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Habilitační práce

Josef Daněk

Plzeň, 2008

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Aplikace kontaktního problému v úlohách biomechaniky a geomechaniky

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Předmluva

Předkládaná habilitační práce je soubor uveřejněných vědeckých prací doplněný komentářem (§72, odst. 3, písm. b Zákona o vysokých školách č. 111/1998 Sb.). Práci lze charakterizovat pomocí několika klíčových slov, která v krátkosti uvádím do souvislosti.

Nosným tématem je *kontaktní problém* obecně více těles, jehož praktické uplatnění lze nalézt například v oblasti *biomechaniky* nebo *geomechaniky*. V prvním případě se může jednat o modelování zatížených kloubů, případně kloubních náhrad v lidském těle. Ve druhém případě může jít o modelování silového působení v okolí tunelu ve skalním masívu, kterým prochází geologický zlom.

Pro numerické řešení problému lze použít *metodu konečných prvků*, která patří v oblasti mechaniky těles k nejvíce používaným nástrojům. Je zřejmé, že počet neznámých diskretizovaného problému může být velký, např. v řádech milionů i více, a je tedy namístě hledat a používat rychlé a efektivní algoritmy, například *metodu rozkladu oblasti*.

Při sestavování modelu je nutné mít informaci o geometrii. V případě modelování kloubů lze *geometrická data* získat např. z rentgenového snímku (pro 2D modely) nebo z magnetické rezonance a z počítačového tomografu (pro 3D modely). Následně se z geometrického popisu generuje vlastní *konečněprvková síť*.

V praktických úlohách nelze vždy přesně určit vstupní parametry modelu jako jsou např. elastické koeficienty materiálů nebo silové zatížení. Pro tato nejistá vstupní data je vhodné použít *metodu nejhoršího scénáře*, která pro vstupní data ze zadané množiny nalezne to řešení, které je podle určitého kritéria nejhorší. Cílem je zaručit, aby míra hodnotícího kritéria nalezeného řešení splňovala předepsané limity.

Při řešení statického kontaktního problému můžeme konečněprvkovou síť vygenerovat tak, že si příslušné uzly na kontaktní hranici odpovídají, tj. mají stejné prostorové souřadnice. Podmínka nepronikání těles potom svazuje neznámé pro uzly z těchto kontaktních dvojic. Pokud řešíme dynamický kontaktní problém, tj. tělesa v jednotlivých časových vrstvách vůči sobě mění polohu, mění polohu i uzly triangulace, a podmínka nepronikání těles se formuluje složitěji. Pro nesouhlasné triangulace můžeme použít tzv. *metodu mortarových konečných prvků*.

Práce sestává z komentáře a ze souboru sedmi původních publikovaných prací (uvedených v přílohách A až G), recenzovaných článků v časopise [21⋆], [68⋆], recenzovaných článků v časopise, publikovaných ve speciálních číslech věnovaných mezinárodním konferencím, [17⋆], [28⋆], oponovaných příspěvků ze sborníků mezinárodních konferencí [19⋆], [101⋆] a příspěvku ze sborníku mezinárodní konference [33⋆]. Pro větší přehlednost jsou v textu odkazy na tyto práce vysázeny s hvězdičkou.

[17⋆] Daněk, J.: *Domain decomposition method for contact problems with small range contact*, J. Mathematics and Computers in Simulation 61/3-6 (2003), 359–373,

[21⋆] Daněk, J., Hlaváček, I., Nedoma, J.: *Domain decomposition for generalized unilateral semi-coercive contact problem with given friction in elasticity*, J. Mathematics and Computers in Simulation 68/3 (2005), 271–300,

[19⋆] Daněk, J., Denk, F., Hlaváček, I., Nedoma, J., Stehlík, J., Vavřík, P.: *On the stress-strain analysis of the knee replacement*, Lecture Notes in Computer Science 3044 (2004), 456–466,

[28⋆] Daněk, J., Nedoma, J., Hlaváček, I., Vavřík, P., Denk, F.: *Numerical Modelling of the Weight-Bearing Total Knee Joint Replacement and Usage in Practice*, J. Mathematics and Computers in Simulation 76/1 (2007), pp. 49–56, Elsevier Science Ltd., North-Holland.

[68⋆] Hlaváček, I., Nedoma, J., Daněk, J.: *Worst Scenario and Domain Decomposition Methods in Geomechanics*, Future Generation Computer Systems 22/4 (2006), Special section: Numerical modelling in geomechanics and geodynamics, 468–483,

[33⋆] Daněk, J., Kutáková, H.: *The mortar finite element method in 2D - Implementation in MATLAB*, In 16th Annual Conference Proceedings of Technical Computing Prague 2008, Praha, Česká republika 2008,

[101⋆] Nedoma, J., Daněk, J.: *Time-space FE-PDAS method for dynamic unilateral contact problem in viscoelasticity*, Lecture Notes in Computer Science 5073 (2008), 707–719.

Komentář se skládá z šesti kapitol, udává stručný přehled studované problematiky a začleňuje jednotlivé články předkládané práce do širšího kontextu. V komentáři se objevují jak odkazy na další původní práce autora, které souvisí s tématem předkládané práce, tak odkazy na práce jiných autorů, které jsou v daných tématech považovány za stěžejní. V první kapitole je podána klasická a variační formulace rovinného kontaktního problému pružných těles bez tření, resp. se zadaným třením, a krátce je zmíněna otázka existence a jednoznačnosti řešení. Druhá kapitola je věnována metodě rozkladu oblasti, zejména verzi vycházející z primární variační formulace. Ve třetí kapitole je popsán způsob získávání sítě konečných prvků pro biomechanické modely např. kolenního kloubu a dále pak metoda nejhoršího scénáře pro úlohy s nejistými vstupními daty. Ve čtvrté kapitole je přiblížena jednak metoda konečných prvků a dále pak metoda mortarových konečných prvků pro nesouhlasné sítě. Pátá kapitola obsahuje numerické výsledky pro některé biomechanické a geomechanické modely, které nejsou obsaženy v souboru článků předkládané práce. V poslední kapitole následuje shrnutí a výhled dalšího zkoumání.

Rád bych zde poděkoval svým kolegům, zejména panu doc. Jiřímu Nedomovi a panu dr. Ivanu Hlaváčkovi, za příjemnou a plodnou spolupráci.

Josef Daněk
prosinec 2008

Obsah

1

Kontaktní problém

Kontaktní problém poprvé formuloval v roce 1882 H. Hertz [57]. Pro speciální případ odvodil analytický vztah mezi působící silou a výsledným posunutím. V 60. letech minulého století popsal A. Signorini kontaktní úlohu ve formě variační nerovnice (viz např. [63]). Otázky existence řešení takto podané úlohy byly poprvé studovány G. Ficherou (viz např. [63]). V současnosti jsou přístupy na řešení kontaktních problémů založeny na variačních metodách, které umožňují problém popsat elegantně a kompaktně. Obsáhlá teorie variačních nerovnic je popsána např. v [44, 53]. V České republice se problematikou teorie variačních nerovnic zabývala zejména skupina prof. J. Nečase, prof. Haslingera a dr. Hlaváčka. Z prací těchto osobností připomeňme alespoň [54, 55, 61, 62, 63, 93]. Teorie existence řešení variačních nerovnic s důrazem na kontaktní úlohy je podána např. v [63].

1.1 Klasická formulace

Uvažujme rovinnou úlohu jednostranného kontaktu s pružných těles, kterým odpovídají oblasti $\Omega^1, \Omega^2, \ldots, \Omega^s \subset \mathbb{R}^2$ s Lipschitzovskou hranicí $\partial\Omega^1$, $\partial\Omega^2$, $\ldots$, $\partial\Omega^s$. Nechť je hranice $\bigcup_{\iota=1}^{s} \partial\Omega^\iota$ rozdělena na disjunktní části

$$\Gamma_u, \Gamma_\tau, \Gamma_c, \Gamma_o, \quad \bigcup_{\iota=1}^{s} \partial\Omega^\iota = \Gamma_u \cup \Gamma_\tau \cup \Gamma_c \cup \Gamma_o,$$

$$\Gamma_u = \bigcup_{\iota=1}^{s} \Gamma_u^\iota, \quad \Gamma_\tau = \bigcup_{\iota=1}^{s} \Gamma_\tau^\iota, \quad \Gamma_o = \bigcup_{\iota=1}^{s} \Gamma_o^\iota, \quad \Gamma_c = \bigcup_{k,l} \Gamma_c^{kl},$$

$$\Gamma_c^{kl} = \overline{\Gamma}_c^k \cap \overline{\Gamma}_c^l, \quad k, l \in \{1, \ldots, s\}, \quad k < l.$$

Na sjednocení oblastí $\Omega = \Omega^1 \cup \Omega^2 \cup \cdots \cup \Omega^s$ budeme hledat vektorové pole posunutí $\mathbf{u} = (u_1, u_2)$, tenzorové pole malých deformací $e_{ij} = e_{ij}(\mathbf{u})$ a tenzor napětí

$\tau_{ij} = \tau_{ij}(\mathbf{u})$, $i, j = 1, 2$. Vztah mezi tenzorem napětí a tenzorem deformací je dán zobecněným Hookovým zákonem (při použití Einsteinovy sumační konvence)

$$\tau_{ij}(\mathbf{u}) = c_{ijkm} e_{km}(\mathbf{u}), \quad i, j = 1, 2, \tag{1.1}$$

a tenzor malých defomací je definován vztahem

$$e_{ij}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad i, j = 1, 2. \tag{1.2}$$

Ze symetrie tenzorů τ_{ij} a e_{ij} plyne pro elastické koeficienty prostředí c_{ijkm} z (1.1), $c_{ijkm} \in L^\infty(\Omega)$, symetrie v první a v druhé dvojici indexů a symetrie dvojic indexů ij, km

$$c_{ijkm} = c_{jikm} = c_{ijmk} = c_{kmij}. \tag{$*$}$$

Pro elastické koeficienty existuje konstanta c_0 taková, že

$$c_{ijkm}(x) e_{ij} e_{km} \geq c_0 e_{ij} e_{km}$$

platí pro všechny symetrické matice e_{ij} skoro všude v Ω.

V případě izotropních těles a rovinné napjatosti platí

$$c_{1112} = \lambda, \quad c_{1212} = \mu,$$

(platí i pro symetrické členy viz $(*)$) a

$$c_{1111} = c_{2222} = \lambda + 2\mu \quad \text{a} \quad c_{ijkm} = 0 \quad \text{v ostatních případech.}$$

V technické praxi se místo *Laméových koeficientů* λ a μ častěji používají *Youngův modul pružnosti* E a *Poissonova konstanta* ν. Pro převod (uvažujeme-li rovinné napětí) platí

$$\mu = \frac{E}{2(1+\nu)}, \qquad \lambda = \frac{E\nu}{1-\nu^2}.$$

Na $\Gamma_c = \bigcup_{k,l} \Gamma_c^{kl}$ definujeme normálovou a tečnou složku vektoru posunutí $\mathbf{u}$ a tenzoru napětí $\boldsymbol{\tau}$

$$u_n^k = u_i^k n_i^k, \quad u_n^l = u_i^l n_i^k = -u_i^l n_i^l, \quad \text{(nesčítáme přes } k, l),$$

$$u_t^k = u_1^k n_2^k - u_2^k n_1^k, \quad u_t^l = u_1^l n_2^k - u_2^l n_1^k,$$

$$\tau_n^k = \tau_{ij}^k n_i^k n_j^k, \quad \tau_t^k = (\tau_{ti}^k), \quad \tau_{ti}^k = \tau_{ij}^k n_j^k - \tau_n^k n_i^k, \quad \tau_t^{kl} \equiv \tau_t^k,$$

kde n_i^k, resp. n_i^l jsou složky jednotkové vnější normály k $\partial\Omega^k$, resp. $\partial\Omega^l$.

Tenzor napětí τ_{ij}^{ι} splňuje v každém tělese Ω^{ι} rovnice rovnováhy

$$\frac{\partial \tau_{ij}^{\iota}}{\partial x_j}(\mathbf{u}) + F_i^{\iota} = 0, \quad i = 1,2, \quad \iota = 1,\ldots,s, \tag{1.3}$$

kde F_i^{ι} jsou složky dané objemové síly.

Na částech hranice předpokládáme splnění následujících okrajových podmínek:

$$u_i = u_{0i} \quad \text{na } \Gamma_u, \quad i = 1,2, \tag{1.4}$$

kde u_{0i} jsou zadané složky vektoru posunutí,

$$\tau_{ij} n_j = P_i \quad \text{na } \Gamma_\tau, \quad i = 1,2, \tag{1.5}$$

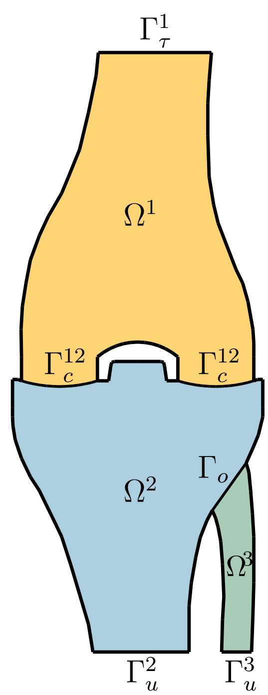
kde P_i jsou zadané složky povrchového zatížení,

$$u_n = 0, \quad \tau_t = 0 \quad \text{na } \Gamma_o, \tag{1.6}$$

představuje podmínku oboustranného kontaktu,

$$u_n^k - u_n^l \leq 0, \quad \tau_n^k = -\tau_n^l \leq 0, \quad (u_n^k - u_n^l)\tau_n^k = 0, \quad \text{na } \Gamma_c^{kl}, \tag{1.7}$$

představuje podmínku jednostranného kontaktu.



Obrázek 1.1: Kontaktní problém - model

Pokud předpokládáme úlohu bez tření, platí podmínka nulového tření

$$\tau_t^{kl} = 0 \quad \text{na } \Gamma_c^{kl}. \tag{1.8}$$

Pokud budeme uvažovat úlohu se zadaným třením, platí podmínky Trescova modelu tření

$$\left.\begin{array}{l} |\tau_t^{kl}| \le g^{kl}, \\ |\tau_t^{kl}| < g^{kl} \quad \Rightarrow \quad u_t^k - u_t^l = 0, \\ |\tau_t^{kl}| = g^{kl} \quad \Rightarrow \quad \exists\vartheta \ge 0, \quad u_t^k - u_t^l = -\vartheta\tau_t^{kl}, \end{array}\right\} \text{ na } \Gamma_c^{kl}, \tag{1.9}$$

kde $g^{kl} \in L^\infty(\Gamma_c^{kl})$ je daná mez tření.

Definice 1.1: Funkci $\mathbf{u}$ splňující (1.1)–(1.8) nazveme klasickým řešením kontaktního problému bez tření.

Definice 1.2: Funkci $\mathbf{u}$ splňující (1.1)–(1.7) a (1.9) nazveme klasickým řešením kontaktního problému se zadaným třením.

Poznámka 1.1: První dva články práce jsou věnovány řešení rovinného kontaktního problému více pružných těles. V článku [17⋆] je uvažována úloha bez tření, zatímco v článku [21⋆] úloha se zadaným třením.

1.2 Variační formulace

Při odvození variační formulace vyjdeme z variačního principu virtuálních posunutí. Definujeme prostor funkcí posunutí s konečnou energií

$$\mathcal{H}^1(\Omega) = \{\mathbf{v}|\mathbf{v} = (\mathbf{v}^1, \mathbf{v}^2, \dots, \mathbf{v}^s) \in [H^1(\Omega^1)]^2 \times [H^1(\Omega^2)]^2 \times \dots \times [H^1(\Omega^s)]^2\},$$

kde

$$H^1(\Omega^\iota) = \{w^\iota \in L_2(\Omega^\iota) : \frac{\partial w^\iota}{\partial x_j} \in L_2(\Omega^\iota),\ j = 1, 2\}$$

je standardní Sobolevův prostor s normou

$$\|w^\iota\|_1^2 = \int_{\Omega^\iota} \Big(w^{\iota 2} + \sum_{j=1}^{2} (\frac{\partial w^\iota}{\partial x_j})^2\Big)\,\mathrm{d}x.$$

Definujeme normu

$$\|\mathbf{v}\|^2 = \|\mathbf{v}\|^2_{\mathcal{H}^1(\Omega)} = \sum_{\iota=1}^{s} \|\mathbf{v}^\iota\|^2_{[H^1(\Omega^\iota)]^2} = \sum_{\iota=1}^{s}\sum_{i=1}^{2} \|v_i^\iota\|_1^2.$$

Dále definujeme seminormu

$$|\mathbf{v}|^2 = \sum_{\iota=1}^{s} \int\limits_{\Omega^\iota} e_{ij}(\mathbf{v}^\iota) e_{ij}(\mathbf{v}^\iota)\, \mathrm{d}x, \tag{1.10}$$

množinu virtuálních posunutí

$$\mathbb{V}_0 \equiv \{\mathbf{v} \in \mathcal{H}^1(\Omega) |\ \mathbf{v} = \mathbf{0} \text{ na } \Gamma_u,\ v_n = 0 \text{ na } \Gamma_o\}, \quad \mathbb{V} = \mathbf{u}_0 + \mathbb{V}_0 \tag{1.11}$$

a množinu přípustných posunutí

$$\mathbb{K} \equiv \{\mathbf{v} \in \mathbb{V} |\ v_n^k - v_n^l \leq 0 \text{ na } \Gamma_c^{kl}\}, \tag{1.12}$$

bilineární formu

$$a(\mathbf{u}, \mathbf{v}) = \sum_{\iota=1}^{s} a^\iota(\mathbf{u}^\iota, \mathbf{v}^\iota) = \sum_{\iota=1}^{s} \int\limits_{\Omega^\iota} c_{ijkm} e_{ij}(\mathbf{u}^\iota) e_{km}(\mathbf{v}^\iota)\, \mathrm{d}x \tag{1.13}$$

a funkcionál

$$L(\mathbf{v}) = \sum_{\iota=1}^{s} L^\iota(\mathbf{v}^\iota) = \sum_{\iota=1}^{s} \int\limits_{\Omega^\iota} F_i^\iota v_i^\iota\, \mathrm{d}x + \int\limits_{\Gamma_\tau^\iota} P_i^\iota v_i^\iota\, \mathrm{d}s \tag{1.14}$$

pro objemové síly $\mathbf{F}^\iota \in [L_2(\Omega^\iota)]^2$ a povrchové zatížení $\mathbf{P}^\iota \in [L_2(\Gamma_\tau^\iota)]^2$.

