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THE RANGES OF NONLINEAR OPERATORS OF THE POLYNOMIAL TYPE

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Abstract: In this paper we prove the existence results for the equation Au + Su = f, where A is a polynomial operator on a reflexive Banach space, S is a strongly continuous nonlinearity.

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1. Introduction. J. Frehse investigated a class of nonlinear functional equations and nonlinear operators of polynomial type (see e.g. [1]). The ranges of these operators are closed linear subspaces with a finite codimension and the equation

(1.1) $$Au = f$$

has at least one solution if f satisfies the Fredholm condition. Further, J. Frehse deals with the solvability of the equation

(1.2) $$Au + Su = f,$$

where S is the Landesman-Lazer type nonlinearity (see e.g.[2]).

This paper continues, in some sense, the works [1],[2] and deals with the solvability of the equation (1.2) in section 2, where S is "subpolynomial-type" nonlinearity. In section 3 the abstract theorems are applied to the examples of polynomial operators, for example, to the problem

$$\begin{cases} (\Delta-\lambda)[(\Delta u-\lambda u)^5+(\Delta u-\lambda u)^3]+ \\ \qquad\qquad +|u|^{\sigma}\operatorname{sign} u = f \text{ in } \Omega, \\ \qquad\qquad u = 0 \text{ on } \partial\Omega. \end{cases}$$

There are also presented results concerning the solvability of (1.2) in section 4, where the operator S has a vanishing strong subasymptote. For example, there is considered the problem

$$\begin{cases} (\Delta-\lambda)[(\Delta u-\lambda u)^5+(\Delta u-\lambda u)^3]+\dfrac{u}{1+u^2} = f \text{ in } \Omega, \\ \qquad\qquad u = 0 \text{ on } \partial\Omega. \end{cases}$$

The proof which is published in [5], is analogous to that contained in the papers [3],[4] where equations with linear non-invertible operators in the main part are considered.

2. Abstract theorems. We shall investigate continuous maps $A:B\to B^*$ where B is a real reflexive Banach space with a norm $\|\cdot\|$, B^* is its dual space. We consider following conditions:

(2.1) There exists $a \geqq 0$ such that it holds

(i) if $\limsup_{t\to+\infty} t^{-a}|\langle A(u+tv),v\rangle| < +\infty$ then $\langle A(u+tv)v\rangle = \langle Au,v\rangle$ whenever $t\in R$, $u,v\in B$,

(ii) if $\limsup_{t\to+\infty} t^{-a}|\langle A(tw),v\rangle| < +\infty$ then $\langle A(tw),v\rangle = \langle A(0),v\rangle$ whenever $t\in R$, $v,w\in B$.

(2.2) If $u,v\in B$, $\varphi(t) = \langle A(u+tv),u+tv\rangle$ and

(i) $\liminf_{t\to+\infty} t^{-1}\varphi(t) \geqq 0$,

(ii) $\limsup_{t\to+\infty} t^{-1}\varphi(t) < +\infty$,

then $\lim_{t\to+\infty} t^{-1}\varphi(t) = 0$.

Any continuous operator A satisfying conditions (2.1) and (2.2) will be said a-polynomial.

An operator A satisfying

(2.3) $\liminf_{\|u\|\to\infty} \|u-v\|^{-1}\langle Au-Av,u-v\rangle \geqq 0$ for each $v\in B$

will be called the asymptotically monotone operator.

(2.4) There exist constants $K,c>0$, $p>1$ and a finite dimensional subspace $V\subset B$ with a bounded linear projection $Q:B\to V$ such that

$$\langle Au,u\rangle \geqq c\|u\|^p - K\|Qu\|^p - K \text{ whenever } u\in B.$$

2.5. Definition. A continuous operator $A:B\to B^*$ is said regular if the variational inequality

$$\langle Au-f,u-v\rangle \leqq 0,\quad v\in K,$$

has a solution $u\in K$ for any bounded closed convex set $K\subset B$ and for every $f\in B^*$.

The main result of Frehse's work [1] is as follows.

