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ON THE DIRICHLET BOUNDARY VALUE PROBLEM FOR NONLINEAR ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS IN SOBOLEV POWER WEIGHT SPACES

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1. INTRODUCTION

Let us consider the nonlinear Dirichlet boundary value problem

$$(1.1)\qquad -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x, u, \nabla u) + a_0(x, u, \nabla u) = f \quad \text{in} \quad \Omega\,,$$

$$u = \varphi \quad \text{on} \quad \partial\Omega\,,$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain. Our aim is to give conditions for the solvability of the problem (1.1) in weighted Sobolev spaces. It is motivated by the two following reasons: First, the behaviour of the right-hand side f near the boundary $\partial\Omega$ may cause non-solvability of the problem (1.1) in a classical (nonweighted) Sobolev space, or the function φ may not possess the suitable trace on $\partial\Omega$. In such cases it is sometimes possible to overcome these difficulties by formulating the problem (1.1) in terms of weighted Sobolev spaces (with weights of a positive power).

Second, from the behaviour of the right-hand side f near the boundary $\partial\Omega$ we should like to deduce the analogous behaviour of the solution. The use of suitable (negative power-type) weights could answer some of such regularity questions.

We shall consider the equations which are elliptic within the classical Sobolev space theory. The case when the coefficients a_i include singularities or are degenerate and when the ellipticity can be regarded only with respect to the weighted Sobolev spaces is discussed in [5].

The problem (1.1) for linear equations is investigated in [1], [2], [9], [10]. The application of the generalized Lax-Milgram lemma (see e.g. [9] or [10], Ch. 6, § 3, p. 294) is essential in these papers. Such a procedure, which would be relatively simpler, cannot be applied to the nonlinear case and thus we transform the problem (1.1) to the operator form usually used in the theory of pseudomonotone operators (see [7], Ch. 2, § 2). Although we cannot prove the pseudomonotonicity of the operator obtained (on the other hand, we do not know of any counterexample),

in Section 2 we use methods of pseudomonotone operators in order to find a sequence convergent to a solution of the operator equation (2.1). Some necessary concepts concerning Sobolev power weight spaces are recalled in Section 3 and the solvability of (1.1) in these spaces is studied in Section 4. Lemma 4.4 appears here to be the key to the verification of the problematical condition (2.8). It is not essential that we work with operators of the second order and with weights related to the whole boundary only.

It remains to remark that our procedure gives the solvability of (1.1) for weighted spaces with small powers only. On the other hand, evidently better results were achieved merely for several special examples with linear operators (see e.g. [2], [10]). The answer as to the uniqueness of the solution is not complete, either.

2. ABSTRACT CONSIDERATIONS

In this part we shall study the solvability of the operator equation

$$(2.1)\qquad Su = g\,,$$

where S is a nonlinear operator acting from a real reflexive Banach space V into its dual V^* and $g \in V^*$.

Let V_m, $m \in \mathbb{N}$, be such closed subspaces of V that $V_m \subset V_n$ for $m \leqq n$ and the set $\bigcup_{m\in\mathbb{N}} V_m$ is dense in V. We define $g_m \in V^*$ by $\langle g_m, v\rangle = \langle g, v\rangle$ for all $v \in V_m$, and by S_m we denote the restriction of S onto the set V_m.

We shall suppose that the equations

$$(2.1)_m\qquad S_m u_m = g_m\,, \quad m \in \mathbb{N}\,,$$

have solutions $u_m \in V_m$ (i.e. $\langle Su_m, v\rangle = \langle g, v\rangle$ for all $v \in V_m$) satisfying the conditions

$$(2.2)\qquad \begin{cases} u_m \to u & \textit{weakly in } V\,,\\ Su_m & \textit{is weakly convergent in } V^* \quad (\textit{for } m \to +\infty)\,. \end{cases}$$

Theorem 2.1. *Let S be an operator acting from a real reflexive Banach space V into its dual V^* and let $g \in V^*$. Further, let solutions u_m of the equations* $(2.1)_m$ *satisfy the conditions* (2.2) *and*

$$(2.3)\qquad \liminf_{m\to+\infty} \langle Su_m, u_m - v\rangle \geqq \langle Su, u - v\rangle \quad \textit{for all} \quad v \in V\,.$$

Then u is a solution of the equation (2.1).

Proof. Since $Su_m = S_m u_m = g_m$, applying (2.2) we obtain that $Su_m \to g$ weakly in V^*. Now we have $\langle Su_m, u_m\rangle = \langle g, u_m\rangle \to \langle g, u\rangle$ and with regard to the convergence $\langle Su_m, u\rangle \to \langle g, u\rangle$ we conclude

$$(2.4)\qquad \langle Su_m, u_m - u\rangle \to 0\,.$$

This fact together with the condition (2.3) implies $\liminf \langle Su_m, u - v\rangle \geqq \langle Su, u - v\rangle$ for all $v \in V$, otherwise $\langle g, u - v\rangle \geqq \langle Su, u - v\rangle$ for all $v \in V$. This inequality yields $Su = g$.

The verification of (2.3) for a particular operator S frequently requires elaborated and rather complicated procedures which are analogous to an investigation of pseudo-monotonicity. Therefore the following modified Leray-Lions conditions (2.5)–(2.8) form important tools in applications (see e.g. [7], Ch. 2, § 2).

We shall suppose that the operator S has the form $Sv = S(v, v)$ where the mapping $(w, v) \to S(w, v)$ acting from $V \times V$ into V^* satisfies the following conditions (here, u_m are solutions of the equations $(2.1)_m$ with the property (2.2)):

(2.5) *For an arbitrary $w \in V$ the mapping $v \to S(w, v)$ is a bounded hemicontinuous operator from V into V^* (i.e., for all $u, h \in V$ and for an arbitrary sequence $\{t_n\}_n$, $t_n \to 0$, we have $S(w, u + t_nh) \to S(w, u)$ weakly in V^*) satisfying $\langle S(w, w) - S(w, v), w - v\rangle \geqq 0$ whenever $v \in V$.*

(2.6) *For an arbitrary $v \in V$ the mapping $w \to S(w, v)$ is a bounded hemicontinuous from V into V^*.*

(2.7) *If $\lim_{m\to+\infty} \langle S(u_m, u_m) - S(u_m, u), u_m - u\rangle = 0$ then there is a subsequence $\{u_{m_k}\}_k \subset \{u_m\}_m$ satisfying $S(u_{m_k}, v) \to S(u, v)$ weakly in V^* for all $v \in V$.*

(2.8) *If $S(u_m, v) \to \psi$ weakly in V^* then $\lim_{m\to+\infty} \langle S(u_m, v), u_m\rangle = \langle \psi, u\rangle$.*

Lemma 2.2. *Let $u_m \in V_m$, $m \in \mathbb{N}$, be solutions of the equations $(2.1)_m$ with the operator S satisfying the conditions (2.2) and (2.5)–(2.8). Then there exists a subsequence $\{u_{m_k}\}_k$ for which*

$$\liminf_{k\to+\infty} \langle Su_{m_k}, u_{m_k} - v\rangle \geqq \langle Su, u - v\rangle \quad \textit{whenever} \quad v \in V.$$

Proof. We remark that (2.2) implies $\langle S(u_m, u_m), u_m - u\rangle \to 0$ (see (2.4)). Since the sequence $\{S(u_m, u)\}_m$ is bounded in V^* we can choose a subsequence (by a small abuse of the notation we denote it in the same way) such that $S(u_m, u) \to \varphi$ weakly in V^*. In virtue of (2.8) we obtain $\langle S(u_m, u), u_m\rangle \to \langle \varphi, u\rangle$, thus $\langle S(u_m, u), u_m - u\rangle \to 0$. This fact together with the condition (2.7) yields $S(u_{m_k}, v) \to S(u, v)$ weakly in V^* for all $v \in V$ ($\{u_{m_k}\}_k$ is a subsequence from (2.7)) and using (2.8) we have

$$\langle S(u_{m_k}, v), u_{m_k} - u\rangle \to 0 \quad \text{for all} \quad v \in V. \tag{2.9}$$

In accordance with (2.5) we have $\langle S(u_{m_k}, u_{m_k}) - S(u_{m_k}, w), u_{m_k} - w\rangle \geqq 0$ for all $w \in V$ and substituting $w = (1 - t)\,u + tv$, $t \in \langle 0, 1\rangle$, in this inequality we obtain

$$t\langle S(u_{m_k}, u_{m_k}), u - v\rangle \geqq - \langle S(u_{m_k}, u_{m_k}), u_{m_k} - u\rangle + \langle S(u_{m_k}, w), u_{m_k} - u\rangle + t\langle S(u_{m_k}, w), u - v\rangle .$$

Now according to (2.4), (2.9) and because $S(u_{m_k}, w) \to S(u, w)$ weakly in V^* we have

$$\liminf \langle S(u_{m_k}, u_{m_k}), u - v\rangle \geqq \liminf \langle S(u_{m_k}, w), u - v\rangle = \langle S(u, w), u - v\rangle .$$

Again using (2.4) we can write

$$\liminf \langle S(u_{m_k}, u_{m_k}), u_{m_k} - v\rangle \geqq \langle S(u, (1 - t)\,u + tv), u - v\rangle .$$

Finally, the convergence $t \to 0_+$ yields the requested inequality.

