Preface

This teaching material is designed for use in a preparatory course of mathematics for those wishing to study at technical universities.

The material is divided into ten chapters. The introduction to each chapter is devoted to the repetition of basic formulae. All chapters contain a variety of worked examples and exercises with their solutions.

I hope you will enjoy working with this material. I am indebted to those who have provided constructive suggestions and contributions. I would like to express my gratitude to Eva Valentová and Janet Rees who have read the entire material for English accurancy.

Petr Tomiczek

Contents

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Algebraic fractions and rationalizations

(A) Basic modifications B) Radical expressions C) Dividing a polynomial by a polynomial)

Notation: $\mathbb{N}$ - Natural numbers, $\mathbb{Z}$ - Integers, $\mathbb{Q}$ - Rational numbers, $\mathbb{R}$ - Real numbers.

Properties of fractions, $a,b,c,d \in \mathbb{R}$:

$$\frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm cd}{bd}\,, \quad \frac{a}{b}\cdot\frac{c}{d} = \frac{ac}{bd} \quad bd \neq 0\,; \qquad \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{ad}{bc} \quad bcd \neq 0\,; \qquad \frac{ac}{bc} = \frac{a}{b} \quad bc \neq 0\,.$$

Properties of exponentials and radicals, $a,b \in \mathbb{R},\ m,n \in \mathbb{N}$:

$$a^m a^n = a^{m+n}\,; \quad (a^m)^n = a^{mn}\,; \quad (ab)^m = a^m b^m\,; \quad \frac{a^m}{a^n} = a^{m-n}\,, \quad \left(\frac{b}{a}\right)^m = \frac{b^m}{a^m} \quad a \neq 0\,;$$

$$a^{1/n} = \sqrt[n]{a} = b \quad \text{if} \quad b^n = a\,; \quad a^{m/n} = (a^{1/n})^m \quad (a > 0 \text{ if } n \text{ is even})\,;$$

$$\sqrt[n]{a^n} = \begin{cases} |a| & \text{if } n \text{ is even} \\ a & \text{if } n \text{ is odd} \end{cases}\,; \quad \sqrt[n]{a}\cdot\sqrt[n]{b} = \sqrt[n]{ab}\,; \quad \frac{\sqrt[n]{b}}{\sqrt[n]{a}} = \sqrt[n]{\frac{b}{a}}\,, \quad \frac{1}{a^n} = a^{-n} \quad a \neq 0\,.$$

Algebraical rules, $a,b \in \mathbb{R},\ n \in \mathbb{N}$:

$$(a+b)^2 = a^2 + 2ab + b^2\,; \quad (a-b)^2 = a^2 - 2ab + b^2\,; \quad a^2 - b^2 = (a-b)(a+b)\,;$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3\,; \quad (a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3\,;$$

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)\,; \quad a^3 + b^3 = (a+b)(a^2 - ab + b^2)\,;$$

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})\,;$$

$$a^n + b^n = (a+b)(a^{n-1} - a^{n-2}b + \cdots - ab^{n-2} + b^{n-1}) \quad \text{if } n \text{ is odd}\,.$$

A) Basic modifications

1) $\left(\frac{1}{a+1} - \frac{2a}{a^2-1}\right)\cdot\left(\frac{1}{a} - 1\right) = \frac{a-1-2a}{(a-1)(a+1)}\cdot\frac{1-a}{a} = \frac{-1-a}{-(a+1)a} = \frac{1}{a}$ [$\frac{1}{a}$]

restriction: $a \neq -1, 0, 1$

2) $\left(\frac{x^3}{y^2} + \frac{x^2}{y} + x + y\right) : \left(\frac{x^2}{y^2} - \frac{y^2}{x^2}\right) = \frac{x^3 + x^2y + xy^2 + y^3}{y^2} : \frac{x^4 - y^4}{y^2x^2} =$ [$\frac{x^2}{x-y}$]

$\frac{x^3 + x^2y + xy^2 + y^3}{y^2}\cdot\frac{y^2x^2}{x^4 - y^4} = \frac{x^2}{x-y}$ restrictions: $x\cdot y \neq 0,\ x \neq \pm y$

3) $\sqrt{a\sqrt[3]{b}} : \sqrt[3]{b\sqrt{a^3}} - \sqrt{b} : \sqrt[3]{b^2} = a^{(\frac{1}{2}-\frac{3}{6})}b^{(\frac{1}{6}-\frac{1}{3})} - b^{(\frac{1}{2}-\frac{2}{3})} = 0$ [0]

restrictions: $a > 0,\ b > 0$

4) $\dfrac{1 + \frac{1}{x-1}}{1 - \frac{1}{x+1}}$ [$\frac{x+1}{x-1}\,;\ x \neq -1, 0, 1$]

5) $\dfrac{m^4-m}{2m^2+2m+2}$ $\quad [\,\frac{m(m-1)}{2}\,;\ m\in\mathbb{R}\,]$

6) $\dfrac{a-2b}{a+b}-\dfrac{2a-b}{b-a}-\dfrac{2a^2}{a^2-b^2}$ $\quad [\,\frac{a-b}{a+b}\,;\ a=\pm b\,]$

7) $\left(\dfrac{1}{n-1}-\dfrac{3}{n^3-1}+\dfrac{3}{n^2+n+1}\right)\cdot\left(n+\dfrac{2n+1}{n-1}\right)$ $\quad [\,\frac{n+5}{n-1}\,;\ n\neq 1\,]$

8) $\dfrac{5a}{3(4-a)}+\dfrac{a+4}{8-3a}\cdot\left(\dfrac{a-1}{a+4}-\dfrac{a-3}{a-4}\right)$ $\quad [\,\frac{-5a+6}{3(a-4)}\,;\ a\neq\pm 4,\frac{8}{3}\,]$

9) $\left(\dfrac{x}{x^2-4}-\dfrac{2}{x^2+2x}\right)\cdot\dfrac{x^2-2x}{12}+\dfrac{x+8}{x-2}$ $\quad [\,\frac{x^3+8x^2+128x+184}{12(x-2)(x+2)}\,;\ x\neq -2,0,2\,]$

10) $2n-\left(\dfrac{2n-3}{n+1}-\dfrac{n+1}{2-2n}-\dfrac{n^2+3}{2n^2-2}\right)\cdot\dfrac{n^3+1}{n^2-n}$ $\quad [\,2\frac{n-1}{n}\,;\ n\neq -1,0,1\,]$

11) $\left(\dfrac{2a^2-2}{a^2+ab}\cdot\dfrac{a+b}{1-a}\right):\dfrac{a^3+1}{a}$ $\quad [\,-\frac{2}{a^2-a+1}\,;\ a\neq -b, a\neq -1, a\neq 0\,]$

12) $\dfrac{\frac{a+b}{a-b}-\frac{a-b}{a+b}}{1-\frac{a^2+b^2}{a^2-b^2}}\cdot\dfrac{2-\frac{1+b^2}{b}}{\frac{1}{b^2}-\frac{2}{b}+1}$ $\quad [\,2a\,;\ a\neq\pm b, b\neq 0,1\,]$

13) $\dfrac{\frac{a^4-b^4}{a^2b^2}}{\left(1+\frac{b^2}{a^2}\right)\cdot\left(1-\frac{2a}{b}+\frac{a^2}{b^2}\right)}$ $\quad [\,\frac{a+b}{a-b}\,;\ a\neq b, a\neq 0, b\neq 0\,]$

14) $\left[b^2-\dfrac{a}{1+(\frac{b-a}{a})^{-1})}\cdot\left(\dfrac{ab}{b-a}-a\right)\right]:\dfrac{a^2+ab+b^2}{b}$ $\quad [\,b-a\,;\ a\neq 0, b\neq 0, a\neq b\,]$

15) $\dfrac{2a+b}{a^2+2ab+b^2}:\left(\dfrac{3}{a-b}+\dfrac{3a}{a^3-b^3}\cdot\dfrac{a^2+ab+b^2}{a+b}\right)$ $\quad [\,\frac{a-b}{3(a+b)}\,;\ a\neq\pm b, a\neq -\frac{b}{2}\,]$

16) $\left(\dfrac{2(a+b)^2}{ab}-8\right):\left(\dfrac{a}{a+b}-\dfrac{b}{b-a}+\dfrac{2ab}{b^2-a^2}\right)$ $\quad [\,\frac{2(a^2-b^2)}{ab}\,;\ a\neq\pm b, a\neq 0, b\neq 0\,]$

17) $\dfrac{a^3+b^3}{a+b}:(a^2-b^2)+\dfrac{a}{a+b}-\dfrac{a^2}{a^2-b^2}$ $\quad [\,\frac{a-b}{a+b}\,;\ a\neq\pm b\,]$

18) $\dfrac{\left(\frac{x}{y}+\frac{y}{x}-1\right)\cdot\left(\frac{x}{y}+\frac{y}{x}+1\right)}{\left(\frac{x^4}{y^2}-\frac{y^4}{x^2}\right):\left(x^2-y^2\right)}$ $\quad [\,1\,;\ x\neq\pm y, x\neq 0, y\neq 0\,]$

19) $\left[\left(\dfrac{3}{x-y}+\dfrac{3x}{x^3-y^3}\cdot\dfrac{x^2+xy+y^2}{x+y}\right):\dfrac{2x+y}{x^2+2xy+y^2}\right]\cdot\dfrac{3}{x+y}$ $\quad [\,\frac{9}{x-y}\,;\ x\neq\pm y, x=-\frac{y}{2}\,]$

20) $\left(\dfrac{2x+y}{x+y}-\dfrac{x-2y}{x-y}-\dfrac{x^2}{x^2-y^2}\right):\dfrac{x^3+y^3}{x^4-y^4}$ $\quad [\,\frac{y^2(x^2+y^2)}{x^3+y^3}\,;\ x\neq\pm y\,]$

21) $\dfrac{6x-3}{2x+2}\cdot\left(\dfrac{2x}{1-4x+4x^2}-\dfrac{4x^2+2x}{8x^3-1}\right)$ $\quad [\,\frac{6x}{8x^3-1}\,;\ x\neq\frac{1}{2},-1\,]$

B) Radical expressions

22) $\dfrac{2+\sqrt{3}}{\sqrt{3}+\sqrt{2+\sqrt{3}}}-\dfrac{2-\sqrt{3}}{\sqrt{3}-\sqrt{2+\sqrt{3}}}=$ $\qquad$ [$(-3+2\sqrt{2+\sqrt{3}})(1+\sqrt{3})$]

$$\frac{(2+\sqrt{3})(\sqrt{3}-\sqrt{2+\sqrt{3}})-(2-\sqrt{3})(\sqrt{3}+\sqrt{2+\sqrt{3}})}{3-(2+\sqrt{3}}=$$

$$\frac{2\sqrt{3}-2\sqrt{2+\sqrt{3}}+3-\sqrt{3}\sqrt{2+\sqrt{3}}-2\sqrt{3}-2\sqrt{2+\sqrt{3}}+3+\sqrt{3}\sqrt{2+\sqrt{3}}}{1-\sqrt{3}}=$$

$$\frac{-4\sqrt{2+\sqrt{3}}+6}{1-\sqrt{3}}=\frac{(-4\sqrt{2+\sqrt{3}}+6)(1+\sqrt{3})}{(1-\sqrt{3})(1+\sqrt{3})}=$$

$$\frac{(-4\sqrt{2+\sqrt{3}}+6)(1+\sqrt{3})}{-2}=(-3+2\sqrt{2+\sqrt{3}})(1+\sqrt{3})$$

23) $\left(\dfrac{a-x}{\sqrt[3]{a}-\sqrt[3]{x}}-\dfrac{a+x}{\sqrt[3]{a}+\sqrt[3]{x}}\right)\cdot\dfrac{1}{2\sqrt[3]{ax}}=$ $\qquad$ [1]

$$\frac{a\sqrt[3]{a}+a\sqrt[3]{x}-x\sqrt[3]{a}-x\sqrt[3]{x}-(a\sqrt[3]{a}-a\sqrt[3]{x}+x\sqrt[3]{a}-x\sqrt[3]{x})}{(\sqrt[3]{a}-\sqrt[3]{x})(\sqrt[3]{a}+\sqrt[3]{x})}\cdot\frac{1}{2\sqrt[3]{ax}}=$$

$$\frac{\sqrt[3]{a}\sqrt[3]{x}(\sqrt[3]{a^2}-\sqrt[3]{x^2})}{\sqrt[3]{a^2}-\sqrt[3]{x^2}}\cdot\frac{1}{\sqrt[3]{ax}}=1$$

restrictions: $a\cdot x\neq 0$, $a\neq\pm x$

24) $(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})+(\sqrt{3}+\sqrt{2})^2+(\sqrt{3}-\sqrt{2})^2$ $\qquad$ [11]

25) $\dfrac{1}{\sqrt{3}+2}+\dfrac{4}{\sqrt{3}-1}-\dfrac{3}{\sqrt{3}}$ $\qquad$ [4]

26) $\dfrac{1}{\frac{1}{\sqrt{1-a^2}}+\frac{1}{1-a^2}}\cdot\left(1+\dfrac{a^2}{\sqrt{1-a^2}}+\sqrt{1-a^2}\right)$ $\qquad$ [$\sqrt{1-a^2}$; $|a|<1$]

27) $\dfrac{x+\sqrt{x^2-1}}{x-\sqrt{x^2-1}}+\dfrac{x-\sqrt{x^2-1}}{x+\sqrt{x^2-1}}$ $\qquad$ [$4x^2-2$; $|x|\geq 1$]

28) $\dfrac{x+\sqrt{3}}{x}+\dfrac{x-\frac{3}{x}}{x+\sqrt{3}}$ $\qquad$ [2 ; $x\neq-\sqrt{3},0$]

29) $\dfrac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}+\dfrac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}$ $\qquad$ [$\frac{2(a+b)}{a-b}$; $a\geq 0, b\geq 0, a\neq b$]

30) $\left(\sqrt{a}+\dfrac{b-\sqrt{ab}}{\sqrt{a}+\sqrt{b}}\right):\left(\dfrac{a}{\sqrt{ab}+b}+\dfrac{b}{\sqrt{ab}}-\dfrac{a+b}{\sqrt{ab}}\right)$ $\qquad$ [$-\frac{a+b}{\sqrt{a}}$; $a>0, b>0$]

31) $\left(\dfrac{a\sqrt{a}+b\sqrt{b}}{\sqrt{a}+\sqrt{b}}-\sqrt{ab}\right):\left(a-b\right)+\dfrac{2\sqrt{b}}{\sqrt{a}+\sqrt{b}}$ $\qquad$ [1 ; $a\geq 0, b\geq 0, a\neq b$]

32) $x^2\left[\dfrac{(\sqrt[4]{x}+\sqrt[4]{y})^2+(\sqrt[4]{x}-\sqrt[4]{y})^2}{x+\sqrt{xy}}\right]^5\cdot\sqrt[3]{\dfrac{\sqrt{x}}{x^{-1}}}$ $[\,2^5\,;\ y\geq 0, x>0\,]$

33) $\dfrac{(x^{\frac{9}{8}}\cdot y^{\frac{5}{4}})^{\frac{2}{3}}\cdot z^{-\frac{3}{4}}}{x^{\frac{2}{3}}\cdot y^{\frac{3}{4}}\cdot z^{-\frac{5}{6}}}$ $[\,(xyz)^{\frac{1}{12}}\,;\ x>0, y>0, z>0\,]$

34) $\sqrt{2xy}\cdot\sqrt[3]{4x^2y^4}\cdot\sqrt[4]{8x^3y^2}\cdot\sqrt[6]{x^5y^7}\cdot\sqrt[12]{2x^3y^9}$ $[\,4\cdot x^3y^{\frac{17}{4}}\,;\ x\geq 0, y\geq 0\,]$

35) $\left[\dfrac{3x^{-\frac{1}{3}}}{x^{\frac{2}{3}}-2x^{-\frac{1}{3}}}-\dfrac{x^{\frac{1}{3}}}{x^{\frac{4}{3}}-x^{\frac{1}{3}}}\right]^{-1}-\left(\dfrac{1-2x}{3x-2}\right)^{-1}$ $[\,\frac{x^2}{(2x-1)}\,;\ x\neq 0,\frac{1}{2},\frac{2}{3},1,2\,]$

36) $\left(\dfrac{a^{-\frac{2}{3}}}{b^{-1}}-\dfrac{b^{-1}}{a^{-\frac{2}{3}}}\right):\left(\dfrac{a^{-\frac{1}{3}}}{b^{-\frac{1}{2}}}-\dfrac{b^{-\frac{1}{2}}}{a^{-\frac{1}{3}}}\right)$ $[\,b^{\frac{1}{2}}a^{-\frac{1}{3}}+b^{-\frac{1}{2}}a^{\frac{1}{3}}\,;\ b>0, a\neq 0, b\neq a^{\frac{2}{3}}\,]$

37) $\left(a^{\frac{1}{4}}+b^{\frac{1}{4}}\right):\left[\left(\dfrac{a\sqrt[3]{b}}{b\sqrt{a^3}}\right)^{\frac{3}{2}}+\left(\dfrac{\sqrt{a}}{a\sqrt[8]{b^3}}\right)^2\right]$ $[\,ab\,;\ a>0, b>0\,]$

38) $\dfrac{\sqrt{x}+1}{x+\sqrt{x}+1}:\dfrac{1}{x\sqrt{x}-1}$ $[\,x-1\,;\ x\geq 0, x\neq 1\,]$

C) Dividing a polynomial by a polynomial

$$(3x^3-x^2+x-4):(x-1)=3x^2+2x+3-\frac{1}{x-1},\quad x\neq 1$$

$$\begin{array}{l}
-(3x^3-3x^2)\\
\hline
\quad 0\ +2x^2\\
\qquad -(2x^2-2x)\\
\hline
\qquad\quad 0\ +3x\\
\qquad\qquad -(3x-3)\\
\hline
\qquad\qquad\quad 0\ -1
\end{array}$$

39) $(2x^4-x^3+x^2-x-1):(x^2+1)$ $[\,2x^2-x-1\,]$

40) $(x^3-5x^2+5x-2):(x-4)$ $[\,x^2-x+1+\frac{2}{x-4}\,;\ x\neq 4\,]$

41) $(2x^5-3x^3+x-5):(x^3+1)$ $[\,2x^2-3+\frac{-2x^2+x-2}{x^3+1}\,;\ x\neq -1\,]$

Equations

A) Linear B) Quadratic and radical C) With an absolute value D) Exponential and logarithmic E) Simultaneous

First we must state the restrictions under which the equation makes sense.

$$\sqrt{5-x}+2=-2(x-1)\,,\qquad \text{restriction: } x\le 5\,.$$

We can solve the equation by using the following steps:

$$\begin{array}{rcllll}
\frac{1}{2}\sqrt{5-x}+1 &=& -(x-1) & / & \cdot 2 & \text{Multiplication with a non-zero number}\\
\sqrt{5-x}+2 &=& -2x+2 & / & -2 & \text{Subtraction}\\
\sqrt{5-x} &=& -2x & / & {}^2 & \text{Even powers - this is a nonequivalent step}\\
5-x &=& 4x & / & +x & \text{Addition}\\
5 &=& 5x & / & :5 & \text{Division by a non-zero number}\\
x &=& 1 & & & \text{We must check the proposed solution:}
\end{array}$$

Verification: Left-hand side: $\frac{1}{2}\sqrt{5-1}+1=2$ Right-hand side: $-(1-1)=0\,.$

Conclusion: This equation has no solution.