2.6. Theorem. Let $A:B\to B^*$ be a regular operator satisfying conditions (2.1)(i) with $a = 0$, (2.2)-(2.4) and let $A(0) = 0$. Then the equation $Au = f$ has at least one solution if and only if $f\perp(R(A))^{\perp}$.

Moreover, $\dim R(A)^{\perp} \leqq \dim V < +\infty$.

We shall use the next lemma in proofs of the following theorems.

2.7. Lemma. Let $A:B\to B^*$ be an asymptotically monotone a-polynomial operator, $A(0) = 0$. Suppose that for some $v\in B$ there exist constants σ, C, $K \geqq 0$ such that the inequality

(2.8) $$\langle Aw,v\rangle \leqq C + K\|w\|^{\sigma}$$

holds for every $w\in B$. If $a \geqq \sigma$ then $v\perp R(A)$.

Proof. The inequality (2.8) implies $\langle A(w+tv),v\rangle \leqq C + K\|w+tv\|^{\sigma}$ and from the asymptotical monotonicity of the operator A (i.e. $\liminf_{|t|\to\infty} |t|^{-1}\langle A(w+tv) - Aw, tv\rangle \geqq 0$) we obtain $\langle A(w+tv),v\rangle \geqq \langle Aw,v\rangle - \varepsilon$ for every $t \geqq t_0$ with some $t_0 > 0$, $\varepsilon > 0$. Together with the supposition (2.8) and the condition (2.1) we have

(2.9) $\langle A(w+tv),v\rangle = \langle Aw,v\rangle$ for every $t \in R$.

Using the inequality $\liminf_{|t|\to+\infty} |t|^{-1}\langle A(w+tv) - A(2w), -w+tv\rangle \geqq 0$ we get that $\limsup_{|t|\to+\infty} |t|^{-1}\langle A(w+tv),w\rangle \leqq K(w)$ with some constant $K(w)$. It yields together with (2.9) $\limsup_{|t|\to+\infty} |t|^{-1}\varphi(t) < +\infty$, where $\varphi(t) = \langle A(w+tv), w+tv\rangle$. From conditions (2.2),(2.3), $A(0) = 0$ it follows that

(2.10) $$\lim_{|t|\to+\infty} t^{-1}\varphi(t) = 0.$$

Let $s \in R$ be fixed. It is obvious that

$$\liminf_{|t|\to+\infty} |t|^{-1}\langle A(w+tv) - A(sw), (1-s)w+tv\rangle \geqq 0$$

and this together with (2.9) yields

$$\liminf_{|t|\to+\infty} |t|^{-1}[(1-s)\varphi(t) + s\langle Aw,tv\rangle - \langle A(sw),(1-s)w+tv\rangle] \geqq 0.$$

According to this fact and with respect to the condition (2.10) we have $s\langle Aw,v\rangle - \langle A(sw),v\rangle \geqq 0$, $-s\langle Aw,v\rangle + \langle A(sw),v\rangle \geqq 0$ and

(2.11) $$s\langle Aw,v\rangle = \langle A(sw),v\rangle, \quad s \in R.$$

If $a<1$ then $0 \leqq \sigma < 1$ and as $s\langle Aw,v\rangle \leqq C + K|s|^{\sigma}\|w\|^{\sigma}$ we get $\langle Aw,v\rangle = 0$, taking the limits $s \longrightarrow \pm\infty$. This completes the proof for $a<1$.

Let $a \geqq 1$. There exists $\vartheta > 0$ such that $\|Au\| \leqq 1 + \|A(0)\| = 1$ for every $u \in B$, $\|u\| \leqq \vartheta$. The inequality

$$\langle Aw,v\rangle = \frac{\|w\|}{\vartheta}\langle A(\frac{w}{\|w\|}\vartheta),v\rangle \geqq -\frac{\|w\|}{\vartheta}\|v\|, \quad w \neq 0,$$

is an immediate consequence of (2.11). Therefore, there exists the constant $L>0$ such that $\langle Aw,v\rangle \geqq -L\|w\|$, $w \in B$. Using the inequality (2.8) and the fact that $a \geqq 1$ we obtain

$$\limsup_{t\to+\infty} t^{-a}|\langle A(tw),v\rangle| < +\infty.$$

From (2.1) we get $\langle Aw,v\rangle = \langle A(0),v\rangle = 0$. It means that $v \perp R(A)$ and the proof of the lemma is complete.

