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A REMARK ON THE SOLVABILITY OF THE DIRICHLET PROBLEM IN SOBOLEV SPACES WITH POWER-TYPE WEIGHTS

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Abstract: For each $\varepsilon \neq 0$ (with $|\varepsilon|$ sufficiently small) such an elliptic partial differential equation is constructed that the corresponding Dirichlet problem is unsolvable in the Sobolev space with a weight given by the ε-th power of the distance to the boundary.

Key words: Dirichlet problem, Sobolev power weight spaces.

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1. Let $\Omega \subset R^N$ be a bounded domain with a smooth boundary $\partial\Omega$ and $M \subset \partial\Omega$ be a closed manifold. The weight d_M is defined in a point $x \in \bar{\Omega}$ by $d_M(x) = \min\{|x-y|; y \in M\}$. The weighted Sobolev space $W_o^{1,2}(\Omega; d_M, \varepsilon)$ is the closure of the set $C_o^\infty(\Omega)$ of smooth functions with a compact support in Ω with respect to the norm

$$\|u\|_\varepsilon = \left(\int_\Omega |\nabla u(x)|^2 \, d_M^\varepsilon(x) \, dx \right)^{\frac{1}{2}}.$$

Let us consider the Dirichlet problem

$$(D) \begin{cases} \sum_{|i|,|j| \le 1} (-1)^{|i|} D^i(a_{ij}(x) \, D^j u(x)) = f(x) \text{ in } \Omega , \\ u(x) = 0 \text{ on } \partial\Omega , \end{cases}$$

where a_{ij} are such that the corresponding bilinear form

$$a(u,v) = \sum_{|i|,|j| \le 1} \int_\Omega a_{ij} \, D^j u \, D^i v \, dx$$

is bounded on $W_o^{1,2}(\Omega) \times W_o^{1,2}(\Omega)$ and, moreover, it is $W_o^{1,2}(\Omega)$-elliptic, i.e.

$$a(u,u) \geq c \|u\|_0^2 \text{ with some } c>0.$$

Here, $W_0^{1,2}(\Omega) = W_0^{1,2}(\Omega; d_M, 0)$ denotes the standard Sobolev space without the weight and D^i denotes the differential operator with the multiindex i.

We shall say that the function $u \in W_0^{1,2}(\Omega; d_M, \varepsilon)$ is a weak solution of the Dirichlet problem (D), if

$$a(u,v) = \int_\Omega f(x)\, v(x)\, dx$$

for every $v \in C_0^\infty(\Omega)$.

Many papers deal with the solvability of the Dirichlet problem in the weighted spaces even in the case if the corresponding equation is nonlinear or of higher power (see e.g. [1],[2],[3]). It is motivated by two following reasons: At first, the behaviour of the right-hand side f near the boundary $\partial\Omega$ may exclude the solvability of the problem (D) in a classical (non-weighted) Sobolev space. At second, if the problem (D) is solvable in a classical Sobolev space then from the behaviour of the right-hand side f near the boundary $\partial\Omega$ we should like to deduce the analogous one of the solution. The use of suitable weights could answer some of such questions. The results obtained are analogous to that from the following theorem.

<u>Theorem</u>. There exists an interval I containing a neighbourhood of 0 that for any $\varepsilon \in I$ and $f \in [W_0^{1,2}(\Omega; d_M, -\varepsilon)]^*$ there exists exactly one weak solution $u \in W_0^{1,2}(\Omega; d_M, \varepsilon)$ of the problem (D).

Such assertion justifies our effort to ask for the maximal interval $I(D)$ in case of each particular Dirichlet problem (D). The aim of our note is to show that there is no chance to obtain any universal $I(D)$ for at least some class of Dirichlet problems.

Namely, we shall prove the following

<u>Proposition</u>. For any $|\varepsilon|$, arbitrarily small, we can find such a Dirichlet problem (D) that $\varepsilon \notin I(D)$.

2. At first, let us remind a certain version of Hardy's inequality. If $\varepsilon \neq 1$ then

$$\text{(H)} \qquad \int_0^1 |u(x)|^2 x^{\varepsilon-2}\, dx \leq \frac{4}{|1-\varepsilon|^2} \int_0^1 |u'(x)|^2 x^{\varepsilon}\, dx,$$

for all $u \in W_0^{1,2}((0,1); x, \varepsilon)$.