A) Linear equations

1) $\dfrac{x+2}{\frac{3}{2}x+1}=\dfrac{2x-1}{3x+1}$ restrictions: $x\neq -\dfrac{1}{3},\ x\neq -\dfrac{2}{3}$ $[\,x=-\frac{6}{13}\,]$

$$\begin{aligned}
(3x+1)(x+2) &= (2x-1)\Big(\frac{3}{2}x+1\Big)\\
3x^2+7x+2 &= 3x^2-\frac{3}{2}x+2x-1\\
\frac{13}{2}x &= -3\\
x &= -\frac{6}{13}
\end{aligned}$$

2) $\dfrac{7+3x}{9}-2+\dfrac{x+2}{4}=5x$ $[\,x=-\frac{26}{159}\,]$

$$\begin{aligned}
4\cdot(7+3x)-2\cdot 4\cdot 9+(x+2)\cdot 9 &= 9\cdot 4\cdot 5\cdot x\\
28+12x-72+9x+18 &= 180x\\
-26 &= 159x\\
x &= -\frac{26}{159}
\end{aligned}$$

3) $x = x - 1$ [no solution]

4) $5 - \dfrac{x}{3} = 2{,}5 - \dfrac{3x+1}{12}$ $[\,x = 31\,]$

5) Find the parameter p such that the equation has only one solution:

$x - 2 = \dfrac{3x - 4 - 2p^2}{p+2}$ $[\,x = -2p,\ p \neq 1, -2\,;\quad x \in \mathbb{R},\ p = 1]$

$$\begin{array}{lrcl}
p \neq -2 & (p+2)(x-2) & = & 3x - 4 - 2p^2 \\
 & px + 2x - 2p - 4 & = & 3x - 4 - 2p^2 \\
 & -2p + 2p^2 & = & x(1-p) \\
p \neq 1 & 2p(p-1) & = & -x(p-1) \\
 & x & = & -2p \\
p = 1 & x & \in & \mathbb{R}
\end{array}$$

6) $\dfrac{2x-3}{3} = \dfrac{2p^2 - x + 1}{p-1}$ $[\,x = 3p\,;\ p \neq -\tfrac{1}{2}, 1\,]$

B) Quadratic equations

A quadratic equation $ax^2 + bx + c = 0\,,\quad a, b, c \in \mathbb{R}\,,$ has two roots:

$x_1 = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}\,,\quad x_2 = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$ if $b^2 - 4ac \geq 0\,.$

It holds: $ax^2 + bx + c = a(x - x_1)(x - x_2)$ and $c = ax_1x_2,\ b = -a(x_1 + x_2).$

If $b^2 - 4ac < 0$ then the equation has no solution for $x \in \mathbb{R}\,.$

7) $x^2 - 7x + 12 = 0$ $[\,x_1 = 3,\ x_2 = 4\,]$

$x_1 = \dfrac{7 + \sqrt{7^2 - 4 \cdot 12}}{2} = 4\,,\quad x_2 = \dfrac{7 - \sqrt{7^2 - 4 \cdot 12}}{2} = 3\,.$

$x^2 - 7x + 12 = (x-3)(x-4)$ and $12 = 3 \cdot 4,\ -7 = -(3+4)\,.$

8) $\dfrac{x+2}{x+3} = \dfrac{2x-1}{3x+1}$ restrictions: $x \neq -3,\ x \neq -\dfrac{1}{3}$ [no solution]

$$\begin{array}{rcl}
(3x+1)(x+2) & = & (2x-1)(x+3) \\
3x^2 + 7x + 2 & = & 2x^2 + 5x - 3 \\
x^2 + 2x + 5 & = & 0 \\
x_{1,2} & = & \dfrac{-2 \pm \sqrt{4 - 20}}{2}
\end{array}$$

9) $2\sqrt{x+2} = x+1$ restriction: $x \geq -2$ $[\, x = 1 \pm \sqrt{8}\,]$

$$\begin{aligned} 4(x+2) &= x^2+2x+1 \\ 0 &= x^2-2x-7 \\ x_{1,2} &= \frac{2 \pm \sqrt{4+28}}{2} \end{aligned}$$

$x_1 = 1+\sqrt{8} \quad x_2 = 1-\sqrt{8}$

10) $\dfrac{4}{\sqrt{3x+1}} + \sqrt{3x+1} = \sqrt{5x}$ restriction: $x \geq 0$ $[\, x = \frac{5}{6}\,]$

$$\begin{aligned} 4+3x+1 &= \sqrt{5x}\cdot\sqrt{3x+1} \\ 9x^2+30x+25 &= 15x^2+5x \\ 0 &= 6x^2+25x+25 \\ x_{1,2} &= \frac{-25 \pm \sqrt{25^2+24\cdot 25}}{12} \end{aligned}$$

$x_1 = \frac{5}{6} \quad x_2 = -5$ (it contradicts our restriction)

11) $\dfrac{\sqrt{2}-\sqrt{x}}{2-x} = \sqrt{\dfrac{1}{2-x}}$ restriction: $2 > x \geq 0$ $[\, x = 0\,]$

$$\begin{aligned} \frac{1}{\sqrt{2}+\sqrt{x}} &= \frac{1}{\sqrt{2-x}} \\ 2-x &= 2+2\sqrt{2x}+x \\ -x &= \sqrt{2x} \\ x^2 &= 2x \end{aligned}$$

$x_1 = 0 \quad x_2 = 2$ (it contradicts our restriction)

12) $\sqrt{2x+5} + \sqrt{5x-9} = \sqrt{7x+2}$ restriction: $x \geq \dfrac{9}{5}$ $[\, x = 2\,]$

$$\begin{aligned} 2x+5+2\sqrt{2x+5}\cdot\sqrt{5x-9}+5x-9 &= 7x+2 \\ 2\sqrt{2x+5}\cdot\sqrt{5x-9} &= 6 \\ (2x+5)(5x-9) &= 9 \\ 10x^2+7x-54 &= 0 \\ x_{1,2} &= \frac{-7 \pm \sqrt{7^2+4\cdot 54\cdot 10}}{20} \end{aligned}$$

$x_1 = 2 \quad x_2 = -\dfrac{27}{10}$ (it contradicts our restriction)

13) $2\sqrt{x+5} = x+2$ $[\, x = 4\,]$

14) $1+\sqrt{x+11} = x$ $[\, x = 5\,]$

15) $\dfrac{x+\sqrt{3}}{x} - \dfrac{2x}{x+\sqrt{3}} = 2$ $[\, x_{1,2} = \pm 1\,]$

16) $\sqrt{3x+1} = x-2$ $[\, x = \frac{7}{2} + \frac{\sqrt{37}}{2}\,]$

17) $\sqrt{1+x\sqrt{x^2+24}} = x+1$ $[\,x_1 = 0\,, x_2 = 5\,]$

18) $x^2 + 4ax + 36 = 0$ $[\,x_{1,2} = -2a \pm 2\sqrt{a^2-9}\,]$

19) Find a number $a \in \mathbb{R}$ such that the following quadratic equation has two coincident solutions. $(a-1)x^2 + 2(a+1)x + a - 2 = 0\,.$ $[\,a = \frac{1}{5}\,]$

20) Determine a number p such that the sum of the second powers of the roots of the following equation is the smallest $x^2 + (p-2)x - p + 1 = 0\,.$ $[\,p = 1\,]$

21) Determine numbers $a, b, c \in \mathbb{R}$ for function $f(x) = ax^2 + bx + c$ such that

$f(-1) = 2,\ f(0) = 1,\ f(2) = 3\,.$ $[\,\frac{2}{3}x^2 - \frac{1}{3}x + 1\,]$

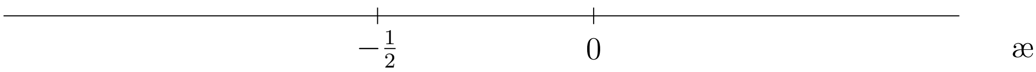
C) Absolute value equations

We define an *absolute value* $|x|$ of the real number $x \in \mathbb{R}$ as $|x| = \max\{x, -x\}\,.$

It holds: $|x| = x$ for $x \ge 0\,,$ $|x| = -x$ for $x \le 0\,.$

We solve absolute value equations on intervals on which expressions in absolute values are nonnegative or nonpositive.

22) $|2x+1| - |2x| + 1 = 2x$ $[\,x = 1\,]$



a) $x \in (-\infty, -\frac{1}{2}\rangle$

$-2x - 1 + 2x + 1 = 2x$

$x = 0$

it contradicts our restriction

b) $x \in \langle -\frac{1}{2}, 0\rangle$

$2x + 1 + 2x + 1 = 2x$

$x = -1$

it contradicts our restriction

c) $x \in \langle 0, \infty)$

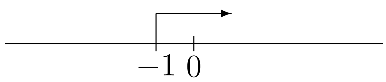
$2x + 1 - 2x + 1 = 2x$

$x = 1$

it is a solution

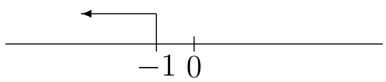
23) $1 - 2|x+1| = 3x + 2$ $[\,x = -\frac{3}{5}\,]$

a) $x + 1 > 0 \Rightarrow x > -1$



$$\begin{aligned} 1 - 2(x+1) &= 3x + 2 \\ -3 &= 5x \\ -\frac{3}{5} &= x \end{aligned}$$

b) $x + 1 < 0 \Rightarrow x < -1$



$$\begin{aligned} 1 + 2(x+1) &= 3x + 2 \\ 1 + 2x + 2 &= 3x + 2 \\ 1 &= x \end{aligned}$$

it contradicts our restriction $x < -1$

24) $1-2|x+1|=3x+2$ $\quad [\,x=-\frac{3}{5}\,]$

25) $|3-2x|+x=1$ $\quad$ [no solution]

26) $1-2|3-x|=3x+1$ $\quad [\,x=-6\,]$

27) $-2x-|3x-2|=2x-1$ $\quad [\,x=-1\,]$

28) $|x-3|=2x+3$ $\quad [\,x=0\,]$

29) $|x|=x-1$ $\quad$ [no solution]

30) $4(x+1)=|2x-1|$ $\quad [\,x=-\frac{1}{2}\,]$

31) $3+4|x-2|=5x$ $\quad [\,x=\frac{11}{9}\,]$

32) $2|x-4|+7=3x+3$ $\quad [\,x=\frac{12}{5}\,]$

33) $4(x+3)-2x=|x+\dfrac{1}{2}|$ $\quad [\,x=-4\frac{1}{6}\,]$

34) $5|x-3|-6=3(x+2)$ $\quad [\,x_1=\frac{27}{2}, x_2=\frac{3}{8}\,]$

35) $|x|=\dfrac{1}{2}x+1$ $\quad [\,x_1=2,\ x_2=-\frac{2}{3}\,]$

36) $4|x-2|+5x=15$ $\quad [\,x=\frac{23}{9}\,]$

37) $3x-|2x-1|=\dfrac{5}{6}$ $\quad [\,x=\frac{11}{30}\,]$

38) $|x-10|=4+2x$ $\quad [\,x=2\,]$

39) $1-2|x+1|=3x$ $\quad [\,x=-\frac{1}{5}\,]$

40) $|7-x|-|2x+1|=2+x$ $\quad [\,x=1\,]$

41) $|x|+|x-1|=1$ $\quad [\,x\in\langle 0,1\rangle\,]$

42) $2x-|7-x|=3+|2-x|$ $\quad [\,x=4\,]$

43) $2|x-3|+|6-2x|=|x+7|$ $\quad [\,x_1=\frac{19}{3}, x_2=1\,]$

44) $|x-1|+|x+1|+1=|x|$ $\quad$ [no solution]

45) $|x|+|x-2|=x+1$ $\quad [\,x_1=1, x_2=3\,]$

46) $|3x+6|-|x-2|=2(x+1)$ $\quad [\,x_1=-\frac{5}{2}, x_2=-1\,]$

47) $|2x+1|-|x+3|=2(1-x)-3$ $\quad [\,x_1=-3, x_2=\frac{1}{3}\,]$

48) $2|5-x|+3(x+1)=2|x|+9$ $\quad [\,x_1=-\frac{4}{3}, x_2=4, x_3=\frac{16}{3}\,]$

49) $x+4-|x-2|=|x|-4$ $\quad [\,x_1=-2, x_2=10\,]$

50) $|x-2|+2|1-x|=2|x|-1$ $\quad [\,x_1=1, x_2=3\,]$

51) $|x-1|+|x+1|-\dfrac{3}{2}=|x|$ $\quad [\,x_1=-\frac{3}{2}, x_2=-\frac{1}{2}, x_3=\frac{1}{2}, x_4=\frac{3}{2}\,]$

52) $|3-2x|+|x-5|=1+2|x-1|$ $\quad [\,x_1=3, x_2=7\,]$

53) $|2x+6|-|4-2x|+|x+7|=4$ for $x\in\langle -2,2\rangle$ $\quad [\,x=-1\,]$

54) $|x+7|-|x-4|=|6-x|$ for $x\in\langle -3,3\rangle$ $\quad [\,x=1\,]$

D) Exponential and logarithmic equations

Exponential function a^x is defined for $a>0$, $x\in\mathbb{R}$. Logarithmic function $\log_b x$ is defined for $b\in(0,1)\cup(1,\infty)$, $x>0$.

Notation: $\log x=\log_{10}x$, $\quad \ln x=\log_{\mathrm{e}}x$.

Basic rules: $a^x\cdot a^y=a^{x+y}$, $\quad \dfrac{a^x}{a^y}=a^{x-y}$, $\quad a^{-x}=\dfrac{1}{a^x}$,

$$\log x+\log y=\log(x\cdot y), \quad \log x-\log y=\log\frac{x}{y}, \quad \log x^n=n\cdot\log x,$$

$$\log_a x=\log_a b\cdot\log_b x, \quad \log_a a=1, \quad \log_a 1=0.$$

55) a) $2^x = 1 \Rightarrow 2^x = 2^0 \Rightarrow x = 0$

b) $2^x = -3$, no solution $(a^x > 0)$

c) $2^x = \frac{1}{4} \Rightarrow 2^x = 2^{-2} \Rightarrow x = -2$

d) $2^x = 3 \Rightarrow \log 2^x = \log 3 \Rightarrow x \log 2 = \log 3 \Rightarrow x = \frac{\log 3}{\log 2} \Rightarrow x = \log_2 3$

56) $4 \cdot 3^x + 2 \cdot 3^{x+1} = \frac{10}{3}$ $[x = -1]$

$4 \cdot 3^x + 2 \cdot 3 \cdot 3^x = \frac{10}{3} \Rightarrow 3^x = \frac{1}{3} \Rightarrow x = -1$

57) $4^{2x+1} = 65 \cdot 4^{x-1} - 1$ $[x_1 = 1, x_2 = -2]$

Substitution: $y = 4^x$, $4y^2 = \frac{65}{4}y - 1 \Rightarrow 16y^2 - 65y + 4 = 0 \Rightarrow$

$(16y - 1)(y - 4) = 0 \Rightarrow x_1 = 1,\ x_2 = -2$.