Theorem 2.2. *Let the assumptions of Lemma 2.2 be fulfilled. Then u is a solution of the equation (2.1).*

Proof. The subsequence $\{u_{m_k}\}_k$ from the assertion of Lemma 2.2 satisfies the conditions (2.2) as well, thus it satisfies all the assumptions of Theorem 2.1.

3. TECHNICAL PRELIMINARIES

Throughout this paper Ω denotes a bounded domain in the Euclidean N-space $\mathbb{R}^N$ with a *Lipschitz boundary* $\partial\Omega$. For a manifold $M \subset \partial\Omega$ we consider the distance $d_M(x) = \inf_{y\in M} |x - y|$ of a point $x \in \bar{\Omega}$ from M. The *Sobolev power weight space* $W^{1,p}(\Omega; d_M, \varepsilon)$ with $p \geqq 1$ is defined to be the set of all functions u defined a.e. on Ω whose (distributional) derivatives $D^\alpha u$ with $|\alpha| \leqq 1$ belong to the *weighted Lebesgue space* $L_p(\Omega; d_M, \varepsilon)$ endowed with the norm

$$|\varphi|_{M,p,\varepsilon} = \left(\int_\Omega |\varphi(x)|^p\, d_M^\varepsilon(x)\, \mathrm{d}x\right)^{1/p} .$$

In order to avoid technical difficulties, we shall deal with the case $M = \partial\Omega$ and $p > 1$ only. Then the space $W^{1,p}(\Omega; d_{\partial\Omega}, \varepsilon)$ with the norm

$$\|u\|_{p,\varepsilon} = \left(\sum_{|\alpha|\leqq 1} \int_\Omega |D^\alpha u(x)|^p\, d_{\partial\Omega}^\varepsilon(x)\, \mathrm{d}x\right)^{1/p} \tag{3.1}$$

is a reflexive Banach space. The set $C^\infty(\bar{\Omega})$ is dense in $W^{1,p}(\Omega; d_{\partial\Omega}, \varepsilon)$ for $\varepsilon \in (-1, p - 1)$ and we can consider *traces of functions* of this space on the boundary $\partial\Omega$ (see e.g. [2]).

The weighted analogue of the Sobolev space $W_0^{1,p}(\Omega)$ is defined by the formula $W_0^{1,p}(\Omega; d_{\partial\Omega}, \varepsilon) = \overline{C_0^\infty(\Omega)}$ where the closure is taken with respect to the norm (3.1). For the sake of brevity we shall denote $d_{\partial\Omega} \equiv d$ and $W_0^{1,p}(\Omega; d, \varepsilon) = V_{p,\varepsilon}$ for $\varepsilon < p - 1$. (If $\varepsilon \leqq -1$ then $W^{1,p}(\Omega; d, \varepsilon) = W_0^{1,p}(\Omega; d, \varepsilon)$.) On $V_{p,\varepsilon}$ we shall consider the norm

$$|||u|||_{p,\varepsilon} = \left(\int_\Omega |\nabla u(x)|^p\, d^\varepsilon(x)\, \mathrm{d}x\right)^{1/p} , \tag{3.2}$$

which is equivalent to the norm (3.1) (see e.g. [10]; it also follows directly from the next lemma).

Let us mention the often used *Hardy inequality*. If $-\infty < a < b \leqq +\infty$, $p > 1$, $\varepsilon < p - 1$ and $f \in L_p((a, b); d_{\{a\}}, \varepsilon)$ then

$$(3.3) \quad \int_a^b (x - a)^{\varepsilon - p} \left[\int_a^x |f(t)| \, dt \right]^p dx \leqq \left(\frac{p}{|\varepsilon - p + 1|} \right)^p \int_a^b (x - a)^\varepsilon |f(x)|^p \, dx$$

(in the case $\varepsilon = 0$ see e.g. [10], for $\varepsilon \neq 0$ the proof is analogous).

Further, we shall work with the set

$$(3.4) \qquad \Omega_n = \left\{ x \in \Omega;\ \operatorname{dist}(x, \partial\Omega) > \frac{1}{n} \right\}$$

which is a domain with a Lipschitz boundary for sufficiently large integer n.

Lemma 3.1. *Let $p > 1$, $\varepsilon < p - 1$. Then there exists a positive constant $c = c(\Omega, p)$ such that the inequality*

$$(3.5) \qquad \int_{\mathcal{O}} |u(x)|^p \, d^{\varepsilon - p}(x) \, dx \leqq c \left(\frac{p}{|\varepsilon - p + 1|} \right)^p \int_{\mathcal{O}} |\nabla u(x)|^p \, d^\varepsilon(x) \, dx$$

holds for all $u \in V_{p,\varepsilon}$, where $\mathcal{O} \equiv \Omega$ or $\mathcal{O} \equiv \Omega \setminus \Omega_n$ with $n \geqq n_0$ (n_0 sufficiently large).

Moreover, if Ω is convex then $c = 1$.

Proof. The application of the partition of unity and the inequality (3.3) give the proof in the case $\mathcal{O} \equiv \Omega$ (see e.g. [2], [10]). If $\mathcal{O} \equiv \Omega \setminus \Omega_n$ we proceed analogously using the fact that the inequality (3.3) is fulfilled for every interval (a, b) with the same constant $(p/|\varepsilon - p + 1|)^p$.

We claim to demonstrate the inequality (3.5) in the case $\mathcal{O} \equiv \Omega$ when Ω is convex. For $\mathcal{O} \equiv \Omega \setminus \Omega_n$ the proof is similar.

The first step. We shall consider (3.5) for $\mathcal{O} \equiv G$ and $u \in C_0^\infty(\operatorname{int} G)$ where G is a closed convex polyhedron. We can decompose this polyhedron with sides $s_1, \ldots, s_n$ into closed polyhedrons $G_1, \ldots, G_n$ such that $x \in G$ is an element of G_i if and only if $\operatorname{dist}(x, \partial G) = \operatorname{dist}(x, s_i)$. Then $G = \bigcup_{i=1}^n G_i$, $\operatorname{int} G_i \cap \operatorname{int} G_j = \emptyset$ for $i \neq j$, G_i contains the side s_i, $i = 1, \ldots, n$ (see Fig. 1).

It is sufficient to establish the inequality (3.5) only for $\mathcal{O} \equiv G_1$. There is an orthonormal matrix $\mathbf{A} \in \mathbb{R}^{N^2}$, a vector $y_0 \in \mathbb{R}^N$ and a local system of coordinates y such that $y = \mathbf{A}x + y_0$, s_1 belongs to the hyperplane $y_1 = 0$ and G_1 belongs to the halfspace $y_1 \geqq 0$. If now $y = (y_1, y') \in \mathbb{R} \times \mathbb{R}^{N-1}$, $Ps_1 = \{y' \in \mathbb{R}^{N-1};\ (0, y') \in s_1\}$, $R(y') = \max\{y_1;\ y = (y_1, y') \in G_1,\ y' \in Ps_1\}$, then using the Hardy inequality (3.3) we can write

$$I_1 = \int_{G_1} |u(x)|^p \, d^{\varepsilon - p}(x) \, dx = \int_{Ps_1} \int_0^{R(y')} |u(y_1, y')|^p \, y_1^{\varepsilon - p} \, dy_1 \, dy' \leqq$$

$$\leqq \left(\frac{p}{|\varepsilon - p + 1|} \right)^p \int_{Ps_1} \int_0^{R(y')} \left| \frac{\partial u}{\partial y_1}(y_1, y') \right|^p y_1^\varepsilon \, dy_1 \, dy' \leqq$$

$$\leqq \left(\frac{p}{|\varepsilon - p + 1|} \right)^p \int_{G_1} \left(\sum_{i=1}^N \left| \frac{\partial u}{\partial y_i}(y_1, y') \right|^2 \right)^{p/2} y_1^\varepsilon \, dy_1 \, dy' .$$