Let us consider the Dirichlet problem

$$(D_\delta) \quad \begin{cases} -u''(x) + \delta(\delta-1)\, x^{-2}\, u(x) = f(x) \text{ for } x \in (0,1), \\ u(0) = u(1) = 0. \end{cases}$$

The corresponding bilinear form

$$a(u,v) = \int_0^1 u'v'dx + \delta(\delta-1) \int_0^1 x^{-2}\, uv\, dx$$

is bounded on $W_0^{1,2}((0,1)) \times W_0^{1,2}((0,1))$ and $W_0^{1,2}((0,1))$-elliptic for $|\delta(\delta-1)| < \frac{1}{4}$. Really, on the basis of (H) and the Hölder inequality we have

$$|a(u,v)| \leq (1+4\,|\delta(\delta-1)|)\, \left(\int_0^1 |u'|^2 dx\right)^{1/2} \left(\int_0^1 |v'|^2 dx\right)^{1/2},$$

$$a(u,u) \geq (1-4\,|\delta(\delta-1)|) \int_0^1 |u'|^2 dx.$$

Further, let $\varepsilon \neq 0$ and $|\varepsilon|$ be sufficiently small. Put $\delta = \frac{1-\varepsilon}{2}$. Let us suppose that the right-hand side of the problem (D_δ) is of the form

$$f(x) = \begin{cases} x^{\delta-2}(-\ln x)^{\omega-1}[2\omega\delta - \omega - \omega(\omega-1)(-\ln x)^{-1}] & \text{for } x \in (0,1/2), \\ -w''(x) + \delta(\delta-1)\, x^{-2} w(x) & \text{for } x \in \langle 1/2, 1), \end{cases}$$

where $\omega \neq 0$, $|\omega| < \frac{1}{2}$ and w is a function with the continuous second derivative in the interval $(\frac{1}{3}, \frac{4}{3})$, $w(1) = 0$, $w(x) = x^{\delta}(-\ln x)^{\omega}$ for $x \in (\frac{1}{3}, \frac{1}{2})$. Then $f \in [W_0^{1,2}((0,1);x,-\varepsilon)]^*$. Indeed,

$$\left|\int_0^1 f(x)\, v(x)\, dx\right| \leqq \text{const}\, \left(\int_0^1 |v'(x)|^2\, x^{-\varepsilon}\, dx\right)^{1/2},$$

for any $v \in W_0^{1,2}((0,1);x,-\varepsilon)$, because the inequality (H) implies

$$\left|\int_0^{1/2} x^{\delta-2}(-\ln x)^{\omega-1} v(x)\, dx\right| \leqq \frac{4}{|1-\varepsilon|^2}\left(\int_0^1 |v'(x)|^2 x^{-\varepsilon} dx\right)^{1/2} \cdot$$

$$\cdot\left(\int_0^{1/2} x^{[illegible]} (-\ln x)^{2\omega-2} dx\right)^{1/2} \leqq \text{const}\, \|v\|_{\varepsilon}.$$

Regarding the uniqueness, the weak solution of the problem (D_{γ}) is of the form

$$u(x) = \begin{cases} 0 & \text{for } x = 0, \\ x^{\delta}(-\ln x)^{\omega} & \text{for } x \in (0, \frac{1}{2}\rangle, \\ w(x) & \text{for } x \in (\frac{1}{2}, 1\rangle \end{cases}$$

However, $2\delta - 2 + \varepsilon = -1$, $2\omega > -1$ and

$$\|u\|_{\varepsilon}^2 = \int_0^1 |u'(x)|^2 x^{\varepsilon}\, dx \geqq \text{const} \int_0^{1/2} x^{2\delta-2+\varepsilon}(-\ln x)^{2\omega} dx = +\infty,$$

i.e. $u \notin W_0^{1,2}((0,1);x,\varepsilon)$.

Therefore $u \notin \mathcal{E}(D_{\frac{1-\varepsilon}{2}})$, for $\varepsilon \neq 0$, and $|\varepsilon|$ sufficiently small.

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