Pro odvození variační formulace kontaktního problému se zadaným třením zavedeme funkcionál

$$j(\mathbf{v}) = \sum_{k,l} \int\limits_{\Gamma_c^{kl}} g^{kl}\, |v_t^k - v_t^l|\, \mathrm{d}s. \tag{1.15}$$

Definice 1.3: Řekneme, že $\mathbf{u} \in \mathbb{K}$ je slabé řešení úlohy (1.1)–(1.8), tj. kontaktního problému bez tření, jestliže

$$a(\mathbf{u}, \mathbf{v} - \mathbf{u}) \geq L(\mathbf{v} - \mathbf{u}) \quad \forall \mathbf{v} \in \mathbb{K}. \tag{1.16}$$

Definice 1.4: Řekneme, že $\mathbf{u} \in \mathbb{K}$ je slabé řešení úlohy (1.1)–(1.7) a (1.9), tj. kontaktního problému se zadaným třením, jestliže

$$a(\mathbf{u}, \mathbf{v} - \mathbf{u}) + j(\mathbf{v}) - j(\mathbf{u}) \geq L(\mathbf{v} - \mathbf{u}) \quad \forall \mathbf{v} \in \mathbb{K}. \tag{1.17}$$

Poznámka 1.2: Zavedeme funkcionál potenciální energie

$$\mathcal{L}_0(\mathbf{v}) = \frac{1}{2} a(\mathbf{v}, \mathbf{v}) - L(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbb{K} \tag{1.18}$$

pro případ s nulovým třením a

$$\mathcal{L}_j(\mathbf{v}) = \frac{1}{2} a(\mathbf{v}, \mathbf{v}) + j(\mathbf{v}) - L(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbb{K} \tag{1.19}$$

pro případ se zadaným třením. Úloha (1.16) je ekvivalentní úloze minimalizace celkové potenciální energie, tj. úloze najít

$$\mathbf{u} = \arg\min_{\mathbf{v}\in\mathbb{K}} \mathcal{L}_0(\mathbf{v}). \tag{1.20}$$

Úloha (1.17) je ekvivalentní úloze minimalizace celkové potenciální energie, tj. úloze najít

$$\mathbf{u} = \arg\min_{\mathbf{v}\in\mathbb{K}} \mathcal{L}_j(\mathbf{v}). \tag{1.21}$$

Poznámka 1.3: V souboru prací jsou obsaženy dva články, ve kterých je použita obecnější formulace kontaktního problému. V článku [68⋆] je kontaktní problém formulován jako kvazi-sdružená (quasi-coupled) úloha v lineární termo-pružnosti a v článku [101⋆] je uvažován dynamický kontaktní problém pro viskopružná tělesa. Pro jednoduchost a názornost se ve výkladu omezíme na řešení problému (1.17), tj. statického kontaktního problému pružných těles v $\mathbb{R}^2$ se zadaným třením.

1.3 Existence a jednoznačnost

Okrajové úlohy lze rozdělit na dva hlavní typy. Pro jejich zavedení definujeme prostor posunutí a otočení tuhých těles

$$\mathcal{R} = \{\mathbf{v} \in \mathcal{H}^1(\Omega) \mid \quad |\mathbf{v}| = 0\}, \tag{1.22}$$

kde seminorma $|\mathbf{v}|$ je definována vztahem (1.10).

Jsou-li zadané části hranic Γ_u^ι a Γ_o^ι takové, že $\mathcal{R} \cap \mathbb{V} = \{0\}$, řekneme že jde o *koercivní* typ úlohy. Pokud $\mathcal{R} \cap \mathbb{V} \neq \{0\}$, jde o *semi-koercivní* typ úlohy.

O koercivní typ se jedná např. když

$$\mathrm{meas}_1 \Gamma_u^\iota > 0 \quad \forall \iota = 1, \ldots, s, \tag{1.23}$$

tj. každá oblast je pevně uchycena. Další jednoduchý případ koercivní úlohy, např. pro 2 oblasti ($s = 2$), je pokud Γ_o^1 a Γ_o^2 mají kladnou 1-rozměrnou míru a obě leží na dvou vzájemně kolmých přímkách.

Uvažujme nyní koercivní úlohu s nulovým třením.

Lemma 1.1: Nechť $\mathcal{R} \cap \mathbb{V} = \{0\}$. Potom platí Kornova nerovnost, tj. existuje kladná konstanta α_0 taková, že

$$|\mathbf{v}|^2 \geq \alpha_0 \|\mathbf{v}\|^2 \quad \forall \mathbf{v} \in \mathbb{V}. \tag{1.24}$$

Důkaz : viz např. [60] nebo [93].

Důsledek 1.1: Nechť $\mathcal{R} \cap \mathbb{V} = \{0\}$. Potom platí

$$a(\mathbf{v}, \mathbf{v})| \geq c_0 \alpha_0 \|\mathbf{v}\|^2 \quad \forall \mathbf{v} \in \mathbb{V}. \tag{1.25}$$

Důkaz : Na základě vlastností elastických koeficientů formulovaných v odstavci 1.1 a (1.13) platí pro všechna $\mathbf{v} \in \mathbb{V}$

$$a(\mathbf{v}, \mathbf{v}) = \sum_{\iota=1}^{s} a^{\iota}(\mathbf{v}^{\iota}, \mathbf{v}^{\iota}) \geq c_0 \sum_{\iota=1}^{s} \int_{\Omega^{\iota}} c_{ijkm} e_{ij}(\mathbf{v}^{\iota}) e_{km}(\mathbf{v}^{\iota}) \, \mathrm{d}x = c_0 |\mathbf{v}|^2. \tag{1.26}$$

Tvrzení je potom důsledkem (1.24).

Věta 1.1: Nechť $\mathcal{R} \cap \mathbb{V} = \{0\}$. Potom existuje právě jedno slabé řešení úlohy (1.1)–(1.8).

Důkaz : Úloha najít slabé řešení úlohy (1.1)–(1.8) je ekvivalentní úloze minimalizace celkové potenciální energie (viz poznámka 1.2), tj.

$$\mathbf{u} = \arg \min_{\mathbf{v} \in \mathbb{K}} \mathcal{L}_0(\mathbf{v}).$$

Díky důsledku 1.1 je funkcionál $\mathcal{L}_0(.)$ koercivní a ryze konvexní ve $\mathbb{V}$. Protože $\mathcal{L}_0$ je kvadratický a množina $\mathbb{K}$ konvexní, uzavřená ve $\mathbb{V}$, existuje jediné řešení úlohy (1.20).

Podívejme se na řešitelnost a jednoznačnost řešení semi-koercivní úlohy se zadaným třením. Označme

$$\mathcal{R}_{\mathbb{V}} = \mathcal{R} \cap \mathbb{V}_0 \quad \text{a} \quad \mathcal{R}_o = \{\mathbf{v} \in \mathcal{R}_{\mathbb{V}} \,|\, v_n^k - v_n^l = 0 \text{ na } \ \cup_{k,l} \Gamma_c^{kl}\}.$$

Věta 1.2 [21⋆, theorem 2.1]: Nechť $\mathcal{R}_o = \{0\}$, $\mathcal{R} \cap K \neq \{0\}$ a

$$L(\mathbf{v}) < j(\mathbf{v}) \quad \forall \mathbf{v} \in \mathcal{R} \cap K - \{0\}.$$

Potom existuje slabé řešení úlohy (1.1)–(1.7) a (1.9). Platí-li

$$|L(\mathbf{v})| > j(\mathbf{v}) \quad \forall \mathbf{v} \in \mathcal{R}_{\mathbb{V}} - \{0\},$$

je řešení jediné. Platí-li

$$|L(\mathbf{v})| \leq j(\mathbf{v}) \quad \forall \mathbf{v} \in \mathcal{R}_{\mathbb{V}},$$

potom pro každé dvě řešení $\mathbf{u}$, $\mathbf{u}^*$ platí

$$\mathbf{u}^* - \mathbf{u} \in \mathcal{R}_{\mathbb{V}} \quad \text{a} \quad L(\mathbf{u}^* - \mathbf{u}) = j(\mathbf{u}^*) - j(\mathbf{u}).$$

Důkaz : viz např. [65, theorem 4.1].

Poznámka 1.4: Následující podmínka je nutnou podmínkou existence slabého řešení úlohy (1.1)–(1.7) a (1.9) (důkaz viz např. [65, lemma 4.4]).

$$L(\mathbf{v}) \leq j(\mathbf{v}) \quad \forall \mathbf{v} \in \mathcal{R} \cap K.$$

2

Metoda rozkladu oblasti

Numerické řešení variačních úloh (například pomocí metody konečných prvků – viz kapitola 4) vede na řešení rozsáhlých soustav algebraických rovnic. Vzhledem k rozměru a číslu podmíněnosti matic těchto soustav může být komplikované použití přímých i iteračních metod. Na druhé straně k řešení stále rozměrnějších úloh mohou být efektivně využívány nové paralelní algoritmy.

Poznámka 2.1: Asi nejjednodušším přístupem k využití paralelních počítačů (avšak ne tak efektivním jako dále zmiňovaná metoda rozkladu oblasti) je cesta modifikace existujích sekvenčních kódů použitím modelu *SPMD* - Single Program Multiple Data. V tomto modelu se relativně jednoduchá činnost provádí nad objemově rozměrnými daty. Zpracovávaná množina dat (typicky homogenní pole prvků) se v tomto případě rozdělí na m částí. Vytvoří se k procesů ($k \leq m$) pracujících podle stejného programu (Single Program) a každý z těchto procesů samostatně zpracuje jednu nebo několik strukturně podobných (ale hodnotami různých) částí dat (Multiple Data). Typickým příkladem využití modelu *SPMD* je násobení matic, kdy každý prvek (nebo řádek či sloupec) výsledné matice lze počítat jedním procesem. Tento způsob paralelizace na úrovni maticových operací pro řešení kontaktního problému je popsán a numericky testován v [15].

2.1 Stručný přehled

Myšlenka metody rozkladu oblasti se poprvé objevila v roce 1869, kdy ji německý matematik H. Schwarz (viz [104]) poprvé použil pro získání analytického řešení Poissonovy rovnice na oblastech se složitější geometrií. Podstatou byla snaha využít znalosti konstrukce přesného řešení úlohy na jednoduchých oblastech (kruhu, čtverci) k určení řešení na oblasti, která je složena z těchto částí. K dalšímu rozvoji metody rozkladu oblastí dochází až v 80-tých letech minulého století v souvislosti s rychlým vývojem moderních superpočítačů.

Metoda rozkladu oblasti je založena na předpokladu, že danou oblast Ω, na které problém řešíme, rozdělíme na podoblasti Ω^ι, $\iota = 1, \ldots, s$, přičemž tyto se mohou nebo nemusí překrývat. Původní problém na Ω můžeme přeformulovat na jednotlivé podoblasti Ω^ι, přičemž tyto menší podproblémy jsou vzájemně vázány hodnotami neznámého řešení na překrývajících se částech, resp. rozhraních podoblastí.

Pro oblasti, které se překrývají, H. Schwarz popsal metodu, která v každém kroku střídavě počítá řešení na jednotlivých oblastech. V této tzv. alternující Schwarzově metodě se v každé iteraci přenáší informace o přibližném řešení na společných částech. Na základě alternující Schwarzovy metody byla odvozena celá třída Schwarzových metod - multiplikativní, aditivní, vícekrokové - (M. Dryja, O. Widlund [45, 46, 47]). Vzhledem ke své iterační podstatě se používají zejména k předpodmiňování. Dá se ukázat, že metody s překrýváním konvergují tím rychleji, čím více se oblasti překrývají. V limitním případě to odpovídá situaci, kdy jsou podoblasti rovny celé oblasti Ω a jako předpodmiňovač vlastně uvažujeme původní matici, která vznikne diskretizací rovnice pro oblast Ω. Prakticky se využívá překryvu v rozsahu 10 až 20 %.

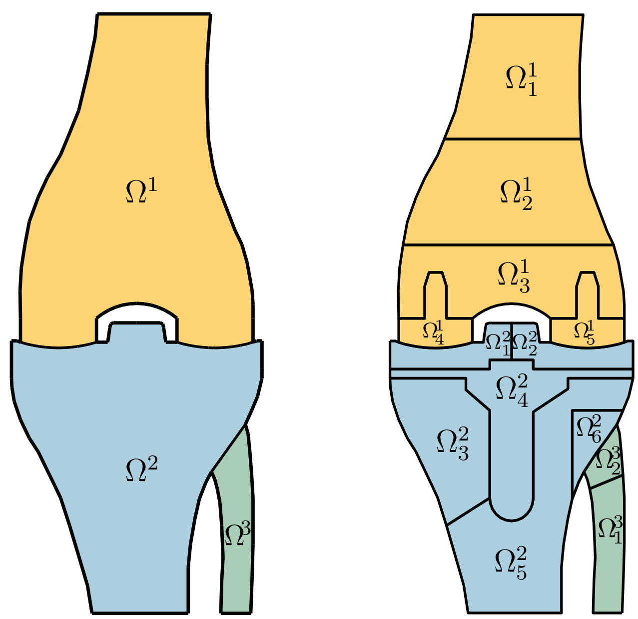
Druhou možností je uvažovat rozklad oblastí bez překrývání. Sousední oblasti mají potom společné pouze rozhraní. Tyto metody jsou označovány jako metody Schurova doplňku nebo substructuring methods, mají podstatu přímých metod a představují cestu k zefektivnění faktorizace matice soustavy. Pokud vyjdeme z primární formulace, získáme tzv. Balancing Domain Decomposition method (BDD) nebo Balancing Domain Decomposition by Constraints (BDDC) (J. Mandel, C. R. Dohrmann, P. Le Tallec [84, 85, 87, 91, 92]), kde je spojitost řešení na rozhraní zajištěna použitím stejné neznámé pro hodnotu řešení na všech sousedních podoblastech. V duálních metodách, které reprezentuje metoda Finite Element Tearing and Interconnecting (FETI) a její další modifikace (C. Farhat, F. X. Roux, J. Mandel [48, 49, 90]), je spojitost řešení na rozhraní zajištěna pomocí Lagrangeových multiplikátorů. Další přístup používá metoda Dual-Primal Finite Element Tearing and Interconnecting (FETI-DP) (C. Farhat [50, 92]), ve které je rovnost řešení na rozhraní podoblastí zaručena pomocí Lagrangeových multiplikátorů s výjimkou rohů podoblastí, které zůstávají v primární formulaci. Problematikou metody rozkladu oblastí se zabývá ještě celá řada dalších autorů, např. A. Quateroni, A. Valli [103], B. F. Smith [105], A. Klawonn [79], R. Tezaur [90, 108], M. Brezina [88], M. Vidrascu [86].

V článku [17⋆], který je součástí práce, je aplikována metoda rozkladu oblasti bez překrývání vycházející z primární formulace na kontaktní problém více těles bez tření ve 2D. V článku [21⋆], který je také součástí práce, je metoda zobecněna pro semi-koercivní kontaktní problémy se zadaným třením. Aplikací duálních metod FETI a jejích variant na kontaktní problémy se zabývá zejména Z. Dostál, D. Horák, F. G. Neto, S. A. Santos [35, 37, 38, 39, 40, 41, 42].