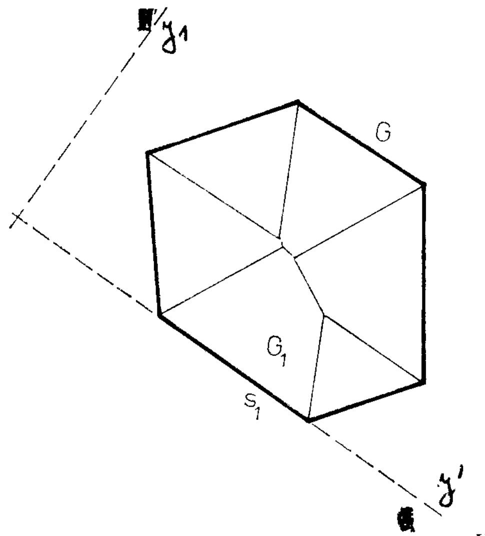

Fig. 1

Since $\mathbf{A}$ is orthonormal, for $u \in C_0^\infty(\Omega)$ we have

$$\sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^2 = \sum_{i=1}^N \left| \frac{\partial u}{\partial y_i} \right|^2$$

in Ω and finally

$$I_1 \leqq \left(\frac{p}{|\varepsilon - p + 1|} \right)^p \int_{G_1} |\nabla u(x)|^p \, d^\varepsilon(x) \, dx .$$

The second step. With regard to the density of $C_0^\infty(\Omega)$ in the space $V_{p,\varepsilon}$ it is sufficient to prove the inequality (3.5) for an arbitrary $u \in C_0^\infty(\Omega)$. Since such a u has a compact support, for any sufficiently large integer n there exists a convex polyhedron $G^{(n)}$ with the properties $\operatorname{supp} u \subset G^{(n)} \subset \Omega$, $\operatorname{dist}(x, \partial\Omega) < 1/n$ for all $x \in \partial G^{(n)}$. According to the first step

$$\int_{G^{(n)}} |u(x)|^p \, d^{\varepsilon - p}_{\partial G^{(n)}}(x) \, dx \leqq \left(\frac{p}{|\varepsilon - p + 1|} \right)^p \int_{G^{(n)}} |\nabla u(x)|^p \, d^\varepsilon_{\partial G^{(n)}}(x) \, dx$$

and applying the *Lebesgue Convergence Theorem* for $n \to +\infty$ we obtain the inequality (3.5) with $\mathcal{O} \equiv \Omega$.

Lemma 3.2. *Let $p > 1$, $\varepsilon < p - 1$. Then the mapping J defined by $J(u) = d^\alpha u$ is an isomorphism of $V_{p,\varepsilon}$ onto $V_{p,\varepsilon-\alpha p}$ whenever $\alpha > (\varepsilon - p + 1)/p$.*

Proof. It is evident that the mapping J is injective and continuous because, in view of Lemma 3.1,

$$\|d^\alpha u\|^p_{p,\varepsilon-\alpha p} \leqq \sum_{i=1}^N \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^p d^\varepsilon \,\mathrm{d}x + N|\alpha|^p \int_\Omega |u|^p d^{\varepsilon-p} \,\mathrm{d}x + \int_\Omega |u|^p d^\varepsilon \,\mathrm{d}x \leqq c_1 \|u\|^p_{p,\varepsilon}$$

(we have $|\partial d/\partial x_i| \leqq 1$, $i = 1, \ldots, N$, a.e. in Ω).

Analogously for any $v \in V_{p,\varepsilon-\alpha p}$ we have

$$\|d^{-\alpha} v\|^p_{p,\varepsilon} \leqq \sum_{i=1}^N \int_\Omega \left|\frac{\partial v}{\partial x_i}\right|^p d^{\varepsilon-\alpha p} \,\mathrm{d}x + N|\alpha|^p \int_\Omega |v|^p d^{\varepsilon-\alpha p-p} \,\mathrm{d}x +$$

$$+ \int_\Omega |v|^p d^{\varepsilon-\alpha p} \,\mathrm{d}x \leqq c_2 \|v\|^p_{p,\varepsilon-\alpha p}$$

whenever $\varepsilon - \alpha p < p - 1$. This completes the proof.

Let us now briefly deal with the *Nemyckij operators* in Sobolev power weight spaces. Suppose that a function $h: \Omega \times \mathbb{R}^s \to \mathbb{R}$ ($h = h(x, \xi_1, \ldots, \xi_s)$) satisfies the *Carathéodory conditions* (i.e., it is measurable on Ω for all $\xi = (\xi_1, \ldots, \xi_s) \in \mathbb{R}^s$ and continuous in ξ for a.a. x in Ω). If the Nemyckij operator $H: (u_1, \ldots, u_s) \to \to h(x, u_1, \ldots, u_s)$ acts from $\prod_{i=1}^s L_{p_i}(\Omega; d, \varepsilon_i)$ into $L_q(\Omega; d, \varepsilon)$, $1 \leqq p_i$, $q < +\infty$, then it is continuous. Actually, because $J: L_q(\Omega) \to L_q(\Omega; d, \varepsilon)$, $J(u) = d^{-\varepsilon/q} u$, $J_i: L_{p_i}(\Omega) \to L_{p_i}(\Omega; d, \varepsilon_i)$, $J_i(u) = d^{-\varepsilon_i/p_i}$, $i = 1, \ldots, s$, are isomorphisms, the formula

$$(\psi_1, \ldots, \psi_s) \to J^{-1} h(x, J_1\psi_1, \ldots, J_s\psi_s)$$

defines a continuous mapping from $\prod_{i=1}^s L_{p_i}(\Omega)$ into $L_q(\Omega)$ (see e.g. [11]). Consequently, the operator

$$H: (J_1\psi_1, \ldots, J_s\psi_s) \to h(x, J_1\psi_1, \ldots, J_s\psi_s)$$

acting on weighted spaces is continuous as well.

Finally, in Section 4 we use functions the existence of which is guaranteed by the following lemma.

Lemma 3.3. *Let $\{a_k\}_k$ be an increasing sequence of integers with a sufficiently large a_1. Then there exist functions $\{\varphi_k\}_k$ satisfying the conditions:*

$\varphi_k \in C^\infty(\mathbb{R}^N)$,

$\varphi_k \equiv 1$ in $\mathbb{R}^N \setminus \Omega_{a_{k+1}}$, $\varphi_k \equiv 0$ *in* $\bar{\Omega}_{a_k}$ *(for Ω_{a_k} see* (3.4)),

$$0 \leqq \varphi_k(x) \leqq 1 \quad \textit{and} \quad |\nabla\varphi_k(x)| \leqq c_3 \frac{a_k a_{k+1}}{a_{k+1} - a_k} \quad \textit{for all} \quad x \in \mathbb{R}^N$$

with a positive constant $c_3 = c_3(\Omega)$.

Proof. We can consider domains Ω'_k, $k = 1, 2, \ldots$, with Lipschitz boundaries which have the following properties:

$$\Omega_{a_k} \subset \Omega'_k \subset \Omega_{a_{k+1}}, \quad \operatorname{dist}(x, \Omega_{a_k}) > \frac{1}{3}\left(\frac{1}{a_k} - \frac{1}{a_{k+1}}\right) \quad \text{for all} \quad x \in \partial\Omega'_k$$

and

$$\operatorname{dist}(x, \Omega'_k) > \frac{1}{3}\left(\frac{1}{a_k} - \frac{1}{a_{k+1}}\right) \quad \text{for all} \quad x \in \partial\Omega_{a_{k+1}}.$$

Let us define functions χ_k on $\mathbb{R}^N$ by

$$\chi_k(x) = \begin{cases} 0 & \text{if} \quad x \in \Omega'_k, \\ 1 & \text{if} \quad x \notin \Omega'_k, \end{cases}$$

and let

$$R_{\gamma_k} u: x \mapsto \frac{1}{\gamma_k^N} \int_\Omega \varrho\left(\frac{x - y}{\gamma_k}\right) u(y) \,\mathrm{d}y, \quad k = 1, 2, \ldots,$$

be mollifiers with a kernel

$$\varrho \in C_0^\infty(\mathbb{R}^N), \quad \int_{\mathbb{R}^N} \varrho(x) \,\mathrm{d}x = 1,$$