58) $\log(x + 3) + \log(x - 2) = 2 - \log 2$ restriction: $x > 2$ $[x = 7]$

$\log(x + 3) \cdot (x - 2) = \log 100 - \log 2 \Rightarrow \log(x^2 + x - 6) = \log 50 \Rightarrow x^2 + x - 56 = 0 \Rightarrow$

$x_1 = 7$, $x_2 = -8$ (it contradicts our restriction)

59) $4^x - 2^{x+1} + 5 = \log_2 16$ $[x = 0]$

$2^{2x} - 2 \cdot 2^x + 5 = \log_2 2^4 \Rightarrow (2^x)^2 - 2 \cdot 2^x + 1 = 0 \Rightarrow y^2 - 2y + 1 = 0 \Rightarrow$
$(y - 1)^2 = 0 \Rightarrow y = 1 \Rightarrow 2^x = 1 \Rightarrow x = 0$

60) $2^{2+x} - 2^{2-x} = 15$ $[x = 2]$

$42^x - 42^{-x} = 15 \Rightarrow 4(2^x)^2 - 152^x - 4 = 0 \Rightarrow 4(y)^2 - 15y - 4 = 0 \Rightarrow$

$(4y + 1)(y - 4) = 0 \Rightarrow -\frac{1}{4} = 2^x,\ 4 = 2^x \Rightarrow x = 2$

61) $\frac{\log(x^2 + 14)}{\log(7 - x)} = 2$ restrictions: $x < 7,\ x \neq 6$ $[x = \frac{35}{14}]$

$\log(x^2 + 14) = 2 \log(7 - x) \Rightarrow \log(x^2 + 14) = \log(7 - x)^2 \Rightarrow$

$\log(x^2 + 14) = \log(49 - 14x + x^2) \Rightarrow x^2 + 14 = 49 - 14x + x^2 \Rightarrow -35 = -14x$

62) $\frac{\log(x^2 + 5)}{2 \log(3 - x)} = 1$ restrictions: $x < 3,\ x \neq 2$ $[x = \frac{2}{3}]$

$x^2 + 5 = 9 - 6x + x^2 \Rightarrow -4 = -6x$

63) $\frac{1}{2} \log(2x + 5) = \log(x - 5)$ restriction: $x > 5$ $[x = 10]$

$2x + 5 = x^2 - 10x + 25 \Rightarrow 0 = x^2 - 12x + 20 \Rightarrow x_{1,2} = \frac{12 \pm \sqrt{144 - 80}}{2} \Rightarrow x = 10$

64) $2\log x - \log\dfrac{1}{x} + \log 2\sqrt{x} = \log x^3 - \log\dfrac{1}{2} - 2$ restriction: $x > 0$ $[\,x = 10^{-4}\,]$

$\log x^2 - \log x^{-1} + \log 2x^{\frac{1}{2}} = \log x^3 - \log\dfrac{1}{2} - \log 100 \;\Rightarrow\; \log\dfrac{x^2}{x^{-1}}\cdot 2x^{\frac{1}{2}} = \log\dfrac{x^3}{\frac{1}{2}}\cdot 100 \;\Rightarrow$

$2x^{\frac{7}{2}} = \dfrac{x^3}{50} \;\Rightarrow\; \sqrt{x} = \dfrac{1}{100} \;\Rightarrow\; x = \dfrac{1}{10^4}$

65) $\log x^3 - \log 2x - \log\sqrt{x} = \dfrac{1}{2}\log x - \log 20 + 2$ restriction: $x > 0$ $[\,x = 10\,]$

$\log\dfrac{x^3}{2x\cdot\sqrt{x}} = \log\dfrac{\sqrt{x}}{20}\cdot 100 \;\Rightarrow\; \dfrac{1}{2}x^{\frac{3}{2}} = 5\sqrt{x} \;\Rightarrow\; x = 10$ ($x = 0$ it contradicts our restriction)

66) $\dfrac{3^{2x-1}}{3^{2+x}} = \dfrac{\log 4}{\log 2}$ $[\,x = \log_3 54\,]$

$3^{2x-1-(2+x)} = \dfrac{2\log 2}{\log 2} \Rightarrow 3^{x-3} = 2 \Rightarrow (x-3)\log_3 3 = \log_3 2 \Rightarrow$

$x = \log_3 2 + 3 = \log_3 2 + \log_3 27 = \log_3 54$

67) $\left(\dfrac{3}{2}\right)^{2x-1}\cdot\left(\dfrac{8}{27}\right)^{x+3} = \dfrac{16}{81}$ $[\,x = 0\,]$

$\left(\dfrac{3}{2}\right)^{2x-1}\cdot\left(\dfrac{2}{3}\right)^{3(x+3)} = \left(\dfrac{2}{3}\right)^4 \;\Rightarrow\; \left(\dfrac{2}{3}\right)^{3x+9-2x+1} = \left(\dfrac{2}{3}\right)^4 \;\Rightarrow\; x + 10 = 4 \;\Rightarrow\; x = -6$

68) $\left(\dfrac{25}{49}\right)^{1-x}\cdot\left(\dfrac{7}{5}\right)^{x} = \left(\dfrac{49}{25}\right)^3$ $[\,x = \frac{8}{3}\,]$

$\left(\dfrac{7}{5}\right)^{2(1-x)}\cdot\left(\dfrac{7}{5}\right)^{x} = \left(\dfrac{7}{5}\right)^3 \;\Rightarrow\; 5 - 2 + 3x = 6 \;\Rightarrow\; x = \dfrac{8}{3}$

69) $2\cdot 3^{x+1} - 6\cdot 3^{x-1} - 3^x = 9$ $[\,x = 1]$

70) $2^{2x+1} + 4^{x+1} + 16^{\frac{x}{2}} = 28$ $[\,x = 1]$

71) $16^{3x-2} - 2\cdot 8^x = 0$ $[\,x = 1]$

72) $2^{x^2-6x-\frac{5}{2}} = 16\sqrt{2}$ $[\,x_1 = 7\,, x_2 = -1]$

73) $\dfrac{1}{2}\log(2x-3) = \log(x-3)$ $[\,x = 6]$

74) $\log(4x-3) + \log(3x-4) - \log(3-x) - \log(x-1) = 1$ $[\,x = 2\,]$

75) $(\log x)^{\log x} = 1$ $[\,x = 10]$

76) $\dfrac{2^{1-5x}}{2^{-2-3x}} = \dfrac{\log 81}{\log 9}$ $[\,x = 1\,]$

E) Simultaneous equations

Methods of solution:

elimination:

$$\begin{array}{rcl} x+y &=& 1 \\ 3x-2y &=& 8 \\ \hline x &=& 1-y \\ 3(1-y)-2y &=& 8 \\ \hline 3-3y-2y &=& 8 \\ -5y &=& 5 \\ y &=& -1 \\ x-1 &=& 1 \\ x &=& 2 \end{array}$$

modification of equations:

$$\begin{array}{rcl} x+y &=& 1 \quad / \cdot 2 \\ 3x-2y &=& 8 \\ \hline 2x+2y &=& 2 \\ 3x-2y &=& 8 \\ \hline 5x+0y &=& 10 \\ x &=& 2 \\ 2+y &=& 1 \\ y &=& -1 \end{array}$$

77) $-x-2y=-1$

$2x+4y=\ 0$ [no solution]

$-x-2y=-1 \ \Rightarrow\ x=1-2y \wedge 2x+4y=0 \ \Rightarrow\ 2(1-2y)+4y=0 \ \Rightarrow\ 2=0$

78) $2x-y=1$
$-6x+3y=-3$ [$y=2x-1, x\in\mathbb{R}$]

79) $x-3y=10$
$3x-2y=9$ [$x=1, y=-3$]

80) $x-3y=8$
$x^2-24y=100$ [$x=4\pm\sqrt{13}$]
[$y=\frac{2}{3}(-2\pm\sqrt{13})$]

81) $x^2+y^2=25$
$4x+3y=25$ [$x=4, y=3$]

82) $x^2+y^2=1$
$2x-y=1$ [$x_1=0, y_1=-1$]
[$x_2=\frac{4}{5}, y_2=\frac{3}{5}$]

83) $x^2+y^2=4$
$y-x+4=0$ [no solution]

84) $x^2+y^2=9$
$3y=x-3$ [$x_1=3, y_1=0$]
[$x_2=-\frac{12}{5}, y_2=-\frac{9}{5}$]

Determine the parameter m such that the system has only one solution.

85) $y=(x-1)^2+2$
$2x-y+m+1=0$ [$m=-2$]
[$x=2, y=3$]

86) $y=(x-1)^2+3$
$x-y+2m=0$ [$m=\frac{7}{8}$]
[$x=\frac{3}{2}, y=\frac{13}{4}$]

87) $y=(1-x)^2+m-3$
$2x+y-10=0$ [$m=12$]
[$x=0, y=10$]

88) $y=(3-x)^2+2m+5$
$x-y+4=0$ [$m=\frac{9}{8}$]
[$x=\frac{7}{2}, y=\frac{15}{2}$]

89) $y=(x-3)^2-3m+5$
$4x-y+7=0$ [$m=-6$]
[$x=5, y=27$]

90) $y=(x+1)^2+m^2+m+4$
$2x+3-y+m=0$ [no solution]

91) Find the solution depending on parameter $a\in\mathbb{R}$.

$3x+(a-1)y=12$ [$a\neq -5, 7; x=\frac{24}{5+a}, y=\frac{12}{5+a}$]

$(a-1)x+12y=24$ [$a=-5$, no solution; $a=7$, $x=4-2y$, $y\in\mathbb{R}$]

Inequalities

We use the same modifications as for equations. Multiplying and dividing each side of an inequality by the same negative number reverses the order of the inequality.

$$\begin{aligned} \frac{-1+2x}{-3} &> -1 && / \cdot (-3) && \text{multiplying by a negative number} \\ -1+2x &< 3 && /+1 \\ 2x &< 4 && / : (+2) && \text{dividing by a positive number} \\ x &< 2 \end{aligned}$$

1) $\dfrac{7x-1}{3} + 6 > 5x - \dfrac{5+3x}{2}$ $[\, x \in (-\infty, 7) \,]$

$14x - 2 + 36 > 30x - 15 - 9x \;\Rightarrow\; 49 > 7x \;\Rightarrow\; x \in (-\infty, 7)$

2) $|x-6| < x^2 - 5x + 9$ $[\, x \in (-\infty, 1) \cup (3, \infty) \,]$

$x \geq 6 \;\Rightarrow\; x - 6 < x^2 - 5x + 9 \;\Rightarrow\; 0 < x^2 - 6x + 15 \;\Rightarrow\; x \geq 6$

$x < 6 \;\Rightarrow\; -x + 6 < x^2 - 5x + 9 \;\Rightarrow\; 0 < x^2 - 4x + 3 \;\Rightarrow\; x \in (-\infty, 1) \cup (3, 6)$

3) $2(1-2x)^2 \geq 2x + 5$ $[\, x \in (-\infty, -\frac{1}{4}\rangle \cup \langle \frac{3}{2}, \infty) \,]$

$2(1 - 4x + 4x^2) \geq 2x + 5 \;\Rightarrow\; 2 - 8x + 8x^2 \geq 2x + 5 \;\Rightarrow\; 8x^2 - 10x - 3 \geq 0 \;\Rightarrow$

$8(x + \frac{1}{4})(x - \frac{3}{2}) \geq 0 \;\Rightarrow\; x \in (-\infty, -\frac{1}{4}\rangle \cup \langle \frac{3}{2}, \infty)$

4) $3 - \dfrac{3x}{2} < \dfrac{5}{8} - \dfrac{4x-3}{6}$ $[\, x \in (\frac{9}{4}, \infty) \,]$

5) $5 - \dfrac{x}{3} < 2{,}5 - \dfrac{3x+1}{12}$ $[\, x \in (31, \infty) \,]$

6) $2x - 3 > 5 \;\wedge\; -x + 5 < 6$ $[\, x \in (4, \infty) \,]$

7) $|x-2| < 5$ $[\, x \in (-3, 7) \,]$

8) $\dfrac{3}{|x+1|} - 1 \geq 0$ $[\, x \in (-4, -1) \cup (-1, 2) \,]$

9) $\dfrac{|3-5x|}{x-2} > 6$ $[\, x \in (2, 9) \,]$

10) $3x - |2x-1| \leq 5 - x$ $[\, x \in (-\infty, 2\rangle \,]$

11) $\sqrt{x+2} > x$ $[\, x \in (-1, 2) \,]$

12) $-\dfrac{x}{2} < 3x - \dfrac{|3+2x|}{4}$ $[\, x \in (\frac{1}{4}, \infty) \,]$

13) $|2x+7| \geq 3$ $[\, x \in (-\infty, -5\rangle \cup \langle -2, \infty) \,]$

14) $\dfrac{3x-5}{x^2+4x-5} \geq \dfrac{1}{2}$ $[\, x \in (-5, 1) \,]$

15) $x^2 + x - 2 < 0$ $[\, x \in (-2, 1) \,]$

16) $|x^2 + x - 2| < x$ $[\, x \in (\sqrt{3} - 1, \sqrt{2}) \,]$

17) $(x-3)^2 < 2x(x+2) - (x+1)^2 + 4$ $[\, x \in (\frac{3}{4}, \infty) \,]$

18) $\log|x+1| < 2$ $[\, x \in (-101, -1) \cup (-1, 99) \,]$

19) $1 \leq \dfrac{2\log x + 3}{3} \leq \dfrac{5}{3}$ $[\, x \in \langle 1, 10\rangle \,]$

20) $1 < \dfrac{\log x + 1}{3} < 2$ $[\, x \in (10^2, 10^5) \,]$

21) $\dfrac{1}{2} < \log x + 1 < \dfrac{5}{3}$ $[\, x \in (10^{-\frac{1}{2}}, 10^{\frac{2}{3}}) \,]$

22) $2 < \log|x| + 3 < 4$ $[\, x \in (-10; -0{,}1) \cup (0{,}1; 10) \,]$

23) $0 < \dfrac{\log|x-1| + 1}{2} \leq 1$ $[\, x \in (\langle -9; 0{,}9) \cup (1{,}1; 11\rangle \,]$

24) $|\log x - 1| < 2$ $[\, x \in (10^{-1}, 10^3) \,]$

25) $|\log \dfrac{x}{2} - 1| \geq 2$ $[\, x \in (0, \frac{1}{5}\rangle \cup \langle 2 \cdot 10^3, \infty) \,]$

Trigonometric functions, equations and inequalities

Common trigonometric identities:

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \sin\beta\cos\alpha\,, \quad \cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\beta\sin\alpha\,.$$

For $\alpha = \beta$ we get: $\cos^2\alpha + \sin^2\alpha = 1\,,\quad \sin 2\alpha = 2\sin\alpha\cos\beta\,,$
$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha,\quad 2\sin^2\alpha = 1 - \cos 2\alpha\,,\quad 2\cos^2\alpha = 1 + \cos 2\alpha\,.$

Verify the following identity:

$$\frac{\sin 2x}{1+\cos 2x}\cdot\frac{\cos x}{1+\cos x} = \mathrm{tg}\frac{x}{2}$$

Restrictions: $1+\cos 2x \neq 0 \;\Rightarrow\; 2x \neq \pi + 2\mathrm{k}\pi,\quad 1+\cos\mathrm{x} \neq 0 \;\Rightarrow\; \mathrm{x} \neq \pi + 2\mathrm{k}\pi,\quad \mathrm{k} \in \mathbb{Z}.$

Solution: $\dfrac{\sin 2x}{1+\cos 2x}\cdot\dfrac{\cos x}{1+\cos x} = \dfrac{2\sin x\cos^2 x}{2\cos^2 x(1+\cos x)} = \dfrac{\sin x}{1+\cos x} = \dfrac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}} = \mathrm{tg}\dfrac{x}{2}\,.$

Common values of trigonometric functions:

function\angle	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\sin x$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1

Simplify:

1) $\dfrac{\sin^2 x - \mathrm{tg}^2 x}{\cos^2 x - \mathrm{cotg}^2 x}$ [$(\frac{\sin x}{\cos x})^6,\quad x \neq \mathrm{k}\frac{\pi}{2}, \mathrm{k} \in \mathbb{Z}$]

Solve:

2) $\sin x\cdot(1+2\cos x) - \mathrm{tg}x = 0$ [$x_1 = \mathrm{k}\pi, \mathrm{x}_2 = \frac{\pi}{3} + 2\mathrm{k}\pi, \mathrm{x}_3 = -\frac{\pi}{3} + 2\mathrm{k}\pi$]

