\tilde{F}) = \emptyset$, $q_2q_3 \in E(F)$, or $q_3q_4 \in E(F)$; however, the last two possibilities are symmetric and we therefore consider only the first of them.

Subcase 3.1: $q_2q_3 \in E(F)$.

Then we have the following possibilities.

$E(F) \cap E(\tilde{F})$	q_2	edges of an $L^{-1}(\Gamma_3)$ in H
$q_2q_3 = ac_2$	$F^1; a, c_1$	$c_2q_1, c_2d_2(c_2b), c_2a, ac_1, c_1q_4, q_4q_5, q_5s_3, q_5s_4$
$q_2q_3 = c_1b$	$F^2; a, b$	$c_1q_1, c_1b, c_1d_1, d_1a, aq_4, q_4q_5, q_5s_3, q_5s_4$

Subcase 3.2: $E(F) \cap E(\tilde{F}) = \emptyset$.

Since one of F^1, F^2 is contracted and d_1, d_2 have no neighbors outside F , the vertices q_2 and q_4 are adjacent in H either to a and b , or to one of a, b and one of c_1, c_2 . So, up to a symmetry, either $q_2a, q_4b \in E(H)$, or $N_H(q_2) \cap \{c_1, c_2\} \neq \emptyset$ and $N_H(q_4) \cap \{a, b\} \neq \emptyset$.

Subcase 3.2.1: $q_2a, q_4b \in E(H)$.

If $F = D^1$, then the edges $q_1s_1, q_1s_2, q_1q_2, q_2a, ac_2, c_2b, bq_4, bc_1$ determine an $L^{-1}(\Gamma_3)$ in H , a contradiction. Hence $F = D^2$.

If bc_1 is a double edge in H , then the edges $q_1s_1, q_1s_2, q_1q_2, q_2a, ad_1, d_1c_1, (c_1b)^1, (c_1b)^2$, and if ad_1 is double in H , then the edges $(ad_1)^1, (ad_1)^2, d_1c_1, c_1b, bq_4, q_4q_5, q_5s_3, q_5s_4$ determine an $L^{-1}(\Gamma_3)$ in H . Thus, $\mu_H(ad_1) = \mu_H(bc_1) = 1$. Since $\{ad_1, bc_1\}$ cannot be an essential edge-cut, c_1 has another neighbor $z \in V(H)$. We show that z cannot be any of the vertices s_1, s_2 (or s_{12}), q_1, q_2, a .

z	edges of an $L^{-1}(\Gamma_3)$ in H
s_1	$q_1s_2, q_1q_2, q_1s_1, s_1c_1, c_1b, ba, ad_1, ad_2$
s_{12}	$(s_{12}q_1)^1, (s_{12}q_1)^2, s_{12}c_1, c_1b, bq_4, q_4q_5, q_5s_3, q_5s_4$
q_1	$q_1s_1, q_1s_2, q_1c_1, c_1b, bq_4, q_4q_5, q_5s_3, q_5s_4$
q_2	$q_2q_1, q_2a, q_2c_1, c_1b, bq_4, q_4q_5, q_5s_3, q_5s_4$
a	$aq_2, ad_1, ac_1, c_1b, bq_4, q_4q_5, q_5s_3, q_5s_4$

But then the edges $q_1s_1, q_1s_2, q_1q_2, q_2a, ad_1, d_1c_1, c_1z, c_1b$ determine an $L^{-1}(\Gamma_3)$ in H , a contradiction.

Subcase 3.2.2: $N_H(q_2) \cap \{c_1, c_2\} \neq \emptyset$ and $N_H(q_4) \cap \{a, b\} \neq \emptyset$.

Let $\bar{q}_2 \in \{c_1, c_2\}$ and $\bar{q}_4 \in \{a, b\}$ denote the neighbor of q_2 or q_4 in F , respectively.

Suppose first that $\bar{q}_2 = c_1$. Then either $\bar{q}_4 = a$, or $\bar{q}_4 = b$.

\bar{q}_4	edges of an $L^{-1}(\Gamma_3)$ in H
a	$q_1s_1, q_1s_2, q_1q_2, q_2c_1, c_1d_1, d_1a, ab, aq_4$
b	$q_1s_1, q_1s_2, q_1q_2, q_2c_1, c_1d_1, d_1a, ab, ad_2(ac_2)$

We get a symmetric contradiction if $\bar{q}_2 = c_2$ and $F = D^2$. Thus, we have $\bar{q}_2 = c_2$ and $F = D^1$ (and either $\bar{q}_4 = a$, or $\bar{q}_4 = b$).

We now show that none of the edges ad_1, bc_1 can be a double edge. Suppose the opposite. Then we have the following possibilities.

double edge	edges of an $L^{-1}(\Gamma_3)$ in H
ad_1	$q_1s_1, q_1s_2, q_1q_2, q_2c_2, c_2b, ba, (ad_1)^1, (ad_1)^2$
bc_1	$q_1s_1, q_1s_2, q_1q_2, q_2c_2, c_2a, ab, (bc_1)^1, (bc_1)^2$

Thus, $\mu_H(ad_1) = \mu_H(bc_1) = 1$. Since $\{ad_1, bc_1\}$ cannot be an essential edge-cut, c_1 must have another neighbor $z \in V(H)$. Clearly $z \neq c_2$ (since then we would have a diamond in H). We show that z cannot be any of the vertices s_1, s_2 (or s_{12}), q_1, q_2 .

z	edges of an $L^{-1}(\Gamma_3)$ in H
s_1	$q_1s_2, q_1q_2, q_1s_1, s_1c_1, c_1b, ba, ad_1, ac_2$
s_{12}	$(s_{12}q_1)^1, (s_{12}q_1)^2, s_{12}c_1, c_1d_1, d_1a, ac_2, c_2q_2, c_2b$
q_1	$q_1s_1, q_1s_2, q_1c_1, c_1d_1, d_1a, ac_2, c_2q_2, c_2b$
q_2	$q_1s_1, q_1s_2, q_1q_2, q_2c_1, c_1d_1, d_1a, ac_2, ab$

But then the edges $q_1s_1, q_1s_2, q_1q_2, q_2c_2, c_2b, bc_1, c_1d_1, c_1z$ determine an $L^{-1}(\Gamma_3)$ in H , a contradiction. ■

The next lemma will describe some structural properties of a minimal counterexample to Theorem 1. Here, we say that a graph is *minimal* with respect to a property \mathcal{P} , if G has \mathcal{P} , but for every vertex $x \in V(G)$, $G - x$ does not have \mathcal{P} .

For an integer $r \geq 2$, $\mathbb{K}_{2,r}^M$ will denote the family of multigraphs that can be obtained from the complete bipartite graph $K_{2,r} = (\{v_1, v_2\}, \{w_1, w_2, \dots, w_r\})$ by replacing at least one of the edges w_iv_1, w_iv_2 with a double edge (for an example, see Fig. 7(b)). Similarly, $\mathbb{K}_{rP_4}^M$ will denote the family of multigraphs obtained by identifying endvertices of r vertex-disjoint paths $P_4^i = v_1w_iz_iv_2$, $i = 1, 2, \dots, r$, and, for each $i = 1, 2, \dots, r$, by replacing at least one of the edges of the P_4^i with a double edge (for an example, see Fig. 7(c)).