2.2 Formulace a popis metody

Uvažujme rovinnou úlohu jednostranného kontaktu s pružných těles, kterým odpovídají oblasti $\Omega^\iota \in \mathbb{R}^2$, $\iota = 1, \dots, s$, s Lipschitzovskou hranicí $\partial\Omega^\iota$ (viz odst. 1.1). Hledáme funkci $\mathbf{u} \in \mathbb{K}$, která je slabým řešením úlohy (1.1)–(1.7) a (1.9), tj. kontaktního problému se zadaným třením. Označme tento problém $(\mathcal{P})$. Každou oblast Ω^ι rozdělíme na $J(\iota)$ podoblastí (viz obr. 2.1), tj.

$$\bar{\Omega}^\iota = \bigcup_{i=1}^{J(\iota)} \bar{\Omega}_i^\iota, \quad \iota = 1, \dots, s.$$



Obrázek 2.1: Rozklad na podoblasti

Dále označme části rozhraní

$$\Gamma_i^\iota = \partial\Omega_i^\iota \setminus \partial\Omega^\iota, \quad \iota = 1, \dots, s, \quad i = 1, \dots, J(\iota).$$

Celé rozhraní je pak

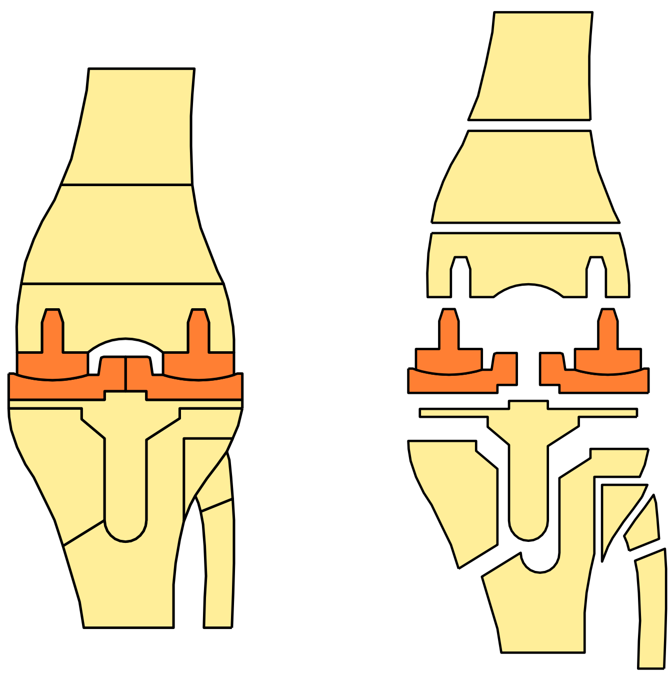
$$\Gamma = \bigcup_{\iota=1}^{s} \bigcup_{i=1}^{J(\iota)} \Gamma_i^\iota.$$

Označme

$$T^\iota = \{j \in \{1, \dots, J(\iota)\} : \quad \overline{\Gamma}_c \cap \overline{\Omega}_j^\iota = \emptyset\}, \qquad \iota = 1, \dots, s \tag{2.1}$$

množinu indexů podoblastí oblasti Ω^ι, které nejsou v kontaktu. Dále označme podoblasti v kontaktu

$$\Omega^{*j} = \bigcup_{[i,\iota]\in\vartheta} \Omega_i^\iota, \qquad \text{kde} \;\; \vartheta = \{[i,\iota]: \; \partial\Omega_i^\iota \cap \Gamma_c \neq \emptyset\}. \tag{2.2}$$



Obrázek 2.2: Rozdělení podoblastí podle typu na podoblasti bez kontaktu a dvojice podoblastí v kontaktu

Předpokládejme, že platí

$$\Gamma \cap \Gamma_c = \emptyset, \tag{2.3}$$

tj. uvažujeme pouze takové rozdělení na podoblasti, jejichž rozhraní neprotne hranici s předepsanou podmínkou jednostranného kontaktu. Potom pro operáror stop $\gamma: \; [H^1(\Omega_i^\iota)]^2 \to [L^2(\partial\Omega_i^\iota)]^2$ dostáváme pro množinu stop funkcí z $\mathbb{K}$, resp. $\mathbb{V}$ na rozhraní Γ:

$$\mathbb{V}_\Gamma = \gamma\mathbb{K}|_\Gamma = \gamma\mathbb{V}|_\Gamma. \tag{2.4}$$

Nechť $\gamma^{-1}: \; \mathbb{V}_\Gamma \to \mathbb{V}$ je libovolné lineární inverzní zobrazení takové, že

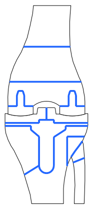
$$\gamma^{-1}\underline{\mathbf{v}} = 0 \quad \text{na} \; \cup_{k,l}\Gamma_c^{kl} \qquad \forall\underline{\mathbf{v}} \in \mathbb{V}_\Gamma. \tag{2.5}$$

Dále zavedeme tyto restrikce $\bar{R}_i^\iota: \mathbb{V}_\Gamma \to \Gamma_i^\iota$; $L_i^\iota: L^\iota \to \Omega_i^\iota$; $j_i^\iota: j^\iota \to \Gamma_c^{kl}$; $a_i^\iota(.,.): a^\iota(.,.) \to \Omega_i^\iota$; $V(\Omega_i^\iota) \to \Omega_i^\iota$ a prostor funkcí s nulovou stopou na Γ_i^ι

$$\mathbb{V}^0(\Omega_i^\iota) = \{\mathbf{v} \in \mathbb{V}| \; \mathbf{v} = 0 \;\; \text{na} \; \overline{(\cup_{\iota=1}^s \Omega^\iota) \setminus \Omega_i^\iota}\}.$$

Věta 2.1 předepisuje podmínky pro řešení kontaktního problému na jednotlivých částech oblasti Ω.

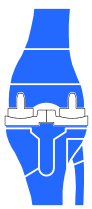
Věta 2.1 [21⋆, theorem 4.1]:
Funkce $\mathbf{u} \in \mathbb{K}$ je řešením globálního problému $(\mathcal{P})$, právě když platí:



1. její stopa $\underline{\mathbf{u}} = \gamma\mathbf{u}|_\Gamma$ na rozhraní Γ splňuje podmínku

$$\sum_{\iota=1}^{s}\sum_{i=1}^{J(\iota)}[a_i^\iota(\mathbf{u}_i^\iota(\underline{\mathbf{u}}), \gamma^{-1}\underline{\mathbf{w}}) - L_i^\iota(\gamma^{-1}\underline{\mathbf{w}})] = 0 \quad \forall\underline{\mathbf{w}} \in V_\Gamma,\ \underline{\mathbf{u}} \in V_\Gamma, \tag{2.6}$$

2. její restrikce $\mathbf{u}_i^\iota(\mathbf{u}) \equiv \mathbf{u}|_{\Omega_i^\iota}$ splňují podmínky

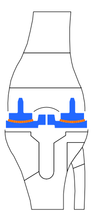


a) pro $i \in T^\iota,\ \iota = 1, \dots, s$ (tj. pro podoblasti bez kontaktu)

$$a_i^\iota(\mathbf{u}_i^\iota(\underline{\mathbf{u}}), \varphi_i^\iota) = L_i^\iota(\varphi_i^\iota) \tag{2.7}$$

$\forall\varphi_i^\iota \in V^0(\Omega_i^\iota),\ \mathbf{u}_i^\iota(\underline{\mathbf{u}}) \in V(\Omega_i^\iota),\ \gamma\mathbf{u}_i^\iota(\underline{\mathbf{u}})|_{\Gamma_i^\iota} = \bar{R}_i^\iota\underline{\mathbf{u}},$

b) pro $[i, \iota] \in \vartheta$ (tj. pro dvojice podoblastí v kontaktu)



$$\sum_{[i,\iota]\in\vartheta} a_i^\iota(\mathbf{u}_i^\iota(\underline{\mathbf{u}}), \varphi_i^\iota) + j^\iota(\mathbf{u}_i^\iota(\underline{\mathbf{u}}) + \varphi_i^\iota) - j^\iota(\mathbf{u}_i^\iota(\underline{\mathbf{u}})) \geq \sum_{[i,\iota]\in\vartheta} L_i^\iota(\varphi_i^\iota) \tag{2.8}$$

$\forall\varphi \equiv (\varphi_i^\iota,\ [i, \iota] \in \vartheta),\ \varphi_i^\iota \in V^0(\Omega_i^\iota),$ takové, že

$$\mathbf{u} + \varphi \in K, \quad \gamma\mathbf{u}_i^\iota(\underline{\mathbf{u}})|_{\Gamma_i^\iota} = \bar{R}_i^\iota\underline{\mathbf{u}} \text{ pro } [i, \iota] \in \vartheta \tag{2.9}$$

Důkaz : viz [21⋆].

Pomocí věty 2.1 jsme původní nelineární problém $(\mathcal{P})$ rozložili na více menších problémů, z nichž nelineárními zůstaly pouze problémy pro dvojice podoblastí v kontaktu, ostatní jsou lineární (viz obr. 2.2). Pomocí metody Schurova doplňku úlohu řešíme pro část řešení na rozhraní Γ (2.6). V článku [21⋆] v odstavci 4.3 jsou definovány příslušné lokální a globální operátory Schurova doplňku a dokázáno lemma 4.1, pomocí něhož můžeme rovnici (2.6) zapsat ve tvaru [21⋆, (4.32)]

$$\mathcal{S}_0\underline{\mathbf{U}} + \mathcal{S}_{CON}\underline{\mathbf{U}} = \mathbf{F}, \tag{2.10}$$

kde $\mathcal{S}_0$ je lineární operátor Schurova doplňku odpovídající podoblastem bez kontaktu a $\mathcal{S}_{CON}$ je nelineární operátor Schurova doplňku odpovídající dvojicím podoblastí v kontaktu. Z důvodu nelinearity $\mathcal{S}_{CON}$ řešíme rovnici (2.10) metodou postupných aproximací

$$\mathcal{S}_0\underline{\mathbf{U}}^k = \mathbf{F} - \mathcal{S}_{CON}\underline{\mathbf{U}}^{k-1}, \quad k = 1, 2, \ldots, \tag{2.11}$$

přičemž počáteční aproximaci $\underline{\mathbf{U}}^0$ určíme jako řešení linearizovaného problému $(\mathcal{P})$, kde původní podmínku jednostranného kontaktu (1.7) na Γ_c^{kl} nahradíme podmínkou oboustranného kontaktu $u_n^k - u_n^l = 0$ a dále uvažujeme nulové tření.

Linerizovaný pomocný problém i každou iteraci metody postupných aproximací řešíme pomocí metody předpodmíněných sdružených gradientů [21⋆, algorithm PCG1 - str. 285, algorithm PCG2 - str. 289]. Předpodmínění typu Neumann-Neumann je popsáno v 5. odstavci článku [21⋆]. Konvegence metody postupných aproximací je dokázána v 7. odstavci článku [21⋆] (viz lemma 7.1, lemma 7.2, theorem 7.1, corollary 7.1).

Poznámka 2.2: V článku [17⋆] je popsána metoda rozkladu oblasti bez překrývání pro jednodušší případ kontaktního problému bez tření.

3

Problematika vstupních dat

Při řešení konkrétního problému je nutné nejprve znát příslušná vstupní data, tj. musíme mít informaci o geometrii těles, o materiálových koeficientech, o silovém zatížení, případně o velikosti zadaného tření. Články v souboru prací obsahují výsledky některých biomechanických a geomechanických modelů. V 5. kapitole je uveden přehled výsledků pro další modely, které nejsou obsaženy v souboru prací. V oblasti biomechaniky se jedná např. o model zatíženého zdravého kolenního kloubu a kolenního kloubu s implantovanou totální náhradou nebo model čelistního kloubu. V oblasti geomechaniky jde např. o model tunelu v masivu, který je narušen geologickým zlomem.

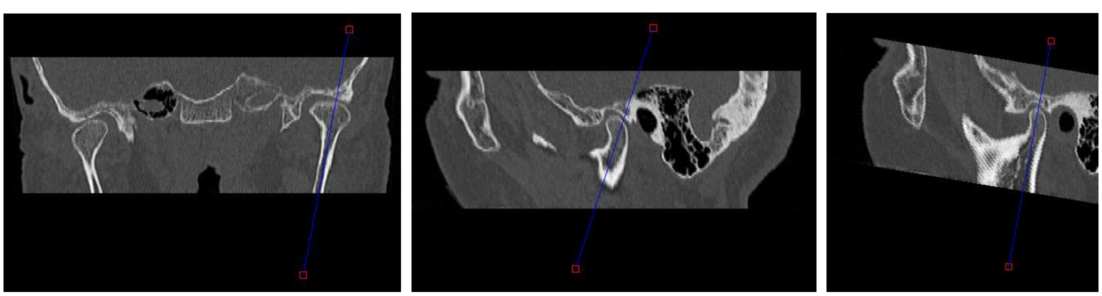
Materiálové koeficienty lidských tkání (kortikální a spongiózní kosti, kostní dřeně, vaziva, menisku, chrupavky) lze nalézt v odborné literatuře (viz např. [1, 10, 78]). Obvykle jsou pro každý typ lidské tkáně předepsány rozsahy, ve kterých by se hodnoty materiálových parametrů měly pohybovat. Pro modely obsahující umělou náhradu jsou mechanické parametry použitých materiálů dány výrobcem (viz např. [120]). Obdobně jsou dostupné i materiálové parametry hornin (např. [119]). Zadávané hodnoty silového zatížení lze v uvažovaných modelech odhadnout, např. v modelu zatíženého kolena předepisujeme působící sílu, která je odvozena z průměrné hmotnosti člověka a z předpokladu, že stojí pouze na jedné noze. U modelu tunelu uvažujeme takové silové působení, které odpovídá množství materiálu nad studovanou oblastí. V dalším odstavci je krátce popsán postup získávání geometrie a následné generování sítě (viz [27]), která je vstupem do metody konečných prvků (viz 4. kapitola) použité pro řešení problému.

3.1 Příprava sítě pro biomechanické modely

Uvažujeme-li kontaktní problém ve dvourozměrném prostoru, je situace relativně jednoduchá. Pro získání geometrie lze využít buď rentgenový snímek (viz obr. 3.1) nebo, pokud např. potřebujeme vytvořit model v řezu obecnou rovinou, lze použít medicínská obrazová data z CT (Computer Tomography) nebo MRI (Magnetic Resonance Imaging). Na obr. 3.2 jsou zobrazeny řezy čelistním (temporomandibulárním) kloubem získané z MRI - (a) přední pohled, (b) boční pohled a (c) je řez použitý k sestavení modelu. Tento pohled je vybrán z řezů rovinami obsahující vyznačenou osu. Pro zpracování obrazových dat z CT nebo MRI je nutný speciální software, který je z velké většiny komerční. Některé softwarové firmy, jako

Obrázek 3.1: Rentgenový snímek kolenního kloubu s náhradou MEDIN univerzál

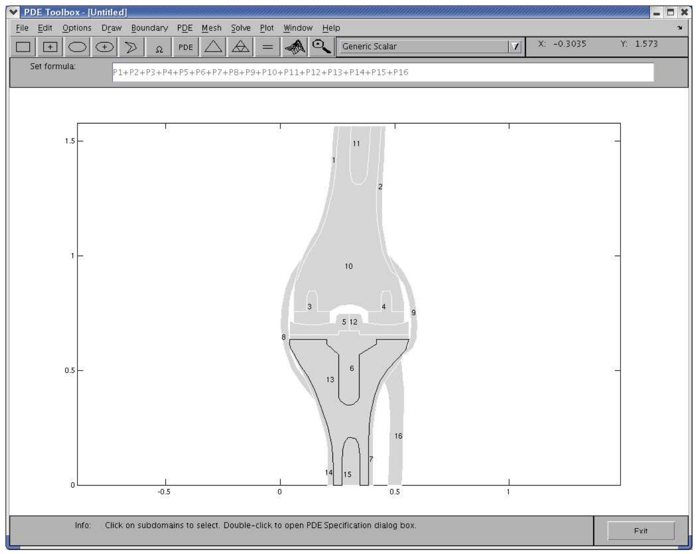
např. Mercury Computer Systems, Inc. (viz [121]), uvolňují alespoň trialové verze produktů, které lze pro zpracování medcínských dat použít, jedná se např. o software Amira (viz [122]). Pokud nejsou k dispozici vlastní medicínská data, lze jako zdroj dat využít snímky např. z The Visible Human Project (viz [123]).



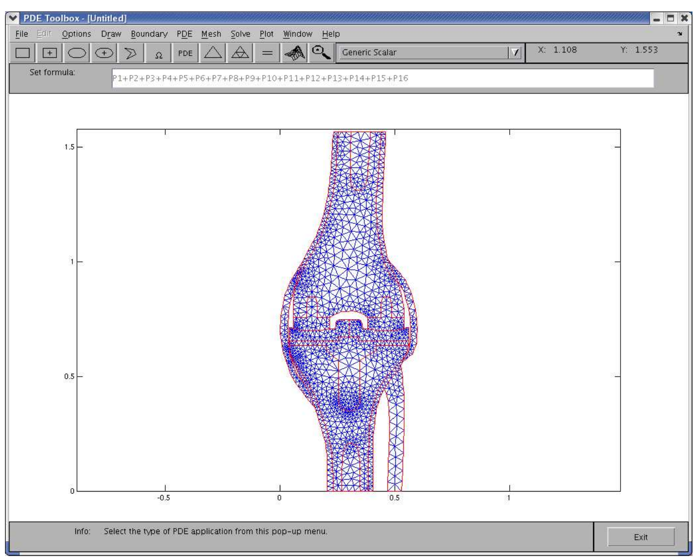
(a) (b) (c)

Obrázek 3.2: Zobrazení čelistního kloubu v různých řezech s využitím dat z MR

Pokud máme dánu geometrii modelu, je třeba vytvořit konečněprvkovou síť. Pro dvourozměrné problémy existuje řada volně dostupných generátorů nebo lze např. použít Partial Differential Equation (PDE) Toolbox v systému MATLAB od společnosti The MathWorks, Inc. (viz [124]). Na obr. 3.3 a 3.4 jsou zobrazeny zadané oblasti a vygenerovaná síť pro model kolenního kloubu s náhradou.