Let $S: B \longrightarrow B^*$ be an operator satisfying conditions

(2.12) $\|Su\|_{B^*} \leqq \alpha + \beta\|u\|^{\sigma}$, $\alpha, \beta, \sigma \geqq 0$,

(2.13) there exist constants $G,H > 0$ such that the inequality

$$\liminf_{\|u_i\|\to+\infty} \|u_i\|^{-1}\langle Su_i - Sw, u_i - w\rangle \geqq -G - H\|w\|^{\sigma}$$

is fulfilled for every $w \in B$.

2.14. __Definition.__ Let V be a closed linear subspace of B, $V_r = \{u \in V, \|u\| \leqq r\}$. A mapping $\Psi: V_1 \longrightarrow R$ will be said a strong subasymptote of the operator S with respect to V if

(2.15) $$\Psi(z) \leqq \liminf_{j\to+\infty} \langle Su_j, \|u_j\|^{-1}(u_j - w)\rangle, \quad w \in B,$$

holds for any sequence $\{u_j\}_{j=1}^{+\infty}$ such that $\|u_j\| \longrightarrow +\infty$ and $\|u_j\|^{-1}u_j \longrightarrow z$ (i.e. weakly) for $j \longrightarrow +\infty$, where $z \neq 0$, $z \in V$.

2.16. __Theorem.__ Let $A,S: B \longrightarrow B^*$ be continuous operators with the following properties

(i) A is an asymptotically monotone a-polynomial operator, $A(0) = 0$ and A satisfies (2.4),

(ii) S satisfies (2.12),(2.13) and $p > 1 + \sigma$, $a \geqq \sigma$,

(iii) A + S is a regular operator.

If $\Psi: (R(A)^{\perp})_1 \longrightarrow R$ is a strong subasymptote of the operator S with respect to $R(A)^{\perp}$ and if

(2.17) $\langle f,z\rangle < \Psi(z)$ for every $z \in (R(A)^{\perp})_1$, $z \neq 0$,

then the equation (1.2) has at least one solution.

__Proof.__ Let us suppose that the equation is not solvable and let u_r be the solution of the variational inequality

(2.18) $\langle Au + Su - f, u - w \rangle \leqq 0, \; w \in B_r.$

Observe that $u_r \in \partial B_r$ and therefore $\|u_r\| = r$. Choose a sequence $\{r_i\}_{i=1}^{+\infty}$ so that $\|u_{r_i}\|^{-1} u_{r_i} \rightharpoonup z$ weakly in B. According to (2.18) with $w = 0$ and in view of the growth of S (see (2.12)) we get the inequality $\langle Au_{r_i}, u_{r_i} \rangle \leqq L \|u_{r_i}\|^{1+\delta}$ for $i \geqq i_0$ with some positive constant L. Since $p > 1 + \delta$ we obtain from (2.4) that $\liminf_{i \to +\infty} \|Qu_{r_i}\|^p \|u_{r_i}\|^{-p} \geqq \frac{c}{K} > 0$. The fact that $\dim R(Q) < +\infty$ implies $Q(u_{r_i} \|u_{r_i}\|^{-1}) \to Qz$ in B for $i \to +\infty$ and $\|Qz\| > 0$, therefore $z \neq 0$.