$\operatorname{supp} \varrho = \{x \in \mathbb{R}^N;\ |x| \leqq 1\}$. If we substitute

$$\gamma_k = \frac{1}{4}\left(\frac{1}{a_k} - \frac{1}{a_{k+1}}\right)$$

then $R_{\gamma_k}\chi_k \in C^\infty(\mathbb{R}^N)$, $(R_{\gamma_k}\chi_k)(x) = 1$ for all $x \in \mathbb{R}^N \setminus \Omega_{a_{k+1}}$ and we have the estimate

$$\left|\frac{\partial(R_{\gamma_k}\chi_k)(x)}{\partial x_i}\right| \leqq \frac{1}{\gamma_k^{N+1}} \max_{z \in \mathbb{R}^N} |\nabla\varrho(z)| \cdot \int_{B_{\gamma_k}(x)} |\chi_k(y)| \,\mathrm{d}y,$$

$i = 1, \ldots, N$, where $B_{\gamma_k}(x) = \{y \in \mathbb{R}^N;\ |x - y| < \gamma_k\}$. Hence we can deduce

$$|\nabla(R_{\gamma_k}\chi_k)(x)| \leqq \sqrt{(N)} \max_{z \in \mathbb{R}^N} |\nabla\varrho(z)| \frac{\operatorname{meas} B_1(0)}{\gamma_k} \quad \text{for all} \quad x \in \Omega,$$

$k = 1, 2, \ldots$. Now, it is sufficient to put $\varphi_k = R_{\gamma_k}\chi_k$.

4. WEAK SOLUTION OF THE NONLINEAR DIRICHLET BOUNDARY VALUE PROBLEM

Let us consider the nonlinear Dirichlet boundary value problem (b.v.p.) (1.1). (We remember that $\Omega \subset \mathbb{R}^N$ is a bounded domain with a Lipschitz boundary.) We assume that functions $a_i: \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, $i = 0, 1, \ldots, N$, satisfy the Carathéodory conditions as well as the following inequalities:

There exist numbers $p > 1$, $\delta \in (0, p-1)$, $\varepsilon \in (-1, p-1)$, *positive functions* $k \in L_{p'}(\Omega; d, \varepsilon)$, $q \in L_1(\Omega; d, \varepsilon)$, *where* $p' > 1$, $(1/p) + (1/p') = 1$, *and positive constants* $\alpha_0, \ldots, \alpha_3$ *such that*

(4.1) $$|a_i(x, \eta, \xi)| \leqq \alpha_0(|\xi|^{p-1} + |\eta|^{p-1} + k(x)), \quad i = 0, 1, \ldots, N,$$

(4.2) $$\sum_{i=1}^N a_i(x, \eta, \xi)\, \xi_i + a_0(x, \eta, \xi)\, \eta \geqq \alpha_1|\xi|^p - \alpha_2|\eta|^{p-\delta} - \alpha_3\, q(x),$$

(4.3) $$\sum_{i=1}^N [a_i(x, \eta, \xi) - a_i(x, \eta, \xi')]\, (\xi_i - \xi_i') > 0,$$

for a.a. $x \in \Omega$, *all* $\eta \in \mathbb{R}$, $\xi, \xi' \in \mathbb{R}^N$, $\xi \neq \xi'$.

Let $\varphi \in W^{1,p}(\Omega; d, \varepsilon)$ and $f \in [V_{p,-\varepsilon(p-1)}]^*$, where the latter symbol denotes the *dual space* to $W_0^{1,p}(\Omega; d, -\varepsilon(p-1))$. Analogously as in the case of classical Sobolev spaces there exist distributions $f_0, \ldots, f_N \in L_{p'}(\Omega; d, \varepsilon)$ such that

$$f = f_0 - \sum_{i=1}^N \frac{\partial f_i}{\partial x_i}.$$

Let us reformulate the b.v.p. (1.1) in the equivalent form

(4.4) $$-\sum_{i=1}^N \frac{\partial}{\partial x_i} b_i(x, v, \nabla v) + b_0(x, v, \nabla v) = f \quad \text{in} \quad \Omega,$$

$$v = 0 \quad \text{on} \quad \partial\Omega,$$

where the functions b_i, $i = 0, 1, \ldots, N$, are defined for a.a. $x \in \Omega$ and for all $\eta \in \mathbb{R}$, $\xi \in \mathbb{R}^N$ by $b_i(x, \eta, \xi) = a_i(x, \eta + \varphi(x), \xi + \nabla\varphi(x))$. It is easy to see that $b_i: \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, $i = 0, \ldots, N$, satisfy the Carathéodory conditions and the inequalities (4.1), (4.3) with suitable constants and that the relation (4.2) is valid with $\delta = 0$ for a sufficiently small constant α_2. However, we shall require the following weaker inequalities:

(4.5) *There exist* $\beta_0, \gamma > 0$ *and a positive function* $h \in L_{p'}(\Omega; d, \varepsilon)$ *such that*

$$|b_0(x, \eta, \xi)| \leqq \beta_0[d^{-1}(x)\, |\xi|^{p-1} + d^{-p}(x)\, |\eta|^{p-1} + d^{-1}(x)\, h(x)],$$

$$|b_i(x, \eta, \xi)| \leqq \beta_0[|\xi|^{p-1} + d^{-(p-1)+\gamma}(x)\, |\eta|^{p-1} + h(x)], \quad i = 1, \ldots, N,$$

for a.a. $x \in \Omega$ *and all* $\eta \in \mathbb{R}$, $\xi \in \mathbb{R}^N$.

(4.6) *There exist* $\beta_1 > 0$, *a positive function* $r \in L_1(\Omega; d, \varepsilon)$ *and for every* $\omega > 0$ *a number* $\beta_2(\omega) > 0$ *such that*

$$\sum_{i=1}^N b_i(x, \eta, \xi)\, \xi_i + b_0(x, \eta, \xi)\, \eta \geqq \beta_1|\xi|^p - \omega d^{-p}(x)\, |\eta|^p - \beta_2(\omega)\, r(x)$$

for a.a. $x \in \Omega$ *and all* $\eta \in \mathbb{R}$, $\xi \in \mathbb{R}^N$.

(4.7) $$\sum_{i=1}^N [b_i(x, \eta, \xi) - b_i(x, \eta, \xi')]\, (\xi_i - \xi_i') > 0$$

for a.a. $x \in \Omega$ *and all* $\eta \in \mathbb{R}$, $\xi, \xi' \in \mathbb{R}^N$ *with* $\xi \neq \xi'$.

Let us define an operator $T: V_{p,\varepsilon} \to [V_{p,-\varepsilon(p-1)}]^*$ by the formula

$$\langle Tv, w\rangle = \sum_{i=1}^N \int_\Omega b_i(x, v, \nabla v)\, \frac{\partial w}{\partial x_i}\, dx + \int_\Omega b_0(x, v, \nabla v)\, w\, dx, \quad w \in V_{p,-\varepsilon(p-1)}.$$

By the use of the Hőlder inequality, the inequalities (4.5) and Lemma 3.1 with $\mathcal{O} \equiv \Omega$ and with $-\varepsilon(p-1)$ instead of ε it is not difficult to verify that

$$\langle Tv, w\rangle \leqq c_4 \cdot (\|v\|_{p,\varepsilon}^{p-1} + |h|_{\partial\Omega,p',\varepsilon}^{p-1}) \cdot \|w\|_{p,-\varepsilon(p-1)}$$

for all $v \in V_{p,\varepsilon}$, $w \in V_{p,-\varepsilon(p-1)}$ with $-1 < \varepsilon < p - 1$. Thus the operator T is bounded and $Tv \in [V_{p,-\varepsilon(p-1)}]^*$ for every $v \in V_{p,\varepsilon}$ (with an admissible ε).