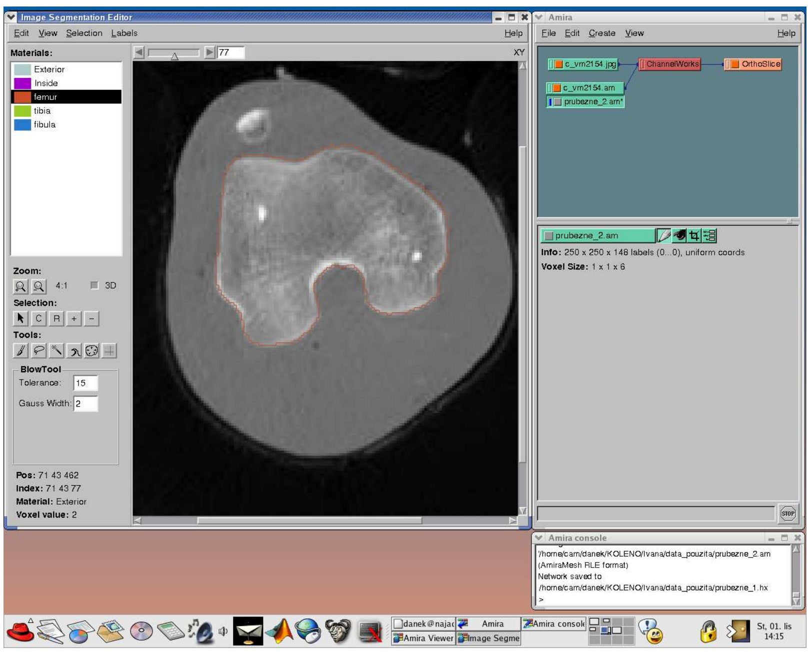
Obrázek 3.3: Oblasti zadané do PDE Toolboxu v systému MATLAB



Obrázek 3.4: Použití PDE Toolboxu v systému MATLAB pro generování sítě

V případě vytváření 3D modelů je situace o mnoho složitější. Problematika tvorby třírozměrných modelů lidských tkání na základě dat z CT nebo MRI pomocí již zmiňovaného sw Amira je popsána v práci [27]. Postup zjednodušeně řečeno spočívá

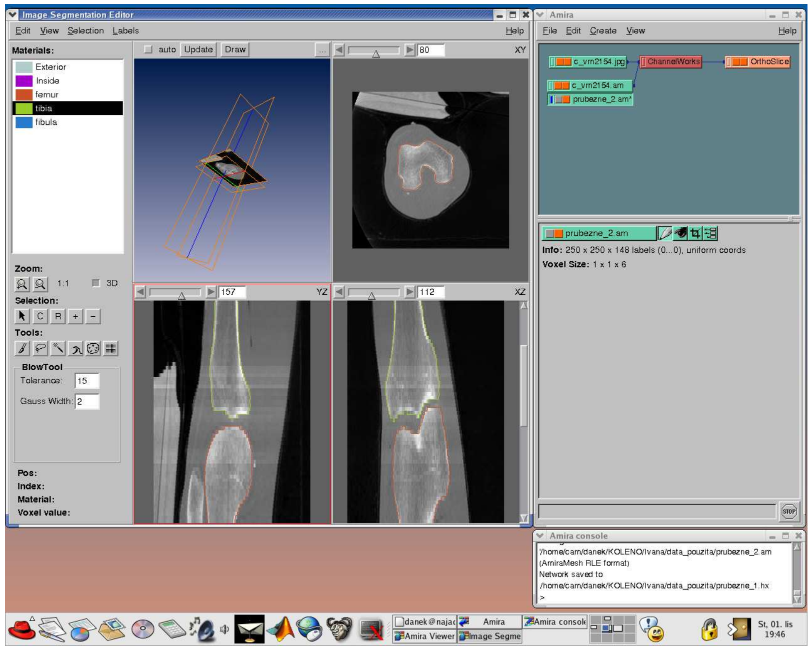
- v načtení souboru obrazových dat - řezů zkoumanou částí těla, jejichž počet může být v řádech stovek v závislosti na velikosti zkoumané oblasti,
- v následném označení jednotlivých typů tkání v každém řezu (viz obr. 3.5) - tato činnost lze usnadnit pomocí řady pomocných nástrojů,



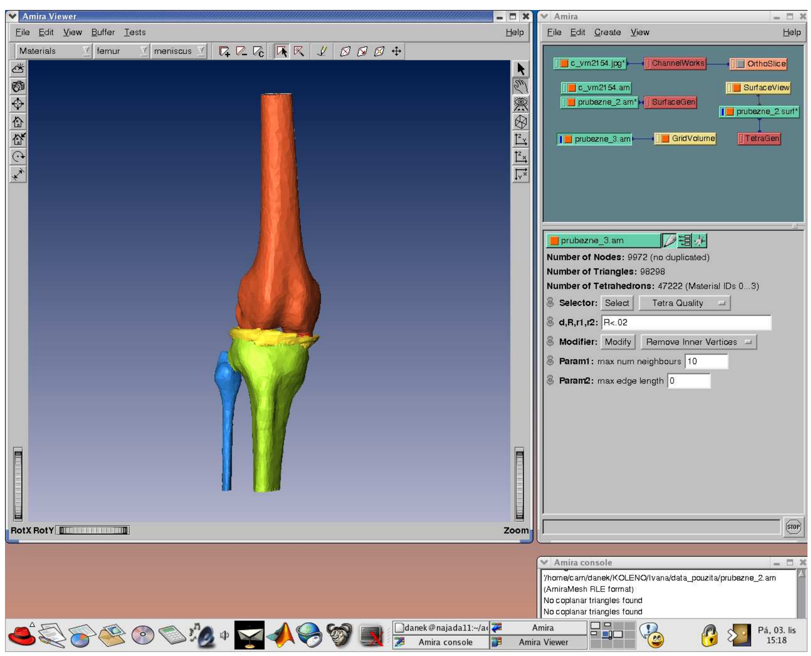
Obrázek 3.5: Označování různých tkání v řezech (sw Amira)

- ve vytvoření povrchového 3D modelu tvořeného trojúhelníky - počet povrchových trojúhelníků je většinou třeba redukovat a dále je důležité zajistit návaznost povrchové sítě sousedících oblastí,
- ve vytvoření objemové sítě tetrahedronů a
- v exportu dat ve vhodném formátu.

Obr. 3.6 ilustruje možnosti vizualizace při zpracovávání dat a obr. 3.7 ukazuje výsledný 3D model kolenního kloubu.

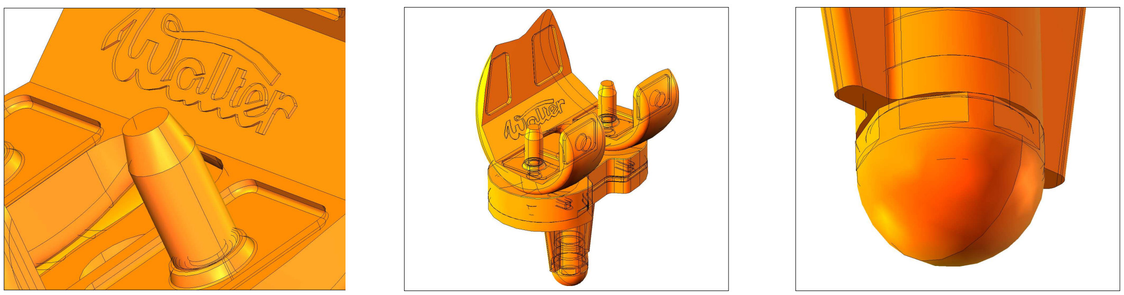


Obrázek 3.6: Vizualizace zpracovávaných dat (sw Amira)

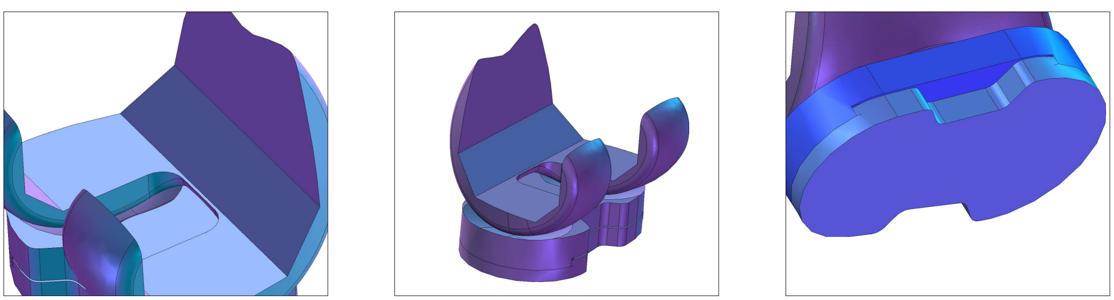


Obrázek 3.7: 3D model kolenního kloubu (sw Amira)

Při vytváření modelů obsahující např. totální náhradu kloubu, můžeme samozřejmě postupovat již zmíněným způsobem (pokud se jedná o pacienta s již implantovanou náhradou), avšak v důsledku méně přesně zadané geometrie náhrady může být výsledný model do určité míry zkreslený. Jsou-li od výrobce k dispozici data s CAD (Computer Aided Design) popisem geometrie náhrady, získáme přesnější model importováním těchto dat do modelu kloubu bez náhrady. Tento způsob již vyžaduje určitou zdatnost a zkušenosti v práci s CAD systémy. Originální soubory s CAD daty popisujícími náhradu obsahují řadu jemných detailů (viz obr. 3.8), které jsou důležité pro výrobu, ovšem pro potřebu vytváření konečněprvkové sítě jsou nevhodné, původní geometrii je tudíž nutné zjednodušit (viz obr. 3.9).



Obrázek 3.8: Originální geometrie náhrady kolenního kloubu Walter Modulár



Obrázek 3.9: Zjednodušená geometrie náhrady kolenního kloubu Walter Modulár

Odkazy na některé volně dostupné programy zpracovávající CAD data lze nalézt na stránkách [125].

Poznámka 3.1: Na rozdíl od biomechanických modelů je rozměr geomechanických modelů mnohonásobně větší, což může znamenat problém ve znalosti vstupních dat, zejména geometrie a materiálových parametrů. Pokud pracujeme s nejistými vstupními daty, lze problém řešit metodou nejhošího scénáře. Situaci lze ilustrovat na příkladu ukládání jaderného odpadu v podzemním tunelu, který je v oblasti geologického zlomu. Cílem je zaručit splnění předepsaných bezpečnostních limitů, které mohou být dány např. pro střední hodnotu posunutí, intenzity smykových napětí, hlavních napětí, atd.

3.2 Nejistá vstupní data a metoda nejhoršího scénáře

První publikací, která pojednává o metodě nejhoršího scénáře (worst scenario method), je zřejmě článek [9] z roku 1940 od B. V. Bulgakova. Po padesáti letech se objevuje monografie [2] autorů Y. Ben-Haima a I. Elishakoffa, ve které je navržena metoda nejhoršího scénáře pro konvexní množinu přípustných dat. Kontaktní problém není jedinou úlohou, na kterou byla aplikována metoda nejhoršího scénáře. V posledních přibližně deseti letech se jednalo např. o lineární eliptické PDR s nejistými koeficienty, kvazi-lineární eliptické okrajové úlohy, parabolické problémy nebo úlohy z teorie plasticity, atd. Stručný přehled lze nalézt v [70].

Metoda nejhoršího scénáře pro kvazi-sdruženou (quasi-coupled) úlohu v termo-pružnosti je studována v článku [68⋆], který je součástí práce. V článku je ve 2. odstavci uvedena formulace kontaktního problému v termo-pružnosti a ve 4. odstavci je studována metoda nejhoršího scénáře. Nejprve jsou zavedeny množiny přípustných dat a poté je uvedena věta [68⋆, proposition 4.2] o existenci a jednoznačnosti slabého řešení kontaktního problému pro každý prvek z množiny přípustných dat (důkaz viz [67]). „Spolehlivým řešením“ úlohy definujeme nejhorší mezi všemi možnými řešeními, kde stupeň nespolehlivosti je měřen jistým funkcionálem, kritériem (criterion functional), který závisí na řešení studované úlohy. V článku je ukázáno 6 možností volby tohoto kritéria (viz [68⋆, (4.29)–(4.34)]. Hlavním výsledkem je [68⋆, theorem 4.6], ve kterém je dokázána existence řešení problému nejhoršího scénáře (worst scenario problem) [68⋆, (4.35)], resp. [68⋆, (4.36)].

V následujícím vyložíme metodu nejhoršího scénáře pro řešení problému (1.1)–(1.7) a (1.9), který byl formulován v 1. kapitole. Za nejistá data budeme považovat materiálové parametry λ a μ, objemové síly $\mathbf{F}$, povrchové zatížení $\mathbf{P}$ a hodnoty zadaného tření g. Podobně jako v odst. 1.2 při odvození variační formulace kontaktního problému zavedeme bilineární formu

$$a(A;\mathbf{u},\mathbf{v}) = \sum_{\iota=1}^{s} \int_{\Omega^\iota} \Big(\lambda \operatorname{div} \mathbf{u}^\iota \operatorname{div} \mathbf{v}^\iota + 2\mu\, e_{ij}(\mathbf{u}^\iota) e_{ij}(\mathbf{v}^\iota)\Big)\, \mathrm{d}x, \tag{3.1}$$

kde $\operatorname{div} \mathbf{u} = \frac{\partial u_i}{\partial x_i}$, tenzor malých deformací $e_{ij}(\mathbf{u})$ je dán (při použití Einsteinovy sumační konvence) vztahem (1.2) a vstupní data $A = \{\lambda, \mu, \mathbf{F}, \mathbf{P}, g\}$ jsou nejistá a jsou z množiny přípustných dat

$$A \in U_{\mathrm{ad}} \;\Leftrightarrow\; \lambda \in U_{\mathrm{ad}}^{\lambda}, \mu \in U_{\mathrm{ad}}^{\mu}, \mathbf{F} \in U_{\mathrm{ad}}^{\mathbf{F}}, \mathbf{P} \in U_{\mathrm{ad}}^{\mathbf{P}}, g \in U_{\mathrm{ad}}^{g}.$$

Dále předpokládejme, že všechny uvažované části Ω^ι jsou po částech homogenní a takové, že existuje dělení oblasti $\overline{\Omega}^\iota$ pro které platí

$$\begin{aligned} \overline{\Omega}^\iota &= \textstyle\bigcup_{i=1}^{J(\iota)} \overline{\Omega}_i^\iota, \quad \Omega_i^\iota \cap \Omega_j^\iota = \emptyset \quad \text{pro} \ \ i \neq j,\ \iota = 1, \dots, s. \\ \overline{\Gamma}_c^{kl} &= \textstyle\bigcup_{q=1}^{Q_{kl}} \overline{\Gamma}_q^{kl}, \quad \Gamma_p^{kl} \cap \Gamma_q^{kl} = \emptyset \quad \text{pro} \ \ p \neq q,\ \forall k, l. \end{aligned} \tag{3.2}$$

Definujeme množiny přípustných dat

$$U_{\text{ad}}^\lambda = \{\lambda \in L^\infty(\Omega) :\ \lambda_{\min}^{\iota,i} \leq \lambda|_{\Omega_i^\iota} = \text{const} \leq \lambda_{\max}^{\iota,i},\ 1 \leq i \leq J(\iota),\ 1 \leq \iota \leq s\}, \tag{3.3}$$

$$U_{\text{ad}}^\mu = \{\mu \in L^\infty(\Omega) :\ \mu_{\min}^{\iota,i} \leq \mu|_{\Omega_i^\iota} = \text{const} \leq \mu_{\max}^{\iota,i},\ 1 \leq i \leq J(\iota),\ 1 \leq \iota \leq s\}, \tag{3.4}$$

kde $0 \leq \lambda_{\min}^{\iota,i} < \lambda_{\max}^{\iota,i},\ 0 < \mu_0 \leq \mu_{\min}^{\iota,i} < \mu_{\max}^{\iota,i},\ \ 1 \leq i \leq J(\iota),\ 1 \leq \iota \leq s$, jsou dané konstanty.

Dále pro nejistá vstupní data A zavedeme funkcionály

$$L(A; \mathbf{v}) = \sum_{\iota=1}^{s} \int_{\Omega^\iota} F_i^\iota v_i^\iota \, \mathrm{d}x + \int_{\Gamma_\tau^\iota} P_i^\iota v_i^\iota \, \mathrm{d}s \tag{3.5}$$

a

$$j(A; \mathbf{v}) = \sum_{k,l} \int_{\Gamma_c^{kl}} g^{kl} \, |v_t^k - v_t^l| \, \mathrm{d}s, \tag{3.6}$$

kde $\mathbf{F} \in U_{\text{ad}}^{\mathbf{F}}$, $\mathbf{P} \in U_{\text{ad}}^{\mathbf{P}}$, $g \in U_{\text{ad}}^g$, $v_t = v_i t_i$ je tečná složka posunutí,

$$\begin{aligned} &U_{\text{ad}}^{\mathbf{F}} = \{\mathbf{F} = [F_1, F_2] \in [L^\infty(\Omega)]^2 : \\ &\quad F_{j,\min}^{\iota,i} \leq F_j|_{\Omega_i^\iota} = \text{const} \leq F_{j,\max}^{\iota,i},\ j = 1, 2,\ \ 1 \leq i \leq J(\iota),\ 1 \leq \iota \leq s\}, \end{aligned} \tag{3.7}$$

$$\begin{aligned} &U_{\text{ad}}^{\mathbf{P}} = \{\mathbf{P} = [P_1, P_2] \in [L^\infty(\Gamma_\tau)]^2 : \\ &\quad P_{j,\min}^{\iota} \leq P_j|_{\Gamma_\tau^\iota} = \text{const} \leq P_{j,\max}^{\iota},\ j = 1, 2,\ 1 \leq \iota \leq s\}, \end{aligned} \tag{3.8}$$

$$\begin{aligned} &U_{\text{ad}}^g = \{g \in L^\infty(\Gamma_c) :\ g|_{\overline{\Gamma}_q^{kl}} \in C^{(0),1}(\overline{\Gamma}_q^{kl}); \\ &\qquad 0 \leq g(s) \leq g_{\max}^{q,kl}, |\mathrm{d}g/\mathrm{d}s| \leq C_g^{kl} \ \ \text{s.v. na}\ \Gamma_q^{kl},\ 1 \leq q \leq Q_{kl}, \forall k, l\}, \end{aligned} \tag{3.9}$$

kde $F_{j,\min}^{\iota,i} \leq F_{j,\max}^{\iota,i}$, $P_{j,\min}^{\iota} \leq P_{j,\max}^{\iota}$, $g_{\max}^{q,kl}$ a C_g^{kl} jsou dané konstanty a $C^{(0),1}$ značí prostor lipschitzovsky spojitých funkcí.