$\sin x\cdot\Big(1 + 2\cos x - \dfrac{1}{\cos x}\Big) = 0$ restrictions: $x \neq \frac{\pi}{2} + \mathrm{k}\pi,\ \mathrm{k} \in \mathbb{Z}$

$\sin x = 0 \;\Rightarrow\; x_1 = \mathrm{k}\pi$ or

$1 + 2\cos x - \frac{1}{\cos x} = 0 \;\Rightarrow\; \cos x + 2\cos^2 x - 1 = 0 \;\Rightarrow\; (y = \cos x)\ 2y^2 + y - 1 = 0 \;\Rightarrow$

$y_1 = -1,\ y_2 = \frac{1}{2} \;\Rightarrow\; x_2 = \frac{\pi}{3} + 2\mathrm{k}\pi,\ \mathrm{x}_3 = -\frac{\pi}{3} + 2\mathrm{k}\pi$

3) $\dfrac{2}{\sqrt{3}}\mathrm{tg}\Big(\dfrac{x}{2} + \dfrac{\pi}{4}\Big) = -\dfrac{2\sqrt{3}}{3}$ [$x = -\pi + 2\mathrm{k}\pi$]

$\mathrm{tg}\Big(\dfrac{x}{2} + \dfrac{\pi}{4}\Big) = -1 \;\Rightarrow\; \dfrac{x}{2} + \dfrac{\pi}{4} = -\dfrac{\pi}{4} + \mathrm{k}\pi \;\Rightarrow\; \mathrm{x} = -\pi + 2\mathrm{k}\pi$

4) $\sqrt{7\sin x - 2} = -\sqrt{2}\cos x$ [$x = \frac{5}{6}\pi + 2\mathrm{k}\pi$]

5) $2\sin^2 x = 3\cos x$ [$x_1 = \frac{5}{3}\pi + 2\mathrm{k}\pi, \mathrm{x}_2 = \frac{\pi}{3} + 2\mathrm{k}\pi$]

6) $\dfrac{1}{\sin^2 x} - \dfrac{2}{\sqrt{3}}\operatorname{cotg} x - 2 = 0$ $[x_1 = \frac{2}{3}\pi + \mathrm{k}\pi, \mathrm{x}_2 = \frac{\pi}{6} + \mathrm{k}\pi, \mathrm{x}_3 = \frac{\pi}{3} + \mathrm{k}\pi, \mathrm{x}_4 = -\frac{\pi}{6} + \mathrm{k}\pi]$

7) $\sin x = 1 - \cos 2x$ $[x_1 = \mathrm{k}\pi, x_2 = \frac{\pi}{6} + 2\mathrm{k}\pi, \mathrm{x}_3 = \frac{5}{6}\pi + 2\mathrm{k}\pi]$

8) $\sin x + \sin 2x = \mathrm{tg} x$ $[x_1 = \frac{\pi}{3} + 2\mathrm{k}\pi,\ x_2 = \frac{5}{3}\pi + 2\mathrm{k}\pi, \mathrm{x}_3 = \mathrm{k}\pi]$

9) $\mathrm{tg}\, x = \sin 2x$ $[x_1 = \frac{\pi}{4} + \mathrm{k}\pi, \mathrm{x}_2 = \frac{3}{4}\pi + \mathrm{k}\pi, \mathrm{x}_3 = \mathrm{k}\pi]$

10) $3 - \operatorname{cotg} x = (2 + \sqrt{3}) \operatorname{cotg} x - \sqrt{3}$ $[x = \frac{\pi}{4} + \mathrm{k}\pi]$

11) $\sin x + \cos x = \dfrac{1}{\cos x}$ $[x_1 = \mathrm{k}\pi, \mathrm{x}_2 = \frac{\pi}{4} + \mathrm{k}\pi]$

12) $\mathrm{tg}^3 x + \mathrm{tg}^2 x - 3\mathrm{tg} x - 3 = 0$ $[x_1 = \frac{1}{3}\pi + \mathrm{k}\pi, \mathrm{x}_2 = \frac{2}{3}\pi + \mathrm{k}\pi, \mathrm{x}_3 = \frac{4}{3}\pi + \mathrm{k}\pi]$

13) $2\cos 2x - 4\cos x = 1$ $[x = \frac{2}{3}\pi + 2\mathrm{k}\pi]$

14) $\sin x + \sin 2x + \sin 3x + \sin 4x = 0$ $[x_1 = \mathrm{k}\pi, \mathrm{x}_2 = \frac{\pi}{2} + \mathrm{k}\pi]$

15) $\dfrac{2}{\sin 2x} = (1 + \mathrm{tg}^2 x) \operatorname{cotg} x$ $[x \in \mathbf{R} \setminus \{\frac{1}{2}\mathrm{k}\pi\}]$

16) $\mathrm{tg}\, 2x - \operatorname{cotg} x = 0$ $[x_1 = \frac{1}{6}\pi + \mathrm{k}\pi, \mathrm{x}_2 = \frac{5}{6}\pi + \mathrm{k}\pi, \mathrm{x}_3 = \frac{1}{2}\pi + \mathrm{k}\pi]$

17) $\mathrm{tg}\, \dfrac{x}{2} = 1 - \cos x$ $[x_1 = 2\mathrm{k}\pi, \mathrm{x}_2 = \frac{1}{2}\pi + 2\mathrm{k}\pi]$

18) $|\sin x| = \sin x + 2\cos x$ $[x_1 = \frac{1}{2}\pi + 2\mathrm{k}\pi, \mathrm{x}_2 = \frac{7}{4}\pi + 2\mathrm{k}\pi]$

19) $\cos^2 x + \sin x \cos \dfrac{x}{2} = 1$ $[x_1 = \mathrm{k}\pi, \mathrm{x}_2 = \frac{1}{3}\pi + 4\mathrm{k}\pi, \mathrm{x}_3 = \frac{5}{3}\pi + 4\mathrm{k}\pi]$

20) $(\cos^2 x - 1) \operatorname{cotg}^2 x = -3\sin x$ $[x_1 = \frac{1}{6}\pi + 2\mathrm{k}\pi, \mathrm{x}_2 = \frac{5}{6}\pi + 2\mathrm{k}\pi]$

21) $\cos 2x - 3\cos x = 4\cos^2 \dfrac{x}{2}$ $[x_1 = \frac{2}{3}\pi + 2\mathrm{k}\pi, \mathrm{x}_2 = \frac{4}{3}\pi + 2\mathrm{k}\pi]$

22) $\cos \dfrac{x}{2} = 1 + \cos x$ $[x_1 = \pi + 2\mathrm{k}\pi, \mathrm{x}_2 = \frac{2}{3}\pi + 4\mathrm{k}\pi, \mathrm{x}_3 = \frac{10}{3}\pi + 4\mathrm{k}\pi]$

23) $2\sin \dfrac{x}{2} = 1 - \cos x$ $[x_1 = 2\mathrm{k}\pi, \mathrm{x}_2 = \pi + 4\mathrm{k}\pi]$

24) $2\cos x - \cos 2x = 1$ $[x_1 = 2\mathrm{k}\pi, \mathrm{x}_2 = \frac{1}{2}\pi + \mathrm{k}\pi]$

25) $\cos x + \cos 2x = 0$ $[x_1 = \frac{1}{3}\pi + 2\mathrm{k}\pi, \mathrm{x}_2 = \frac{5}{3}\pi + 2\mathrm{k}\pi, \mathrm{x}_3 = \pi + 2\mathrm{k}\pi]$

26) $\cos^2 x - \sin^2 x - \cos x = 0$ $[x = \frac{2}{3}\mathrm{k}\pi]$

27) $\cos x + \sin 2x = 0$ $[x_1 = \frac{1}{2}\pi + \mathrm{k}\pi, \mathrm{x}_2 = \frac{7}{6}\pi + 2\mathrm{k}\pi, \mathrm{x}_3 = \frac{11}{6}\pi + 2\mathrm{k}\pi]$

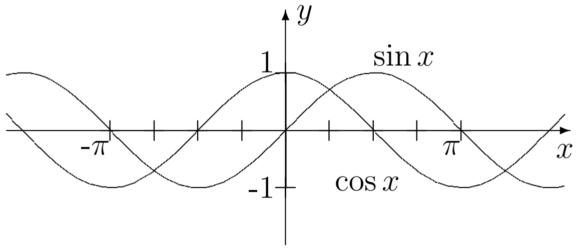
28) $\sin x + \cos 2x > 1$ $[x \in (0 + 2\mathrm{k}\pi, \frac{\pi}{6} + 2\mathrm{k}\pi) \cup (\frac{5}{6}\pi + 2\mathrm{k}\pi, \pi + 2\mathrm{k}\pi)]$

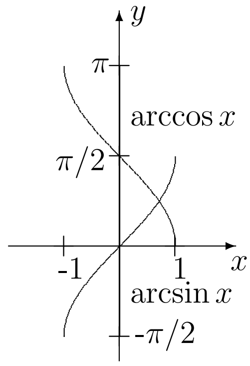
29) $\cos x > \dfrac{1}{2}$ $[x \in (-\frac{\pi}{3} + 2\mathrm{k}\pi, \frac{\pi}{3} + 2\mathrm{k}\pi)]$

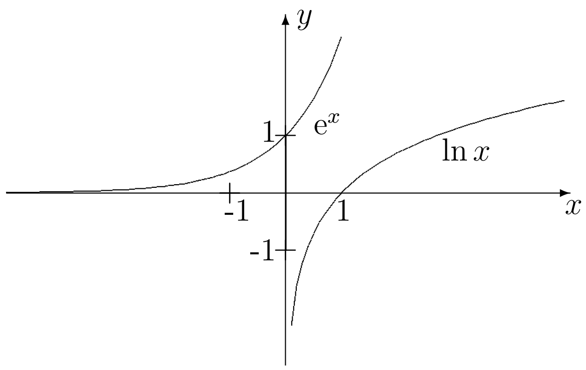
30) $\sin x > \cos x$ $[x \in (\frac{\pi}{4} + 2\mathrm{k}\pi, \frac{5}{4}\pi + 2\mathrm{k}\pi)]$

Functions, domain sets, graphs of functions

Function	Domain set	Function	Domain set
$\sin x$	$\mathbb{R}$	$\arcsin x$	$\langle -1, 1 \rangle$
$\cos x$	$\mathbb{R}$	$\arccos x$	$\langle -1, 1 \rangle$
e^x	$\mathbb{R}$	$\ln x$	$(0, \infty)$
$\mathrm{tg}\, x$	$\mathbb{R} \setminus \{\frac{\pi}{2} + k\pi\}$, $k \in \mathbb{Z}$	$\mathrm{arctg}\, x$	$\mathbb{R}$
$\mathrm{cotg}\ x$	$\mathbb{R} \setminus \{k\pi\}$, $k \in \mathbb{Z}$	$\mathrm{arcotg}\, x$	$\mathbb{R}$
x^2	$\mathbb{R}$	$\sqrt{x}$	$\langle 0, \infty)$







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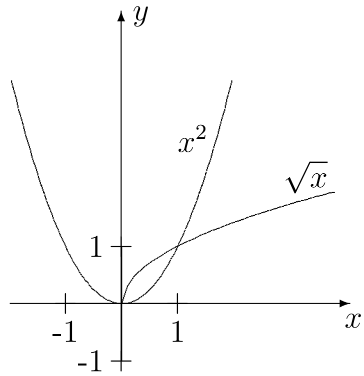

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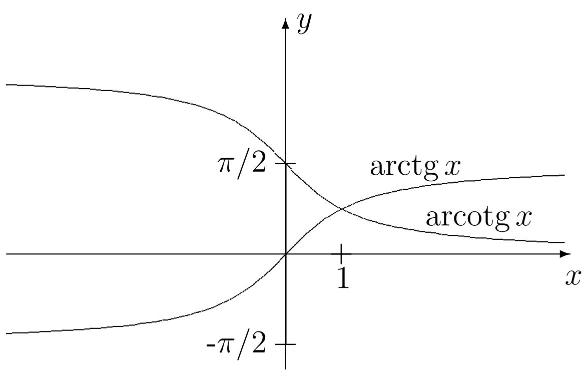

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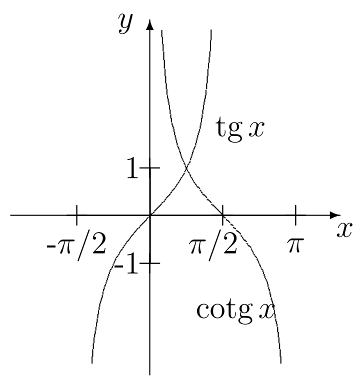

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Find the domain set:

1) $y = \sqrt{\dfrac{1-\operatorname{cotg} x}{1+\operatorname{cotg} x}}$ $\qquad [\, D(y) = \langle \frac{\pi}{4} + \mathrm{k}\pi, \frac{3}{4}\pi + \mathrm{k}\pi) \,]$

$$\frac{1-\operatorname{cotg} x}{1+\operatorname{cotg} x} \geq 0 \quad \wedge \quad 1+\operatorname{cotg} x \neq 0 \quad \Rightarrow$$

$$1-\operatorname{cotg} x \geq 0 \quad \wedge \quad 1+\operatorname{cotg} x > 0 \qquad \text{or} \qquad 1-\operatorname{cotg} x \leq 0 \quad \wedge \quad 1+\operatorname{cotg} x < 0$$

$$1 \geq \operatorname{cotg} x \quad \wedge \quad \operatorname{cotg} x > -1 \qquad \text{or} \qquad 1 \leq \operatorname{cotg} x \quad \wedge \quad \operatorname{cotg} x < -1$$

$$x \in \langle \tfrac{\pi}{4} + \mathrm{k}\pi, \tfrac{3}{4}\pi + \mathrm{k}\pi) \qquad\qquad \text{no solution}$$

2) $y = \sqrt{x+1} + \log(x^2 - 3x)$ $\qquad [\, D(y) = \langle -1, 0) \cup (3, \infty) \,]$

$x + 1 \geq 0 \ \wedge \ x(x-3) > 0 \ \Rightarrow \ x \in \langle -1, \infty) \ \wedge \ x \in (-\infty, 0) \cup (3, \infty)$

3) $y = x + \sqrt{1-x}$ $\qquad [\, D(y) = (-\infty, 1\rangle \,]$

4) $y = \sqrt{\dfrac{x-2}{1-3x}}$ $\qquad [\, D(y) = (\frac{1}{3}, 2\rangle \,]$

5) $y = \dfrac{x}{x^2 + 8x + 7}$ $\qquad [\, D(y) = \mathbb{R} \setminus \{-1, -7\} \,]$

6) $y = \dfrac{1}{\sqrt{|x| - x - 1}}$ $\qquad [\, D(y) = (-\infty, -\frac{1}{2}) \,]$

7) $y = \log \dfrac{x+2}{x-3}$ $\qquad [\, D(y) = (-\infty, -2) \cup (3, \infty) \,]$

8) $y = \sqrt{x^2 - 4x + 3}$ $\qquad [\, D(y) = (-\infty, 1\rangle \cup \langle\, 3, \infty) \,]$

9) $y = \sqrt{x+2} + \dfrac{1}{\log(1-x)}$ $\qquad [\, D(y) = \langle -2, 0) \cup (0, 1) \ ]$

10) $y = \log(2x^2 + 4x - 6)$ $\qquad [\, D(y) = (-\infty, -3) \cup (1, \infty) \,]$

11) $y = \sqrt{1 + 2\sin x} - \sqrt{1 - 2\sin x}$ $\qquad [\, D(y) = \langle \frac{5}{6}\pi + 2\mathrm{k}\pi, \frac{7}{6}\pi + 2\mathrm{k}\pi\rangle \cup \langle \frac{11}{6}\pi + 2\mathrm{k}\pi, \frac{13}{6}\pi + 2\mathrm{k}\pi\rangle \,]$

Draw graphs of the following functions:

1) $y = \dfrac{\sqrt{x^2 + 6x + 9}}{x+3}$

2) $y = |x+1| - |1-x|$

3) $y = \dfrac{x + |x|}{2x^2}$

4) $y = \dfrac{|x| + x}{x}$

5) $y = \operatorname{tg} x\,,\ y = \operatorname{tg}\Big(x + \dfrac{\pi}{3}\Big)\,,\ y = |\operatorname{tg} x|$

6) $y = \sin\Big(x + \dfrac{\pi}{4}\Big)\,,\ y = \sin\Big(x + \dfrac{\pi}{4}\Big) + 2$

7) $y = \log x\,,\ y = \log(x+2)\,,\ y = \log(x+2) - 3$

Sequences

We denote a *sequence* as $\{a_n\}_{n=1}^{\infty} = \{a_1, a_2, a_3, \cdots, a_n, \cdots\}$, $a_n \in \mathbb{R}$.
We define $s_n = a_1 + a_2 + \cdots + a_n$ of what we call partial sums s_n.