Definition. Let $p > 1$, $\varepsilon \in (-1, p-1)$. A function $u \in V_{p,\varepsilon}$ is said to be a *weak solution* of the problem (4.4) if

$$\langle Tu, w\rangle = \langle f, w\rangle \quad \text{for every} \quad w \in V_{p,-\varepsilon(p-1)}.$$

As in Section 3, the operator

$$J: u \to d^\varepsilon u$$

is an isomorphism of the space $V_{p,\varepsilon}$ onto $V_{p,-\varepsilon(p-1)}$ for $\varepsilon \in (-1, p-1)$ and therefore its dual mapping $J^*: [V_{p,-\varepsilon(p-1)}]^* \to [V_{p,\varepsilon}]^*$ is an isomorphism as well. The equation $Tu = f$ now has at least one solution for $f \in [V_{p,-\varepsilon(p-1)}]^*$ if and only if the equation

(4.8) $$J^*Tu = J^*f$$

has a solution. Since we cannot prove the pseudomonotonicity of the mapping $J^*T: V_{p,\varepsilon} \to [V_{p,\varepsilon}]^*$ we shall study its range in a similar way as in Section 2. In what follows, let us put

$$S = J^*T, \quad g = J^*f$$

and let us denote

$$V_p^n = \{u \in V_{p,\varepsilon};\ \operatorname{supp} u \subset \bar{\Omega}_n\} \quad (\text{for } \Omega_n \text{ see } (3.4)).$$

The space V_p^n is a classical (non-weighted) Sobolev space and J is an isomorphism of V_p^n onto itself. Further, the equations

$$(4.8)_n \qquad J^*Tu = J^*f|_{V_p^n},$$

where the solvability is investigated in spaces V_p^n, correspond to $(2.1)_n$. From Lemmas 4.1 and 4.2 below we obtain that the equations $(4.8)_n$ have solutions satisfying the conditions (2.2). According to Theorem 2.1 and Lemma 2.2, the forthcoming investigation of the solvability of (4.4) in Sobolev power weight spaces is reduced to the verification of the conditions (2.5)–(2.8). To verify the validity of (2.5)–(2.7), it is more or less sufficient to follow Lions' approach (see [7], Ch. 2, § 2). On the other hand, when verifying the condition (2.8) we shall essentially employ the assertion of Lemma 4.4 which concerns the behaviour of the solutions u_n of $(4.8)_n$ near the boundary.

Lemma 4.1. *There exists an interval I, $0 \in \operatorname{int} I$, such that the operator $S: V_{p,\varepsilon} \to [V_{p,\varepsilon}]^*$ is coercive for every $\varepsilon \in I$.*

Proof. By means of the Hölder inequality and of (3.5) we obtain for $u \in V_{p,\varepsilon}$

$$\langle Su, u\rangle = \langle Tu, Ju\rangle = \sum_{i=1}^{N}\left[\int_\Omega b_i(x, u, \nabla u)\frac{\partial u}{\partial x_i} d^\varepsilon\,dx + \right.$$

$$\left. + \varepsilon \int_\Omega b_i(x, u, \nabla u)\, u \frac{\partial d}{\partial x_i} d^{\varepsilon-1}\,dx\right] + \int_\Omega b_0(x, u, \nabla u)\, u d^\varepsilon\,dx \geqq$$

$$\geqq \beta_1 \int_\Omega |\nabla u|^p d^\varepsilon\,dx - \omega \int_\Omega |u|^p d^{\varepsilon-p}\,dx - \beta_2(\omega)\int_\Omega r d^\varepsilon\,dx -$$

$$- |\varepsilon| \cdot N \cdot \beta_0 \left[\int_\Omega |\nabla u|^{p-1} |u|\, d^{\varepsilon-1}\,dx + \int_\Omega |u|^p d^{\varepsilon-p+\gamma}\,dx + \int_\Omega h|u|\, d^{\varepsilon-1}\,dx\right] \geqq$$

$$\geqq [\beta_1 - 2\omega c_5 - |\varepsilon|\, N\beta_0(c_5^{1/p} + c_5 \max_{x\in\bar\Omega} d^\gamma(x))]\, |||u|||_{p,\varepsilon}^p -$$

$$- \beta_2(\omega)\int_\Omega r d^\varepsilon\,dx - c_6(\omega)\int_\Omega h^{p'} d^\varepsilon\,dx,$$

where $c_5 = c(p/|\varepsilon - p + 1|)^p$ is the constant from (3.5). (We have used the inequality $h|u|\, d^{\varepsilon-1} \leqq 1/p(\omega^{1/p}p^{1/p}|u|\, d^{(\varepsilon/p)-1})^p + 1/p'(\omega^{-1/p}p^{-1/p}hd^{\varepsilon/p'})^{p'} = \omega|u|^p d^{\varepsilon-p} + 1/p'(\omega^{-1/(p-1)}p^{1/(p-1)}h^{p'}d^\varepsilon)$ with $\omega > 0$.) Since we can choose $\omega > 0$ arbitrarily small the operator S will be coercive if

$$(4.9) \qquad \beta_1 - |\varepsilon|\, N\beta_0\left(c^{1/p}\frac{p}{|\varepsilon - p + 1|} + c\left(\frac{p}{|\varepsilon - p + 1|}\right)^p \max_{x\in\bar\Omega} d^\gamma(x)\right) > 0\,.$$

Obviously, this inequality is valid for the values ε from a suitable interval I with $0 \in \operatorname{int} I$.

Lemma 4.2. *There exists an integer $n_1 > 0$ such that every equation $(4.8)_n$ with $n \geqq n_1$ has a solution $u_n \in V_p^n$.*

Moreover, if $\varepsilon \in I$ (for the interval I see Lemma 4.1), *then for a suitable $c_7 > 0$,*

$$(4.10) \qquad \|u_n\|_{p,\varepsilon} \leqq c_7 \quad \textit{whenever} \quad n \geqq n_1\,.$$

Proof. There is an integer n_1 such that Ω_n, $n \geqq n_1$, is a nonempty domain with a Lipschitz boundary. Since J is an isomorphism of the space V_p^n onto itself, the function $u_n \in V_p^n$ is a solution of $(4.8)_n$ if and only if $\langle Tu_n, w\rangle = \langle f, w\rangle$ for all $w \in V_p^n$. However, viewing the operator T as a mapping acting from V_p^n into its dual space we can use the well known results concerning pseudomonotone coercive operators. (See e.g. [7], Ch. 2, § 2, Theorem 2.8. The assumptions of this assertion result from (4.5)–(4.7).) Thus the equation $(4.8)_n$ has at leas one solution. Finally, from

$$\langle Su_n, u_n\rangle = \langle Tu_n, Ju_n\rangle = \langle f, Ju_n\rangle \leqq c_8\|f\|_{[V_{p,-\varepsilon(p-1)}]^*}\|u_n\|_{p,\varepsilon}$$

and from Lemma 4.1 we derive the estimate (4.10).

With regard to the estimate (4.10), to the boundedness of the operator S and the reflexivity of $V_{p,\varepsilon}$, for $\varepsilon \in I$ we can consider a subsequence $\{u_{m_j}\}_j$ of solutions of $(4.8)_{m_j}$ (in what follows, we shall omit the index j) with the property

$$(4.11) \qquad \begin{cases} u_m \to u \textit{ weakly in } V_{p,\varepsilon}, \\ Su_m \textit{ is weakly convergent in } [V_{p,\varepsilon}]^*. \end{cases}$$

For all $w, v, z \in V_{p,\varepsilon}$ we put

$$\langle S(w, v), z\rangle = \langle S_1(w, v), z\rangle + \langle S_2 w, z\rangle,$$

where

$$\langle S_1(w, v), z\rangle = \sum_{i=1}^{N}\int_\Omega b_i(x, w, \nabla v)\frac{\partial z}{\partial x_i} d^\varepsilon\,dx,$$

$$\langle S_2 w, z\rangle = \int_\Omega b_0(x, w, \nabla w)\, z d^\varepsilon\,dx + \varepsilon\sum_{i=1}^{N}\int_\Omega b_i(x, w, \nabla w)\, z d^{\varepsilon-1}\frac{\partial d}{\partial x_i}\,dx\,.$$

Further, we shall verify the validity of the conditions (2.5)–(2.8) for the operator $(w, v) \to S(w, v)$. Applying Theorem 2.2 we get existence results for the b.v.p. (4.4) and (1.1) which will be formulated later in Theorems 4.6 and 4.7.

Condition (2.5). Analogously as for the operator T it is possible to show that the operator $v \to S(w, v)$ is bounded. To verify its hemicontinuity we have to deduce

$$\langle S(w, v_1 + tv_2), z\rangle \to \langle S(w, v_1), z\rangle \quad \text{for} \quad t \to 0$$

with any $w, v_1, v_2, z \in V_{p,\varepsilon}$. But the properties of the Nemyckij operators (see Section 3) yield

$$b_i(x, w, \nabla v_1 + t\nabla v_2) \to b_i(x, w, \nabla v_1) \quad \text{for} \quad t \to 0$$

strongly in $L_{p'}(\Omega; d, \varepsilon)$, $i = 1, \ldots, N$.