Pro daná vstupní data $A \in U_{\text{ad}} \equiv U_{\text{ad}}^\lambda \times U_{\text{ad}}^\mu \times U_{\text{ad}}^{\mathbf{F}} \times U_{\text{ad}}^{\mathbf{P}} \times U_{\text{ad}}^g$ dostáváme problém $(\mathcal{P}_A)$: najít $\mathbf{u}(A) \in \mathbb{K}$ tak, aby

$$a(A; \mathbf{u}(A), \mathbf{v} - \mathbf{u}(A)) + j(A; \mathbf{v}) - j(A; \mathbf{u}(A)) \geq L(A; \mathbf{v} - \mathbf{u}(A)) \quad \forall \mathbf{v} \in \mathbb{K}. \tag{3.10}$$

Platí následující odhady (viz např. [64, lemma 2.1], [67, lemma 1], [68⋆, lemma 4.1]).

Lemma 3.1: Existují kladné konstanty c_i, $i = 0, 1, 2, 3$ nezávislé na $A \in U_{\mathrm{ad}}$, takové, že

$$a(A; \mathbf{v}, \mathbf{v}) \geq c_0 \|\mathbf{v}\|_1^2 \quad \forall \mathbf{v} \in \mathbb{V}_0, \tag{3.11}$$

$$|a(A; \mathbf{v}, \mathbf{w})| \leq c_1 \|\mathbf{v}\|_1 \, \|\mathbf{w}\|_1 \quad \forall \mathbf{v}, \mathbf{w} \in \mathcal{H}^1(\Omega), \tag{3.12}$$

$$|L(A; \mathbf{v})| \leq c_2 (\|\mathbf{v}\|_{0,\Omega} + \|\mathbf{v}\|_{0,\Gamma_\tau}) \quad \forall \mathbf{v} \in \mathcal{H}^1(\Omega), \tag{3.13}$$

$$|j(A; \mathbf{v}) - j(A; \mathbf{w})| \leq c_3 \|\mathbf{v} - \mathbf{w}\|_{0,\Gamma_c} \quad \forall \mathbf{v}, \mathbf{w} \in \mathcal{H}^1(\Omega). \tag{3.14}$$

Na základě těchto odhadů lze dokázat existenci a jednoznačnost řešení problému $(\mathcal{P}_A)$:

Věta 3.1: Existuje jediné řešení $\mathbf{u}(A) \in \mathbb{K}$ úlohy (3.10) pro každé $A \in U_{\mathrm{ad}}$.
Důkaz : analogický jako v [64, proposition 2.1] nebo [67, proposition 2].

Mezi všemi možnými řešeními problému $(\mathcal{P}_A)$ vybereme to „nejhorší“, kde nespolehlivost měříme jistým funkcionálem, který závisí na řešení $\mathbf{u}(A)$ studované úlohy. Ukážeme několik možností volby funkcionálu kritéria.

Nechť $G_k \subset \Omega$, $k = 1, \dots, \overline{k}$, jsou například (malé) podoblasti přilehlé k hranicím $\partial\Omega^\iota$. Potom jedním příkladem funkcionálu kritéria může být

$$\Phi_1(A; \mathbf{u}) = \max_{1 \leq k \leq \overline{k}} \Big[(\mathrm{meas}_2\, G_k)^{-1} \int_{G_k} u_i n_i(X_k)\, \mathrm{d}x \Big], \tag{3.15}$$

kde $\mathbf{n}(X_k)$ je jednotková vnější normála v pevném bodě $X_k \in \partial\Omega^\iota \cap \partial G_k$ (je-li $G_k \subset \Omega^\iota$) k hranici $\partial\Omega^\iota$. Druhým příkladem funkcionálu kritéria je

$$\Phi_2(A; \mathbf{u}) = \max_{1 \leq k \leq \overline{k}} \Big[(\mathrm{meas}_1\, G'_k)^{-1} \int_{G'_k} u_i n_i(X_k)\, \mathrm{d}s \Big], \tag{3.16}$$

kde $G'_k \subset \bigcup_{\iota=1}^{s} \partial\Omega^\iota \setminus \Gamma_u$. Další možností, jak volit funkcionál kritéria, je

$$\Phi_3(A; \mathbf{u}) = \max_{1 \leq k \leq \overline{k}} \Big[(\mathrm{meas}_2\, G_k)^{-1} \int_{G_k} I_2^2(\boldsymbol{\tau}(A; \mathbf{u}))\, \mathrm{d}x \Big], \tag{3.17}$$

kde $I_2(\boldsymbol{\tau}) = (\tau_{ij}^D \tau_{ij}^D)^{\frac{1}{2}}$ je intenzita smykových napětí a $\tau_{ij}^D = \tau_{ij} - \frac{1}{3}\tau_{kk}\delta_{ij}$ je deviátor napětí, tj.

$$I_2^2(\boldsymbol{\tau}) = \frac{2}{3}[\tau_{11}^2 + \tau_{22}^2 + \tau_{33}^2 - (\tau_{11}\tau_{22} + \tau_{11}\tau_{33} + \tau_{22}\tau_{33}) + 3\tau_{12}^2],$$

$$\tau_{ij}(A;u) = \lambda\delta_{ij}\,\mathrm{div}\,\mathbf{u} + 2\mu e_{ij}(\mathbf{u}), \quad i,j = 1,2,$$
$$\tau_{33}(A;u) = \lambda\,\mathrm{div}\,\mathbf{u}.$$

Úlohy nejhoršího scénáře (worst scenario problems) potom formulujeme takto:

hledáme

$$A^{0i} = \arg\max_{A\in U_{\mathrm{ad}}} \Phi_i(A;\mathbf{u}(A)), \quad i = 1,2,3, \tag{3.18}$$

kde $\mathbf{u}(A)$ je řešení problému $(\mathcal{P}_A)$, tj. (3.10).

Abychom dokázali řešitelnost problému (3.18), musíme ukázat spojitost zobrazení $A \mapsto \mathbf{u}(A)$ definovaného ve větě 3.1 na množině U_{ad}. Definujeme prostor

$$U = [\mathbb{R}^{\overline{J}}]^6 \times \sqcap_{k,l} \sqcap_{q=1}^{Q_{kl}} C(\overline{\Gamma}_q^{kl}), \tag{3.19}$$

kde $\overline{J} = \sum_{\iota=1}^{s} J(\iota)$, s normou

$$\|A\|_U = \|\{\lambda,\mu,\mathbf{F},\mathbf{P},g\}\|_U = \|\lambda\|_{0,\infty} + \|\mu\|_{0,\infty} + \sum_{i=1}^{2}\|F_i\|_{0,\infty} + \sum_{i=1}^{2}\|P_i\|_{0,\infty} + \|g\|_{0,\infty}.$$

Lemma 3.2: Nechť $A_n \in U_{\mathrm{ad}}$, $A_n \to A$ v U a $\mathbf{u}_n \rightharpoonup \mathbf{u}$ slabě v $\mathcal{H}^1(\Omega)$. Potom

$$a(A_n;\mathbf{u}_n,\mathbf{v}) \to a(A;\mathbf{u},\mathbf{v}) \quad \forall \mathbf{v} \in \mathcal{H}^1(\Omega), \tag{3.20}$$
$$L(A_n;\mathbf{u}_n) \to L(A;\mathbf{u}), \tag{3.21}$$
$$j(A_n;\mathbf{u}_n) \to j(A;\mathbf{u}). \tag{3.22}$$

Důkaz : analogický důkazu [64, lemma 3.1] nebo [67, lemma 2].

Věta 3.2: Nechť $A_n \in U_{\mathrm{ad}}$, $A_n \to A$ v U pro $n \to \infty$. Potom

$$\mathbf{u}(A_n) \to \mathbf{u}(A) \ \text{ v } \mathcal{H}^1(\Omega). \tag{3.23}$$

Důkaz : analogický důkazu [64, proposition 3.1] nebo [67, theorem 2].

Lemma 3.3: Nechť $\Phi_i(A;\mathbf{u})$, $i = 1,2,3$, je definován v (3.15) - (3.17), nechť $A_n \in U_{\mathrm{ad}}$, $A_n \to A$ v U a $\mathbf{u}_n \to \mathbf{u}$ v $\mathcal{H}^1(\Omega)$. Potom

$$\lim_{n\to\infty} \Phi_i(A_n;\mathbf{u}_n) = \Phi_i(A;\mathbf{u}) \tag{3.24}$$

Důkaz : analogický důkazu [64, lemma 3.2] nebo [67, lemma 4, lemma 5].

Věta 3.3: Existuje aspoň jedno řešení maximalizačního problému (3.18).
Důkaz : V důsledku lemmatu 3.3 a věty 3.2 je funkcionál $J(A) = \Phi_i(A;\mathbf{u}(A))$ spojitý na množině U_{ad}. Pomocí Arzela-Ascoliho věty lze ukázat, že je U_{ad} kompaktní v U. Důsledkem je, že na množině U_{ad} existuje maximum.

4

Metoda konečných prvků

Obecně platí, že analytické řešení problémů popsaných parciálními diferenciálními rovnicemi je známé pouze ve velmi málo případech a navíc na speciálních oblastech. Je tudíž nevyhnutelné použít některou numerickou metodu, pomocí které získáme aproximaci řešení. Mezi nejužitečnější patří Galerkinova metoda (viz např. [12]), ze které speciální volbou bázových funkcí získáme metodu konečných prvků (MKP).

Metodu konečných prvků poprvé navrhl R. L. Courant v roce 1943 (viz [13]), bohužel ale v době vzniku nebyla rozpoznána její významnost a tak byla myšlenka metody zapomenuta. Na začátku padesátých let 20. století byla metoda znovuobjevena leteckými inženýry, ovšem první matematická analýza aproximace řešení metodou konečných prvků byla podána až na konci šedesátých let 20. století M. Zlámalem (viz [116]). Od té doby se metoda konečných prvků i díky velkému rozvoji počítačů stala jedním z nejpoužívanějších a nejsilnějších nástrojů pro numerické řešení parciálních diferenciálních rovnic.

4.1 Základní idea

Myšlenku metody ukážeme na lineárním eliptickém problému. Vyjdeme z variační formulace, tj. řešíme problém: najít $\mathbf{u} \in \mathbb{V}$ tak, aby

$$a(\mathbf{u}, \mathbf{v}) = L(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbb{V}, \tag{4.1}$$

kde bilineární forma $a(\mathbf{u}, \mathbf{v})$ je dána vztahem (1.13), funkcionál $L(\mathbf{v})$ vztahem (1.14), množiny $\mathbb{V}_0$ a $\mathbb{V}$ vztahem (1.11) a dále předpokládáme, že $\Gamma_c = \emptyset$, $\Gamma_o = \emptyset$ a $\mathbf{u}_0 = \mathbf{0}$ ($\Rightarrow \mathbb{V} = \mathbb{V}_0$). Jedná se o problém lineární pružnosti, tj. hledáme posuny tuhého tělesa, které je zatíženo objemovými silami $\mathbf{F}$, povrchovým zatížením $\mathbf{P}$, a pro které je předepsán nulový posun na části hranice Γ_u.

Zkonstruujeme konečnědimenzionální podprostor konečných prvků $\mathbb{V}_h$ prostoru testovacích funkcí $\mathbb{V}$ tak, že funkce $\mathbf{v}_h \in \mathbb{V}_h$ jsou po částech polynomiální (daného stupně) vzhledem k dělení výpočetní oblasti. Zvolené dělení je konečným sjednocením určitých disjunktních podmnožin, např. trojúhelníků, čtverců, obdélníků atd. V případě trojúhelníkových prvků se jedná o triangulaci (viz obr. 3.4). Diskretizační parametr h vyjadřuje maximální rozměr všech částí zvoleného dělení.

Problém (4.1) nahradíme aproximovaným problémem: najít $\mathbf{u}_h \in \mathbb{V}_h$ tak, aby

$$a(\mathbf{u}_h, \mathbf{v}_h) = L(\mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbb{V}_h. \tag{4.2}$$

Řešení $\mathbf{u}_h$ diskretizovaného problému (4.2) hledáme ve tvaru lineární kombinace bázových funkcí prostoru $\mathbb{V}_h$. Bázi $\{\mathbf{v}^i\}_{i=1}^N$ prostoru $\mathbb{V}_h$, kde $N = \dim \mathbb{V}_h$, volíme tak, aby její funkce měly malý nosič. Dosazením

$$\mathbf{u}_h = \sum_{j=1}^{N} c_j \mathbf{v}^j \tag{4.3}$$

do (4.2) a volbou bázových funkcí $\mathbf{v}^i$, $i = 1, \ldots, N$, za $\mathbf{v}_h$ dostaneme

$$a\Big(\sum_{j=1}^{N} c_j \mathbf{v}^j, \mathbf{v}^i\Big) = L(\mathbf{v}^i) \quad i = 1, \ldots, N.$$

Hledané koeficienty lineární kombinace získáme vyřešením soustavy lineárních algebraických rovnic

$$\sum_{j=1}^{N} a(\mathbf{v}^j, \mathbf{v}^i)\, c_j = L(\mathbf{v}^i) \quad i = 1, \ldots, N. \tag{4.4}$$

V důsledku V-elipticity bilineární formy $a(.,.)$ je matice $\mathbf{A} = [a(\mathbf{v}^j, \mathbf{v}^i)]_{i,j=1}^N$ regulární. Díky předpokladu, že bázové funkce mají malý nosič, je matice $\mathbf{A}$ navíc řídká. Pro libovolné 2 bázové funkce $\mathbf{v}^i$ a $\mathbf{v}^j$, jejichž nosiče mají prázdný průnik, totiž platí, že $a(\mathbf{v}^i, \mathbf{v}^j) = 0$.

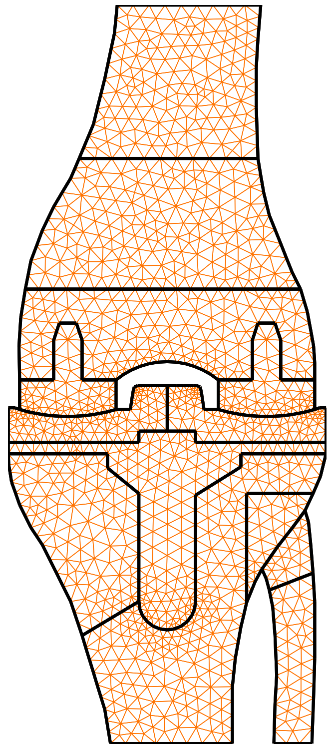
Poznámka 4.1: Matice $\mathbf{A}$ je často nazývána maticí tuhosti a vektor pravé strany $\mathbf{L} = [L(\mathbf{v}^i)]_{i=1}^N$ vektorem zatížení.

4.2 Aproximace kontaktního problému

Vraťme se k řešení kontaktního problému se zadaným třením (1.1)–(1.7) a (1.9). Hledáme slabé řešení $\mathbf{u} \in \mathbb{K}$ problému $(\mathcal{P})$ tak, aby platilo (viz (1.17))

$$a(\mathbf{u}, \mathbf{v} - \mathbf{u}) + j(\mathbf{v}) - j(\mathbf{u}) \geq L(\mathbf{v} - \mathbf{u}) \quad \forall \mathbf{v} \in \mathbb{K}. \tag{4.5}$$

Oblast $\Omega = \bigcup_{\iota=1}^{s} \Omega^\iota$ aproximujeme pomocí $\Omega_h = \bigcup_{\iota=1}^{s} \Omega_h^\iota$ s polygonální hranicí $\partial\Omega_h = \bar{\Gamma}_{uh} \cup \bar{\Gamma}_{\tau h} \cup \bar{\Gamma}_{ch} \cup \bar{\Gamma}_{oh}$, kde $\bar{\Gamma}_{uh}, \bar{\Gamma}_{\tau h}, \bar{\Gamma}_{ch}, \bar{\Gamma}_{oh}$ jsou po částech lineární.



Obrázek 4.1: Triangulace oblasti Ω_h

Oblast $\Omega_h = \bigcup_{\iota=1}^{s} \Omega_h^\iota$ triangulujeme (viz obr. 4.1) a uzly triangulace označíme q_i. Označme $\mathcal{T}_h^\iota$, $\iota = 1,\dots,s$, regulární triangulace polygonálních oblastí Ω_h^ι, $\iota = 1,\dots,s$, a $\mathcal{T}_h = \{\mathcal{T}_h^\iota,\ \iota = 1,\dots,s\}$. Předpokládáme, že $\mathcal{T}_h^\iota$, $\iota = 1,\dots,s$, jsou konzistentní s příslušnými hranicemi $\partial\Omega_h^\iota$, $\iota = 1,\dots,s$, a že uzly ležící na Γ_c^{kl} náleží do triangulací odpovídajících sousedních oblastí v kontaktu Ω^k a Ω^l.

Definujeme konečnědimenzionální množinu virtuálních posunutí $\mathbb{V}_h$

$$\mathbb{V}_h = \{\mathbf{v}_h \mid \mathbf{v}_h \in [C(\Omega^1)]^2 \times \cdots \times [C(\Omega^s)]^2, \mathbf{v}_h|_{T_{hi}} \in [P_1(T_{hi})]^2, \forall T_{hi} \in \mathcal{T}_h;$$

$$v_{hn}(q_i) = 0, q_i \in \Gamma_o; \mathbf{v}_h(q_i) = \mathbf{u}_0(q_i), q_i \in \Gamma_u\}$$

a konečnědimenzionální množinu přípustných posunutí $\mathbb{K}_h$

$$\mathbb{K}_h = \{\mathbf{v}_h | \mathbf{v}_h \in \mathbb{V}_h,\ (v_{hn}^k - v_{hn}^l)(q_i) \leq 0,\ q_i \in \Gamma_c^{kl}\}.$$

Poznámka 4.2: V obecném případě platí $\mathbb{K}_h \not\subset \mathbb{K}$.