For an *arithmetic sequence* (AS) it holds:

$a_{n+1} = a_n + d$, where $a_1, d \in \mathbb{R}, n \in \mathbb{N}$, d is common *difference*, $s_n = \dfrac{(a_1 + a_n) \cdot n}{2}$.

For a *geometric sequence* (GS) it holds:

$a_{n+1} = a_n \cdot q$, where $a_1, q \in \mathbb{R}, n \in \mathbb{N}$, q is common *ratio*, $s_n = a_1 \dfrac{1 - q^n}{1 - q} \quad (q \neq 1)$,

$s_n = na_1 \quad (q = 1)\,,\ \lim\limits_{n\to\infty} s_n = s_\infty = \dfrac{a_1}{1-q} \quad$ for $|q| < 1$.

We prove that sequence $\left(\dfrac{n}{n+1}\right)_{n=1}^{\infty}$ is neither arithmetic nor geometric:

$d = a_{n+1} - a_n = \dfrac{n+1}{n+2} - \dfrac{n}{n+1} = \dfrac{(n+1)^2 - n(n+2)}{(n+2)(n+1)} = \dfrac{1}{(n+2)(n+1)}$, d is not a constant.

$q = \dfrac{a_{n+1}}{a_n} = \dfrac{\frac{n+1}{n+2}}{\frac{n}{n+1}} = \dfrac{(n+1)^2}{(n+2)n} = 1 + \dfrac{1}{n^2 + 2n}\,, \quad q$ is not a constant.

1) Determine AS for which the sum of the first three numbers is 3 and the sum of the square of the first three numbers is 35. $[\,a_1 = 5, d = -4;\ a_1 = -3, d = 4\,]$

$a_1 + a_2 + a_3 = 3$

$a_1^2 + a_2^2 + a_3^2 = 35$

$a_1 + a_1 + d + a_1 + 2d = 3 \ \Rightarrow\ 3a_1 + 3d = 3 \ \Rightarrow\ a_1 + d = 1$

$(1-d)^2 + 1^2 + (1+d)^2 = 35 \ \Rightarrow\ 3 + 2d^2 = 35 \ \Rightarrow\ d = \pm 4$

2) Determine all GS for which the sum of the first and fourth terms is 18, the sum of the second and third terms is 12. $[\,a_1 = 2, q = 2;\ a_1 = 16, q = \frac{1}{2}\,]$

$a_1 + a_4 = 18$

$a_2 + a_3 = 12$

$a_1 + a_1 \cdot q^3 = 18$

$a_1 \cdot q + a_1 \cdot q^2 = 12$

$a_1 = \dfrac{18}{1+q^3} \wedge a_1 = \dfrac{12}{q(1+q)} \quad q \neq -1 \ \Rightarrow\ \dfrac{18}{1+q^3} = \dfrac{12}{q(1+q)} \ \Rightarrow$

$3q = 2(1 - q + q^2) \ \Rightarrow\ 0 = 2q^2 - 5q + 2 \ \Rightarrow\ 0 = (2q-1)(q-2)$

3) Prove that the sequence $\left(\dfrac{2n-1}{3}\right)_{n=1}^{\infty}$ is an arithmetic sequence. $[\,d = \frac{2}{3}\,]$

4) Divide the sum of 1225 crowns among several persons so that the first person receives 100 crowns and each succeeding person receives a sum higher by 5 crowns than the sum receved by the preceding person. How many persons receive some money and how many crowns does the last person receive? [$n = 10, a_{10} = 145$ crowns]

5) Determine AS for which the sum of the first three numbers is 27 and the sum of the square of the first three numbers is 275. [$a_1 = 5, d = 4;\ a_1 = 13, d = -4$]

6) Determine AS such that $s_n = 7n^2 - 3n$. [$a_1 = 4, d = 14$]

7) Determine AS such that $a_2 + a_5 - a_3 = 10$, $a_1 + a_6 = 17$. [$a_1 = 1, d = 3$]

8) Determine AS such that $a_1 + a_7 = 22$, $a_3 \cdot a_4 = 88$. [$a_1 = 2, d = 3$]

9) Determine AS such that $a_7 = 21$, $s_7 = 105$. [$a_1 = 9, d = 2$]

10) Determine a_1 in AS, if $a_{10} = 37$, $s_{10} = 190$. [$a_1 = 1$]

11) It holds for AS that $a_2 - a_3 + a_5 = 30$, and $a_1 + a_6 = 38$. Determine a_{12}, s_{12}. [$a_{12} = 206, s_{12} = 1020$]

12) It holds for increasing AS that $a_1 + a_2 = 4$, $a_1^2 + a_2^2 = 10$. Determine a_{10}, s_{10}. [$a_{10} = 19, s_{10} = 100$]

13) It holds for AS that $a_1 + a_4 = -14$, $a_2 + a_5 = -10$. Determine a_{10}, s_{10}. [$a_{10} = 8, s_{10} = -10$]

14) It holds for AS that $a_1 + a_4 = 22$, $a_2 + a_5 = 26$. Determine a_5, s_5. [$a_5 = 16, s_5 = 60$]

15) For AS determine a_{10} if $s_n = n(3n - 5)$. [$a_{10} = 52$]

16) We have an initial deposit of 100 dollars in a bank account. The bank pays interest at an annual percentage rate of 5%. Find the value of our account after 4 years. [$100 \cdot (1 + \frac{5}{100})^4$]

17) If we add the same number to the numbers $2, 7, 17$, we get the first three numbers of GS. What number have we added? [$x = 3$]

18) For $x \in \langle 0, 2\pi \rangle$ solve the following equation: $1 + \sin^2 x + \sin^4 x + \sin^6 x + \cdots = 2\,\mathrm{tg} x$. [$x_1 = \frac{\pi}{4}, x_2 = \frac{5\pi}{4}$]

19) Solve the equation in $\mathbb{R}$: $1 - \dfrac{2}{x} + \dfrac{4}{x^2} - \dfrac{8}{x^3} + \cdots = \dfrac{6}{x+5}$. [$x_1 = 4, x_2 = -3$]

20) Determine a_5 in GS for which it holds: $a_1 = 3$, $a_3 = 21$. [$q = \pm\sqrt{7}, a_5 = 147$]

21) Determine GS for which it holds: $a_1 = 2$, $a_p = 13122$, $s_p = 19682$, $p \in \mathbb{N}$. [$q = 3$]

22) Determine a_1, a_7 in GS for which it holds: $q = 2$, $s_7 = 635$. [$a_1 = 5, a_7 = 320$]

23) Determine q, s_5 in GS for which it holds: $a_1 = 3$, $a_5 = 12\,288$. [$q = 8, s_5 = 14043$]

24) Determine a_5 in GS for which it holds: $a_4 - a_1 = 14$, $a_3 - a_2 = 4$. [$a_5 = 32, a_5 = -1$]

25) Determine a_5 in GS for which it holds: $a_1 + a_2 + a_3 = -2$, $a_1 + 4 = a_2$. [$a_5 = -\frac{1}{6}, a_5 = -2$]

Combinations, binomial theorem, mathematical induction

The number n *factorial* is given as $n! = 1 \cdot 2 \cdot 3 \cdots n$. We define $0! = 1$.

A *permutation* is an arrangement of distinct objects in a definite order. The number of all permutations of n objects is $n!$
(Six distinct people to be placed in six distinct positions - $6! = 720$).

The number of *permutations of n distinct objects taken k at a time* is $P(n,k) = \frac{n!}{(n-k)!}$
(Six distinct people to be placed in four positions - $P(6,4) = \frac{6!}{(6-4)!} = 360$).

The number of *combinations of n objects taken k at a time* is $C(n,k) = \frac{n!}{(n-k)! \cdot k!} = \binom{n}{k}$
(The number of different committees of four people selected from six people - $C(6,4) = \frac{6!}{(6-4)!4!} = 15$).

Binomial theorem for $n \in \mathbb{N}$:

$$(a+b)^n = \binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b^1 + \cdots + \binom{n}{n}a^0 b^n = \sum_{i=1}^{n} \binom{n}{i} a^{n-i} b^i$$

Principle of *mathematical induction*:

Let $V(n)$ be a statement about a positive integer n. If
1) $V(1)$ is true, and 2) the truth of $V(k)$ implies the truth of $V(k+1)$
then $V(n)$ is true for all $n \in \mathbb{N}$.

1) We will prove that $s_n = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.

1) For $n = 1$: $s_1 = 1 = \frac{1(1+1)}{2}$. The statement is true for $n = 1$.

2) Assume the statement is true for some positive integer k.

$s_k = 1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}$ (Induction hypothesis).

Now verify that the formula is true when $n = k + 1$.

That is, verify that $s_{k+1} = \frac{(k+1)(k+1+1)}{2} = \frac{(k+1)(k+2)}{2}$ (Goal of the induction).

We note that $s_{k+1} = s_k + a_{k+1}$ and $a_{k+1} = k + 1$.

Hence $s_{k+1} = \frac{k(k+1)}{2} + k + 1 == \frac{k(k+1)+2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$.

We have verified the two parts of the principle of mathematical induction.

We can conclude that the statement is true for all positive integer values.

2) $2\binom{x-1}{x-2} = \binom{x}{0} + \binom{x}{2}$ [$x_1 = 2, x_2 = 3$]

$$\begin{aligned}
2 \cdot \frac{(x-1)!}{(x-1-(x-2))! \cdot (x-2)!} &= \frac{x!}{(x-0)! \cdot 0!} + \frac{x!}{(x-2)! \cdot 2!} \\
2(x-1) &= 1 + \frac{x(x-1)}{2!} \\
4x - 4 &= 2 + x^2 - x \\
0 &= x^2 - 5x + 4
\end{aligned}$$

3) $12\binom{x}{x-1} + \binom{x+4}{x+2} = 162$ [$x = 8$]

4) $\binom{n}{2} + \binom{n-1}{2} = 4$ [$n = 3$]

5) $\binom{x}{2} = x + 9$ [$x = 6$]

6) How many 4-digit numbers can be created, if each digit is used only once. [4536]

7) A student must answer six out of twelve questions on an exampaper. How many different choices can the student make? [924]

8) We want to create 272 pairs of persons. How many persons do we need? [17]

9) A company has more than 676 employees. Explain why there must be at least 2 employees who have the same first and last initials. [26 letters and $26^2 = 676$]

10) A chess tournament has twelve participants. How many games must be scheduled if every player must play every other player exactly once? [66]

11) A state lottery game requires a person to select six different numbers from forty numbers. The order of the selection is not important. In how many different ways can this be done? [3 838 380]

12) Five cards are chosen at random from an ordinary pack of playing cards. In how many ways can the cards be chosen so that: a) all are hearts; b) all are the same suit; c) exactly three are kings; d) two or more are aces?
[a) 1287; b) 5148; c) 4512; d) 108 336]

13) Seven identical balls are randomly placed in seven available containers in such a way that two balls are in one container. Of the remaining six containers, each receives at most one ball. Find the number of ways this can be accomplished. [42]

14) Using the binomial theorem, calculate: $(\sqrt{2} + i\sqrt{3})^5$ [$-11\sqrt{2} - i31\sqrt{3}$]

15) Find the fourth term in the expansion of $(2x^3 - 3y^2)^5$ [$-1080x^6y^6$]

Using mathematical induction prove that:

16) $\sum_{k=0}^{n} q^k = 1 + q + q^2 + \cdots + q^{n-1} = \dfrac{1-q^n}{1-q}$

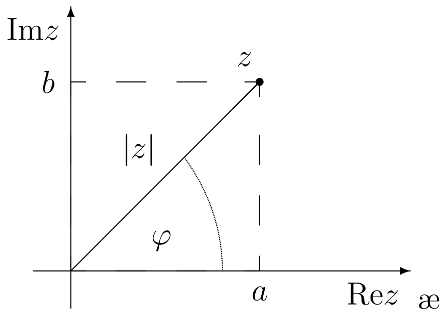
17) $\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

18) $\sum_{k=1}^{n} \dfrac{1}{k(k+1)} = \dfrac{1}{1\cdot 2} + \dfrac{1}{2\cdot 3} + \dfrac{1}{3\cdot 4} + \cdots + \dfrac{1}{n\cdot(n+1)} = \dfrac{n}{n+1}$

Complex numbers

A *complex number* is $z = a + ib$, where $a, b \in \mathbb{R}$, and a is the real part of z, b is the imaginary part of z, and i is the imaginary unit such that $i^2 = -1$.
The *trigonometric form* of a complex number: $z = |z|(\cos\varphi + i\sin\varphi)$, $|z|$ is a *modulus* of the complex number, and φ is an *argument* of the complex number. The complex *conjugate* is $\overline{z} = a - ib$; it holds: $|z|^2 = z \cdot \overline{z}$, $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$, $|\frac{z_1}{z_2}| = \frac{|z_1|}{|z_2|}$.

Argand diagram:



It holds: $|z| = \sqrt{a^2 + b^2}$, $\quad \mathrm{tg}\varphi = \dfrac{b}{a}$,

$$\sin\varphi = \frac{b}{|z|}, \quad \cos\varphi = \frac{a}{|z|}.$$

Moivre theorem: $(\cos\varphi + i\sin\varphi)^n = \cos n\varphi + i\sin n\varphi$.

Euler identity: $\mathrm{e}^{i\varphi} = \cos\varphi + i\sin\varphi$.

1) Find the trigonometric forms of the complex numbers $z = \sqrt{3}+i$, $z = \sqrt{3}-i$, $z = -1+i\sqrt{3}$.

[$z = 2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}), z = 2(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}), z = 2(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3})$]

$$z = \sqrt{3} + i \;\Rightarrow\; |z| = \sqrt{(\sqrt{3})^2 + 1^2} = 2,$$

$$\left.\begin{array}{l} \sin\varphi = \dfrac{1}{2} \;\Rightarrow\; \varphi = \dfrac{\pi}{6} + 2\mathrm{k}\pi \quad \text{or} \quad \varphi = \dfrac{5\pi}{6} + 2\mathrm{k}\pi \\ \cos\varphi = \dfrac{\sqrt{3}}{2} \;\Rightarrow\; \varphi = \dfrac{\pi}{6} + 2\mathrm{k}\pi \quad \text{or} \quad \varphi = -\dfrac{\pi}{6} + 2\mathrm{k}\pi \end{array}\right\} \Rightarrow\; \varphi = \frac{\pi}{6} + 2\mathrm{k}\pi$$

2) Find the real and imaginary parts of the complex number $2\sqrt{2}(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4})$.