We claim $z \perp R(A)$. Observe that

$$\liminf_{i \to +\infty} \|u_{r_i}\|^{-1} \langle Au_{r_i} - Aw, u_{r_i} - w \rangle \geqq 0,$$

$$\liminf_{i \to +\infty} \|u_{r_i}\|^{-1} \langle f - Au_{r_i} - Su_{r_i}, u_{r_i} - w \rangle \geqq 0$$

and therefore

(2.19) $$\liminf_{i \to +\infty} \|u_{r_i}\|^{-1} \langle f - Su_{r_i} - Aw, u_{r_i} - w \rangle \geqq 0.$$

From (2.13) we have

$$\liminf_{i \to +\infty} \|u_{r_i}\|^{-1} \langle f - Aw - Sw, u_{r_i} - w \rangle \geqq -G - H\|w\|^{\delta}$$

and this gives the estimate

$$\liminf_{i \to +\infty} \langle -Aw, u_{r_i} \|u_{r_i}\|^{-1} \rangle \geqq -G - H\|w\|^{\delta} - (\alpha + \beta\|w\|^{\delta}) - -|\langle f, z \rangle| .$$

Consequently, $\langle Aw, z \rangle \leqq G + |\langle f, z \rangle| + \alpha + (\beta + H)\|w\|^{\delta}$ and the Lemma 2.7 implies $z \perp R(A)$.

Observe that the inequality (2.19) yields

$$\langle f, z \rangle - \langle Aw, z \rangle - \liminf_{i \to +\infty} \langle Su_{r_i}, \|u_{r_i}\|^{-1}(u_{r_i} - w) \rangle \geqq 0.$$

As Ψ is the strong subasymptote of the operator S we get $\langle f, z \rangle - \Psi(z) \geqq 0$, which is the contradiction with (2.17) and the proof is complete.

2.20. <u>Proposition</u>. The condition (2.17) is necessary for the solvability of (1.2), if $\langle Su, z \rangle < \Psi(z)$ for every $u \in B$, $z \neq 0$, $z \in (R(A)^{\perp})_1$.

<u>Proof</u>. If $Au + Su = f$ then $\langle f, z \rangle = \langle Su, z \rangle < \Psi(z)$ for $z \in (R(A)^{\perp})_1$.

In the case $\delta < 1$, the strong subasymptote of the operator S can be replaced by more verifiable conditions:

(2.21) $$\liminf_{\|u_i\| \to +\infty} \|u_i\|^{-1} \langle Su_i - Sw_i, u_i - w_i \rangle \geqq -G$$

for every bounded sequence $\{w_i\}_{i=1}^{+\infty}$.

(2.22) For every $z \in R(A)^{\perp}$, $z \neq 0$, there exist $t_z \in R$, $v_z \in B$ such that $\langle S(t_z z + v_z), z \rangle > G$, where G is the constant from (2.21).

(2.23) $$\liminf_{i \to +\infty} \langle S(tz_i + v), -z_i \rangle \leqq \langle S(tz + v), -z \rangle$$

holds for any $t \in R$, $v \in B$ and any sequence $\{z_i\}_{i=1}^{+\infty} \subset B$, $z_i \rightharpoonup z$ weakly for $i \to +\infty$, $z \in R(A)^{\perp}$, $z \neq 0$.

A strongly continuous operator S satisfies the condition (2.23).

2.24. <u>Theorem</u>. Let $A, S : B \to B^*$ be continuous operators with the following properties

(i) A is an asymptotically monotone a-polynomial operator satisfying (2.4), $A(0) = 0$,

(ii) S satisfies (2.12),(2.21)-(2.23) and $p > 1 + \delta$, $a \geqq \delta$, $\delta < 1$,

(iii) $A + S$ is a regular operator.

Then the equation $Au + Su = 0$ has at least one solution.

<u>Proof</u>. The condition (2.21) implies (2.13). Let us suppose that the equation $Au + Su = 0$ is not solvable. Analogously as in the proof of Theorem 2.16 there exists a sequence

$\{u_{r_i}\}_{i=1}^{+\infty}$, $\|u_{r_i}\| \to +\infty$, $\|u_{r_i}\|^{-1} u_{r_i} \rightharpoonup z$ weakly in B for $i \to +\infty$, $z \in R(A)^{\perp}$, $z \neq 0$, and $\langle Au_{r_i} + Su_{r_i}, u_{r_i} - w\rangle \leqq 0$ for every $w \in B_{r_i}$. As the operator S satisfies (2.21) and (2.22) we have