Definice 4.1: Funkce $\mathbf{u}_h \in \mathbb{K}_h$ je řešení problému $(\mathcal{P})_h$, jestliže

$$a(\mathbf{u}_h, \mathbf{v}_h - \mathbf{u}_h) + j(\mathbf{v}_h) - j(\mathbf{u}_h) \geq L(\mathbf{v}_h - \mathbf{u}_h) \qquad \forall \mathbf{v}_h \in \mathbb{K}_h. \tag{4.6}$$

Souvislost mezi problémy $(\mathcal{P})$ a $(\mathcal{P})_h$ pro $h \to 0_+$ ukazuje následující věta. Předpokladem je dostatečná hladkost řešení kontaktního problému.

Věta 4.1 [21⋆, theorem 3.1]:
Nechť jsou hranice $\partial\Omega$ a její části Γ_u, Γ_τ, Γ_o, Γ_c po částech polygonální. Nechť řešení problému $(\mathcal{P})$ $\mathbf{u} \in \mathbb{K} \cap [H^2(\Omega)]^2$, $\tau_{ij}(\mathbf{u}^\iota) \in H^1(\Omega^\iota)$, $i, j = 1, 2$ a $\iota = 1, \dots, s$, $\tau_n^{kl}(\mathbf{u}) \in L^\infty(\Gamma_c^{kl})$, u_n^k, $u_n^l \in H^2(\Gamma_c^{kl})$, $k, l = 1, \dots, s$. Nechť je dále $\mathbf{u}_h \in \mathbb{K}_h$ řešení problému $(\mathcal{P})_h$, $\mathbb{K}_h \subset \mathbb{K}$ a počet bodů, ve kterých dochází ke změně $u_n^k - u_n^l < 0$ na $u_n^k - u_n^l = 0$ je konečný.
Potom v semi-koercivním případě

$$|\mathbf{u} - \mathbf{u}_h| \ = O(h), \text{ kde seminorma } |.| \text{ je definována v } (1.10),$$

a v koercivním případě

$$\|\mathbf{u} - \mathbf{u}_h\| = O(h).$$

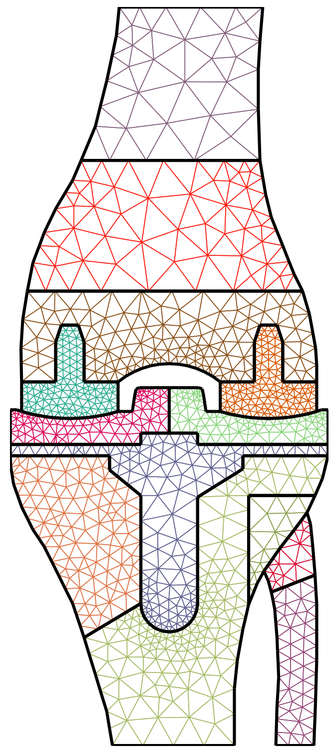
Důkaz : viz [65].

Poznámka 4.3: Při generování sítě pro oblast Ω_h jsme již mohli zohlednit rozdělení oblastí Ω_h^ι, $\iota = 1, \dots, s$, na další polygonální podoblasti, tj. $\bar{\Omega}_h^\iota = \bigcup_{i=1}^{J(\iota)} \bar{\Omega}_{hi}^\iota$ (viz metoda rozkladu oblasti). Uvažované podoblasti mohou představovat např. různé komponenty náhrady kloubu, které mají různé materiálové koeficienty. V důsledku toho můžeme předpokládat, že každé podoblasti Ω_{hi}^ι odpovídá homogenní materiál, což usnadní generování příslušných matic tuhosti. Důležitým předpokladem bylo, že použitá triangulace je konformní, tj. nemůže nastat případ, kdy uzel jednoho prvku leží uvnitř strany jiného prvku. Splnění tohoto předpokladu jsme požadovali na celé triangulaci oblasti Ω_h, tj. i na rozhraních mezi podoblastmi $\partial\Omega_{hi}^\iota \setminus \partial\Omega_h^\iota$, $\iota = 1, \dots, s$, $i = 1, \dots, J(\iota)$, i na částech hranice Γ_{ch} a Γ_{oh}, kde byla předepsána podmínka jednostranného a oboustranného kontaktu (viz obr. 4.1).

4.3 Mortarové prvky

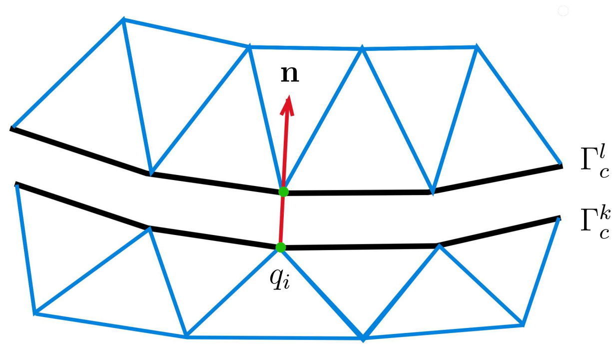
Pří generování sítě nejprve musíme pro každou podoblast Ω_{hi}^ι zadat popis geometrie, tj. v $\mathbb{R}^2$ souřadnice vrcholů odpovídajícího polygonu. Pokud nevyžadujeme, aby byla výsledná síť konformní, můžeme si generování usnadnit tak, že každou podoblast triangulujeme nezávisle (viz obr. 4.2). Výsledná triangulace potom bude nekonformní, tj. uzly triangulací sousedních podoblastí si obecně nebudou odpovídat. Výhodou tohoto přístupu může být i fakt, že sítě podoblastí, ve kterých nás hodnoty řešení zajímají nejvíce, mohou být mnohem jemnější než v ostatních podoblastech.

Další a zřejmě největší výhoda tohoto přístupu nastane v případě, když budeme řešit dynamický kontaktní problém. Zatímco pro statický kontaktní problém má



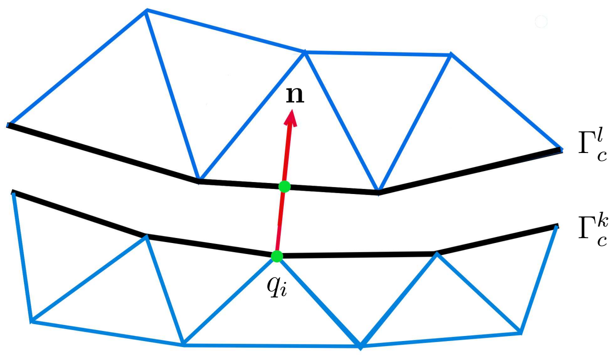
Obrázek 4.2: Nekonformní triangulace oblasti Ω_h

použití konformní sítě výhodu ve snadné implementaci podmínky nepronikání $(u_{hn}^k - u_{hn}^l)(q_i) \leq 0$ (viz obr. 4.3), v dynamické úloze se již ve druhé časové



Obrázek 4.3: Ilustrace podmínky nepronikání pro konformní triangulaci

vrstvě musíme vypořádat se změnou původně stejných souřadnic příslušného uzlu q_i. Jednou z cest by mohlo být přegenerování sítě (tak, aby byla opět konformní), což by mimo jiné znamenalo generování nových matic tuhosti a velkou výpočetní náročnost. Druhým, efektivnějším, způsobem je použití tzv. metody mortarových konečných prvků, která používá nekonformní sítě.



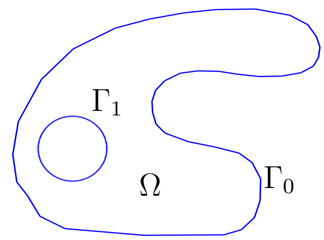
Obrázek 4.4: Ilustrace podmínky nepronikání pro nekonformní triangulaci

Metodu mortarových konečných prvků poprvé uvedli C. Bernardy, Y. Maday a A. T. Patera začátkem 90. let 20. století v pracech [6] a [7]. V článku [3] F. B. Belgacem a Y. Maday uvedli prostorou verzi metody a X. C. Cai, M. Dryja a M. Sarkis v článku [11] rozšířili použití metody pro překrývající se oblasti. Různým typům mortarových prvků se věnuje B. I. Wohlmuth např. v [112, 113]. Konečně na kontaktní problém aplikují metodu autoři F. B. Belgacem, P. Hild a P. Laborde v práci [4], S. Hüeber a B. I. Wohlmuth v [75], I. Hlaváček v [71] a Z. Dostál, D. Horák a D. Stefanica v [43].

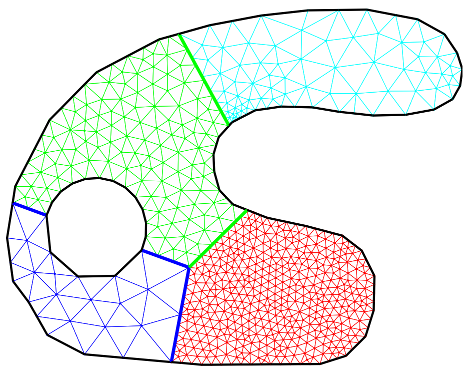
V případě, že máme síť vygenerovanou na každé podoblasti Ω_{hi}^{ι} nezávisle (viz obr. 4.2), dostáváme 2 typy podmínek, které musíme splnit.

1) Na rozhraních podoblastí $\partial\Omega_{hi}^{\iota} \setminus \partial\Omega_{h}^{\iota}$, $\iota = 1, \ldots, s$, $i = 1, \ldots, J(\iota)$, musíme zajistit spojitost řešení ve slabém smyslu. Tento případ je studován v článku [33⋆], který je součástí práce. Zde je metoda mortarových konečných prvků použita pro řešení lineárních eliptických problémů ve 2D, pro jednoduchost je uvažován Poissonův problém. Podmínku slabé spojitosti řešení na rozhraní vyjadřuje rovnice [33⋆,(19)]. Jelikož je rozhraní určeno vždy částmi hranic dvou podoblastí, je nutné označit společnou hranici příslušnou jedné podoblasti jako mortarovou a společnou hranici příslušnou sousední podoblasti jako nemortarovou (závislou). Použitý způsob označování mortarových a nemortarových hran je znázorněn na obr. [33⋆, Fig. 7]). Numerické výsledky uvedené na konci článku ukazují vlastnosti metody mortarových konečných prvků, zejména slabou spojitost řešení na rozhraní oblastí. Pro ilustraci řešme následující problém:

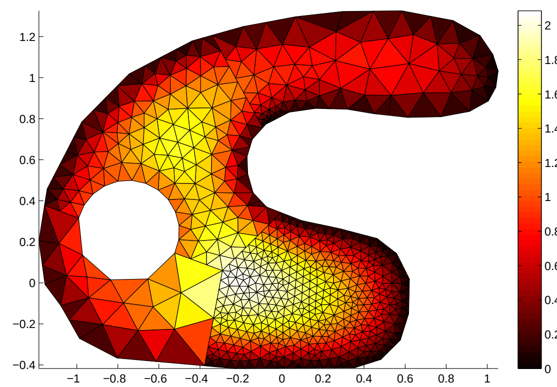
$$\begin{aligned} -\triangle u &= 30 \quad \text{on } \Omega, \\ u &= 0 \quad \text{on } \Gamma_0, \\ u &= 1 \quad \text{on } \Gamma_1. \end{aligned}$$



Na obr. 4.5 je znázorněno zvolené dělení na podoblasti a použitá triangulace. Na obr. 4.6 je vykresleno řešení.



Obrázek 4.5: Diskretizace oblasti Ω



Obrázek 4.6: Řešení

2) Podobně jako v předchozím bodu, budeme požadovat na Γ_{ch} a Γ_{oh} splnění kontaktních podmínek v integrálním (slabém) smyslu. Podmínkou oboustranného kontaktu se nebudeme detailně zabývat, neboť je to speciální případ podmínky jednostranného kontaktu, kde je podmínka nepronikání splněna ve tvaru rovnosti (lineární problém). Mortarový přístup je použit v článku [101⋆], který je poslední součástí souboru prací. Diskrétní tvar slabé podmínky jednostranného kontaktu představuje vztah [101⋆,(15)].

Při odvození mortarové metody vycházíme ze smíšené variační formulace (kontaktního) problému, kdy „slepujeme“ (z angl. překladu slova mortar … *malta*) aproximace řešení na jednotlivých podoblastech - viz body **1)** a **2)**. V lineárním případě je odvození odpovídající diskretizované úlohy sedlového bodu [33⋆,(22)] obsaženo v [33⋆,odst. 3.2]. V případě použití mortarové metody na kontaktní problém odpovídá tvar diskretizované úlohy sedlového bodu problému [101⋆,(13)]. V obou případech vyjadřuje druhá rovnice podmínku na „slepení“ řešení. Pro aproximaci Lagrangeových multiplikátorů použijeme tzv. duální bázi, která je biortogonální vzhledem k standardní bázi spojitých po částech lineárních funkcí, tj. stop standardního prostoru konečných prvků na podoblasti s mortarovou hranicí. Tato duální báze se skládá z nespojitých po částech lineárních funkcí (viz [75, 114]). „Slepující“ podmínka vždy svazuje dvě nekonformní triangulace. Abychom podmínku vyjádřili pouze pomocí stop konečných prvků z podoblasti s mortarovou hranicí, je třeba provést transformaci celkové báze (viz [71, 75, 115], [101⋆]). Tím také umožníme snadnou eliminaci Lagrangeových multiplikátorů. V nové bázi se potom problém sedlového bodu zjednoduší v maticovém tvaru na soustavu [101⋆,(26)]. Konečně pro řešení této soustavy lze využít algoritmus primárně duální metody aktivních množin (primal-dual active set strategy - PDAS), který se dá ztotožnit se zobecněnou iterační Newtonovou metodou pro řešení nelineárních rovnic a je popsán např. v [58, 59, 69, 71, 75].

5

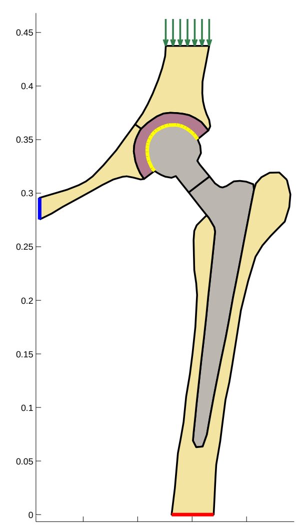
Numerické výsledky

V článcích [21$\star$], [19$\star$], [28$\star$] a [68$\star$] jsou obsaženy numerické výsledky pro některé biomechanické a geomechanické modely. Obsahem této kapitoly je stručný přehled výsledků autora pro další vybrané problémy, které nejsou součástí předkládaného souboru prací.

5.1 Biomechanické modely

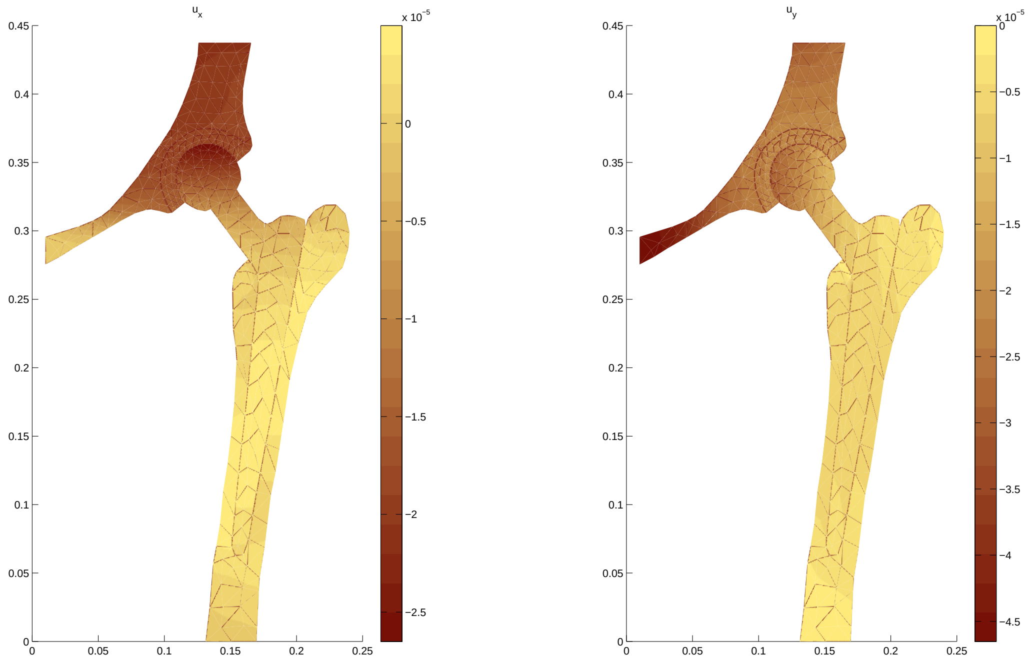
5.1.1 Model náhrady kyčelního kloubu

Na obr. 5.1 je zobrazena geometrie modelu náhrady kyčelního kloubu. Zadané elastické koeficienty pro kost jsou Youngův modul pružnosti $E = 2 \times 10^9$ [Pa], Poissonova konstanta $\nu = 0,3$ a pro dřík a jamku $E = 2 \times 10^{10}$ [Pa], $\nu = 0,3$. Červenou barvou je označena část hranice, kde je stehenní kost upevněna (nulová Dirichletova podmínka). Modrou barvou je označena část hranice, na které předepisujeme podmínky oboustranného kontaktu (podmínka symetrie). Žlutou barvou je označena část hranice, kde předepisujeme podmínky jednostranného kontaktu (mezi kyčelní jamkou a dříkem). Na horní části hranice předpokládáme zatížení tlakem $0,4 \times 10^6$ [Pa] (naznačeno zelenými šipkami). Na obr. 5.2 jsou vykresleny horizontální a vertikální složka výsledného posunutí a na obr. 5.3 je vykreslen detail hlavních napětí (červené šipky →← představují tlaky, modré šipky ←→ popisují tahy). Právě ze zobrazených hlavních napětí si lze udělat představu o rozložení silového



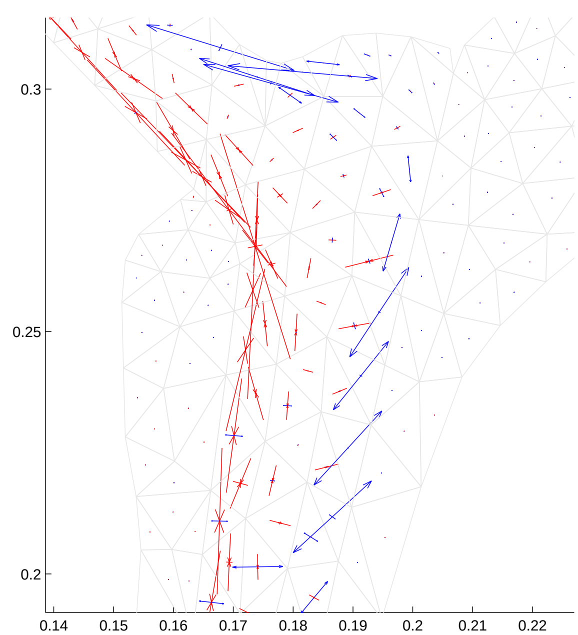
Obrázek 5.1: Model náhrady kyčelního kloubu

působení. Uvedený model lze nalézt v práci [15].