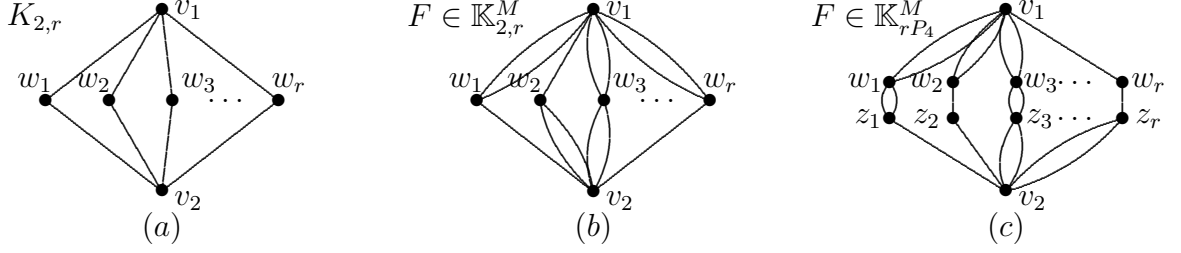


Figure 7: The graph $K_{2,r}$, a multigraph from $\mathbb{K}_{2,r}^M$ and a multigraph from $\mathbb{K}_{rP_4}^M$.

Lemma 3. *Let G be a minimal 3-connected $\{K_{1,3}, \Gamma_3\}$ -free non-Hamilton-connected graph, let \bar{G} be its Γ_3 -closure, and let $H = L^{-1}(\bar{G})$. Then*

- (i) every vertex $x \in V(H)$ is incident with at most two pendant edges,
- (ii) every edge $e \in E(H)$ that is in a cycle of length at least three has multiplicity $\mu(e) \leq 2$,
- (iii) H does not contain as a subgraph the graph $K_{2,4}$ such that its vertices of degree 2 are of degree 2 in H ,
- (iv) H does not contain as a subgraph a multigraph from $\mathbb{K}_{2,4}^M$ such that every vertex of $\bigcup_{i=1}^4 N_H(w_i) \setminus \{v_1, v_2\}$ has degree 1 in H ,
- (v) H does not contain as a subgraph a multigraph from $\mathbb{K}_{4P_4}^M$ such that $N_H(w_i) = \{v_1, z_i\}$ and $N_H(z_i) = \{v_2, w_i\}$, $i = 1, 2, 3, 4$.

Proof. (i) By the assumption of the lemma, H is essentially 3-edge-connected, not containing an $L^{-1}(\Gamma_3)$, and H does not have an (e, f) -IDT for some $e, f \in E(H)$. Suppose that H contains $s \geq 3$ pendant edges at a vertex $x \in V(H)$, choose $s - 2$ of them such that none of them is e or f , and let H' be obtained from H by removing the chosen $s - 2$ pendant edges. Then clearly H' is still essentially 3-edge-connected, not containing an $L^{-1}(\Gamma_3)$, and H' does not have an (e, f) -IDT. Since $|V(L(H'))| = |V(G)| - (s - 2) < |V(G)|$, we have a contradiction with the minimality of G .

(ii) Similarly, suppose that $\mu_H(e) = s \geq 3$ for some edge $e = ab \in E(H)$, let $e, f \in E(H)$ be such that H has no (e, f) -IDT, choose the notation such that none of the edges $(ab)^3, \dots, (ab)^s$ is any of e, f , and let H' be obtained by from H by removing the edges $(ab)^3, \dots, (ab)^s$. Then clearly H' contains no $L^{-1}(\Gamma_3)$, no (e, f) -IDT, and since $e = ab$ is in a cycle of length at least 3, H' is also essentially 3-edge-connected. Thus, the graph $G' = L(H')$ contradicts the minimality of G .

(iii) Suppose that H contains a subgraph $F \simeq K_{2,4} = (\{v_1, v_2\}, \{w_1, w_2, w_3, w_4\})$ such that $d_H(w_i) = 2$, $i = 1, 2, 3, 4$. Then in $\text{co}(H)$ we have $\mu(v_1v_2) = 4$, hence H is not UM-closed by Theorem G(vi). Thus, some of the edges w_iv_j is contractible. Choose the notation such that v_1w_1 is contractible and set $H_1 = H|_{v_1w_1}$ and $G_1 = L(H_1)$. Then, in G_1 , $\langle N_{G_1}(L(v_1v_2)) \rangle_{G_1}$ is 2-connected by Lemma L, hence v_1v_2 is contractible by Theorem K. Set $H_2 = H_1|_{v_1v_2}$ and $G_2 = L(H_2)$. Then, in H_2 , the whole F contracts to a single vertex plus 8 pendant edges, and G_2 is still 3-connected and not Hamilton-connected.

Let now H' be obtained from H by replacing the subgraph F with the graph $F' = K_{2,3} = (\{v_1, v_2\}, \{w_1, w_2, w_3\})$. Then, analogously, v_1v_2 is a triple edge in $\text{co}(H')$, v_1w_1 (say) is contractible in H' , and $\langle N_{G'_1}(L(v_1v_2)) \rangle_{G'_1}$ is 2-connected in $G'_1 = L(H'_1)$, where $H'_1 = H'|_{v_1w_1}$. In

$H'_2 = H'_1|_{v_1v_2}$ then F' contracts to a single vertex plus 6 pendant edges. Moreover, $G' = L(H')$ is 3-connected, and G' is Hamilton-connected if and only if $G'_2 = L(H'_2)$ is Hamilton-connected.

Since H_2 and H'_2 differ only in number of pendant edges at the vertex resulting from contracting F or F' , respectively, G'_2 is also not Hamilton-connected, implying G' is not Hamilton-connected. Since G' is an induced subgraph of G , G' is Γ_3 -free. Thus, G' contradicts the minimality of G .

(iv) Suppose that H contains a submultigraph $F \in \mathbb{K}_{2,4}^M$ satisfying the assumptions of the lemma. Note that, by part (ii), every edge of F has multiplicity 1 or 2, and, by the definition of $\mathbb{K}_{2,4}^M$, at least one of the edges w_iv_1, w_iv_2 has multiplicity 2, $i = 1, 2, 3, 4$. For each $i = 1, 2, 3, 4$, let f_i be one of the edges w_iv_1, w_iv_2 with $\mu(f_i) = 2$.

By Theorem G(vi)(β), some three of the edges f_1, f_2, f_3, f_4 are contractible, and we choose the notation such that the contractible edges are f_2, f_3 and f_4 .

Let H' be the multigraph obtained from H by removing the vertex w_4 and possibly the (by (i) at most two) its neighbors of degree 1. Observe that H' is also essentially 3-edge-connected.

Set $H_1 = H|_{\{f_2, f_3, f_4\}}$ and $H'_1 = H'|_{\{f_2, f_3\}}$. Then in H the submultigraph F contracts to a graph with vertices v_1, v_2, w_1 , one multiple edge v_1v_2 , and two edges w_1v_1 and w_1v_2 , at least one of them being multiple. In H' , F' contracts to the same structure with the only difference that $\mu_{H'}(v_1v_2) < \mu_H(v_1v_2)$. More specifically, since the multiple edge v_1v_2 results from 3 edges in H but 2 edges in H' , we have $\mu_H(v_1v_2) \geq \mu_{H'}(v_1v_2) + 1$.

Again by Theorem G(vi)(β), one of the multiple edges in contractible, resulting in an edge with multiplicity at least 3 (both in H and in H'), and another application of Theorem G(vi) contracts the whole F in H (or F' in H') to a single vertex with some pendant edges. Thus, the multigraphs $H|_F$ and $H'|_{F'}$ are isomorphic up to a different number of pendant edges at the vertex resulting from contracting F (or F'). Consequently, $L(H|_F)$ is Hamilton-connected if and only if $L(H'|_{F'})$ is Hamilton-connected, implying that G' is not Hamilton-connected. Since H' is essentially 3-edge-connected, G' is 3-connected, and G' is Γ_3 -free since $G' \stackrel{\text{IND}}{\subset} G$. Thus, the graph G' contradicts the minimality of G .