$$- G \leqq \liminf_{i\to+\infty} \|u_{r_i}\|^{-1} \langle S(t_z u_{r_i} \|u_{r_i}\|^{-1} + v_z) - Su_{r_i}, t_z u_{r_i} \|u_{r_i}\|^{-1} + v_z - u_{r_i}\rangle =$$

$$= \liminf_{i\to+\infty} \langle S(t_z u_{r_i} \|u_{r_i}\|^{-1} + v_z) - Su_{r_i}, -u_{r_i} \|u_{r_i}\|^{-1}\rangle$$

because $\sigma < 1$. The operator A + S is regular and therefore we get $\langle Au_{r_i} + Su_{r_i}, -u_{r_i}\rangle \geqq 0$ and

$$\liminf_{i\to+\infty} \langle S(t_z u_{r_i} \|u_{r_i}\|^{-1} + v_z) + Au_{r_i}, -u_{r_i} \|u_{r_i}\|^{-1}\rangle \geqq -G.$$

Further, A is asymptotically monotone, e.g.

$$\liminf_{i\to+\infty} \langle -Au_{r_i}, -u_{r_i} \|u_{r_i}\|^{-1}\rangle \geqq 0$$

and

$$\liminf_{i\to+\infty} \langle S(t_z u_{r_i} \|u_{r_i}\|^{-1} + v_z), -u_{r_i} \|u_{r_i}\|^{-1}\rangle \geqq -G.$$

From (2.23) we obtain $\langle S(t_z z + v_z), z\rangle \leqq G$, which is the contradiction with (2.22).

3. Examples. Let $P_j : R^s \to R$, $j = 1,2,\dots,s$, be polynomials satisfying the following conditions (with $C,K,c > 0$)

(3.1) $\quad |P_j(\xi)| \leqq C(1 + |\xi|^{p-1})$ for every $\xi \in R^s$,

(3.2) $\quad \sum_{j=1}^{s} P_j(\xi)\xi_j \geqq c|\xi|^p - K$ for every $\xi \in R^s$,

(3.3) $\quad \sum_{j=1}^{s} (P_j(\xi) - P_j(\eta))(\xi_j - \eta_j) \geqq 0$ for all $\xi, \eta \in R^s$.

Let $\Omega \subset R^N$ be a bounded domain with a smooth boundary and let $V = W^{2m,p}(\Omega) \cap W_0^{m,p}(\Omega)$, $p > 1$. We define

$$L_j u = \sum_{|r|,|q| \leqq m} (-1)^r D^r(a_{rq}^{(j)}(x) D^q u), \ j = 1,\dots,s,$$

for every $u \in V$ where $a_{rq}^{(j)} \in C^\infty(\bar\Omega)$ $(|r|,|q| \leqq m$, $j = 1,\dots,s)$. Let

$$\sum_{|r|,|q|=m} (-1)^m a_{rq}^{(j)}(x)\, \xi^{r+q} \geqq \alpha|\xi|^{2m}, \ j = 1,\dots,s,$$

hold with some $\alpha > 0$ for every $\xi \in R^N$. Let us define the operator $A : V \to V^*$ by

$$\langle Au, v\rangle = \sum_{j=1}^{s} \int_\Omega P_j(L_1 u,\dots,L_s u)\, L_j v, \ v \in V.$$

Using the Theorem 2.6, we see that the equation $Au = f$ is solvable if $(f - A(0)) \perp (R(A) - A(0))^{\perp}$. Let us remark that for $s = 1$ it is possible to show: if we consider the operator $A : V/_{Ker[L_1]} \to (V/_{Ker[L_1]})^*$ then this result follows from the theory of monotone operators and $(R(A) - A(0))^{\perp} = Ker[L_1]$.