[$z = 2 - i2$]

Calculate:

3) $|3 - i4| + \left|\dfrac{-2 - i3}{3 - i2}\right|$ [6]

$$|3 - i4| + \left|\frac{-2 - i3}{3 - i2}\right| = \sqrt{9 + 16} + \frac{\sqrt{4 + 9}}{\sqrt{9 + 4}} = 6$$

4) $\left(\dfrac{1+i}{1-i}\right)^6$ [-1]

$$\left(\frac{1+i}{1-i}\right)^6 = \left(\frac{1+i}{1-i} \cdot \frac{1+i}{1+i}\right)^6 = \left(\frac{1+2i+i^2}{1-i^2}\right)^6 = \left(\frac{2i}{2}\right)^6 = -1$$

Calculate:

5) $\left|\dfrac{i^{10}-i}{i2+1}\right|$ $[\,\frac{\sqrt{10}}{5}\,]$

6) $\dfrac{1}{i}-\dfrac{1}{1+i}+\dfrac{1}{1-i}+\left|\dfrac{1+i}{1-i}-\dfrac{1-i}{1+i}\right|$ $[\,2\,]$

Calculate the modulus of the following numbers:

7) $\dfrac{7+i3}{2+i5}$ $[\,\sqrt{2}\,]$

8) $\dfrac{1}{(1+i)(2-i)(3+i)}$ $[\,\frac{1}{10}\,]$

9) $2-i-\dfrac{3+i3}{1-i}$ $[\,\sqrt{20}\,]$

10) $\dfrac{3-i2}{3+i2}$ $[\,1\,]$

Calculate the product z_1z_2 and the quotient $\frac{z_1}{z_2}$:

11) $z_1=3(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6})$,
$z_2=16(\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3})$.
$[\,z_1\cdot z_2=48(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6})\,]$
$[\,\frac{z_1}{z_2}=\frac{3}{16}(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2})\,]$

12) $z_1=3-i, z_2=2+i3$.
$[\,z_1\cdot z_2=9+i7, \frac{z_1}{z_2}=\frac{1}{13}(3-i11)\,]$

Calculate:

13) $(1+i)^{20}$ $[\,-1024\,]$

14) $\left(\dfrac{1+i}{\sqrt{2}}\right)^{26}$ $[\,i\,]$

15) $(\sqrt{3}-i)^{20}$ $[\,2^{20}(\cos\frac{2}{3}\pi+i\sin\frac{2}{3}\pi)\,]$

16) $(1-i)^{12}$ $[\,-2^6\,]$

17) $(1+i)^8$ $[\,16\,]$

Find all the complex numbers satisfying:

18) $x^3-2=0$ $[\,x_k=\sqrt[3]{2}(\cos\frac{2k\pi}{3}+i\sin\frac{2k\pi}{3}), k=0,1,2\,]$

Instruction: $x^3=|x|^3(\cos 3\varphi+\sin 3\varphi)\,,\ 2=2(\cos(0+2\mathrm{k}\pi)+\sin(0+2\mathrm{k}\pi))\,.$

19) $\dfrac{x+2}{x+3}=\dfrac{2x-1}{3x+1}$ $[\,x=-1\pm i2\,]$

20) $(2+i3)z+iz=1-i$ $[\,z=-\frac{1}{10}-i\frac{3}{10}\,]$

21) $x^6+64=0$ $[\,x_k=2(\cos\frac{-\pi+2k\pi}{6}+i\sin\frac{-\pi+2k\pi}{6}), k=0,1,2,3,4,5\,]$

22) $x^4=-8+i8\sqrt{3}$ $[\,x_k=2(\cos\frac{\pi+3k\pi}{6}+i\sin\frac{\pi+3k\pi}{6}), k=0,1,2,3\,]$

23) $x^5=-16\sqrt{3}-i16$ $[\,x_k=2(\cos\frac{\pi+6k\pi}{30}+i\sin\frac{\pi+6k\pi}{30}), k=0,1,2,3,4\,]$

24) $x=\sqrt[3]{-1}$ $[\,x_k=(\cos\frac{\pi+2k\pi}{3}+i\sin\frac{\pi+2k\pi}{3}), k=0,1,2\,]$

25) $x=\sqrt{i}$ $[\,x_k=(\cos\frac{\pi+4k\pi}{4}+i\sin\frac{\pi+4k\pi}{4}), k=0,1\,]$

26) $x^6+i8=0$ $[\,x_k=\sqrt{2}(\cos\frac{\pi+2k\pi}{6}+i\sin\frac{\pi+2k\pi}{6}), k=0,1,2,3,4,5\,]$

27) $(5+i)\overline{z}+2z=i22\,,\quad z=x+iy\,.$ $[\,z=1-i7\,]$

28) Using the Moivre theorem find the rule for $\cos 3\varphi$ and $\sin 3\varphi$:
$[\,\cos 3\varphi=\cos^3\varphi-3\cos\varphi\sin^2\varphi;\ \sin 3\varphi=3\cos^2\varphi\sin\varphi-\sin^3\varphi\,]$

29) Using the Moivre theorem find the rule for $\cos 4\varphi$ and $\sin 4\varphi$:
$[\,\cos 4\varphi=\cos^4\varphi-6\cos^2\varphi\sin^2\varphi+\sin^4\varphi;\ \sin 4\varphi=4\cos^3\varphi\sin\varphi-4\cos\varphi\sin^3\varphi\,]$

Geometry

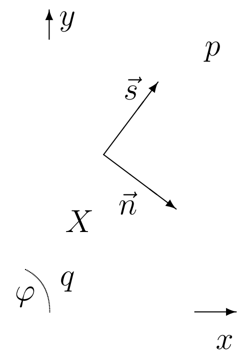
Analytic geometry in the plane:

Notation: points A, B; straight lines p, q; circle $k \equiv (S; r = 2)$ with centre S and radius 2; curve κ; planes σ, ϱ; angles $\alpha, \beta\ \angle ABC$; determination of a straight line $p = AB$; determination of a plane $\sigma = (ABS)$, $\sigma = (p, q)$; magnitude of a segment $|AB|$; intersection point of two lines $A \equiv p \cap q$; intersection line of two planes $q \equiv \varrho \cap \sigma$; intersection of two curves $B \equiv \kappa \cap k$; perpendicularity of lines, planes $p \perp q$, $\sigma \perp \varrho$; parallelism of lines, planes $p \parallel q$, $\sigma \parallel \varrho$.

Let S be a set of four objects $\{O, x, y, j\}$, where O is an arbitrary point in the plane; x, y are two oriented perpendicular lines passing through the point O and j is the length of the unit segment (usually $j = 1$). Each point X in the plane has its coordinates related to the system of coordinates. It is an ordered pair $[x_1, x_2]$ of real numbers. A set S is called the *System of Cartensian Coordinates*. Each point X in the plane determines one radius vector $\overrightarrow{OX} = (x_1, x_2)$.

The *scalar product* $\vec{a} \cdot \vec{b}$ of two vectors $\vec{a} = (a_1, a_2)$, $\vec{b} = (b_1, b_2)$ is given by the formula $\vec{a} \cdot \vec{b} = a_1 \cdot b_1 + a_2 \cdot b_2$. Two vectors are *perpendicular*, if $\vec{a} \cdot \vec{b} = 0$; they are *parallel*, if there exists a constant $c \in \mathbb{R}$ such that $\vec{a} = c \cdot \vec{b}$. The *norm* of vector $\vec{a}$ is defined by $\|\vec{a}\| = \sqrt{a_1^2 + a_2^2}$. If α is the *angle* between two nozero vectors $\vec{a}, \vec{b}$, then $\cos\alpha = \dfrac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \cdot \|\vec{b}\|}$. The *distance* $|AB|$ between two points $A = [a_1, a_2]$, $B = [b_1, b_2]$ is given by the formula $|AB| = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}$. The *distance* $d(A, p)$ between the point $A = [a_1, a_2]$ and the straight line $p\colon ax + by + c = 0$ is given by the formula $d(A, p) = \dfrac{|aa_1 + ba_2 + c|}{\|(a, b)\|}$.

Straight line p:



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General form of p: $ax + by + c = 0\,,$	$\vec{n} = (a, b)$ is the normal vector, $\vec{n} \perp p$.
Slope-intercept form: $y = kx + q\,,$	k is the gradient of the straight line p, $k = \operatorname{tg}\varphi$; q is the vertical intercept.
Point-slope form: $y - y_1 = k(x - x_1)\,,$	$X = [x_1, y_1]$ is a point lying on straight line p.
Two-point form: $y - y_1 = \dfrac{y_2 - y_1}{x_2 - x_1}(x - x_1)\,,$	$X = [x_1, y_1]\,,\ Y = [x_2, y_2]$ are points lying on straight line p.
Parametric form: $x = x_1 + s_1 t$, $y = x_2 + s_2 t\,,$	$X = [x_1, y_1]$ is a point lying on straight line p and $\vec{s} = (s_1, s_2)$ is the direction vector of p.

Conic curves:

A *parabola* is the set of all points in the plane that are equidistant from a fixed line (the *directrix*) and a fixed point (the *focus*) not on the directrix.

An *ellipse* is the set of all points in the plane the sum of whose distances from two fixed points (*foci*) is a positive constant.

A *hyperbola* is the set of all points in the plane the difference of whose distances from two fixed points (*foci*) is a positive constant.

A *circle* is the set of all points in the plane that are a fixed distance (*radius*) from a specified point (*centre*).

parabola:	$2p(y-v_2)=(x-v_1)^2\,,$	$V[v_1,v_2]$ is the vertex of the parabola, $\lvert p/2\rvert$ is the distance between the focus of the parabola and its vertex.
ellipse:	$\dfrac{(x-s_1)^2}{a^2}+\dfrac{(y-s_2)^2}{b^2}=1\,,$	
hyperbola:	$\dfrac{(x-s_1)^2}{a^2}-\dfrac{(y-s_2)^2}{b^2}=1\,,$	$S[s_1,s_2]$ is the centre of a conic curve, $2a$ is the length of the major axis, $2b$ is the length of the minor axis of the conic curve.
circle:	$(x-s_1)^2+(y-s_2)^2=r^2\,,$	r is the radius of a circle.

A *tangent* to a curve is a straight line that touches the curve at one point.

A graph is *symmetric with respect to a line* L if for each point P on the graph there is a point P' on the graph such that the line L is the perpendicular bisector of the line segment PP'.

A graph is *symmetric with respect to a point* Q if for each point P on the graph there is a point P' on the graph such that Q is the midpoint of the line segment PP'.

Triangles:

C

v_c

S_b S_a

T

t_b

t_a

A B

The *altitude* (v_C)is a line segment joins the vertex and is perpendicular to the opposite side.
The *median* (t_a) is a line segment joins the vertex and the midpoint of the opposite side.
The *centre of graviti* (*centroid* T) is the intersection of three medians.
The intersection of side bisectors is the centre of the *circumcircle*.
The intersection of angle bisectors is the centre of the *incircle*.

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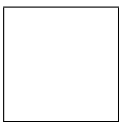
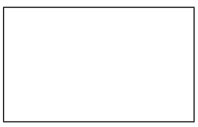
Angles

Triangles

Right Acute Obtuse Right Isosceles Equilateral æ

Quadrilaterals





Square Rectangle Parallelogram Rhombus Trapezoid

Analytic geometry in space:

Plane ρ: a general form: $ax + by + cz + d = 0\,,$ $\vec{n} = (a, b, c)$ is a normal vector, $\vec{n} \perp \rho$.

a parametric form:
$x = x_1 + r_1 u + s_1 v$
$y = x_2 + r_2 u + s_2 v$
$z = x_3 + r_3 u + s_3 v$

$X[x_1, x_2, x_3]$ is an arbitrary point in a plane ρ, $\vec{r} = (r_1, r_2, r_3)$ and $\vec{s} = (s_1, s_2, s_3)$ are two direction vectors of the plane ρ, $\vec{r} \not\parallel \vec{s}$; u, v are parameters.

Straight line p: a parametric form:
$x = x_1 + s_1 t$
$y = x_2 + s_2 t$
$z = x_3 + s_3 t$

$X[x_1, x_2, x_3]$ is an arbitrary point on a straight line p and $\vec{s} = (s_1, s_2, s_3)$ is the direction vector p; t is the parameter.

1) The triangle has the vertex $A = [3, -4]$, and equations of the altitudes $v_b:\ 7x - 2y - 1 = 0$ and $v_c:\ 2x - 7y - 6 = 0$. Find the coordinates of the vertex C.

[$C[-4; -2]$]

Solution: The normal vector of line v_b is $\vec{n_b} = (7, -2)$ and it is the direction vector of line b. The parametric form of line b (containing side b) is given by: $x = 3 + 7t \quad y = -4 - 2t\,.$

Point C is the intersection of the lines b and v_c:

$2(3 + 7t) - 7(-4 - 2t) - 6 = 0 \ \Rightarrow\ t = -1 \ \Rightarrow\ x = -4\,,\ y = -2\,.$

2) Find the intersection point X of two lines p and q, where $p = AB$, $A = [1, \frac{9}{2}]$, $B = [-4, 3]$ and $q: y = x$.

[$X = [6, 6]$]

Solution:

$$P:\ A + t\cdot \overrightarrow{AB} \Rightarrow \begin{array}{l} x = 1 + t(-5) \\ y = 4.5 + t(-1.5) \end{array} \ \wedge\ y = x \ \Rightarrow$$

$1 - 5t = 4.5 - 1.5t \ \Rightarrow\ -3.5t = 3.5 \ \Rightarrow\ t = -1 \ \Rightarrow\ [x, y] = [6, 6]$

3) Find the focus and directrix of the parabola given by the equation $3x + 2y^2 + 8y - 4 = 0$.

[$F = [\frac{29}{8}, -2]$, $x = \frac{35}{8}$]

Solution:

y

-2 0 2 4 x

-2 $xF = \frac{35}{8}$

-4

$3x + 2y^2 + 8y - 4 = 0 \ \Rightarrow\ 3x + 2(y + 2)^2 - 12 = 0 \ \Rightarrow$
$\frac{3}{2}(x - 4) = -(y + 2)^2 \ \Rightarrow$
vertex $V = [4, -2]$, parameter $\frac{p}{2} = \frac{3}{8} \ \Rightarrow$
focus $F = [4 - \frac{3}{8}, -2]$, directrix $x = 4 + \frac{3}{8}$

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4) Find the equation of the ellipse with the major axis of length 10, foci $F_1 = [1, -2]$, $F_2 = [7, -2]$. $[\frac{(x-4)^2}{25} + \frac{(y+2)^2}{16} = 1]$

Solution:

y

2

0 2 4 6 8 x

-2 F_1 a S F_2

-4 b a

-6

centre $S = \frac{F_1+F_2}{2} \Rightarrow S = [4, -2] \Rightarrow |F_1S| = 3$

major axis $2a = 10 \wedge |F_1S|^2 = a^2 - b^2 \Rightarrow$

$3^2 = 5^2 - b^2 \Rightarrow b = 4$

5) The coordinates of the foci of the hyperbola are $F_1[c, 0]$ and $F_2[-c, 0]$, the length of the major axis is $2a$ and the length of the minor axis is $2b$. Find the relation between constants c, a and b.

$[c^2 = a^2 + b^2]$

Solution:

y

$X = [x, y]$

$F_1 = [-c, 0]$

$F_2 = [c, 0]$

b

F_1 a F_2 x

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$$|XF_1| - |XF_2| = 2a$$

$$\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = 2a$$

$$(x+c)^2 + y^2 - 2\sqrt{(x+c)^2 + y^2}\sqrt{(x-c)^2 + y^2} + (x-c)^2 + y^2 = 4a^2$$

$$2x^2 + 2c^2 + 2y^2 - 4a^2 = 2\sqrt{(x+c)^2 + y^2}\sqrt{(x-c)^2 + y^2}$$

$$(x^2 + c^2 + y^2 - 2a^2)^2 = [(x+c)^2 + y^2][(x-c)^2 + y^2]$$

$$(x^2 + c^2 + y^2)^2 - 4a^2(x^2 + c^2 + y^2) + 4a^2 = (x^2 + c^2 + y^2 + 2xc)(x^2 + c^2 + y^2 - 2xc)$$

$$(x^2 + c^2 + y^2)^2 + 4a^2(a^2 - c^2) - 4a^2x^2 - 4a^2y^2 = (x^2 + c^2 + y^2)^2 - 4x^2c^2$$

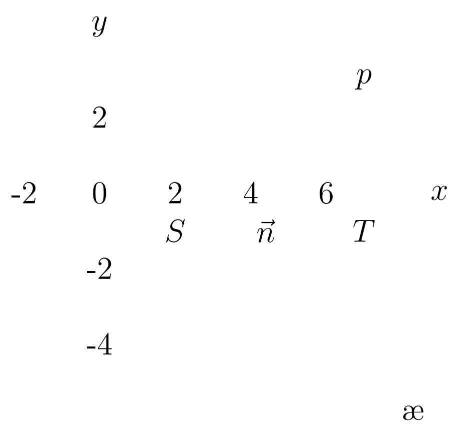
$$x^2(c^2 - a^2) - a^2y^2 = a^2(c^2 - a^2)$$

$$\frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} = 1 \Rightarrow c^2 - a^2 = b^2$$

6) Write an equation for a tangent line to the circle $(x-2)^2 + (y+1)^2 = 25$ at the point $T = [6, 2]$.

$[\,4x + 3y - 30 = 0\,]$

Solution:



tangent line $p:\ ax + by + c = 0,$

$\vec{n} = (a, b) = \overrightarrow{TS}, \quad S = [2, -1], \quad T = [6, 2]$

$\Rightarrow\ \vec{n} = (4, 3)\ \wedge\ T \in p\ \Rightarrow$

$4 \cdot 6 + 3 \cdot 2 + c = 0\ \Rightarrow\ c = -30$

7) Write the equation for the plane which passes through points $A = [1, -1, 2]$, $B = [2, 1, 2]$, $C = [1, 1, 3]$.