(v) Suppose that H contains a subgraph $F \in \mathbb{K}_{4P_4}^M$. First observe that if, say, w_1z_1 is a double edge, then the edges $(w_1z_1)^1, (w_1z_1)^2, w_1v_1, v_1w_2, w_2z_2, z_2v_2, v_2z_3, v_2z_4$ determine an $L^{-1}(\Gamma_3)$ in H , a contradiction. Thus, by symmetry, all edges w_iz_i , $i = 1, 2, 3, 4$, are simple edges.

By Theorem G(vi)(β), all multiple edges in F , except for possibly one, are contractible. Choose the notation such that the (possibly) noncontractible double edge is the edge v_1w_1 , and let \tilde{F} be the subgraph of F consisting of P_4^2, P_4^3 and P_4^4 . Let F_c be the set of contractible edges of F , and set $H_1 = H|_{F_c}$. Then, in H_1 , each of the paths P_4^i , $i = 2, 3, 4$, contracts either to an edge v_1v_2 plus 4 pendant edges (if $|E(P_4^i) \cap F_c| = 2$), or to a (v_1, v_2) -path of length 2 with interior vertex of degree 2, plus 2 pendant edges (if $|E(P_4^i) \cap F_c| = 1$). Thus, the subgraph $\tilde{F}_1 = \tilde{F}|_{F_c} \subset H_1$ corresponds in $\text{co}(H_1)$ to a triple edge v_1v_2 . By Theorem G(vi), H_1 is not UM-closed, hence some of the edges of \tilde{F}_1 , say, e , is contractible.

If $e = v_1v_2$, then in $H_2 = H_1|_e$ the whole \tilde{F} contracts to a single vertex plus some pendant edges. Otherwise, e is an edge of a (v_1, v_2) -path of length 2, and then in $H_2 = H_1|_e$ the (v_1, v_2) -path is replaced with an edge v_1v_2 (plus a pendant edge); and repeating the argument, we are in the first case. Thus, in each of the cases, the contractions result in the graph H_2 , in which \tilde{F} is contracted to a single vertex (plus pendant edges). The graph $G_2 = L(H_2)$ is

3-connected since clearly H_2 is essentially 3-edge-connected, and, by Theorem G(iii), G_2 is not Hamilton-connected.

Let now H' be obtained from H by replacing the subgraph F with $F' \in \mathbb{K}_{3P_4}^M$, and choose again the notation such that the (possibly) noncontractible multiedge is the double edge v_1w_1 . Let \tilde{F}' be the subgraph of F' consisting of P_4^2 and P_4^3 . Then clearly H' is also essentially 3-edge-connected, hence $G' = L(H')$ is 3-connected. Moreover, $H'|_{F'}$ is the same multigraph as H_2 , with only different number of pendant edges at the vertex resulting from contracting \tilde{F} (or \tilde{F}' , respectively). Consequently, $L(H'|_{F'})$ is not Hamilton-connected, hence $G' = L(H')$ is also not Hamilton-connected (since $L(H|_{F'})$ was obtained from G' by a series of local completions). Since G' is an induced subgraph of G , the graph G' is Γ_3 -free, hence G' contradicts the minimality of G . \blacksquare

The following lemma will be crucial in the proof of Theorem 1 for graphs containing a small cycle. In the lemma, C_9^M denotes the multigraph obtained from the cycle $C_9 = x_0x_1 \dots x_8$ by adding one parallel edge to each of the edges x_0x_1 , x_3x_4 and x_6x_7 , and \mathcal{F} denotes the finite family consisting of all multigraphs listed in the file `F.txt` available at [24]. We will follow the labeling of vertices of some special graphs as introduced in Figures 6 and 7.

Lemma 4. *Let G be a Γ_3 -free line graph of a multigraph and let $H = L^{-1}(G)$. Then $H \in \mathcal{F}$ if and only if H satisfies all conditions (1), \dots , (8) and every subgraph F of H satisfies each of conditions (9), \dots , (14):*

- (1) each vertex of H has at most two neighbors of degree 1,
- (2) each multiedge of H has multiplicity at most 2,
- (3) H contains C_k as a subgraph for some $k \in \{7, \dots, 10\}$,
- (4) H does not contain D as a subgraph,
- (5) H does not contain C_9^M as a flat subgraph,
- (6) at least 10 vertices of H have degree at least 3,
- (7) H is essentially 3-edge-connected,
- (8) H is essentially 2-connected,
- (9) if $F \simeq K_3$, then every vertex of F has at least three neighbors in H ,
- (10) if $F \simeq D^1$, then every vertex of F has at least three neighbors in H ,
- (11) if $F \simeq K_{2,4}$, then $\sum_{i=1}^4 |N_H(w_i)| > 8$,
- (12) if $F \simeq D^2$, then $|N_H(c_1)| + |N_H(c_2)| > 4$ and $|N_H(d_1)| + |N_H(d_2)| > 4$,
- (13) if $F \in \mathbb{K}_{4,P_4}^M$, then $\sum_{u \in V(F) \setminus \{v_1, v_2\}} |N_H(u)| > 16$,
- (14) if $F \in \mathbb{K}_{2,4}^M$, then some vertex of $\cup_{i=1}^4 N_H(w_i) \setminus \{v_1, v_2\}$ has degree at least 2 in H .

Proof. We prove the lemma with the help of a computer. To this end, we design an algorithm that essentially checks all possible candidates for the multigraph H and generates a plain text full proof of the lemma. The source code and the full proof can be found at [24]. Here, we explain the logic behind the algorithm and show that, indeed, it proves the lemma.

The general approach of the algorithm is to start with a small graph and test all relevant extensions, step by step, until it is clear that no multigraph obtained by further extensions can possibly satisfy the lemma. Given a multigraph M and sets $U \subset V(M)$ and $R \subset E_S(M)$, we consider two types of extensions denoted by $\mathcal{A}(M, U)$ and $\mathcal{M}(M, R)$ as follows:

- $\mathcal{A}(M, U)$ is the family of all possible multigraphs M^+ obtained from M by adding a vertex v such that v is incident with no multiedge in M^+ and $|N_{M^+}(v) \cap U| \geq 1$, and if $d_{M^+}(v) = 1$ then v has at most one twin in M^+ ,
- $\mathcal{M}(M, R)$ is the family of all possible multigraphs M^+ obtained from M by multiplying an edge of R .

We write $\mathcal{A}(M)$ as a short for $\mathcal{A}(M, V(M))$, and similarly $\mathcal{M}(M)$ for $\mathcal{M}(M, E_S(M))$.

For every $k \in \{7, \dots, 10\}$, the computer considers all graphs on k vertices containing C_k (but containing no C_ℓ subgraph where $7 \leq \ell \leq k - 1$). For each of these graphs, it tests all relevant extensions by calling `INVESTIGATE()`, see Algorithm 1.

Algorithm 1 Recurrent investigation of M , the algorithm branches on $\mathcal{B} \subset \mathcal{A}(M) \cup \mathcal{M}(M)$

```

1: procedure INVESTIGATE( $M$ )
2:   if  $L(M)$  is  $\Gamma_3$ -free and  $M$  contains no subgraph  $D$  and no flat subgraph  $C_9^M$  then
3:     if conditions (6),  $\dots$ , (14) are all satisfied then
4:       add  $M$  to F.txt
5:       set  $\mathcal{B} = \mathcal{A}(M) \cup \mathcal{M}(M)$ 
6:     else
7:       choose a violated condition  $p$  from (6),  $\dots$ , (14) at random
8:       set  $\mathcal{B} = \text{GET\_ALL\_SOLUTION\_ATTEMPTS}(M, p)$  ▷ see Algorithm 2
9:     for each  $M^+ \in \mathcal{B}$  do
10:      INVESTIGATE( $M^+$ ) ▷ recurrence on extended multigraphs

```

We should note that considering all extensions of $\mathcal{A}(M) \cup \mathcal{M}(M)$ in each iteration is not desirable since it leads to infinite families of multigraphs satisfying the condition on line 2 of Algorithm 1 (and hence the process never finishes). The key idea of the proof is to choose a particular violation of a condition from (6), \dots , (14) and to test just all extensions potentially helping to solve this violation (see lines 7 and 8 of Algorithm 1). It turns out that every branch of this investigation is finite since at some point each $M^+ \in \mathcal{B}$ fails to satisfy the condition on line 2 of Algorithm 1. The relevant extensions are obtained by calling `GET_ALL_SOLUTION_ATTEMPTS()`, see Algorithm 2.