Obrázek 5.2: Horizontální a vertikální složka vektoru posunutí

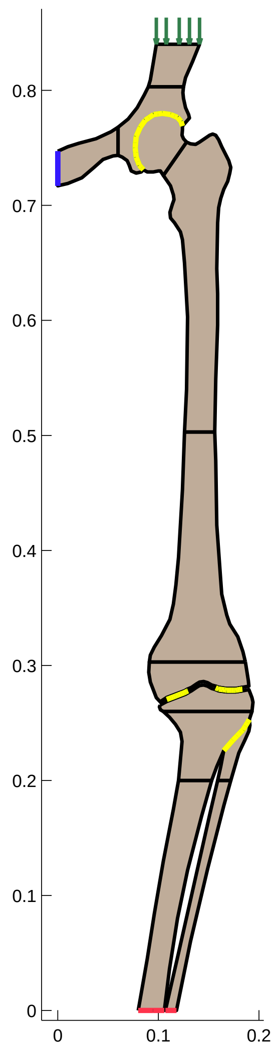
Uvedený model odpovídá situaci, kdy je náhrada pevně spojena s kostní tkání. Pokud bychom chtěli např. modelovat stav, kdy je dřík uvolněný, museli bychom na uvažované části rozhranní dříku a stehenní kosti předepsat podmínku jednostranného kontaktu. Po zatížení by na vnější části již nebyl dřík a stehenní kost v kontaktu, důsledkem čehož by se na část stehenní kosti nepřenášelo žádné napětí a odpovídající část kostní tkáně by postupně ztrácela pevnost.



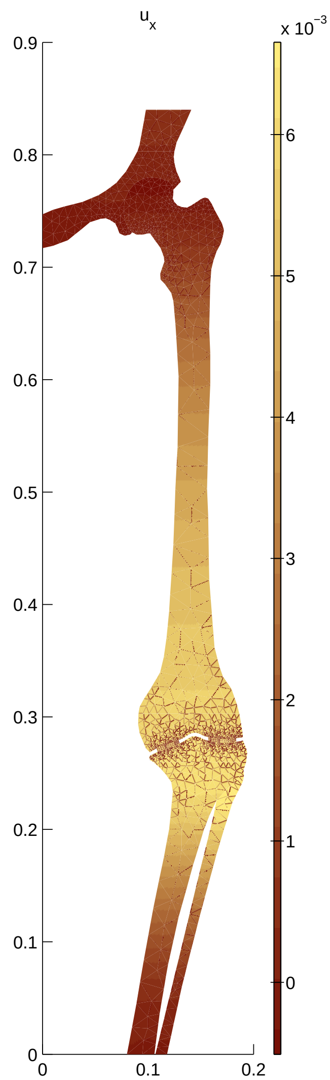
Obrázek 5.3: Hlavní napětí - detail

5.1.2 Model dolní končetiny

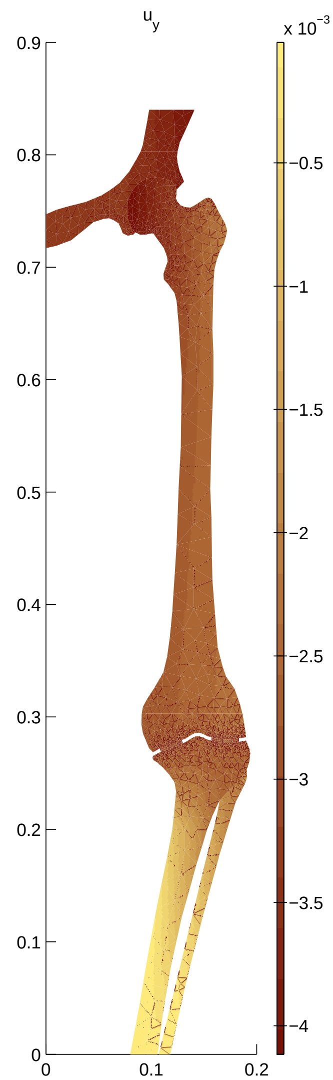
Geometrie modelu dolní končetiny je vykreslena na obr. 5.4. Použité materiálové parametry jsou pro kost $E = 1,71 \times 10^{10}$ [Pa], $\nu = 0,25$, pro chrupavku $E = 0,492 \times 10^{9}$ [Pa], $\nu = 0,1$. Podobně jako v modelu náhrady kyčelního kloubu



Obrázek 5.4: Model dolní končetiny



Obrázek 5.5: Horizontální složka posunutí

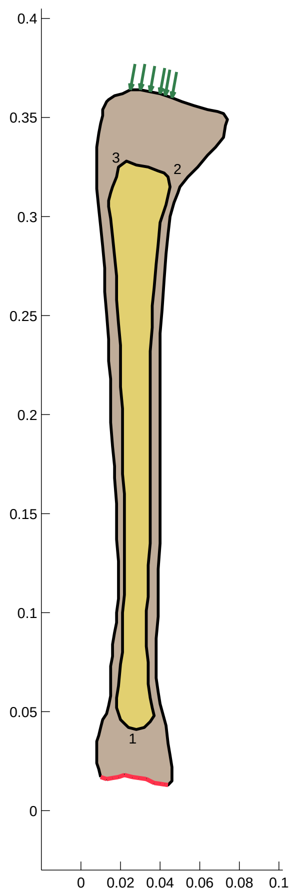


Obrázek 5.6: Vertikální složka posunutí

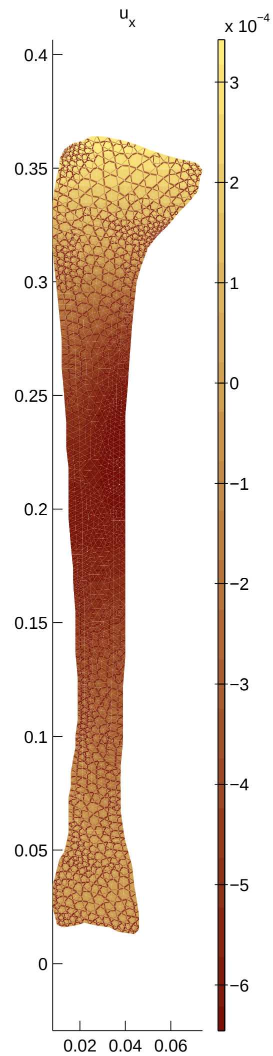
použijeme toto barevné značení: na červené části hranice je předepsán nulový posun, na modré části hranice předepisujeme podmínky oboustranného kontaktu a žlutě jsou označeny části hranice, kde předepisujeme podmínky jednostranného kontaktu. Na horní části hranice předpokládáme zatížení tlakem 1×10^6 [Pa] (naznačeno zelenými šipkami). Na obr. 5.5, resp. 5.6 je vykreslena horizontální, resp. vertikální složka posunutí. Podrobněji je model popsán v článku [25].

5.1.3 Model holenní kosti

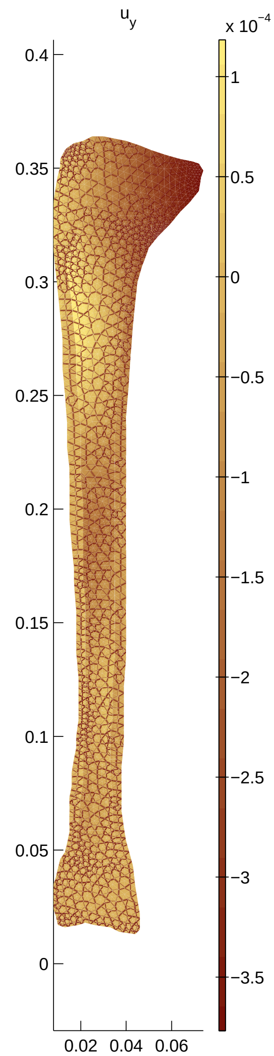
V předchozích modelech nebyla uvažována kostní dřeň, která má jiné materiálové vlastnosti než vlastní kost. Na základě numerických výsledků pro model holenní kosti lze získat představu o tom, jak se napětí v kostní dřeni přenáší. Geometrie modelu holenní kosti je zobrazena na obr. 5.7. Červeně označená hranice je fixo-



Obrázek 5.7: Model holenní kosti

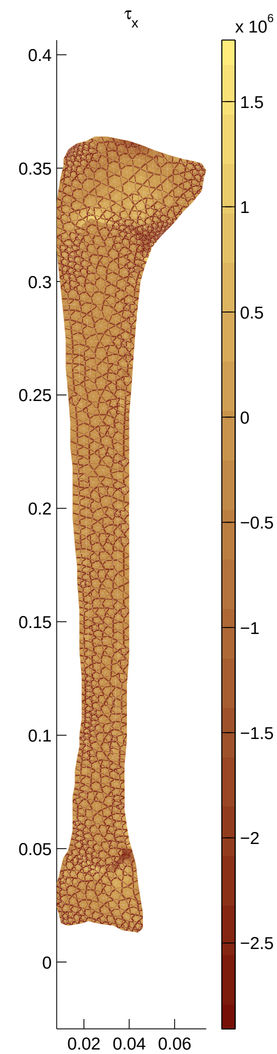


Obrázek 5.8: Horizontální složka posunutí

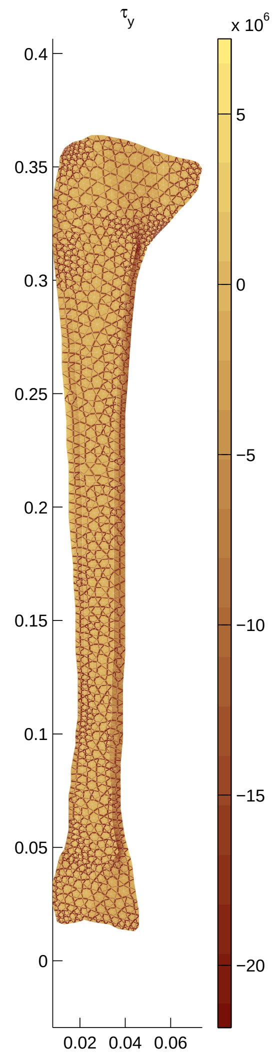


Obrázek 5.9: Vertikální složka posunutí

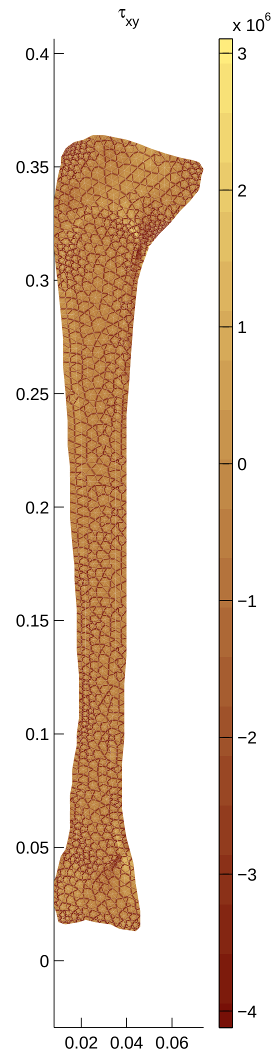
vána, na horní části hranice je předepsáno zatížení $0,2 \times 10^7$ [Pa] (viz zelené šipky) a mezi kostní dření a kostí uvažujeme podmínku oboustranného kontaktu. Elastické koeficienty jsou pro kost $E = 1,71 \times 10^{10}$ [Pa], $\nu = 0,25$ a pro kostní dřeň $E = 2 \times 10^6$ [Pa], $\nu = 0,49$. Složky napětí τ_x, τ_y a τ_{xy} jsou vykresleny na obr. 5.10, 5.11 a 5.12.



Obrázek 5.10: Složka napětí τ_x

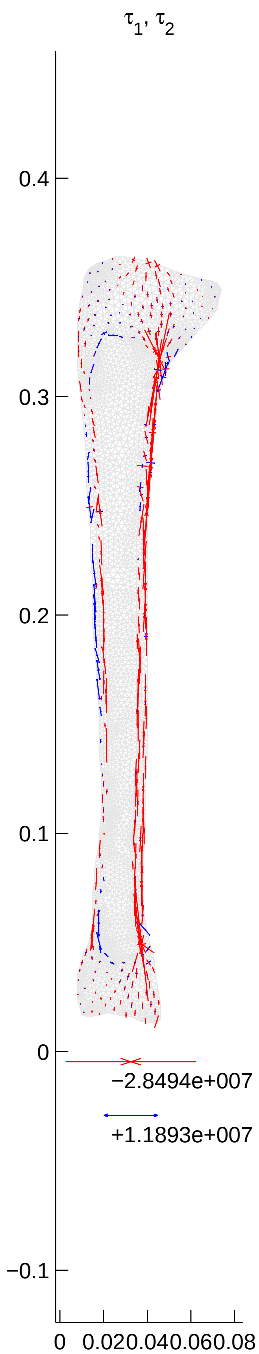


Obrázek 5.11: Složka napětí τ_y

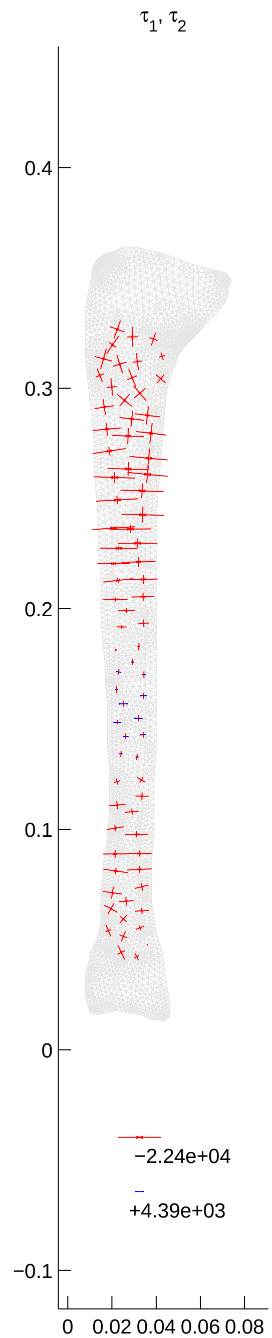


Obrázek 5.12: Složka napětí τ_{xy}

Na obr. 5.13 a 5.14 jsou vykreslena hlavní napětí. Je zřejmé, že se kostní dření napětí přenáší také, ovšem jeho velikost je o 3 až 4 řády menší než je napětí přenášené vlastní kostí (Youngův modul pružnosti pro kost je přibližně $4\times$ větší než Youngův modul pružnosti pro kostní dřeň). Numerické výsledky jsou pro model holenní kosti uvedeny v [110].