$[\,-2x + y - z + 5 = 0\,]$

Solution: plane ϱ: $A + t \cdot \overrightarrow{AB} + + s \cdot \overrightarrow{AC}$

$$\varrho:\ \begin{aligned} x &= 1 + t \cdot 1 + s \cdot 0 \quad \Rightarrow\ t = x - 1 \\ y &= -1 + t \cdot 2 + s \cdot 2 \\ z &= 2 + t \cdot 0 + s \cdot 1 \quad \Rightarrow\ s = z - 2 \end{aligned}$$

$y + 1 = (x - 1) \cdot 2 + (z - 2) \cdot 1\ \Rightarrow\ -2x + y - z + 5 = 0$

8) Find the slope of the line passing through the points whose coordinates are given a) $[1, 2]$ and $[3, 6]$, b) $[-3, 4]$ and $[1, -2]$, c) $[1, 2]$ and $[5, 10]$. $[\,$a) $k = 2$, b) $k = \frac{-3}{2}$, c) $k = 2\,]$

9) Find the equation of a line with slope -3 that passes through $[-1, 4]$. Write your answer in the slope-intercept form. $[\,y = -3x + 1\,]$

10) Find the equation in a slope-intercept form of a line that passes through points $[-3, 2]$ and $[2, -4]$. $[\,y = -\frac{6}{5}x - \frac{8}{5}\,]$

11) Find the general form of the equation of the line p that passes through the point $[-2, 3]$ and is parallel to the graph of $3x + 4y - 12 = 0$. $[\,3x + 4y - 6 = 0\,]$

12) Find the general form of the equation of the line p that passes through the point $[2, 1]$ and is perpendicular to the graph of $3x + y - 6 = 0$. $[\,x - 3y + 1 = 0\,]$

13) Find the intersection point of the straight lines
p: $3x + 4y - 3 = 0$, q: $x = 1 + 2t$, $y = 2 - t$, $t \in \mathbb{R}$. $[\,P = [-7, 6]\,]$

14) Find a point B which is symmetric with point $A = [5, -1]$ with respect to a straight line $3x + 4y + 14 = 0$. $[\,B = [-1, -9]\,]$

15) Write equations for all the straight lines q that pass through point $A = [-1, 3]$ where the angle between q and line $p : 4x - 2y - 1 = 0$ is equal $\frac{\pi}{4}$.

$$\left[\ q_1 : \begin{array}{l} x = -1 + 3t \\ y = 3 + t \end{array} \qquad q_2 : \begin{array}{l} x = -1 - t \\ y = 3 + 3t \end{array}\ \right]$$

16) Find the general form of the equation of the line p that passes through the point $[2, 1]$ and is perpendicular to the vector $\vec{v} = (2, 7)$. $[\,2x + 7y - 11 = 0\,]$

17) Find the number a such that point $A = [-7; a\,]$ belongs to the line $p = BC$ where $B = [2, 3]$, $C = [4, -3]$. $[\,a = 30\,]$

18) Find a point P which is symmetric with a point $Q = [10, 21]$ with respect to a straight line $2x + 5y - 38 = 0$. $[\,P = [-2, -9]\,]$

19) Find the straight line p which passes through the point $A = [\frac{5}{2}, 1]$ and is perpendicular to the line $q :\ 2x + y - 3 = 0$. $[\,-x + 2y + 0{,}5 = 0\,]$

20) In the triangle with vertices $A = [4, 6]\,,\ B = [-4, 0]\,,\ C = [-1, -4]$ find equations of a line, which contains altitude v_a. $[\,-3x + 4y - 12 = 0\,]$

21) The triangle has vertices $A = [2, 1]\,,\ B = [-4, 3]\,,\ C = [-3, -2]$. Find the general equation of a line which contains the median t_a and calculate its magnitude.

$[\,-2x + 22y - 18 = 0,\ \frac{\sqrt{122}}{2}\,]$

22) Prove that triangle $A = [4, -2.7]\,,\ B = [0.4, -2.3]\,,\ C = [2, -4, 3]$ is right triangle!

$[\,\overrightarrow{AC} \cdot \overrightarrow{BC} = 0\,]$

23) We have a triangle ABC in the plane and we know the equations of the side $a :\ 3x + 7y - 3 = 0$, the side $b :\ 2x + 3y + 2 = 0$ and the intersection $P = [2, -3\,]$ of the altitude v_c and the side c. Write general equations of the side c. $[\,2x - 8y - 28 = 0\,]$

24) Count the angles of the triangle whose vertices are created by the numbers 2, 6, 9 of a clock.

$[\,45°, 60°, 75°\,]$

25) The base of the isosceles triangle has vertices $A = [-3, 1]$, $B = [5, -7]$. Find the vertex C which belongs to the line $2x - 3y = 5$. $[\,C = [7, 3]\,]$

26) We have vertices $A = [-2, -4]$, $B = [2, -2]$ and the intersection of altitudes $V = [1, 2]$. Find the vertex C. $[\,C = [-2, 4]\,]$

27) The ratio of the altitude and parallel sides of a trapezoid is $2 : 3 : 5$, the area of the trapezoid is $P = 512\,\text{cm}^2$. Calculate the length of the parallel sides and the altitude. $[\,16, 24, 40\,]$

28) We have an isosceles trapezoid with a circumscribe. Calculate the length c of the non-parallel side of the trapezoid, if the lengths of the parallel sides are a, b. $[\,c = \frac{a+b}{2}\,]$

29) We have points $A = [2, 3]$, $B = [4, 7]$, $C = [6, 2]$. Find the point D such that $ABCD$ is a parallelogram.

$[\,D = [4, -2]\,]$

30) A sheet of A0 paper is a rectangle where the ratio of its sides is $1 : \sqrt{2}$ and its area is $1\,\mathrm{m}^2$. Calculate the sides of the rectangle. $[\,2^{-\frac{1}{4}}\,\mathrm{m},\ 2^{\frac{1}{4}}\,\mathrm{m}\,]$

31) We have point $A = [1, 1]$, straight line $p:\ 4x - 3y + 9 = 0$ and point $C = [-3, -1]$, $C \in p$. The line p forms one side of the rectangle $ABCD$. Write the remaining general equations of the lines which form the sides of the rectangle $ABCD$.
$[\,4x - 3y - 1 = 0,\ 3x + 4y - 7 = 0,\ 3x + 4y + 13 = 0\,]$

32) Prove that the quadrilateral $ABCD$ where $A = [-1, -3]$, $B = [-4, 1]$, $C = [-8, -2]$, $D = [-5, -6]$ is a square. $[\,\overrightarrow{AB} \cdot \overrightarrow{BC} = \overrightarrow{BC} \cdot \overrightarrow{CD} = \overrightarrow{CD} \cdot \overrightarrow{AB} = 0\,]$

33) Find the focus and directrix of the parabola given by the equation $y = -\dfrac{1}{2}x^2$.
$[\,F = [0, -\frac{1}{2}],\ y = \frac{1}{2}\,]$

34) Find the equation of the parabola in standard form with vertex at the origin and focus at $[-2, 0]$. $[\,y^2 = -8x\,]$

35) Find the equation of the parabola in standard form with directrix $x = -1$ and focus at $[3, 2]$. $[\,(y - 2)^2 = 8(x - 1)\,]$

36) The coordinates of the foci of the ellipse are $F_1[c, 0]$ and $F_2[-c, 0]$, the major axis is $2a$ and the minor axis is $2b$. Find the relation between constants c, a and b. $[\,c^2 = a^2 - b^2\,]$

37) Find the length of the major axis and foci of the ellipse given by the equation $\dfrac{x^2}{25} + \dfrac{y^2}{49} = 1$.
$[\,a = 7,\ F_1 = [0, 2\sqrt{6}],\ F_2 = [0, -2\sqrt{6}]\,]$

38) Find the equation of the ellipse with the major axis of the length 10, foci $F_1 = [3, 0]$, $F_2 = [-3, 0]$ and centre $S = [0, 0]$. $[\,\frac{x^2}{25} + \frac{y^2}{16} = 1\,]$

39) Find the length of the major axis and foci of the ellipse given by the equation $4x^2 + 9y^2 - 8x + 36y + 4 = 0$. $[\,a = 3,\ F_1 = [1 + \sqrt{5}, -2],\ F_2 = [1 - \sqrt{5}, -2]\,]$

40) Find the foci and vertices of the hyperbola given by the equation $\dfrac{y^2}{9} - \dfrac{x^2}{4} = 1$.
$[\,V_1 = [0, 3],\ V_2 = [0, -3],\ F_1 = [0, \sqrt{13}],\ F_2 = [0, -\sqrt{13}]\,]$

41) Find the foci and vertices of the hyperbola given by the equation $4x^2 - 9y^2 - 16x + 54y - 29 = 0$.
$[\,V_1 = [2, 5],\ V_2 = [2, 1],\ F_1 = [2, 3 + \sqrt{13}\,],\ F_2 = [2, 3 - \sqrt{13}\,]\,]$

42) Find the straight lines which pass through the point $H = [2, -5]$ and are parallel with asymptotes of hyperbola $x^2 - 4y^2 = 4$. $[\,2y = x - 12\,,\ -2y = x + 8\,]$

43) Find the equation of a parabola whose vertical axis $x = 0$ and which passes through the points $A = [2, 5]\,,\ B = [1, -1]$. $[\,(y + 3) = 2x^2\,]$

44) Find the equation in standard form of the ellipse with foci $F_1 = [-2, 1]\,,\ F_2 = [4, 1]$ and major axis of length 10. $[\,(\frac{x-1}{5})^2 + (\frac{y-1}{4})^2 = 1\,]$

45) Find the equation of the circle whose centre is at the point $S = [5, -1]$ and which touches the straight line $3x + 4y + 14 = 0$. $[\,(x - 5)^2 + (y + 1)^2 = 25\,]$

46) Find the equation of the vertex tangent of the parabola $y^2 + 3x + 4y - 8 = 0$. $[\, x = 4\,]$

47) Calculate the cordinates of the foci of a curve given by the equation $4x^2+3y^2+4x-6y = 8$. $[\, F_1 = [-\frac{1}{2}, 2]\,,\ F_2 = [-\frac{1}{2}, 0]\,]$

48) Find the equations of the tangent of the circle $x^2 + y^2 = 5$ which is parallel with straight line $p : 2x - y + 1 = 0$. $[\, 2x - y \pm 5 = 0\,]$

49) Find the equation of a parabola which passes through the point $A = [3, -6]$ whose vertex is at the origin and horizontal axis of symmetry $y = 0$. Find the focus and directrix. $[\, y^2 = 12x\,,\ F = [3; 0]\,,\ x = -3\,]$

50) What kind of curve is given by the equation $2x^2 + 2y^2 + 6x - 10y = 1$? [circle]

51) Find the centre and radius of the circle which passes through points $A = [2, 1]$, $B = [1, 4]$, $C = [6, 9]$. $[\, S = [6, 4]\,,\ r = 5\,]$

52) The axes of the hyperbola are $x = 0$ and $y = 0$ and this hyperbola passes through the points $M = [4, \sqrt{3}\,]$ and $N = [2\sqrt{2}, -1\,]$. Find the foci of this hyperbola. $[\, F_1 = [-\sqrt{5}, 0]\,,\ F_1 = [\sqrt{5}, 0]\,]$

53) Find the equation of an ellipse such that the major axis is parallel with the axis x and the centre is $S = [1, 3]$, the focus is $F = [-4; 3\,]$ and the magnitude of the minor axis is $b = 4$. $[\, \frac{(x-1)^2}{41} + \frac{(y-3)^2}{16} = 1\,]$

54) For which value of the number K is $x^2 + y^2 + 2x - 6y + K = 0$ an equation of the circle? $[\, K < 10\,]$

55) Prove that a hyperbola is given by the equation $x^2 - 4y^2 - 6x - 16y - 11 = 0$ and find its centre and the equations of the asymptotes. $[\, S = [3, -2], 2y = -x - 1, 2y = x - 7\,]$

56) We have two circles, the first with centre S_1 and radius 17 cm, the second with centre S_2 and radius 10 cm. The distance between the centres of these circles is 21 cm. The tangent line to both circles intersects the line S_1S_2 at the point P. Calculate the distances S_1P and S_2P. $[\, 30, 51\,]$

57) We have a circle with centre S and radius r = 5 cm. The line p intersects the circle at points A and B. The distance between the line p and the centre S is 3 cm. Calculate the length of the segment line AB. [8 cm]

58) Determine the equation of the hyperbola which passes through the point $A = [4.5, 1]$ whose has major axis is given by the equation $y = 0$ and its asymptotes are given by the equations $3y = -2x$ and $3y = 2x$. $[\, \frac{x^2}{18} - \frac{y^2}{8} = 1\,]$

59) We have a circle with the centre S and radius r =15 cm. The line p intersects the circle at points A and B. The length of the segment line AB is 10 cm. Calculate the distance between the line p and the centre S. $[\, \sqrt{2} \cdot 22.5\,]$

60) We have a circle with the centre $S = [-1, 2]$ and radius $r = 5\,\mathrm{cm}$. The line p given by the equation $y = x - 4$ intersects the circle at points A and B. Calculate the length of the segment line AB. $[\sqrt{2}]$

61) Determine the equation of the circle with the centre $S = [1, 3]$ which touches the straight line $p\colon 7x + y = 0$ at the point T. Find the coordinates of the point T.
$[(x-1)^2 + (y-3)^2 = 2,\ T = [-0.4, 2.8]]$

62) We have the circle $k(S = [-1, 3],\ r = 4\,\mathrm{cm})$ and the straight line AB. Calculate the intersection point $k \cap AB$, where $A = [-2, -4]$, $B = [1, 5]$. $[P = [-1, -1]]$

63) Find the equation of an ellipse which passes through the point $A = [3, 1]$ whose centre is at the origin and one focus is at the point $F = [4, 0]$. $[x^2 + 9y^2 = 18]$

64) Calculate the coordinates of the point T of a circle given by the equation $x^2 + y^2 = 1$ which lies next to the point $B = [1, 2]$. $[T = [\frac{1}{\sqrt{5}}; \frac{2}{\sqrt{5}}]]$

65) Are the following straight lines parallel or not: $p_1\colon x = 3 + 2t,\ y = 2 + 4t,\ z = 2 + 6t$;
$p_2\colon x = 4 + s,\ y = 5 + 2s,\ z = 1 + 3s$? [parallel]

66) We have the points $M = [3, 0, 4]$ and $N = [5, 6, 9]$. Find the equation of the plane which passes through point M and is perpendicular to the vector $\overrightarrow{MN}$.
$[2x + 6y + 5z - 26 = 0]$

67) Find the equation of the plane which passes through the origin and is perpendicular to the vector $\vec{n} = (2, 1, -3)$. $[2x + 1y - 3z = 0]$

68) Find the equation of the smallest sphere which contains points $A = [1, 0, 0]$,
$B = [1, -3, 0]$, $C = [6, 2, 1]$ and whose centre is $S = [1, 2, 3]$.
$[(x-1)^2 + (y-2)^2 + (z-3)^2 = 34]$

69) The surface of a block $S = 166\,\mathrm{cm}^2$, the volume $V = 140\,\mathrm{cm}^2$ and the length of one edge is $4\,\mathrm{cm}$. Determine the length of all the edges of the block. $[5, 7]$

70) The square pyramid $ABCDV$ has a base whose one edge has length a and the other $2a$. Calculate the length of the line segment AS, where S is the midpoint of the line segment CV. $[d = a\sqrt{2}]$

71) The surface of a cylinder is P, the circumference of its base is o. Calculate the volume of this cylinder. $[\frac{o \cdot P}{4\pi} - \frac{o^3}{8\pi^2}]$

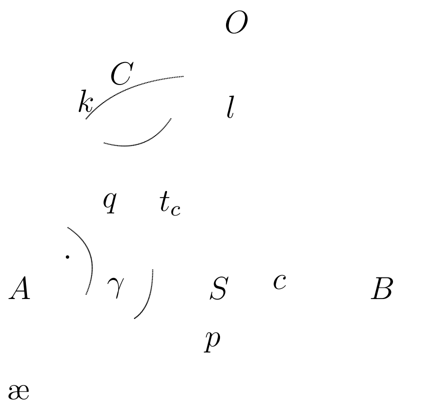
72) A sphere and a cube have the same surface. Find the ratio of their volumes.
$[\sqrt{\frac{6}{\pi}}]$

73) We create one sphere from three spheres with the radii $r_1 = 3\,\mathrm{cm}$, $r_2 = 4\,\mathrm{cm}$, $r_3 = 5\,\mathrm{cm}$. What radius does the new sphere have? $[6]$

74) From the point P we can see only 1% of the surface of the earth. Calculate the distance h between the point P and the earth. (The earth is a sphere with radius $r = 6370\,\mathrm{km}$).
$[h = 6370(\frac{1}{\cos\frac{\pi}{100}} - 1)]$

Constructions

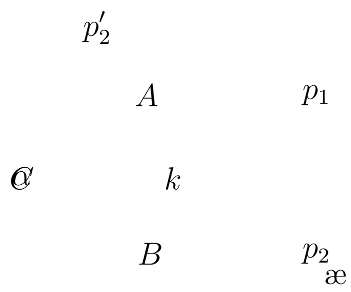
75) Construct a triangle ABC, if $c = 8\,\mathrm{cm}$, $t_c = 7\,\mathrm{cm}$, $\gamma = 60°$.



Construction:

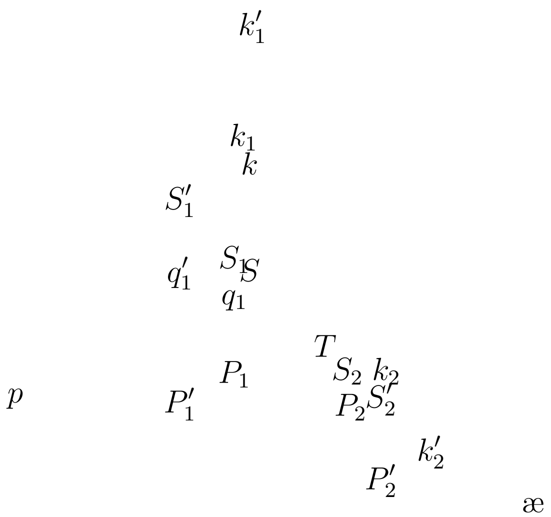
1. c: $c = \overline{AB} = 8\,\mathrm{cm}$ - side c
2. S: S - the midpoint of the segment $\overline{AB}$
3. k: $k \equiv (S; r = t_c)$
4. p: $A \in p$, $\angle(p, c) \equiv \gamma$
5. q: $A \in q$, $q \perp p$
6. l: $S \in l$, $l \perp c$
7. O: $O = l \cap q$
8. m: circle $m \equiv (O; r = OA)$
9. C: $C = m \cap k$
10. $\triangle ABC$

76) Construct an equilateral triangle ABC such that one vertex is the point C, vertex A belongs to the line p_1 and vertex B belongs to the line p_2, and $p_2 || p_1$ (p_2 is parallel to p_1).