We now show that the algorithm, indeed, proves the lemma. It suffices to show that a multigraph H is added to `F.txt` by the algorithm if and only if H completely satisfies the hypothesis of the lemma.

We first consider a multigraph H added to `F.txt` and we show that it satisfies the hypothesis of the lemma. We recall that H is obtained from a simple graph on k vertices containing C_k where $k \in \{7, \dots, 10\}$ by iteratively applying extensions $\mathcal{A}()$ and $\mathcal{M}()$, and we note that H has the following properties:

- H contains neither loops nor pendant multiedges,
- H satisfies conditions (1), (2) and (3).

In particular, we note that pendant simple edges of H precisely correspond to simplicial vertices of $L(H)$, and thus H is a preimage of a line graph of multigraph. Since H is added to `F.txt` at line 4 of Algorithm 1, H satisfies the conditions on lines 2 and 3 of Algorithm 1. We conclude that H is a preimage of a Γ_3 -free line graph of multigraph and H satisfies all conditions (1), \dots , (14).

Algorithm 2 Choose a particular problem of type p at random and return all relevant solution attempts

```

1: procedure GET_ALL_SOLUTION_ATTEMPTS( $M, p$ )
2:   if  $p$  is (6) then
3:     set  $R$  as the set of all edges from  $E_S(M)$  incident with a vertex of degree 2 in  $M$ 
4:     return  $\mathcal{A}(M) \cup \mathcal{M}(M, R)$ 
5:   if  $p$  is (7) then
6:     choose an essential 2-edge-cut  $\{e, f\}$ 
7:     if  $\{e, f\} \subset E_S(M)$  then
8:       choose a non-trivial component  $C$  of  $M - \{e, f\}$ 
9:       return  $\mathcal{A}(M, V(C)) \cup \mathcal{M}(M, \{e, f\})$ 
10:    else ▷  $M$  has an essential cutvertex
11:      reset  $p = (8)$ 
12:  if  $p$  is (8) then
13:    choose an essential cutvertex  $u$  and a non-trivial component  $C$  of  $M - u$ 
14:    return  $\mathcal{A}(M, V(C))$ 
15:  if  $p$  is (9) then
16:    choose a vertex  $u$  of a  $K_3$  subgraph such that  $|N_M(u)| = 2$ 
17:    return  $\mathcal{A}(M, \{u\})$ 
18:  if  $p$  is (10) then
19:    choose a vertex  $u$  of a  $D^1$  subgraph such that  $|N_M(u)| = 2$ 
20:    return  $\mathcal{A}(M, \{u\})$ 
21:  if  $p$  is (11) then
22:    choose a  $K_{2,4}$  subgraph such that  $\sum_{i=1}^4 |N_M(w_i)| = 8$ 
23:    return  $\mathcal{A}(M, \{w_1, \dots, w_4\})$ 
24:  if  $p$  is (12) then
25:    choose a  $D^2$  subgraph such that  $|N_M(c_1)| + |N_M(c_2)| = 4$  ▷ or  $d_1, d_2$  by symmetry
26:    return  $\mathcal{A}(M, \{c_1, c_2\})$ 
27:  if  $p$  is (13) then
28:    choose a subgraph  $F \in \mathbb{K}_{4, P_4}^M$  such that  $\sum_{u \in V(F) \setminus \{v_1, v_2\}} |N_M(u)| = 16$ 
29:    return  $\mathcal{A}(M, V(F) \setminus \{v_1, v_2\})$ 
30:  if  $p$  is (14) then
31:    set  $U = \bigcup_{i=1}^4 N_M(w_i) \setminus \{v_1, v_2\}$ 
32:    choose a subgraph  $F \in \mathbb{K}_{2,4}^M$  such that each vertex of  $U$  has degree 1 in  $M$ 
33:    return  $\mathcal{A}(M, U \cup \{w_1, \dots, w_4\})$ 

```

Next, we let H be a multigraph which satisfies the hypothesis of the lemma, and we show that H is added to `F.txt`. Since H is the preimage of a line graph of multigraph, H has neither loops nor pendant multiedges. We show the following four claims.

Claim 1. *Every flat subgraph of H satisfies the condition on line 2 of Algorithm 1.*

Claim 2. *Let M be a flat subgraph of H and let $U \subset V(M)$ such that for every $u \in U$, all vertices of $N_M(u)$ of degree 1 in M also belong to U . If there is a vertex x of $V(H) \setminus V(M)$ such that $|N_H(x) \cap U| \geq 1$, then H contains a multigraph M^+ from $\mathcal{A}(M, U)$ as a flat subgraph.*

Claim 3. *Let M be a flat subgraph of H . If M is distinct from H , then H contains a multigraph M^+ from $\mathcal{A}(M) \cup \mathcal{M}(M)$ as a flat subgraph (possibly $M^+ \simeq H$).*

Claim 4. *Let M be a flat subgraph of H . If M violates a condition p from (6), \dots , (14), then H contains some multigraph M^+ given by `GET_ALL_SOLUTION_ATTEMPTS`(M, p) as a flat subgraph.*

Proof of Claim 1. For the sake of a contradiction, we suppose that H has a flat subgraph M which fails to satisfy the condition on line 2 of Algorithm 1. Hence, M contains a subgraph X such that $L(X) \simeq \Gamma_3$ or contains D as a subgraph or M contains C_9^M as a flat subgraph. Since M is a flat subgraph of H , every (flat) subgraph of M is also a (flat) subgraph of H . Hence, H also fails to satisfy the condition on line 2 of Algorithm 1. In other words, $L(H)$ is not Γ_3 -free or H violates condition (4) or (5), a contradiction. \square

Proof of Claim 2. We let M' be the multigraph obtained from M by adding a new vertex adjacent by simple edges to precisely the vertices of $N_H(x) \cap V(M)$. Clearly, M' is a flat subgraph of H . If M' belongs to $\mathcal{A}(M, U)$, then we are done. Hence, we can assume that M' does not belong to $\mathcal{A}(M, U)$, and it follows that the new vertex is of degree 1 in M' and has more than one twin in M' . We let A be a set consisting of the new vertex and its two twins in M' . Since H satisfies condition (1), there exists a vertex, say y , of $V(H) \setminus V(M')$ such that $|N_H(y) \cap A| \geq 1$. We choose a vertex $u \in A$ such that $|N_H(y) \cap A \setminus \{u\}| \geq 1$, and we consider the multigraph $M' - u$ (we note that it is isomorphic to M). We let M^+ be the multigraph obtained from $M' - u$ by adding a new vertex adjacent by simple edges to precisely the vertices of $N_H(y) \cap V(M' - u)$. Clearly, M^+ is a flat subgraph of H . Finally, we note that the hypothesis of Claim 2 implies that $|A \cap U| \geq 2$. Since the vertices of A are twins in M' and the new vertex of M^+ is adjacent to a vertex of A , we conclude that $M^+ \in \mathcal{A}(M, U)$. \square

Proof of Claim 3. We discuss two cases based on $V(M)$. For the first case, we suppose that $V(M) = V(H)$. We recall that H has no loops, no pendant multiedges and no multiedges of multiplicity greater than 2. Since M is a flat subgraph of H and M is distinct from H , there is a simple non-pendant edge e of M which corresponds to a multiedge in H . We let M^+ be the multigraph obtained from M by multiplying e , and we conclude that M^+ is a flat subgraph of H and M^+ belongs to $\mathcal{M}(M)$.