Further, the inequality

$$\langle S(w, w) - S(w, v), w - v\rangle = \langle S_1(w, w), w - v\rangle - \langle S_1(w, v), w - v\rangle \geqq 0$$

is a direct consequence of (4.7).

Condition (2.6). Using Lemma 3.1, the Hölder inequality and (4.5) we have

$$\left|\sum_{i=1}^{N}\int_\Omega b_i(x, w, \nabla w)\, z d^{\varepsilon-1}\frac{\partial d}{\partial x_i}\,dx\right| \leqq \frac{c_8}{|\varepsilon - p + 1|}\left(\|w\|_{p,\varepsilon}^{p-1} + |h|_{\partial\Omega,p',\varepsilon}\right)\|z\|_{p,\varepsilon}.$$

Now, it is easy to see that the operator $w \to S(w, v)$ is bounded. Its hemicontinuity follows again from the properties of the Nemyckij operators.

Condition (2.7). Let us put

$$G_m(x) = \sum_{i=1}^{N}\left[b_i(x, u_m(x), \nabla u_m(x)) - b_i(x, u_m(x), \nabla u(x))\right]\left(\frac{\partial u_m}{\partial x_i}(x) - \frac{\partial u}{\partial x_i}(x)\right)d^\varepsilon(x).$$

Lemma 4.3. *If* $\int_\Omega G_m(x)\,dx \to 0$ *then there exists a subsequence* $\{u_k\}_k$ *of* $\{u_m\}_m$ (in what follows, our notation will not distinguish between a sequence and its subsequences) *satisfying the condition*

$$b_i(x, u_k, \nabla u_k) \to b_i(x, u, \nabla u) \text{ weakly in } L_{p'}(\Omega; d, \varepsilon), \quad i = 0, \ldots, N.$$

Proof. With regard to (4.7) we have $G_m \geqq 0$. As $u_m \to u$ strongly in $L_p(\Omega; d, \varepsilon)$ (the imbedding $V_{p,\varepsilon} \hookrightarrow L_p(\Omega; d, \varepsilon)$ is compact) we can choose a subsequence $\{k\}$ of integers such that

(4.12) $$u_k(x) \to u(x), \quad G_k(x) \to 0 \quad \text{for all} \quad x \in \Omega \setminus Z, \quad \text{meas } Z = 0.$$

Let r, h be the functions from (4.5) and (4.6), $r(x), h(x) < +\infty$ for a fixed $x \in \Omega \setminus Z$. Let us put $\eta_k = u_k(x)$, $\eta = u(x)$, $\xi = \nabla u(x)$. Further, let ξ^* be a cluster point of the sequence $\{\xi_k\}_k$, where $\xi_k = \nabla u_k(x)$. We have

(4.13) $$|\xi^*| < +\infty$$

since

$$G_k(x) \geqq d^\varepsilon(x)\left[\sum_{i=1}^{N} b_i(x, \eta_k, \xi_k)\,\xi_{ki} - N\beta_0|\xi|\left(|\xi_k|^{p-1} + \right.\right.$$
$$+ d^{-(p-1)+\gamma}(x)|\eta_k|^{p-1} + h(x)) - N\beta_0(|\xi| + |\xi_k|)\cdot(|\xi|^{p-1} +$$
$$+ d^{-(p-1)+\gamma}(x)|\eta_k|^{p-1} + h(x)) + b_0(x, \eta_k, \xi_k)\,\eta_k -$$
$$- \beta_0|\eta_k|\left(d^{-1}(x)|\xi_k|^{p-1} + d^{-p}(x)|\eta_k|^{p-1} + d^{-1}(x)\,.\,h(x)\right)\Big] \geqq$$
$$\geqq d^\varepsilon(x)\,\beta_1|\xi_k|^p - c_9(|\xi_k|^{p-1} + |\xi_k| + 1)$$

and since $G_k(x) \to 0$ by (4.12). In view of (4.12), (4.13) and of the continuity of the functions b_i in η and ξ we obtain

$$d^\varepsilon(x)\sum_{i=1}^{N}\left[b_i(x, \eta, \xi^*) - b_i(x, \eta, \xi)\right].(\xi_i^* - \xi_i) = 0,$$

therefore, by (4.7), $\xi^* = \xi$.

Finally, the Carathéodory conditions yield

$$b_i(x, u_k(x), \nabla u_k(x)) \to b_i(x, u(x), \nabla u(x)) \text{ a.e. in } \Omega \quad \text{for} \quad i = 0, \ldots, N.$$

Because the sequence $\{b_i(x, u_k, \nabla u_k)\}_k$, $i = 0, \ldots, N$, are bounded in $L_{p'}(\Omega; d, \varepsilon)$ and this space is reflexive we can write

$$b_i(x, u_k, \nabla u_k) \to b_i(x, u, \nabla u) \quad \text{weakly in} \quad L_{p'}(\Omega; d, \varepsilon).$$

(The weak limit is independent of the selection of a subsequence of $\{u_k\}_k$.) The assertion of the lemma is proved.

Now, in our case the condition (2.7) means that $\int_\Omega G_m(x)\,dx \to 0$ and we can consider the subsequence $\{u_k\}_k$ from Lemma 4.3. The convergence $u_k \to u$ a.e. in Ω yields $b_i(x, u_k(x), \nabla v(x)) \to b_i(x, u(x), \nabla v(x))$ for a.a. $x \in \Omega$, $i = 0, \ldots, N$, and in virtue of the boundedness of the sequences $\{b_i(x, u_k, \nabla v)\}_k$, $i = 0, \ldots, N$, in $L_{p'}(\Omega; d, \varepsilon)$ we get

(4.14) $$b_i(x, u_k, \nabla v) \to b_i(x, u, \nabla v) \quad \text{weakly in} \quad L_{p'}(\Omega; d, \varepsilon).$$

Further, if $z \in C_0^\infty(\Omega)$ then

$$d^\varepsilon\frac{\partial z}{\partial x_i}, \quad d^\varepsilon z, \quad d^{\varepsilon-1}\frac{\partial d}{\partial x_i} z \in [L_{p'}(\Omega; d, \varepsilon)]^* = L_p(\Omega; d, -\varepsilon(p-1)),$$
$$i = 1, \ldots, N,$$

and from Lemma 4.3 and from (4.14) we obtain

$$\langle S(u_k, v), z\rangle \to \langle S(u, v), z\rangle \quad \text{for all} \quad z \in C_0^\infty(\Omega).$$

However, the set $C_0^\infty(\Omega)$ is dense in $V_{p,\varepsilon}$ and so

$$S(u_k, v) \to S(u, v) \quad \text{weakly in} \quad [V_{p,\varepsilon}]^*.$$

Condition (2.8). Let u_m satisfy the assumption of (2.8), i.e. $S(u_m, v) \to \psi$ weakly in $[V_{p,\varepsilon}]^*$. The first condition of (4.11) implies the strong convergence $u_m \to u$ in $L_p(\Omega; d, \varepsilon - p + \gamma p')$ (for $\gamma > 0$ see (4.5)) since the imbedding $V_{p,\varepsilon} \hookrightarrow L_p(\Omega; d, \varepsilon - p + \gamma p')$ is compact (see e.g. [10]). By the growth conditions, for a fixed

$v \in V_{p,\varepsilon}$ the mappings $z \to b_i(x, z, \nabla v)$, $i = 1, \ldots, N$, act from $L_p(\Omega; d, \varepsilon - p + \gamma p')$ into $L_{p'}(\Omega; d, \varepsilon)$ and so they are continuous due the properties of Nemyckij operators. Hence we obtain

$$b_i(x, u_m, \nabla v) \to b_i(x, u, \nabla v) \quad \text{strongly in} \quad L_{p'}(\Omega; d, \varepsilon), \quad i = 1, \ldots, N,$$

and so

$$\text{(4.15)} \qquad \langle S_1(u_m, v), u_m \rangle \to \langle S_1(u, v), u \rangle \quad \text{for an arbitrary} \quad v \in V_{p,\varepsilon}.$$

The most complicated part of the verification of (2.8) is to show that

$$\text{(4.16)} \qquad \langle S_2 u_m, u_m - u \rangle \to 0 ;$$

we shall postpone it for a while.

