Matematické Metody v Ekonometrii 8.

Generalised Linear Normal Regression Model (GLM)

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zimní semestr 2016/2017

Generalised Linear Normal Regression Model (GLM)

- linear regression model $\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}, \mathbf{Var}(\varepsilon) = \sigma^2 \boldsymbol{I}$
- normal linear regression model $\boldsymbol{Y} \sim N\left(\boldsymbol{X}\boldsymbol{\beta}, \sigma^2 \boldsymbol{I}\right)$
- general linear regression model $\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}, \mathbf{Var}(\varepsilon) = \boldsymbol{\Sigma}$
- general normal linear regression model $\boldsymbol{Y} \sim N\left(\boldsymbol{X}\boldsymbol{\beta}, \sigma^2 \boldsymbol{\Sigma}\right)$, where Σ is a positive-definite matrix
- general linear regression model = Aitken linear regression model
- Heteroskedasticity arises if different error terms do not have identical variances, so that the diagonal elements of the covariance matrix are not identical.
- Autocorrelation almost exclusively arises in cases where the data have a time dimension. It implies that the covariance matrix is non diagonal such that different error terms are correlated.

Some properties of the positive-definite matrix

- $\boldsymbol{x}^T \boldsymbol{\Sigma} \boldsymbol{x} > 0$ for $\boldsymbol{x} \in \mathbf{R}^n$
- diagonal elements $\sigma_{ii}^2 > 0$ represents variance of ε
- $\det(\boldsymbol{\Sigma}) > 0$
- $\boldsymbol{\Sigma}$ is regular and therefore invertibile matrix exists
- there exists a regular matrix $\boldsymbol{P}$ such that $\boldsymbol{P}^T \boldsymbol{P} = \boldsymbol{\Sigma}^{-1}$ and $\boldsymbol{P} \boldsymbol{\Sigma} \boldsymbol{P}^T = \boldsymbol{I}$
- the eigenvalues of positive-define matrix are real and positive

Deriving an Alternative Estimator for model $\boldsymbol{Y} = \boldsymbol{X}\beta + \varepsilon$

- for classical OLS method

$$\boldsymbol{b} = \left(\boldsymbol{X}^T\boldsymbol{X}\right)^{-1}\boldsymbol{X}^T\boldsymbol{Y} = \beta + \left(\boldsymbol{X}^T\boldsymbol{X}\right)^{-1}\boldsymbol{X}^T\varepsilon$$

  estimator is still unbias , but computed variance, standard errors, confidence intervals . . . will be based on $\mathrm{E}\left(\boldsymbol{b}|\boldsymbol{X}\right) = \beta$ and $\mathbf{Var}\left(\boldsymbol{b}|\boldsymbol{X}\right) = \sigma^2\left(\boldsymbol{X}^T\boldsymbol{X}\right)^{-1}\boldsymbol{X}^T\boldsymbol{\Sigma}\boldsymbol{X}\left(\boldsymbol{X}^T\boldsymbol{X}\right)^{-1}$

- we can transform the entire model by this $\boldsymbol{P}$ matrix to obtain $\boldsymbol{P}\boldsymbol{Y} = \boldsymbol{P}\boldsymbol{X}\beta + \boldsymbol{P}\varepsilon$ or $\boldsymbol{Y}^* = \boldsymbol{X}^*\beta + \varepsilon^*$ and

$$\mathrm{E}\left(\varepsilon^*|\boldsymbol{X}\right) = \mathrm{E}\left(\boldsymbol{P}\varepsilon|\boldsymbol{X}\right) = \mathrm{E}\,\boldsymbol{P}\left(\varepsilon|\boldsymbol{X}\right) = 0$$

$$\mathbf{Var}\left(\varepsilon^*|\boldsymbol{X}\right) = \mathbf{Var}\left(\boldsymbol{P}\varepsilon|\boldsymbol{X}\right) = \boldsymbol{P}\mathbf{Var}\left(\varepsilon|\boldsymbol{X}\right)\boldsymbol{P}^T = \boldsymbol{P}\sigma^2\boldsymbol{\Sigma}\boldsymbol{P}^T = \sigma^2\boldsymbol{I}$$

In other words, $\boldsymbol{P}\varepsilon$ satisfies the Gauss–Markov conditions.

Aitken estimator - Generalised least squares method (GLS)

Let $\boldsymbol{Y} = \boldsymbol{X}\beta + \varepsilon, \mathbf{Var}(\varepsilon) = \boldsymbol{\Sigma}$ be a generalised linear regression model. Then the following holds:

1. The vector ${}_{(GLS)}\widehat{\boldsymbol{b}}_{\Sigma} = (\boldsymbol{X}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{Y}$ is BLUES of β
2. The variance matrix of ${}_{(GLS)}\boldsymbol{b}$ is given as

$$\mathbf{Var}\left({}_{(GLS)}\boldsymbol{b}\right) = \sigma^2(\boldsymbol{X}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{X})^{-1}$$

3. The unbiased estimator of σ^2 is

$${}_{(GLS)}RSE_{\Sigma} = \frac{\left(\boldsymbol{Y} - {}_{(GLS)}\widehat{\boldsymbol{Y}}\right)^T\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{Y} - {}_{(GLS)}\widehat{\boldsymbol{Y}}\right)}{n-p}$$

Normal Generalised least squares method (NGLS)

For normal general linear model $\boldsymbol{Y} \sim N\left(\boldsymbol{X}\beta, \sigma^2\boldsymbol{\Sigma}\right)$ it holds

1. the vector ${}_{(GLS)}\widehat{\boldsymbol{b}}_{\Sigma}$ has normal distribution

$$ {}_{(GLS)}\widehat{\boldsymbol{b}}_{\Sigma} \sim N\left(\beta, \sigma^2(\boldsymbol{X}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{X})^{-1}\right); $$

2. $\frac{{}_{(GLS)}RSE_{\Sigma}}{\sigma^2}$ has χ^2 distribution with degree of freedom $n-p$,
3. RSE and ${}_{(GLS)}\widehat{\boldsymbol{b}}_{\Sigma}$ are independent.

GLM estimator

- In practice, decomposition of $\boldsymbol{\Sigma}^{-1}$ can be constructed as $\boldsymbol{P} = \boldsymbol{D}^{-\frac{1}{2}}\boldsymbol{V}$, where $\boldsymbol{D}$ is a diagonal matrix composed of the matrix $\boldsymbol{\Sigma}$ eigenvalues and $\boldsymbol{V}$ is a matrix which consists of the corresponding eigenvectors.
- It is important to emphasize that the estimator $_{(GLS)}\boldsymbol{b}$ contains the unknown $n \times n$ matrix $\boldsymbol{\Sigma}$, which means a maximum of $n(n+1)/2$(due to its symmetry) unknown parameters.
- These parameters cannot be estimated using n observations and hence the so called *feasible consistent estimator* $\widehat{\boldsymbol{\Sigma}}$ of the matrix can be estimated only in several special cases which shall be discussed later.
- It can be shown that if we used OLS instead of GLS in the generalised linear regression model, the estimator $\boldsymbol{b}$ would remain unbiased, but it would not be the best unbiased estimator. Moreover, estimator s^2 would not remain unbiased and consequently, all other techniques using unbiasedness of s^2 would be incorrect.