1. rotation of the line p_2 around the point C with angle $\alpha = 60° \rightarrow p_2'$
2. B: $B \equiv p_2' \cap p_1$
3. k: $k \equiv (C;\ r = \overline{CB})$
4. A: $A \equiv k \cap p_2$
5. triangle CBA is equilateral

77) Construct a circle k which touches a straight line p and a circle $k_1(S, r), r = ST$ at the point T.



1. k_1': $k_1' \equiv (S_1';\ r \text{ (arbitrary)})$, $T \equiv k \cap k_1'$
2. q_1': $q \perp p$, $S_1' \in q_1'$
3. P_1': $P_1' \equiv q_1' \cap k_1'$
4. P_1: $P_1 \equiv P_1'T \cap p$
5. q_1: $q_1 \parallel q_1'$, $P_1 \in q_1$
6. S_1: $S_1 \equiv q_1 \cap S_1'T$
7. k_1: $k_1 \equiv (S_1;\ \overline{S_1P_1})$
8. similarly k_2

78) Take two arbitrary circles with the same centre. Construct a straight line p which cuts both circles such that the lengths of the line segments created by straight line p and the circles have the ratio 1 : 2 : 1.

p

$C k'$

$k S\ r\ a\ A\ a\ S'\ B\ q$

æ

1. k_2: $k_2 \equiv (S;\ 2r)$
2. straight line q: $S \in q$
3. A, B: $A \equiv q \cap k$, $B \equiv q \cap k_2$
4. S': $S' \equiv \frac{A+B}{2}$
5. k': $k' \equiv (S';\ \frac{r}{2})$
6. C: $C \equiv k' \cap k_1$
7. p: $p = AC$

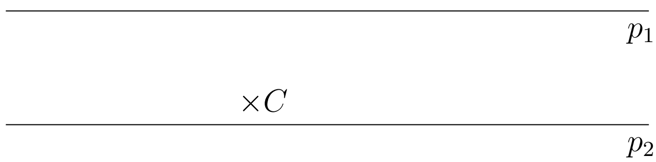
79) Construct a right triangle ABC, if $a + b =$10 cm, $\alpha = 30°$. Evaluate the ratio of its legs.

80) Construct a triangle ABC, if $a = 8$ cm, $v_a = 6$ cm, $\alpha = 60°$ and evaluate the radius of its circumscribed circle.

81) Take a segment AS with the length $|AS| = 5$ cm. Construct a triangle ABC with median AS such that $|AC| = 4$ cm, $|AB| = 6$ cm.

82) Construct a triangle ABC if side c, altitude v_c and median t_c are given.

83) Construct a right triangle ABC such that the point C is the vertex of the right angle, vertex A belongs to the line p_1 and vertex B belongs to the line p_2, and $p_2||p_1$ (p_2 is parallel to p_1).



84) Construct a triangle ABC, if $v_a = 7$ cm, $v_b = 6$ cm, $\alpha = 45°$ and inscribe a circle within this triangle.

85) A circle $k \equiv (S; r = 15\,\text{mm})$ and a point M such that $|MS| > 15$ are given. Construct an equilateral triangle ABC such that the circle k is its circumscribed circle and whose one side contains the point M.

86) Construct a triangle ABC if $a : b : c = 3 : 5 : 6$, $v_c = 4$ cm.

87) Construct a triangle ABC if $a = 8$ cm, $v_a = 5$ cm, $t_a = 7$ cm and its circumcircle.

88) Construct a triangle ABC if $\alpha = 60°$, $v_b = 9$ cm, $\rho = 2.5$ cm (radius of an inscribed circle).

89) Take the angle $KAM = \alpha$ with the length $\alpha = 30°$. Construct all triangles ABC so that vertex B belongs to the line AK, vertex C belongs to the line AM, the altitude $v_c = 2$ cm and $|BC| + |CA| = 6.5$ cm.

90) Construct a triangle ABC if $\gamma = 60°$, $v_a = 6\,\mathrm{cm}$, $c = 8\,\mathrm{cm}$. Calculate the length of v_a so that the triangle ABC is an equilateral triangle.

91) Construct a triangle ABC if γ, v_c, c are given. Describe the construction.

92) Construct a triangle ABC in which $t_c = 8\,\mathrm{cm}$, $c = 10\,\mathrm{cm}$, $r = 6\,\mathrm{cm}$ (radius of circumcircle). Evaluate $\sin\gamma$.

93) The point A is the intersection point of two circles $k_1(S_1; r_1 = 5\,\mathrm{cm})$, $k_2(S_2; r_2 = 4\,\mathrm{cm})$, $|S_1S_2| = 6\,\mathrm{cm}$. Draw all straight lines p so that point A belongs to p and the chords created on the circle k_1 and the circle k_2 by p have the same length.

94) We have a circle $k(S; r = 4\,\mathrm{cm})$ and a point P with $|SP| = 10\,\mathrm{cm}$. Construct a straight line p such that $P \in p$ and the chord created by the line p and the circle k has the length $t = 5\,\mathrm{cm}$. Evaluate the distance between the centre of this chord and the point S.

95) We have parallel lines p, q $10\,\mathrm{cm}$ apart and a point $M \in q$. Draw a circle k such that it touches the line p, $M \in k$ and the line segment created by line q and the circle k has the length $6\,\mathrm{cm}$.

96) We have parallel lines p_1, p_2 and an intersecting line p_3. Construct a square $ABCD$ such that vertex $A \in p_1$, vertex $C \in p_2$ and diagonal $BD \in p_3$.

97) We have a cube $ABCDA'B'C'D'$. Construct a straight line p such that $A' \in p$ and $p \perp AB'D'$ (p is perpendicular to the plane $AB'D'$). Denote by M the intersection $p \cap AB'D'$. Construct the segment line $A'M$.

98) We have a rectangle $ABCD$. Draw (without calculating) a square with the same length as rectangle $ABCD$.

99) Construct a regular octagon whose sides have the length $10\,\mathrm{mm}$. Describe the method of construction.

100) Draw the contour of a cube which we look at along its internal diagonal.

101) We have a sector ASB with angle $ASB = 75°$ and radius $|AS| = |BS| = r = 7{,}5\,\mathrm{cm}$. Inscribe a square into this sector such that one side of this square is parallel with line AB.

102) Construct a circle which touches a given straight line p at the point T and a given circle $k(S; r = 5\,\mathrm{cm})$.

103) We have two intersecting circles. Draw tangent lines to these circles.

104) Construct all points M on the straight line AB such that $|AM| : |AB| = \sqrt{2} : 2$.

Miscellaneous exercises

1) Simplify $\dfrac{n!}{(n-1)} + \dbinom{n}{2}$. [$n((n-2)! + \frac{n-1}{2})$]

2) Determine a_1 and d for the arithmetic sequence in which $a_3 + a_7 = 46$, $a_2 : a_6 = 2 : 7$.
[$a_1 = 3,\ d = 5$]

3) Calculate $\dfrac{2}{\sqrt{3}}\mathrm{tg}\left(\dfrac{x}{2} + \dfrac{\pi}{4}\right) = -\dfrac{2\sqrt{3}}{3}$. [$x = -\pi + 2k\pi$]

4) Calculate $|x-4| \le |6-x|$. [$x \in (-\infty, 5\rangle$]

5) Calculate $x^{\log x} = 1000x^2$. [$x = 1000$]

1) Solve the inequality $|2x+2| < |x-2|$ in $\mathbb{R}$. [$x \in (-4, 0)$]

2) Solve the inequality $\log(x-3) > 0$ in $\mathbb{R}$. [$x \in (4, \infty)$]

3) The value of $\sqrt{3 - 2\sqrt{2}}$ a) is equal to $\frac{1}{\sqrt{2}-1}$ b) is equal to $1-\sqrt{2}$ c) is equal to $\sqrt{2}-1$ d) another answer is correct. [c]

4) If $x \ne 0$ then term $(x + x^2 + x^3)^{-1}(x^{-1} + x^{-2} + x^{-3})$ a) is equal to 1 b) is equal to $\frac{1}{x^3}$ c) is equal to $\frac{1}{x^4}$ d) another answer is correct. [c]

5) Solve the equation $\dfrac{1}{\sin^2 x} - \dfrac{2}{\sqrt{3}}\mathrm{cotg}\ x - 2 = 0$ in $\mathbb{R}$. [$x_1 = \frac{\pi}{6} + k\pi, x_2 = \frac{2\pi}{3} + k\pi$]

6) Find the complex number z, if $(5+i)\overline{z} - 22i = -2z$. [$z = 1 - i7$]

7) How many members of the geometric sequence $3, 12, 48, \ldots$ do we need to add to get the sum 4095?
[6]

1) Solve the inequality $|x-5| < 1$ in $\mathbb{R}$. [$x \in (4, 6)$]

2) Solve the equation $5^x = 2$ in $\mathbb{R}$. [$x = \log_5 2$]

3) The value $\dfrac{1}{\sqrt{3} - \sqrt{2}}$ a) is equal to $\frac{\sqrt{3}+\sqrt{2}}{2}$ b) is equal to $\sqrt{3} + \sqrt{2}$ c) is equal to 1 d) another answer is correct.
[b]

4) The term $3\sin x \cos^2 x - \sin^3 x$ a) is equal to $\sin x(3 - 4\sin^2 x)$ b) is equal to $\sin x(3 - 2\sin^2 x)$ c) is equal to $\sin x(1 - 2\sin^2 x) - \cos x \sin 2x$ d) another answer is correct. [a]

5) For $x \in \langle 0, \pi\rangle$ solve the inequality $\cos 2x - \cos^2 x > \sin^2 x - \sin x$. [$x \in (0, \frac{\pi}{6}) \cup (\frac{5\pi}{6}, \pi)$]

6) Use the Moivre theorem and find the real and imaginary parts of the complex number $(-1 + i\sqrt{3})^7$.
[$2^7(-1 + i\sqrt{3})$]

7) Solve the equation $2 \cdot 3^{x+2} - 135 = 2(3^x + 3^{x-1} + 3^{x-2} + \cdots)$. [$x = 2$]

1) Solve the inequality $\sqrt{2x-1} \le x$ in $\mathbb{R}$. [$x \in \langle \frac{1}{2}, \infty)$]

2) Solve the equation $1 - 2\log x = 0$ in $\mathbb{R}$. [$x = \sqrt{10}$]

3) The value $\left(\sqrt{\sqrt{3}+2} - \sqrt{2-\sqrt{3}}\right)^2$ a) is equal to 2 b) is equal to 4 c) is equal to $\sqrt{3}$ d) another answer is correct.

[a]

4) For $x \neq k\pi$ the term $\dfrac{\cos x + 1}{\sin x}$ a) is equal to $\dfrac{\cos^2 x + 2\cos x + 1}{\sin^2 x}$ b) is equal to $\dfrac{\sin x}{1 - \cos x}$ c) is equal to $\dfrac{\sin x}{\cos x - 1}$ d) another answer is correct. [b]

5) For $x \in \mathbb{R}$ solve the equation $\sin(2x + \frac{\pi}{12}) = -\frac{\sqrt{2}}{2}$. [$x_1 = \frac{7\pi}{12} + k\pi, x_2 = \frac{10\pi}{12} + k\pi$]

6) Use the Moivre theorem and prove that $\sin 3x = 3\sin x \cos^2 x - \sin^3 x$.
[$(\cos x + i\sin x)^3 = \cos^3 x + i3\cos^2 x \sin x - i3\cos x \sin^2 x - i\sin^3 x = \cos 3x + i\sin 3x$]

7) Solve the equation $\dfrac{(n-1)! + 2(n-1)! + 3(n-1)! + \cdots + n(n-1)!}{n! + \frac{n!}{2} + \frac{n!}{4} + \frac{n!}{8} + \cdots} = \dfrac{(n+1)!}{4x}$.

[$x = n!$]

1) Solve the equation $|2x - 1| = 1$ in $\mathbb{R}$. [$x_1 = 0,\ x_2 = 1$]

2) Solve the equation $1 + 2\log x = 0$ in $\mathbb{R}$. [$x = 10^{-\frac{1}{2}}$]

3) The value $\dfrac{(n+1)!}{n!} - \dfrac{n!}{(n-1)!}$ a) is equal to n b) is equal to $n+1$ c) is equal to 1 d) another answer is correct. [c]

4) For $a \neq 0 \wedge b \neq 0 \wedge a \neq b$ the term $\left(\dfrac{1}{a^{-1} - b^{-1}}\right)^{-1}$ a) is equal to $\dfrac{ab}{a-b}$ b) is equal to $\dfrac{a-b}{ab}$ c) is equal to $a - b$ d) another answer is correct. [d]

5) For $x \in \langle 0, \pi \rangle$ solve the equation $1 + \sin^2 x + \sin^4 x + \sin^6 x + \cdots = 2\,\mathrm{tg} x$.

[$x_1 = \frac{\pi}{4}$]

6) Use the Moivre theorem and find the real and imaginary parts of the complex number $\left(\dfrac{1+i}{\sqrt{2}}\right)^{26}$

[$z = i$]

7) Determine the first member and the common difference for the arithmetic sequence in which
$a_3 + a_7 = 46\,, \quad a_2 : a_6 = 2 : 7$. [$a_1 = 3, d = 5$]

1) Solve the inequality $\dfrac{3}{x-2} < 1$ in $\mathbb{R}$. [$x \in (-\infty, 2) \cup (5, \infty)$]

2) Solve the equation $\log(x-3)^2 = 6$ in $\mathbb{R}$. [$x = 1003$]

3) The value $\dfrac{n!}{(n-1)!} + \dbinom{n}{2}$ a) is equal to $(n-1)!$ b) is equal to $\dfrac{n^2+2n}{2}$ c) is equal to $\dfrac{3}{2}n$ d) is equal to $\dfrac{n^2+n}{2}$. [d]

4) For $x \neq 0$ the term $(x^{-1} + x^{-2} + x^{-3})^{-1}$ a) is equal to $\dfrac{1}{x^3+x^2+x}$ b) is equal to $x^3 + x^2 + x$ c) is equal to $x^3(x^2+x+1)^{-1}$ d) another answer is correct. [c]

5) For $x \in \mathbb{R}$ solve the inequality $\mathrm{tg}^2 \frac{x}{2} - \sin^2 \frac{x}{2} - \cos^2 \frac{x}{2} \leq 0$. [$x \in (-\frac{\pi}{2}, \frac{\pi}{2})$]

6) Calculate $\left| \dfrac{5(i^{10} - i)}{2i+1} \right|$. [$\sqrt{10}$]

7) For $x \in \mathbb{R}$ solve the equation $1 - \dfrac{2}{x} + \dfrac{4}{x^2} - \dfrac{8}{x^3} + \cdots = \dfrac{6}{x+5}$. [$x_1 = 4, x_2 = -3$]

1) Solve the inequality $\dfrac{x-1}{x+2} < 1$ in $\mathbb{R}$. [$x \in (-2, \infty)$]

2) Solve the equation $2^x + 2^x + 2^{x+1} = 16$ in $\mathbb{R}$. [$x = 2$]

3) For $a \neq -\frac{1}{3} \wedge a \neq -\frac{1}{4}$ the value $\dfrac{a-3}{1+3a} - \dfrac{a-4}{1+4a}$ a) is equal to $\dfrac{7(a^2-1)}{1+7a+12a^2}$ b) is equal to $\dfrac{1+a^2}{(1+3a)(1+4a)}$ c) is equal to $\dfrac{a^2-7}{1+12a^2}n$ d) is equal to $\dfrac{a^2+1}{1+12a^2}$. [b]

4) The domain set of function $y = 3\log(x+2)$ a) is equal to the interval $(0, \infty)$ b) is equal to the interval $(-2, \infty)$ c) is equal to the interval $(-3, \infty)$ d) another answer is correct. [b]

5) For $x \in \mathbb{R}$ solve the equation $(\sin x)(1 + 2\cos x) - \mathrm{tg}x = 0$.
[$x_1 = k\pi, x_2 = \frac{\pi}{3} + 2k\pi, , x_3 = \frac{5\pi}{3} + 2k\pi$]

6) Find the real numbers x, y for which $(x + iy)^2 = 3 + 4i$.
[$x = 2, y = 1$ or $x = -2, y = -1$]

7) Find the geometric series such that $a_1 = 1$ and its sum is equal to the sum of the series $6 - \dfrac{6}{5} + \dfrac{6}{25} - \dfrac{6}{125} + \cdots$. [$a_1 = 1, q = \frac{4}{5}$]