For the second case, we suppose that $V(M)$ is a proper subset of $V(H)$. Since H is connected, there is a vertex x of $V(H) \setminus V(M)$ such that $|N_H(x) \cap V(M)| \geq 1$. We observe that we can apply Claim 2 with $U = V(M)$ and obtain a desired multigraph M^+ . \square

Proof of Claim 4. We discuss nine cases based on p .

First, we consider the case where p is (6) which stands for the fact that the multigraph M has few vertices of degree at least 3. We use that H is connected, has no pendant multiedge and satisfies condition (6) and that M is a flat subgraph of H , and we observe that at least one of the following is true:

- some vertex of $V(H) \setminus V(M)$ is adjacent to a vertex of $V(M)$ in H , or
- some edge of R corresponds to a multiedge in H .

For the first item, we use Claim 2 with $U = V(M)$ and observe that H contains a multigraph from $\mathcal{A}(M)$ as a flat subgraph. For the second item, we note that H contains a multigraph from $\mathcal{M}(M, R)$ as a flat subgraph. The obtained multigraph from $\mathcal{A}(M) \cup \mathcal{M}(M, R)$ is included at line 4 of Algorithm 2.

For the case (7), we consider the condition on line 7 of Algorithm 2 and discuss the two options. We first suppose that the condition is satisfied, that is, $\{e, f\} \subset E_S(M)$. Since H satisfies condition (7) and M is a flat subgraph of H , at least one of the following is true:

- some vertex of $V(H) \setminus V(M)$ is adjacent to a vertex of C , or
- some edge of $\{e, f\}$ corresponds to a multiedge in H .

For the first item, we consider an arbitrary vertex u of C and note that all vertices of $N_M(u)$ of degree 1 in M also belong to C . Hence, we can apply Claim 2 with $U = V(C)$. For the second item, we note that H contains a multigraph from $\mathcal{M}(M, \{e, f\})$ as a flat subgraph. It follows that H contains a multigraph from $\mathcal{A}(M, V(C)) \cup \mathcal{M}(M, \{e, f\})$ as a flat subgraph, and this multigraph is included at line 9 of Algorithm 2.

Next, we suppose that one of the edges, say e , is a pendant edge in M . We observe that at least one of the vertices incident with f is an essential cutvertex in M . Hence, the algorithm can reset $p = (8)$ and continue with lines 12, 13 and 14.

For the case (8), we use that H satisfies condition (8) and M is a flat subgraph of H , and we note that some vertex of $V(H) \setminus V(M)$ is adjacent to a vertex of C . We apply Claim 2 with $U = V(C)$, and we conclude that H contains a multigraph from $\mathcal{A}(M, V(C))$ as a flat subgraph, and it is included at line 14 of Algorithm 2.

For the case (9), we use that H satisfies condition (9), and hence some vertex of $V(H) \setminus V(M)$ is adjacent to u . Since $|N_M(u)| = 2$, the vertex u has no neighbor of degree 1 in M . Thus, we can apply Claim 2 with $U = \{u\}$ and conclude that H contains a multigraph from $\mathcal{A}(M, \{u\})$ as a flat subgraph. This multigraph is included at line 17 of Algorithm 2.

We note that the cases (10), \dots , (13) are similar to (9). In each case, M contains a set, say U , of vertices (possibly of size 1) such that no vertex of U has neighbor of degree 1 in M and at least one vertex of U has an additional neighbor in H . Hence, Claim 2 yields that H contains a multigraph from $\mathcal{A}(M, U)$ as a flat subgraph, and it is included at the respective line of Algorithm 2.

Lastly, for the case (14) we note that H has a vertex of degree at least 2 adjacent to at least one of the vertices w_1, \dots, w_4 . In particular, this vertex has at least two neighbors in H since H has no pendant multiedge. Hence, there exists a vertex x of $V(H) \setminus V(M)$ such that at least one of the following is true:

- x is adjacent to at least one of w_1, \dots, w_4 , or

- x is adjacent to a vertex u such that $d_M(u) = |N_M(u) \cap \{w_1, \dots, w_4\}| = 1$ (in particular, adding x increases the degree of u to 2).

In other words, x is adjacent to a vertex of $U = \bigcup_{i=1}^4 N_M[w_i] \setminus \{v_1, v_2\}$ since each vertex of $N_M(w_i) \setminus \{v_1, v_2\}$ has degree 1 in M . Hence, Claim 2 yields that H contains a multigraph from $\mathcal{A}(M, U)$ as a flat subgraph, and this multigraph is included at line 33 of Algorithm 2. \square

With Claims 1, 3 and 4 on hand, we now show the desired implication. For the sake of a contradiction, we suppose that there exists a multigraph H which satisfies the hypothesis of the lemma but is not added to `F.txt`. We consider all flat subgraphs M of H for which the algorithm calls `INVESTIGATE(M)`, and we choose such a multigraph M maximizing $\sum_{u \in V(M)} d_M(u)$. We should also say that such M clearly exists since H satisfies condition (3) and the algorithm calls `INVESTIGATE()` for the graphs on k vertices containing C_k where $k \in \{7, \dots, 10\}$. Furthermore, since M is obtained by recurrently extending one of these graphs by $\mathcal{A}()$ and $\mathcal{M}()$, we note that M has no pendant multiedges.

Since M is a flat subgraph of H , M satisfies the condition on line 2 of Algorithm 1 by Claim 1. We now discuss M subject to the condition on line 3 of Algorithm 1 and we obtain a multigraph M^+ as follows. If M satisfies this condition, then M is added to `F.txt` at line 4 of Algorithm 1. Hence, M is distinct from H (since H is not added to `F.txt`), and thus we can apply Claim 3 to M ; and we let M^+ be the obtained multigraph. Otherwise, M violates a condition from (6), \dots , (14), and we can apply Claim 4 to M ; and we let M^+ be the obtained multigraph.

We consider the obtained multigraph M^+ , and we note that M^+ belongs to \mathcal{B} due to line 5 or 8 of Algorithm 1. Finally, line 10 of Algorithm 1 calls `INVESTIGATE(M^+)`, which contradicts the choice of M . Thus, every multigraph satisfying the hypothesis of the lemma is added to `F.txt` which concludes the proof of the equivalence.

In order to improve runtime, our implementation of the algorithm is slightly more involved. In the remainder of the proof, we outline details of the implementation (an interested reader is also invited to have a look at the code and the commentary therein).

In the implementation, we keep track of solved cases (that is, we save multigraphs whose all extension branches are finished). Later, when the computer investigates a different case, it tests whether this is already solved and then perhaps not investigate it again (it tests whether some of the saved multigraphs appears as a flat subgraph of the multigraph currently investigated). The list of solved multigraphs can be easily kept short and apt in the recurrence scheme.

In each branch, we keep track of the extensions. In particular, an extension is not investigated again if it is known to lead to violating the condition on line 2 of Algorithm 1 or to a case already solved.

Lastly, a violated condition p and a particular violation are chosen at random, but the choice favors small $|\mathcal{B}|$ (it is not uniformly random, and we consider only violations whose sets \mathcal{B} are inclusion minimal). In fact, if there is a particular violation such that $|\mathcal{B}| = 0$, then this choice is always preferred (this means that the violation cannot be fixed even with further extensions and the branch finishes). \blacksquare

Proof of Theorem 1. Let, to the contrary, G be a minimal 3-connected $\{K_{1,3}, \Gamma_3\}$ -free graph that is not Hamilton-connected, let \bar{G} be one of its Γ_3 -closures, and let $H = L^{-1}(\bar{G})$.