Let the function φ be continuous, odd, increasing, $\lim_{t\to+\infty} \varphi(t) = +\infty$ and $|\varphi(t)| \leqq \bar\alpha + \bar\beta|t|^{\sigma}$, $t \in R$, with some $\bar\alpha, \bar\beta, \sigma > 0$. Let $2mp > N$. We define the operator $S : V \to V^*$ by

$$\langle Su, v\rangle = \int_\Omega \varphi(u)\, v, \ v \in V.$$

We note that the inequality (2.12) holds with some constants α, β. Let us assume the conditions

(3.4) $\quad \limsup_{t\to+\infty} \varphi(\omega t)[\varphi(t)]^{-1} = \chi(\omega) < +\infty$

for every $\omega \geqq 1$, where χ is a continuous function with $\lim_{\omega\to 1+} \chi(\omega) = 1$,

(3.5) $\quad \operatorname{meas}\Omega > 2 \operatorname{meas}\{x \in \Omega; z(x) = 0\}$

for every $z \in (R(A) - A(0))^{\perp}$, $z \neq 0$.

3.6. Proposition. The mapping $\Psi : ((R(A) - A(0))^{\perp})_1 \to \{K\}$, where K is a real number, is a strong subasymptote of the operator S defined above with respect to $(R(A) - A(0))^{\perp}$.

Proof. We assume that $A(0) = 0$ and that for a sequence $\{u_n\}_{n=1}^{+\infty} \subset V$ it is $\|u_n\|^{-1} \to +\infty$, $(u_n - w)\|u_n\|^{-1} \to z$ weakly for $n \to +\infty$, $z \in R(A)^{\perp}$, $z \neq 0$, $w \in V$. It suffices to show that

$$\liminf_{n\to+\infty} \int_{\Omega} \varphi(u_n) \frac{u_n - w}{\|u_n\|} - K \geqq 0.$$

As $W^{2m,p}(\Omega)$ is compactly imbedded into $C(\bar{\Omega})$ we have $u_n\|u_n\|^{-1} \to z$ and $(u_n - w)\|u_n\|^{-1} \to z$ in $L_{\infty}(\Omega)$. If we denote $\Omega_{\varepsilon}^{+} = \{x \in \Omega ; z(x) \geqq \varepsilon\}$, $\Omega_{\varepsilon}^{-} = \{x \in \Omega ; z(x) \leqq -\varepsilon\}$, $\Omega_{\varepsilon} = \Omega_{\varepsilon}^{-} \cup \Omega_{\varepsilon}^{+}$ then according to (3.4) and (3.5) there exist $\varepsilon > 0$, an integer $k_o > 1$ such that the inequality

$$\text{(3.6)} \qquad \operatorname{meas} \Omega_{\varepsilon} - \frac{k+1}{k-1} \chi\left(\frac{k+1}{k-1}\right) \operatorname{meas}(\Omega \setminus \Omega_{\varepsilon}) > 0$$

holds for every $k \geqq k_o$. There exists a natural number n_o such that

$$z(x) - \frac{\varepsilon}{k_o} \leqq \frac{u_n(x) - w(x)}{\|u_n\|} \leqq \frac{\varepsilon}{k_o} + z(x) \quad \text{a.e. in } \Omega,$$

$$z(x) - \frac{\varepsilon}{k_o} \leqq \frac{u_n(x)}{\|u_n\|} \leqq \frac{\varepsilon}{k_o} + z(x) \quad \text{a.e. in } \Omega$$

for every $n \geqq n_o$. So we get

$$\int_{\Omega} \varphi(u_n) \frac{u_n - w}{\|u_n\|} \geqq \int_{\Omega_{\varepsilon}^{+}} \varphi(u_n) \frac{u_n - w}{\|u_n\|} + \int_{\Omega_{\varepsilon}^{-}} \varphi(-u_n) \frac{-u_n + w}{\|u_n\|} -$$

$$- \int_{\Omega\setminus\Omega_{\varepsilon}} \varepsilon \frac{k_o + 1}{k_o} \varphi\left(\varepsilon \frac{k_o + 1}{k_o} \|u_n\|\right) \geqq$$

$$\geqq \int_{\Omega_{\varepsilon}} \varepsilon \frac{k_o - 1}{k_o} \varphi\left(\varepsilon \frac{k_o - 1}{k_o} \|u_n\|\right) - \int_{\Omega\setminus\Omega_{\varepsilon}} \varepsilon \frac{k_o + 1}{k_o} \varphi\left(\varepsilon \frac{k_o + 1}{k_o} \|u_n\|\right) \geqq$$