Now, the condition (4.15) yields

$$\langle S_2 u_m, u \rangle = \langle S(u_m, v), u \rangle - \langle S_1(u_m, v), u \rangle \to \langle \psi, u \rangle - \langle S_1(u, v), u \rangle$$

and according to (4.16) we deduce

$$\langle S_2 u_m, u_m \rangle \to \langle \psi, u \rangle - \langle S_1(u, v), u \rangle .$$

Finally, we have

$$\langle S(u_m, v), u_m \rangle = \langle S_1(u_m, v), u_m \rangle + \langle S_2 u_m, u_m \rangle \to \langle \psi, u \rangle ,$$

which is the assertion of (2.8).

In the proof of (4.16) we cannot employ the imbedding of $V_{p,\varepsilon}$ into $L_p(\Omega; d, \varepsilon - p)$ since it is not compact. However, we can use the following

Lemma 4.4. *Let $\varepsilon \in I$, where I is the interval from Lemma 4.1. Then there exist a constant $c_{10} = c_{10}(f, \Omega, p, \varepsilon, \beta_0, \beta_1, \gamma, h, r)$ and an increasing sequence $\{a_k\}_k$ of integers such that the inequalities*

$$\int_{\Omega \setminus \Omega a_{k+1}} |\nabla u_m(x)|^p \, d^\varepsilon(x) \, dx \leq \frac{c_{10}}{k}, \quad k \geq 1,$$

hold for all solutions u_m of the equations $(4.8)_m$ with $m \geq n_1$ (for n_1 see Lemma 4.2).

Remark 4.1. This fact together with Lemma 3.1 implies

$$\int_{\Omega \setminus \Omega a_{k+1}} |u_m(x)|^p \, d^{\varepsilon - p}(x) \, dx \leq \frac{c_{11}}{k}, \quad k \geq 1, \quad m \geq n_1,$$

where c_{11} is a positive constant.

Proof. Since meas $(\Omega \setminus \Omega_n) \to 0$ for $n \to +\infty$, $h, f_i \in L_{p'}(\Omega; d, \varepsilon)$, $i = 0, \ldots, N$, $r \in L_1(\Omega; d, \varepsilon)$, there exists an increasing sequence $\{a_k\}_k$ of integers satisfying the conditions

$$\text{(4.17)} \qquad \begin{cases} \displaystyle\sum_{k=1}^{\infty} \int_{\Omega \setminus \Omega a_k} r(x) \, d^\varepsilon(x) \, dx < 1, \quad \sum_{k=1}^{\infty} \int_{\Omega \setminus \Omega a_k} |h(x)|^{p'} \, d^\varepsilon(x) \, dx < 1, \\ \displaystyle\sum_{k=1}^{\infty} R_k(f) < 1, \quad \frac{a_{k+1}}{a_k} \geq 2 \quad \text{for} \quad k = 1, 2, \ldots, \end{cases}$$

where

$$R_k(f) = \sum_{i=0}^{N} \left(\int_{\Omega \setminus \Omega a_k} |f_i(x)|^{p'} \, d^\varepsilon(x) \, dx \right)^{1/p'} .$$

Let us consider the functions $\{\varphi_k\}_k$ from Lemma 3.3 corresponding to the sequence $\{a_k\}_k$. We have $\varphi_k u_m d^\varepsilon \in V_p^m$ and the equality $\langle Tu_m, \varphi_k u_m d^\varepsilon \rangle = \langle f, \varphi_k u_m d^\varepsilon \rangle$ yields

$$\text{(4.18)} \qquad \sum_{i=1}^{N} \int_\Omega b_i(x, u_m, \nabla u_m) \frac{\partial(\varphi_k u_m d^\varepsilon)}{\partial x_i} \, dx + \int_\Omega b_0(x, u_m, \nabla u_m) \, \varphi_k u_m d^\varepsilon \, dx =$$

$$= \int_\Omega f_0 \varphi_k u_m d^\varepsilon \, dx + \sum_{i=1}^{N} \int_\Omega f_i \frac{\partial(\varphi_k u_m d^\varepsilon)}{\partial x_i} \, dx$$

for all $m \geq n_1$, $k = 1, 2, \ldots$.

We denote by $L(P)$ the left-hand (right-hand) side of (4.18). Then $L = I_1 + I_2 + I_3$ where

$$I_1 = \int_{\Omega \setminus \Omega a_{k+1}} \left[\sum_{i=1}^{N} b_i(x, u_m, \nabla u_m) \frac{\partial u_m}{\partial x_i} + b_0(x, u_m, \nabla u_m) \, u_m \right] \varphi_k d^\varepsilon \, dx +$$

$$+ \varepsilon \sum_{i=1}^{N} \int_{\Omega \setminus \Omega a_{k+1}} b_i(x, u_m, \nabla u_m) \, u_m \varphi_k d^{\varepsilon - 1} \frac{\partial d}{\partial x_i} \, dx ,$$

$$I_2 = \int_{\Omega a_{k+1}} \left[\sum_{i=1}^{N} b_i(x, u_m, \nabla u_m) \frac{\partial u_m}{\partial x_i} + b_0(x, u_m, \nabla u_m) \, u_m \right] \varphi_k d^\varepsilon \, dx +$$

$$+ \varepsilon \sum_{i=1}^{N} \int_{\Omega a_{k+1}} b_i(x, u_m, \nabla u_m) \, u_m \varphi_k d^{\varepsilon - 1} \frac{\partial d}{\partial x_i} \, dx ,$$

$$I_3 = \sum_{i=1}^{N} \int_\Omega b_i(x, u_m, \nabla u_m) \, u_m \frac{\partial \varphi_k}{\partial x_i} \, d^\varepsilon \, dx .$$

Using Lemma 3.1 for $\mathcal{O} \equiv \Omega \setminus \Omega_{a_{k+1}}$ we obtain analogously as in the proof of Lemma 4.1

$$I_1 \geq [\beta_1 - 2\omega c_5 - |\varepsilon| \, N\beta_0(c_5^{1/p} + c_5 \max_{x \in \bar\Omega} d^\gamma(x))] \int_{\Omega \setminus \Omega a_{k+1}} |\nabla u_m|^p \, d^\varepsilon \, dx -$$

$$- \beta_2(\omega) \int_{\Omega \setminus \Omega a_{k+1}} r \, d^\varepsilon \, dx - c_6(\omega) \int_{\Omega \setminus \Omega a_{k+1}} h^{p'} d^\varepsilon \, dx ,$$

where the term in the square brackets is positive for $\varepsilon \in I$. (I is the interval from Lemma 4.1.) The inequality (for the properties of φ_k see Lemma 3.3)

$$\left|\sum_{i=1}^{N}\int_{\Omega a_{k+1}\setminus\Omega a_k} v_i \frac{\partial\varphi_k}{\partial x_i} d^\varepsilon \,dx\right| \leqq \frac{1}{a_k} c_3 \frac{a_k a_{k+1}}{a_{k+1}-a_k}\,.$$

$$\int_{\Omega a_{k+1}\setminus\Omega a_k} \sum_{i=1}^{N}|v_i|\, d^{\varepsilon-1}\,dx \leqq 2c_3 \int_{\Omega a_{k+1}\setminus\Omega a_k} \sum_{i=1}^{N}|v_i|\, d^{\varepsilon-1}\,dx$$

holds for all $v_i \in L_1(\Omega; d, \varepsilon-1)$, $i = 1, \ldots, N$, and using (4.5) and the inequality

$$ab \leqq \frac{1}{p}|a|^p + \frac{1}{p'}|b|^{p'}\,, \quad a, b \in \mathbb{R}\,, \tag{4.19}$$

we obtain

$$|I_2| + |I_3| \leqq c_{12}\int_{\Omega a_{k+1}\setminus\Omega a_k}(d^\varepsilon|\nabla u_m|^p + d^{\varepsilon-p}|u_m|^p + d^\varepsilon h^{p'})\,dx\,.$$