Obviously, H is essentially 3-edge-connected since G is 3-connected. If H has a cutvertex, we can apply our considerations to each of its (nontrivial) blocks, hence we can assume that H is essentially 2-connected. Finally, it is straightforward to verify that if $H \simeq W$ (see Fig. 2), then G is Hamilton-connected, hence $H \not\cong W$. By Theorem E(i), H contains a cycle of length at least 9.

Moreover, we have the following fact.

Claim 1. *If $\text{co}(H) \in \{W \cup \mathbb{W}\}$, then H contains as a subgraph an $L^{-1}(\Gamma_3)$, or $G = L(H)$ is Hamilton-connected.*

Proof. Set $\tilde{H} = \text{co}(H)$. First observe that if $\tilde{H} \in \{W \cup \mathbb{W}\}$, then, by Lemma 3(ii), the multiple edge of \tilde{H} is of multiplicity 2, hence $\tilde{H} \in \{W_1, W_2\}$, where W_1 and W_2 are the multigraphs shown in Fig. 8 (in which v denotes the only vertex of degree 4).

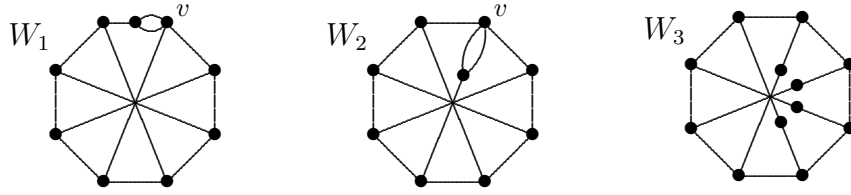


Figure 8: The multigraphs W_1 and W_2 , and the graph W_3 .

Let thus $\tilde{H} \in \{W, W_1, W_2\}$, and let V_1 be the set of all vertices of \tilde{H} that are not incident to a double edge.

It is easy to see that if a vertex $x \in V_1$ is incident in H to neither a pendant edge nor an edge containing a vertex of degree 2, then x is not necessarily visited by an IDT in H (all edges incident to x can be dominated without visiting x), and then it is straightforward to verify that H has an (e, f) -IDT for any $e, f \in E(H)$, a contradiction. (Note that this fact can be also alternatively seen by applying Lemma 5 from [12] to \tilde{H} and setting $A = V_1 \setminus \{x\}$ if $\tilde{H} \simeq W$, or $A = V_1 \cup \{v\} \setminus \{x\}$ if $\tilde{H} \in \{W_1, W_2\}$).

Thus, in H , every vertex in V_1 is incident to a pendant edge or to an edge containing a vertex of degree 2. To prove the claim, it suffices to show that every such multigraph either contains an $L^{-1}(\Gamma_3)$, or its line graph is Hamilton-connected. This will be done in the next claim, since it is straightforward to verify that $L(W_3)$ (where W_3 is the graph shown in Fig. 8) is Hamilton-connected. \square

Claim 2. *Let $M \in \{W, W_1, W_2\}$, let $E \subset E(M)$, and let V be the set of all vertices from $V(M)$ that are incident to neither an edge from E nor to a multiedge in M . Let N be the multigraph obtained from M by subdividing each edge of E with one vertex of degree 2 and by adding one pendant edge to each vertex of V . If N does not contain a subgraph isomorphic to $L^{-1}(\Gamma_3)$, then $N \simeq W_3$.*

Proof. We prove the lemma with the help of a computer. For each choice of M and E , we test the obtained multigraph N . If N does not contain $L^{-1}(\Gamma_3)$ as a subgraph, then the program outputs N . The only such N is the graph W_3 . The source code of the proof can be found at [25]. \square

Thus, by Theorem E(ii) and Theorem F(v), we have $|V(\text{co}(H))| \geq 10$, i.e., H has at least 10 vertices of degree at least 3.

Let now C be a cycle in H such that C is a shortest cycle of length at least 7.

Case 1: $7 \leq |V(C)| \leq 8$, or $9 \leq |V(C)| \leq 10$ and C is not chordless.

By the assumptions and by Lemmas 2 and 3, H satisfies the assumptions of Lemma 4. Thus, by Lemma 4, $H \in \mathcal{F}$. To reach a contradiction, it remains to show that for each $H \in \mathcal{F}$, $L(H)$ is Hamilton-connected. This was done with the help of a computer. For each $H \in \mathcal{F}$, and for each pair of vertices of $L(H)$, we find a hamiltonian path by using the function ‘hamiltonian_path’ in SageMath. As a certificate of Hamilton-connectedness, we also provide a list of these hamiltonian paths along with a simple program that verifies this certificate. The verification is faster (less than a minute) than finding the hamiltonian paths (a few hours). The source codes are available at [24].

Case 2: $9 \leq |V(C)| \leq 10$ and C is chordless, or $|V(C)| \geq 11$.

Note that, by the choice of C and by Case 1, H contains no cycle C' of length $7 \leq |V(C')| < |V(C)|$. Denote $|V(C)| = r$, $V(C) = \{x_0, x_1, \dots, x_{r-1}\}$, and $R = V(H) \setminus V(C)$.

Claim 3. *The cycle C is chordless and for any vertices $x_i, x_j \in V(C)$ with $\text{dist}_C(x_i, x_j) \geq 3$, $N_H(x_i) \cap N_H(x_j) = \emptyset$.*

Proof. For $9 \leq r \leq 10$, C is chordless by the assumption of the case. If $r \geq 11$ and $x_i x_j$ is a chord in C (i.e., $\text{dist}_C(x_i, x_j) \geq 2$ and $x_i x_j \in E(H)$), then the edge $x_i x_j$ creates with one of the two parts of C joining x_i and x_j a cycle C' of length $7 \leq |V(C')| \leq r - 1$, a contradiction. Thus, C is chordless.

If $r \geq 9$, $\text{dist}_C(x_i, x_j) \geq 3$ and x_i, x_j have a common neighbor $y \in R$, then similarly the path $x_i y x_j$ creates with one of the two parts of C joining x_i and x_j a cycle C' of length $7 \leq |V(C')| \leq |V(C)| - 1$, a contradiction. \square

The next several claims will be proved in a slightly more general setting for a cycle in H satisfying the conditions given in Claim 3 (and will be therefore true also for the cycle C). Let thus $C' = y_0 y_1 \dots y_{t-1}$, $t = |V(C')| \geq 9$, be a cycle in H such that C' is chordless and $N_H(y_i) \cap N_H(y_j) = \emptyset$ for any $y_i, y_j \in V(C')$ with $\text{dist}_{C'}(y_i, y_j) \geq 3$. Let $R' = V(H) \setminus V(C')$. If $N_{R'}(y_i) \neq \emptyset$, we will sometimes use \bar{y}_i for (some) neighbor of y_i in R' .

Claim 4. $R' \neq \emptyset$.

Proof. Let, to the contrary, $V(C') = V(H)$. If there are two nonconsecutive edges $e_1, e_2 \in E(C') \cap E_S(H)$, then $\{e_1, e_2\}$ is an essential edge-cut of H (recall that C' is chordless), contradicting the connectivity assumption. Thus, since $t = |V(C')| \geq 9$, we can choose the notation such that, say, $\{y_0 y_1, y_5 y_6\} \subset E_M(H)$. Then the edges $(y_0 y_1)^1, (y_0 y_1)^2, y_1 y_2, y_2 y_3, y_3 y_4, y_4 y_5, (y_5 y_6)^1, (y_5 y_6)^2$ determine an $L^{-1}(\Gamma_3)$ in H , a contradiction. Thus, $R' = V(H) \setminus V(C') \neq \emptyset$. \square

Claim 5. $E(C') \cap E_M(H) \neq \emptyset$.