$$\geqq \varepsilon \frac{k_o - 1}{k_o} \varphi\left(\varepsilon \frac{k_o - 1}{k_o} \|u_n\|\right) \operatorname{meas} \Omega_{\varepsilon} -$$

$$- \frac{k_o + 1}{k_o} \varepsilon \left[\chi\left(\frac{k_o + 1}{k_o - 1}\right) + \vartheta_n\right] \operatorname{meas}(\Omega \setminus \Omega_{\varepsilon}) \varphi\left(\varepsilon \frac{k_o - 1}{k_o} \|u_n\|\right),$$

where $\vartheta_n \to 0$ for $n \to +\infty$. Observe that

$$\int_{\Omega} \varphi(u_n) \frac{u_n - w}{\|u_n\|} \geqq \varepsilon \frac{k_o - 1}{k_o} \varphi\left(\varepsilon \frac{k_o - 1}{k_o} \|u_n\|\right) \Big[\operatorname{meas} \Omega_{\varepsilon} -$$

$$- \frac{k_o + 1}{k_o - 1} \left(\chi\left(\frac{k_o + 1}{k_o - 1}\right) + \vartheta_n\right) \operatorname{meas}(\Omega \setminus \Omega_{\varepsilon})\Big].$$

Denote the expression in the square brackets by c_n. It follows from (3.6) that $\lim_{n\to+\infty} c_n > 0$ and therefore

$$\lim_{n\to+\infty} \varepsilon \frac{k_o - 1}{k_o} \varphi\left(\varepsilon \frac{k_o - 1}{k_o} \|u_n\|\right) c_n = +\infty.$$

The proof is finished.

If the operator A satisfies the condition (3.5) then the Theorem 2.16 can be applied. If $\sigma < 1$ then the operator $S - f$ satisfies the conditions (2.21)-(2.23) and the Theore 2.24 can be used. In these cases, if $p > 1 + \sigma$, $a \geqq \sigma > 0$ then the equation $Au + Su = f$ has at least one solution.

For example, the problem

$$(\Delta - \lambda)[(\Delta u - \lambda u)^5 + (\Delta u - \lambda u)^3] + |u|^{\sigma} \operatorname{sign} u = f \text{ in } \Omega,$$
$$u = 0 \text{ on } \partial\Omega$$

has at least one weak solution $u \in W_o^{1,6}(\Omega) \cap W^{2,6}(\Omega)$ for $0 < \sigma < 3$.

4. Problems with a bounded nonlinearity. Let B be a linear closed subspace of $W^{k,p}(\Omega)$, $kp > N$, $p > 1$, $A(0) = 0$,

$$\text{(4.1)} \qquad \langle Su, v \rangle = \int_{\Omega} \varphi(u)\, v, \text{ for } u, v \in B,$$

where the function φ is continuous, odd, $\lim_{|t|\to+\infty} \varphi(t) = 0$. Then $\|Su\|_{B^*} \leqq \wp$ for every $u \in B$ with some constant $\wp$. Further, we shall assume the following conditions be satisfied

(4.2) for all $w \in R(A)^{\perp}$, $t \in R$, $v \in B$ it is $A(v + tw) = Av$,

(4.3) there exists a bounded linear projection $Q: B \longrightarrow R(A)^{\perp}$ and $\langle Au,u\rangle \geq C \|u\|^{p} - K\|Qu\|^{p} - L$ for every $u \in B$, where $p>1$, $C,K,L>0$.

4.4. Proposition. Let the function $t \mapsto \langle A(u + tv),w\rangle$ be a polynomial for any fixed $u,v,w \in B$. If A is regular and satisfies (2.3),(2.4), $A(0) = 0$, then the condition (4.2) is fulfilled.

The proof can be found in Frehse's papers or in [5].