(Evidently $\varphi_k \equiv 0$ in Ω_{a_k} and $\varphi_k \equiv 1$ in $\Omega\setminus\Omega_{a_{k+1}}$.) Via the Hölder inequality we deduce, for the right-hand side of (4.18),

$$|P| \leqq \int_{\Omega\setminus\Omega a_k}|f_0|\,|u_m|\,d^\varepsilon\,dx + \sum_{i=1}^{N}\int_{\Omega\setminus\Omega a_k}|f_i|\,|\nabla u_m|\,d^\varepsilon\,dx +$$

$$+ |\varepsilon|\sum_{i=1}^{N}\int_{\Omega\setminus\Omega a_k}|f_i|\,|u_m|\,d^{\varepsilon-1}\,dx + \frac{1}{a_k}c_3\frac{a_k a_{k+1}}{a_{k+1}-a_k}\sum_{i=1}^{N}\int_{\Omega a_{k+1}\setminus\Omega a_k}|f_i|\,|u_m|\,d^{\varepsilon-1}\,dx \leqq$$

$$\leqq c_{13}\sum_{i=0}^{N}\left(\int_{\Omega\setminus\Omega a_k}|f_i|^{p'}\,d^\varepsilon\,dx\right)^{1/p'}\left(\int_\Omega|\nabla u_m|^p\,d^\varepsilon\,dx\right)^{1/p}.$$

Finally, from these inequalities and from (4.10), (4.17), (4.18) we have

$$\int_{\Omega\setminus\Omega a_{k+1}}|\nabla u_m|^p\,d^\varepsilon\,dx \leqq c_{14}\left[\int_{\Omega\setminus\Omega a_{k+1}} r d^\varepsilon\,dx + \int_{\Omega\setminus\Omega a_k} h^{p'} d^\varepsilon\,dx + \right.$$

$$\left. + R_k(f) + \int_{\Omega a_{k+1}\setminus\Omega a_k}(d^\varepsilon|\nabla u_m|^p + d^{\varepsilon-p}|u_m|^p)\,dx\right]$$

and

$$\sum_{k=1}^{\infty}\int_{\Omega\setminus\Omega a_{k+1}}|\nabla u_m|^p\,d^\varepsilon\,dx \leqq c_{14}\left[3 + \int_\Omega(d^\varepsilon|\nabla u_m|^p + d^{\varepsilon-p}|u_m|^p)\,dx\right] \leqq c_{10}\,.$$

Since $\Omega\setminus\Omega_{a_i} \subset \Omega\setminus\Omega_{a_l}$ for $i > l$ and since the constant c_{10} is independent of m, the proof is complete.

The proof of (4.16). The inequalities (4.5), (4.19) together with Lemma 4.4 and Remark 4.1 yield

$$X_k(m) = \left|\int_{\Omega\setminus\Omega a_{k+1}} b_0(x, u_m, \nabla u_m)(u_m - u)\,d^\varepsilon\,dx + \right.$$

$$\left. + \varepsilon\sum_{i=1}^{N}\int_{\Omega\setminus\Omega a_{k+1}} b_i(x, u_m, \nabla u_m)(u_m - u)\,d^{\varepsilon-1}\frac{\partial d}{\partial x_i}\,dx\right| \leqq$$

$$\leqq c_{15}\left[\int_{\Omega\setminus\Omega a_{k+1}}|\nabla u_m|^p\,d^\varepsilon\,dx + \int_{\Omega\setminus\Omega a_{k+1}}|u_m|^p\,d^{\varepsilon-p}\,dx + \right.$$

$$\left. + \int_{\Omega\setminus\Omega a_{k+1}}|u|^p\,d^{\varepsilon-p}\,dx + \int_{\Omega\setminus\Omega a_{k+1}} h^{p'} d^\varepsilon\,dx\right] \leqq \chi(k)\,,$$

where χ is independent of m and $\chi(k) \to 0$ for $k \to +\infty$. Similarly, using the Hölder inequality, (4.5) and (4.10) we estimate

$$Y_k(m) = \left|\int_{\Omega a_{k+1}} b_0(x, u_m, \nabla u_m)(u_m - u)\,d^\varepsilon\,dx + \right.$$

$$\left. + \varepsilon\sum_{i=1}^{N}\int_{\Omega a_{k+1}} b_i(x, u_m, \nabla u_m)(u_m - u)\,d^{\varepsilon-1}\frac{\partial d}{\partial x_i}\,dx\right| \leqq$$

$$\leqq c_{16}\left(\int_{\Omega a_{k+1}}|u_m - u|^p\,d^{\varepsilon-p}\,dx\right)^{1/p} \leqq c_{16}a_{k+1}^{\beta/p}\left(\int_\Omega|u_m - u|^p\,d^{\varepsilon-p+\beta}\,dx\right)^{1/p}$$

with a number $\beta > 0$. In virtue of the compactness of the imbedding $V_{p,\varepsilon} \hookrightarrow L_p(\Omega; d, \varepsilon - p + \beta)$ we obtain the convergence $Y_k(m) \to 0$ for $m \to +\infty$, where k is arbitrary.

Finally, given $\alpha > 0$ we find integers $k > 0$ and $n_2 \geqq n_1$ such that

$$|\langle S_2 u_m, u_m - u\rangle| \leqq X_k(m) + Y_k(m) < 2\alpha \quad \text{for all} \quad m \geqq n_2\,,$$

which completes the proof.

Let us now summarize the results concerning the solvability of the problems (1.1) and (4.4) into the following theorems.

Theorem 4.6. *Let functions $b_i: \Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$, $i = 0, \ldots, N$, satisfy the Carathéodory conditions and the inequalities (4.5)–(4.7). Then there exists an interval I with $0 \in \operatorname{int} I$ such that if $\varepsilon \in I$, then the b.v.p. (4.4) has at least one weak solution $u \in W_0^{1,p}(\Omega; d, \varepsilon)$ whenever*

$$f = f_0 - \sum_{i=1}^{N}\frac{\partial f_i}{\partial x_i}\,, \quad f_0, f_1, \ldots, f_N \in L_{p'}(\Omega; d, \varepsilon)\,.$$

Theorem 4.7. *Let functions $a_i: \Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$, $i = 0, \ldots, N$, satisfy the Carathéodory conditions and the inequalities (4.1)–(4.3). Then there exists an* *interval I with $0 \in \operatorname{int} I$ such that if $\varepsilon \in I$, then the b.v.p. (1.1) has at least one weak solution $u \in W^{1,p}(\Omega; d, \varepsilon)$ whenever*

$$f = f_0 - \sum_{i=1}^{N} \frac{\partial f_i}{\partial x_i}, \quad f_0, f_1, \ldots, f_N \in L_{p'}(\Omega; d, \varepsilon),$$

and $\varphi \in W^{1,p}(\Omega; d, \varepsilon)$.

Remark 4.2. In the case of $\varepsilon < 0$ and of a degenerate right-hand side the weak solution of the problem (1.1) or (4.4) belonging to the space $W^{1,p}(\Omega; d, \varepsilon)$ or $W_0^{1,p}(\Omega; d, \varepsilon)$, respectively, will be a weak solution of the same problem in the corresponding classical Sobolev space as well. Therefore, if the b.v.p. (1.1) or (4.4) has a unique weak solution in the classical sense (for example, if the operator T is strongly monotone) then this will be the unique solution in the corresponding Sobolev power weight space.

An open problem, however, is to find reasonable conditions of the uniqueness for $\varepsilon > 0$.

Remark 4.3. It would require rather lengthy and purela technical considerations to get analogous results for Sobolev power weight spaces $W^{1,p}(\Omega; d_M, \varepsilon)$, $M \subset \partial\Omega$ being a manifold with $\dim M \leqq N - 1$, and the same is true for operators of higher orders. One can make use of estimates similar to that in Lemma 4.4 to verify the condition (2.8).

Remark 4.4. Finally, it remains to discuss the situation from Lemma 4.1 where the interval I obtained in the course of the proof determines the choice of the suitable weight.

Very often, the situation met in particular cases is such that I can be larger than the interval which we get from (4.9). For example, a finer estimate guarantees the solvability of the b.v.p.

$$(4.20) \qquad -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(|\nabla u|^{p-2} \frac{\partial u}{\partial x_i}\right) = f \quad \text{in} \quad \Omega,$$

$$u = 0 \quad \text{on} \quad \partial\Omega,$$

(with $p > 1, f \in [V_{p,-\varepsilon(p-1)}]^*$) in the Sobolev power weight space for

$$\varepsilon \in J = \left(\frac{-p+1}{c^{1/p}p - 1}, \frac{p-1}{c^{1/p}p + 1}\right),$$

where c is the constant from (3.5); note that $c = 1$ if Ω is convex.

However, there is still another interesting problem to be solved. Namely, the problem (4.20) and the corresponding b.v.p. with non-zero boundary data can also be formulated in spaces $V_{p,\varepsilon}$ for $\varepsilon \in (-1, p - 1) \setminus J$ (because a suitable trace theorem is available), and an existence theorem would be desirable.

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