Proof. Let, to the contrary, $E(C') \subset E_S(H)$. Since $R' \neq \emptyset$, we can choose the notation such that $N_{R'}(y_0) \neq \emptyset$. Then $N_{R'}(y_4) = \emptyset$ for otherwise the edges $y_{t-1}y_0, \bar{y}_0y_0, y_0y_1, y_1y_2, y_2y_3, y_3y_4, y_4\bar{y}_4, y_4y_5$ determine an $L^{-1}(\Gamma_3)$ in H . By the connectivity assumption, there cannot be two consecutive vertices of degree 2 on C' , hence $N_{R'}(y_3) \neq \emptyset$ and $N_{R'}(y_5) \neq \emptyset$. Now, if $N_{R'}(y_1) \neq \emptyset$, then the edges $y_0y_1, \bar{y}_1y_1, y_1y_2, y_2y_3, y_3y_4, y_4y_5, y_5\bar{y}_5, y_5y_6$ determine an $L^{-1}(\Gamma_3)$ in H , hence $N_{R'}(y_1) = \emptyset$, implying (by the connectivity assumption) that $N_{R'}(y_2) \neq \emptyset$. Then $N_{R'}(y_6) = \emptyset$ for otherwise the edges $y_1y_2, \bar{y}_2y_2, y_2y_3, y_3y_4, y_4y_5, y_5y_6, y_6\bar{y}_6, y_6y_7$ determine an $L^{-1}(\Gamma_3)$ in H . By the connectivity assumption, $N_{R'}(y_7) \neq \emptyset$, but then the edges $y_2y_3, \bar{y}_3y_3, y_3y_4, y_4y_5, y_5y_6, y_6y_7, y_7\bar{y}_7, y_7y_8$ determine an $L^{-1}(\Gamma_3)$ in H , a contradiction. \square

Thus, C' contains at least one multiple edge.

Claim 6. C' contains an edge y_iy_{i+1} such that $y_iy_{i+1} \in E_M(H)$ and at least one of y_i, y_{i+1} has a neighbor in R' .

Proof. Suppose, to the contrary, that $N_{R'}(y_i) = N_{R'}(y_{i+1}) = \emptyset$ for any multiple edge y_iy_{i+1} of C' , and choose the notation such that $y_0y_1 \in E_M(H)$. Since $\{y_{t-1}y_0, y_1y_2\}$ cannot be a cutset, by symmetry, we can assume that $y_1y_2 \in E_M(H)$. Now $\{y_{t-1}y_0, y_2y_3\}$ cannot be a cutset, implying, by symmetry, $y_2y_3 \in E_M(H)$. Repeating the argument, we have $y_5y_6 \in E_M(H)$, but then the edges $(y_0y_1)^1, (y_0y_1)^2, y_1y_2, y_2y_3, y_3y_4, y_4y_5, (y_5y_6)^1, (y_5y_6)^2$ determine an $L^{-1}(\Gamma_3)$ in H , a contradiction. \square

Claim 7. Let $C' = y_0y_1 \dots y_{t-1}$ be a cycle in H such that $t = |V(C')| \geq 9$, C' is chordless, and $N_H(y_i) \cap N_H(y_j) = \emptyset$ for any two vertices $y_i, y_j \in V(C')$ with $\text{dist}_{C'}(y_i, y_j) \geq 3$. Then $t \equiv 0 \pmod{3}$, and the notation can be chosen such that

- (i) if $i \equiv 0 \pmod{3}$, then $N_{R'}(y_i) \neq \emptyset$ and $y_iy_{i+1} \in E_M(H)$, and
- (ii) if $i \equiv 1 \pmod{3}$ or $i \equiv 2 \pmod{3}$, then $N_{R'}(y_i) = \emptyset$ and $y_iy_{i+1} \in E_S(H)$.

Recall that, specifically, the cycle C satisfies the assumptions of Claim 7 by Claim 3.

Proof. By Claim 6, choose the notation such that $y_0y_1 \in E_M(H)$ and $N_{R'}(y_0) \neq \emptyset$. Then immediately $N_{R'}(y_4) = N_{R'}(y_5) = \emptyset$ and $\{y_4y_5, y_5y_6\} \subset E_S(H)$ (since otherwise, in all cases, we have an $L^{-1}(\Gamma_3)$ in H). By the connectivity assumption, necessarily $y_3y_4 \in E_M(H)$ and $N_{R'}(y_6) \neq \emptyset$ or $y_6y_7 \in E_M(H)$. Then necessarily $N_{R'}(y_2) = \emptyset$ and $y_1y_2 \in E_S(H)$ (otherwise we have an $L^{-1}(\Gamma_3)$ in H). Since $\{y_1y_2, y_4y_5\}$ is not a cutset, $N_{R'}(y_3) \neq \emptyset$.

Summarizing, we conclude that $N_{R'}(y_3) \neq \emptyset$, $y_3y_4 \in E_M(H)$, $N_{R'}(y_4) = N_{R'}(y_5) = \emptyset$, and $\{y_4y_5, y_5y_6\} \subset E_S(H)$, i.e., we have the requested statement for $i = 3, 4, 5$. Repeating the argument, starting with $N_{R'}(y_3) \neq \emptyset$ and $y_3y_4 \in E_M(H)$, we get the statement for $i = 6, 7, 8$. The claim then follows by induction. \square

Now we can apply Claim 7 to the cycle C (recall that, by Claim 3, C satisfies the assumptions of Claim 7). Thus, we have $|V(C)| = r \equiv 0 \pmod{3}$, which specifically implies that $r = 9$ or $r \geq 12$.

For $\ell = 0, 1, 2$, we denote $X_\ell = \{x_i \in V(C) \mid i \equiv \ell \pmod{3}\}$ and $E_\ell = \{x_ix_{i+1} \in E(C) \mid i \equiv \ell \pmod{3}\}$. Thus, specifically, by Claim 7, $E_0 \subset E_M(H)$ and $E_1 \cup E_2 \subset E_S(H)$. By the

connectivity assumption, the edges in $E_1 \cup E_2$ cannot form a cutset of H , hence there is an (x_α, x_β) -path with endvertices $x_\alpha, x_\beta \in X_0$ and with interior vertices in R .

For the purpose of this proof, by an *arc* we will mean a path that is chordless and any two vertices at distance at least 3 have no common neighbor. Clearly, if for some vertices $x_\alpha, x_\beta \in X_0$ there is an (x_α, x_β) -path with interior vertices in R , there is also an (x_α, x_β) -arc with interior vertices in R .

Let T be an (x_α, x_β) -arc with interior vertices in R for some $x_\alpha, x_\beta \in X_0$. Since for any two vertices $x_\alpha, x_\beta \in X_0$, $\text{dist}_C(x_\alpha, x_\beta) \geq 3$ and C is shortest, T is of length at least 3. The vertices x_α, x_β divide C into two paths (called *segments*), and each of them, together with T , creates in H a cycle of length at least 6. Let C^T be the shorter one of these cycles (or any of them if both segments of C are of the same length).

Suppose first that the (x_α, x_β) -arc T can be chosen such that $|V(C^T)| \geq 7$, and, subject to this condition, choose T such that C^T is shortest possible. By the choice of C and by the previous arguments, $9 \leq |V(C)| \leq |V(C^T)|$. Choose the notation such that, in the natural orientation of C given by increasing indices, the segment of C that is in C^T is oriented from x_α to x_β , and set $T = z_0 z_1 \dots z_t$, where $z_0 = x_\alpha$ and $z_t = x_\beta$.