Let $\Psi:(0,+\infty) \longrightarrow (0,+\infty)$ be the increasing function satisfying

$$\sup_{\substack{w \in R(A)^{\perp}\\ \|w\|_{C(\bar{\Omega})}=1}} \int_{\Omega_{\varepsilon}(w)} |w| \leq \Psi(\varepsilon),$$

where $\Omega_{\varepsilon}(w) = \{x \in \Omega\,; 0<|w(x)|<\varepsilon\}$ and such that

$$\limsup_{\varepsilon \to 0_+} [\Psi(\varepsilon)]^{-1}\, \Psi(\omega\varepsilon) < +\infty \text{ for every } \omega \in (0,+\infty).$$

4.5. Theorem. Let a regular asymptotically monotone 0-polynomial operator A satisfy the conditions (4.2),(4.3), $A(0) = 0$ and let S be given by (4.1). If

$$(4.6) \qquad \lim_{\xi \to +\infty} [\Psi(\tfrac{1}{\xi})]^{-1} \min_{\tau \in \langle a,\xi\rangle} \varphi(\tau) = +\infty$$

for some $a>0$ then the equation $Au + Su = f$ has at least one solution for an arbitrary $f \perp R(A)^{\perp}$.

Sketch of the proof. Let us consider the function

$$\tilde{\varphi}: \xi \longmapsto \begin{cases} \varphi(\xi) & \text{for } |\xi| \leq b, \\ \varphi(b) & \text{for } \xi > b, \\ \varphi(-b) & \text{for } \xi < -b, \end{cases}$$

and the corresponding equation $Au + \tilde{S}u = f$. From the Theorem 2.16 this equation has a solution u because

$$0 = \sup_{\substack{w \in R(A)^{\perp}\\ \|w\|_{C(\bar{\Omega})}=1}} |\langle f,w\rangle| < |\tilde{\varphi}(b)| \inf_{\substack{w \in R(A)^{\perp}\\ \|w\|_{C(\bar{\Omega})}=1}} \int_{\Omega} |w|.$$

Using the condition (4.2) we can obtain a priori estimate

$$\|Q^{c}u\|_{C(\bar{\Omega})} \leq c_1 = c_1(\|f\|_{B^*}).$$

Further, methods from [3],[4] give a priori estimate

$$\|Qu\|_{C(\bar{\Omega})} \leq c_3 = c_3(a,\tilde{\varphi},f),$$

where $a>0$,

$$c_3 = \frac{a + c_1}{\Psi^{-1}\left(c_2\left(\inf_{\xi \geq a} \tilde{\varphi}(\xi) + \sup_{\xi \in R} |\tilde{\varphi}(\xi)|\right)^{-1}\right)},$$

$$c_2 = c_2(a,\tilde{\varphi},f) = \inf_{\substack{w \in R(A)^{\perp}\\ \|w\|_{C(\bar{\Omega})}=1}} \left(\inf_{\xi \geq a} \tilde{\varphi}(\xi) \int_{\Omega} |w|\right).$$

If there exist numbers $a,b \in R$, $0<a<b$, such that $b > c_1(\tilde{\varphi},f) + c_3(a,\tilde{\varphi},f)$ then the solution u of the equation $Au + \tilde{S}u = f$ is also the solution of the equation $Au + Su = f$ because $\tilde{S}u = Su$. The condition (4.6) guarantees the existence of such numbers a, b.

For example, the problem

$$\begin{cases} (\Delta - \lambda)[(\Delta u - \lambda u)^5 + (\Delta u - \lambda u)^3] + \dfrac{u}{1+u^2} = f \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

has at least one weak solution $u \in W_0^{1,6}(\Omega) \cap W^{2,6}(\Omega)$ if $f \perp \mathrm{Ker}[\Delta - \lambda\, \mathrm{id}]$.

It is also possible to apply the abstract results to the existence of solution of the Neuman problem

$$\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left[(\alpha + |\nabla u|^2)^{\frac{p}{2}-1} \frac{\partial u}{\partial x_i}\right] + \frac{u}{1+|u|^k} = f \text{ in } \Omega$$

$$\frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega,$$

where $c>0$, $p>1$, $k\geq 2$. If $f\in L_1(\Omega)$, $\int_\Omega f(x)dx = 0$, this problem has at least one weak solution $u\in W^{1,p}(\Omega)$.

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