We verify that the cycle C^T satisfies the assumptions of Claim 7. This is clear for $y_i, y_j \in V(C)$ (by Claim 3), and for $y_i, y_j \in V(T)$ (by the choice of T); so, let $y_i \in V(C^T) \setminus V(T)$ and $y_j \in V(C^T) \setminus V(C)$. Since $y_i \in X_0$ by Claim 7, $\text{dist}_C(y_i, x_\alpha) \geq 3$ and $\text{dist}_C(y_i, x_\beta) \geq 3$, hence $\text{dist}_C(x_\alpha, x_\beta) \geq 6$ (by the choice of C^T as the shorter one determined by x_α, x_β and T), implying $|V(C^T)| \geq 12$. Thus, any chord $y_i y_j \in E(H)$ or a vertex in $N_R(y_i) \cap N_R(y_j)$ yields a cycle C' of length $7 \leq |V(C')| < |V(C^T)|$, contradicting the choice of C^T . Thus, C^T satisfies the assumptions of Claim 7. This implies that $t \equiv 0 \pmod{3}$ and $z_2 z_3 \in E_M(H)$. Since also $x_{\alpha-3} x_{\alpha-2} \in E_M(H)$, the edges $(x_{\alpha-3} x_{\alpha-2})^1, (x_{\alpha-3} x_{\alpha-2})^2, x_{\alpha-2} x_{\alpha-1}, x_{\alpha-1} x_\alpha, x_\alpha z_1, z_1 z_2, (z_2 z_3)^1, (z_2 z_3)^2$ determine an $L^{-1}(\Gamma_3)$ in H , a contradiction.

Thus, for every (x_α, x_β) -arc T in H with interior vertices in R , we have $|V(C^T)| = 6$, i.e., $t = 3$ and $\text{dist}_C(x_\alpha, x_\beta) = 3$. Then, replacing in C the (x_α, x_β) -segment (of length 3) by the (x_α, x_β) -arc T , we get a cycle C' of the same length as C , and, by Claim 7, we have $z_0 z_1 \in E_M(H)$, $\{z_1 z_2, z_2 z_3\} \subset E_S(H)$, and $N_{R'}(z_1) = N_{R'}(z_2) = \emptyset$ (where $R' = V(H) \setminus V(C')$). Note that, by the connectivity assumption, there is at most one (x_α, x_β) -segment of C of length 3 without any (x_α, x_β) -arc with interior vertices in R . Then it is straightforward to verify that H has an (e, f) -IDT for any $e, f \in E(H)$, hence $G^{\Gamma_3} = L(H)$ is Hamilton-connected by Theorem D (for an example, see Fig. 9(a)). Alternatively viewed, $L(H)$ is the graph that can be obtained from a cycle by replacing each vertex with a clique and each edge, except at most one, with at least 2 vertices of degree 3 attached to both cliques (see Fig. 9(b)), and it is straightforward to verify that G^{Γ_3} is Hamilton-connected. ■

6 Concluding remarks

1. Theorem 1 admits a slight extension as follows. For $s \geq 0$, a graph G is *s-Hamilton-connected* if the graph $G - M$ is Hamilton-connected for any set $M \subset V(G)$ with $|M| \leq s$. Obviously, an *s-Hamilton-connected* graph must be $(s + 3)$ -connected. Since an induced

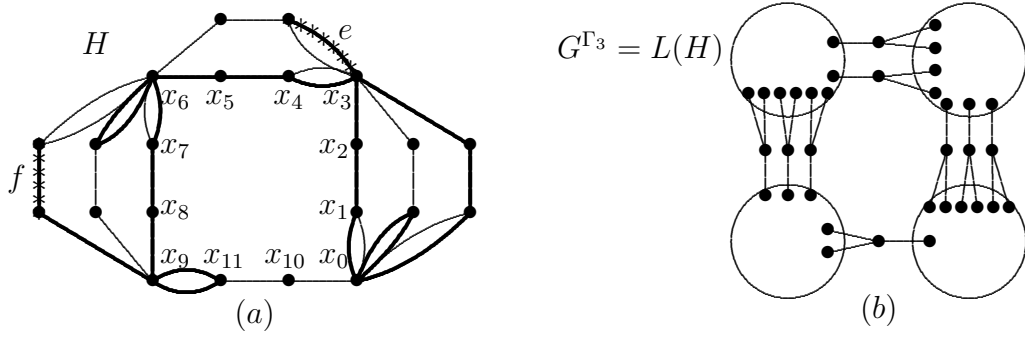


Figure 9: An (e, f) -IDT in H and the graph $G^{\Gamma_3} = L(H)$.

subgraph of a $\{K_{1,3}, \Gamma_3\}$ -free graph is also $\{K_{1,3}, \Gamma_3\}$ -free, we immediately have the following fact, showing that, in $\{K_{1,3}, \Gamma_3\}$ -free graphs, the obvious necessary condition is also sufficient.

Corollary 5. *Let $s \geq 0$ be an integer, and let G be a $\{K_{1,3}, \Gamma_3\}$ -free graph. Then G is s -Hamilton-connected if and only if G is $(s + 3)$ -connected.*

2. A Γ_3 -closure of a graph G , as defined in Section 4, is not unique in general. However, in view of Theorem 1, it is unique on 3-connected $\{K_{1,3}, \Gamma_3\}$ -free graphs since each such graph is Hamilton-connected, hence has complete closure.

3. We can now update the discussion of potential pairs X, Y of connected graphs that might imply Hamilton-connectedness of a 3-connected $\{X, Y\}$ -free graph, as summarized in [20].

As shown in [6], up to a symmetry, necessarily $X = K_{1,3}$, and, summarizing the discussions from [3], [6], [7] and [13], there are the following possibilities for Y (see Fig. 1):

- (i) $Y = P_i$ with $4 \leq i \leq 9$,
- (ii) $Y = Z_i$ with $i \leq 6$, or $Y = Z_7$ for $n = |V(G)| \geq 21$,
- (iii) $Y = B_{i,j}$ with $i + j \leq 7$,
- (iv) $Y = N_{i,j,k}$ with $i + j + k \leq 7$,
- (v) $Y \in \{\Gamma_1, \Gamma_3\}$, or $Y = \Gamma_5$ for $n = |V(G)| \geq 21$.

Best known results in the direction of each of these subgraphs are summarized in Theorem A, and we summarize the current status of the problem in the following table.

Y	Possible	Best known	Reference	Open
P_i	$4 \leq i \leq 9$	P_9	[3]	—
Z_i	$i \leq 6$; Z_7 for $n \geq 21$	Z_6 ; Z_7 for $G \not\cong L(W^+)$	[19]	—
$B_{i,j}$	$i + j \leq 7$	$i + j \leq 7$	[20]	—
$N_{i,j,k}$	$i + j + k \leq 7$	$i + j + k \leq 7$	[12, 13, 14]	—
Γ_i	$\Gamma_1, \Gamma_3, \Gamma_5$ for $n \geq 21$	Γ_1, Γ_3	[6], this paper	Γ_5 for $n \geq 21$

Thus, the only remaining case is the case $Y = \Gamma_5$ for $n \geq 21$ (or possibly Γ_5 for $G \not\cong L(W^+)$). We believe that this case is also true, and we think that it could be doable with the techniques of this paper; however, the proof would be too technical to be reasonably handled even with the help of a computer.

4. The source codes of our proof-assisting programs are available at [24] and [25]. The codes are written in Python 3.8 and use functions imported from SageMath 9.6. We thank the SageMath community [21] for developing a valuable open-source mathematical software.

7 Acknowledgement

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Computing source codes

- [24] The proof of Lemma 4 and of Case 1 in the proof of Theorem 1:
<https://arxiv.org/src/2410.18309/anc>
- [25] The proof of Claim 2 in the proof of Theorem 1:
<https://arxiv.org/src/2410.18309/anc>