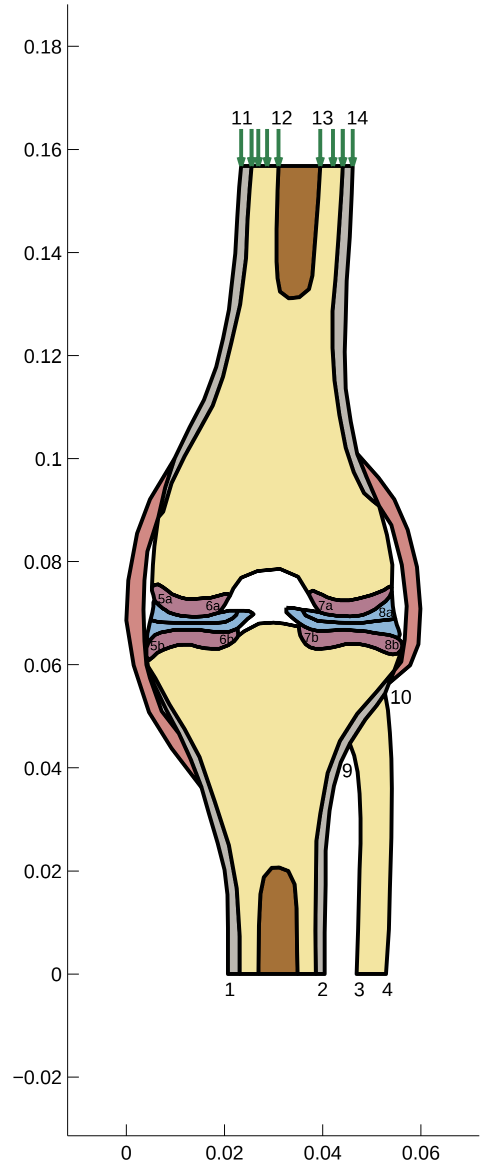


Obrázek 5.13: Hlavní napětí - kost



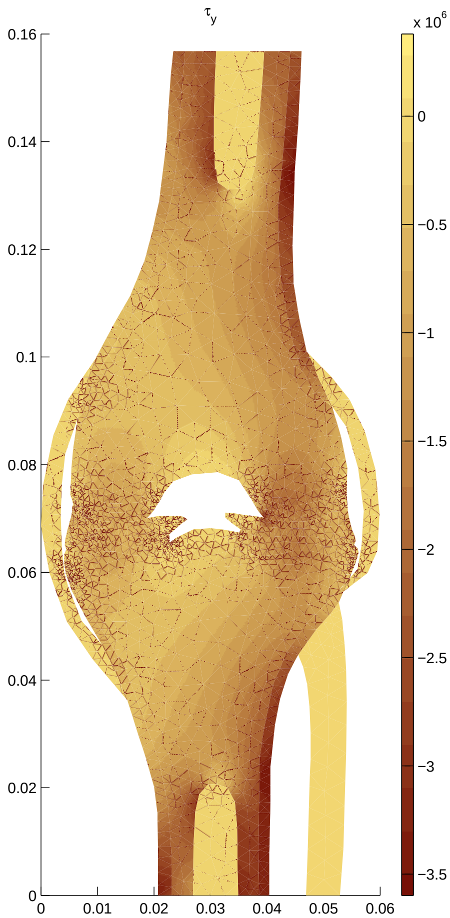
Obrázek 5.14: Hlavní napětí - kostní dřeň

5.1.4 Model kolenního kloubu

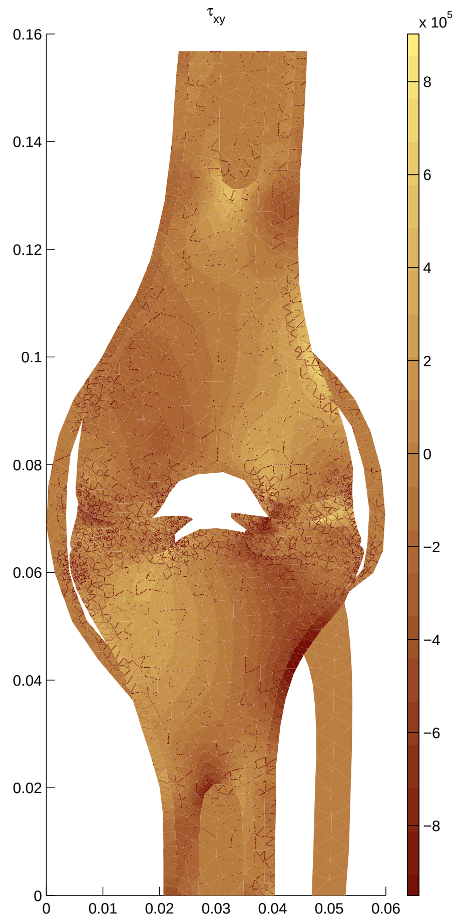


Obrázek 5.15: Model kolenního kloubu

Geometrie modelu kolenního kloubu je zobrazena na obr. 5.15. V modelu uvažujeme detailnější strukturu obsahující kortikální kost ($E = 1.71 \times 10^{10}$ [Pa], $\nu = 0.25$ - označena šedou barvou), spongiózní kost ($E = 1.4 \times 10^{10}$ [Pa], $\nu = 0.3$ - označena žlutou barvou), kostní dřeň ($E = 2 \times 10^6$ [Pa], $\nu = 0.45$ - označena hnědou barvou), vazivo ($E = 7 \times 10^4$ [Pa], $\nu = 0.49$ - označeno růžovou barvou), chrupavku ($E = 2 \times 10^9$ [Pa], $\nu = 0.2$ - označena fialovou barvou) a meniskus ($E = 4.92 \times 10^8$ [Pa], $\nu = 0.1$ - označena modrou barvou). Na částech hranice 1-2 a 3-4 jsou lýtková kost (fibula) a holenní kost (tibie) fixovány, na rozhraní mezi chrupavkou a meniskem, tj. 5a-6a, 5b-6b, 7a-8a a 7b-8b, a na části hranice 9-10 mezi fibulou a tibií předepisujeme podmínku jednostranného kontaktu, na částech 11-12 a 13-14 předepisujeme zatížení odpovídající statické vlastní hmotnosti pacienta, tj. $0,215 \times 10^7$ [Pa] a na zbývajících částech hranice předepisujeme nulové zatížení. Na obr. 5.16 je vykreslena vertikální složka napětí τ_y a na obr. 5.17 je vykresleno smykové napětí τ_{xy}. Z obrázku 5.16 je patrné, že je tlakové napětí více koncentrováno do oblasti vnějšího kondylu a méně pak do oblasti vnitřního kondylu. Podrobnější numerické výsledky pro model kolenního kloubu jsou uvedeny v [26].



Obrázek 5.16: Složka napětí τ_y

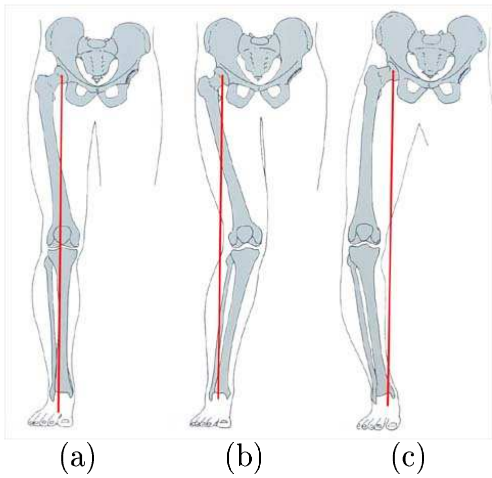


Obrázek 5.17: Složka napětí τ_{xy}

5.1.5 Model náhrady kolenního kloubu

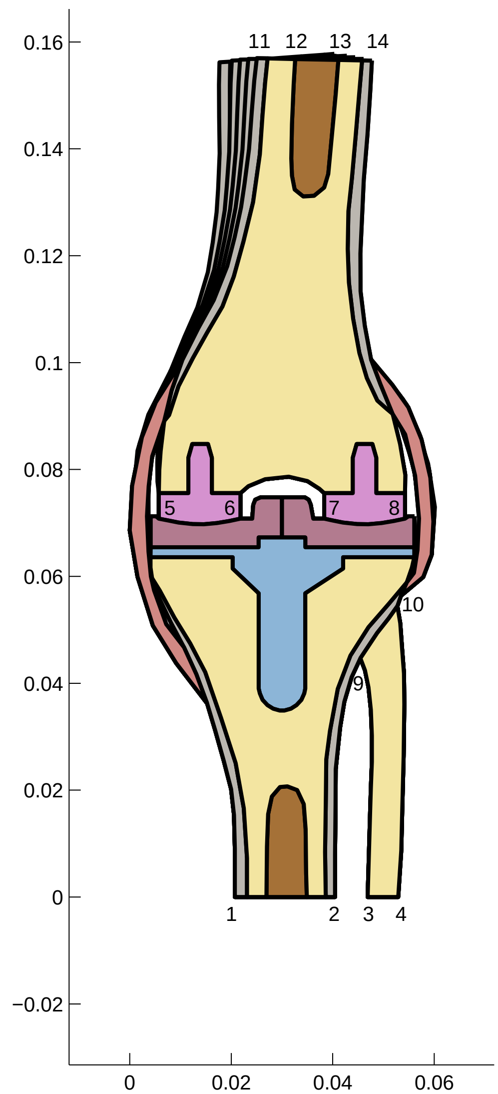
V článku [28⋆], který je součástí práce, jsou uvedeny numerické výsledky pro model náhrady kolenního kloubu jednak ve frontální rovině a také v sagitální rovině. Výpočty byly provedeny pro několik různých materiálů, které byly použity na jednotlivé části náhrady.

Jedním z podstatných faktorů ovlivňujících úspěch operace kolenního kloubu je výsledná geometrie dolní končetiny. Hodnocení osových poměrů skeletu dolní končetiny před operací protézy kolenního kloubu se běžně provádí z rtg. snímku dolní končetiny, kde musí být zachycen nejen kyčelní kloub, ale i část pánve s acetabulem, horní konec stehenní kosti, kolenní kloub a bérec s hlezennýmm kloubem. V případě kolenního kloubu je třeba určit tzv. mechanickou osu dolní končetiny (viz obr. 5.18), což je spojnice středu hlavice stehenní kosti se středem hlezenného

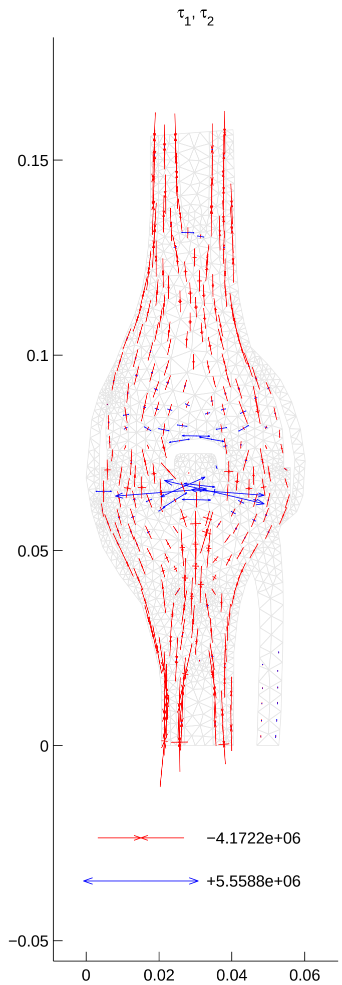


Obrázek 5.18: Mechanická osa dolní končetiny

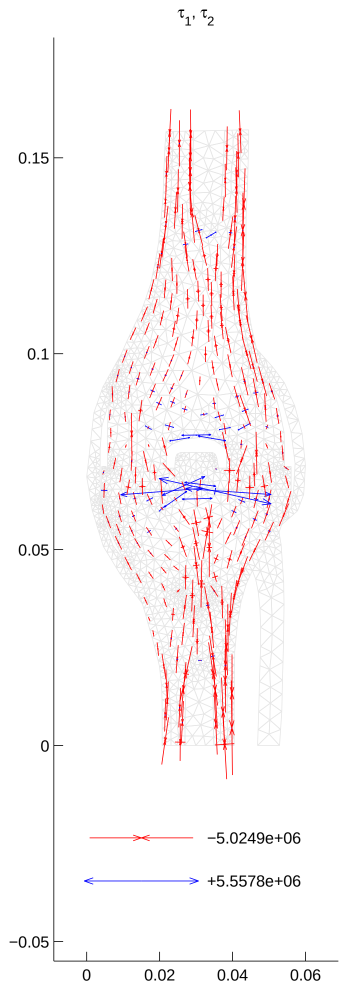
kloubu. Úhel, který svírá mechanická osa s anatomickou osou stehenní kosti určuje stupeň fyziologické valgozity, ve kterém by měla být provedena resekce dolního konce stehenní kosti. Dodržením správné techniky implantace, především stranově vyrovnaného napětí měkkých tkání a odstraněním zvýšeného tlaku na zadní část tibiálního plata a respektováním mechanické osy končetiny se vytvoří základní předpoklady pro správné biomechanické fungování totální náhrady. V ideálním případě mechanická osa prochází středem kolenního kloubu (viz obr. 5.18a). Pokud tomu tak není (viz obr. 5.18b,c), lze nápravy dosáhnout pomocí resekce stehenní kosti pod větším či menším úhlem. V článku [19⋆], který je součástí práce, jsou uvedeny výsledky pro model náhrady kolenního kloubu ve frontální rovině, kde je femorální komponenta implantována na konec stehenní kosti s různými hodnotami úhlu resekce. Výpočty byly opět provedeny pro několik různých materiálů použitých na výrobu jednotlivých částí náhrady. Numerické výsledky pro podobné modely jsou uvedeny v [26]. V těchto modelech jsou, podobně jako v odst. 5.1.4, zahrnuty kortikální a spongiózní kost, kostní dřeň i vazivo. Geometrie modelů pro úhly resekce 0°, 3°, 5°, 7° a 9° je na obr. 5.19. Na obr. 5.20, 5.21 a 5.22 jsou vykreslena hlavní napětí pro úhly 0°, 5° a 9°.



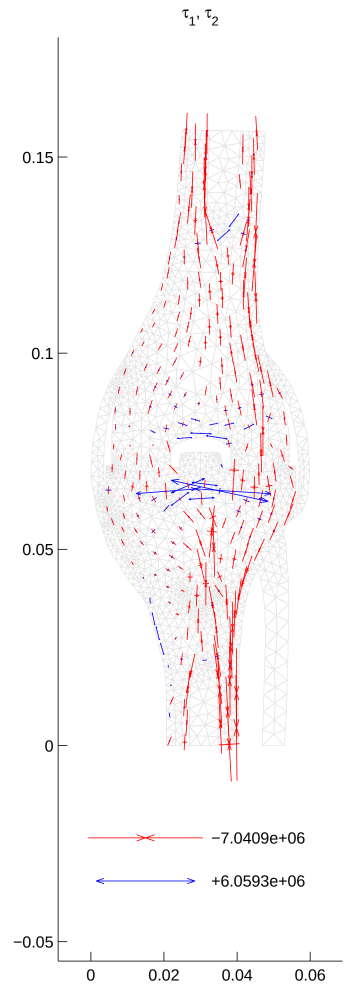
Obrázek 5.19: Model náhrady kolenního kloubu



Obrázek 5.20: Hlavní napětí (úhel resekce 0°)



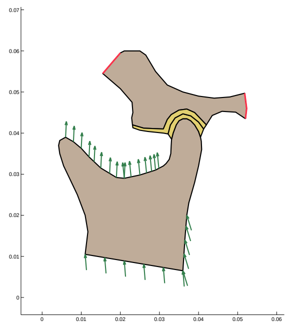
Obrázek 5.21: Hlavní napětí (úhel resekce 5°)



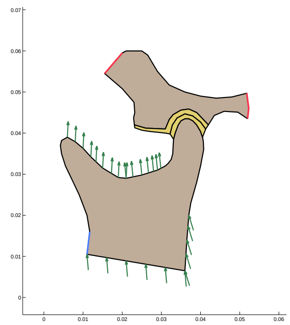
Obrázek 5.22: Hlavní napětí (úhel resekce 9°)

5.1.6 Model čelistního kloubu

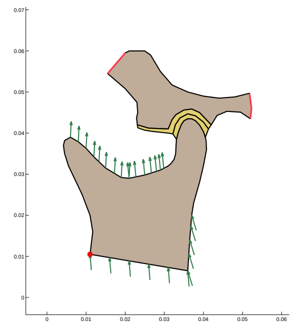
Analýza rozložení napětí v čelistním kloubu je obsahem článku [73]. Geometrie modelu byla získána postupem zmíněným v 3. kapitole a její popis vychází z obr. 3.2. Pro tuto geometrii bylo vytvořeno několik modelů pomocí různé volby okrajových podmínek. Na obr. 5.23, 5.24 a 5.25 jsou znázorněny 3 modely. Červeně označená část hranice (na horní čelisti) je fixována, zelené šipky značí povrchové zatížení na příslušné části hranice a podmínky jednostranného kontaktu předepisujeme na společné hranici mezi chrupavkami, které jsou vykresleny žlutou barvou.



Obrázek 5.23: Čelistní kloub - model 1

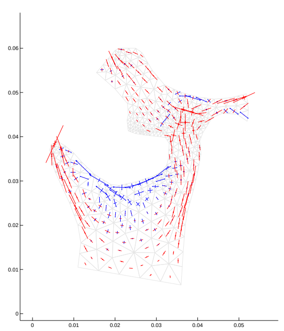


Obrázek 5.24: Čelistní kloub - model 2

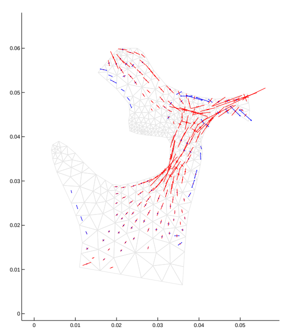


Obrázek 5.25: Čelistní kloub - model 3

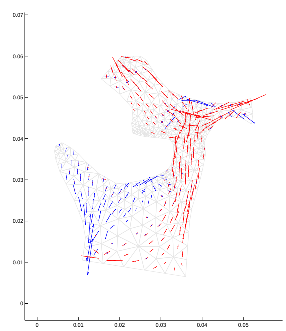
Tyto předpoklady jsou společné pro všechny uvedené modely. V modelu 1 je pro všechny uzlové body dolní čelisti připojena podmínka omezující maximální posun ve vodorovném i svislém směru na ± 1 mm. V modelu 2 je připojena podmínka oboustranného kontaktu pro modře označenou část hranice dolní čelisti. V modelu 3 je pro červeně označený bod dolní čelisti (levý dolní roh) předepsán posun $[-1 \text{ mm}, 1 \text{ mm}]$. Z obr. 5.26, 5.27 a 5.28, na kterých jsou zobrazena hlavní napětí pro model 1, 2 a 3, jsou potom patrné rozdíly v rozložení přenášeného zatížení.



Obrázek 5.26: Hlavní napětí - model 1



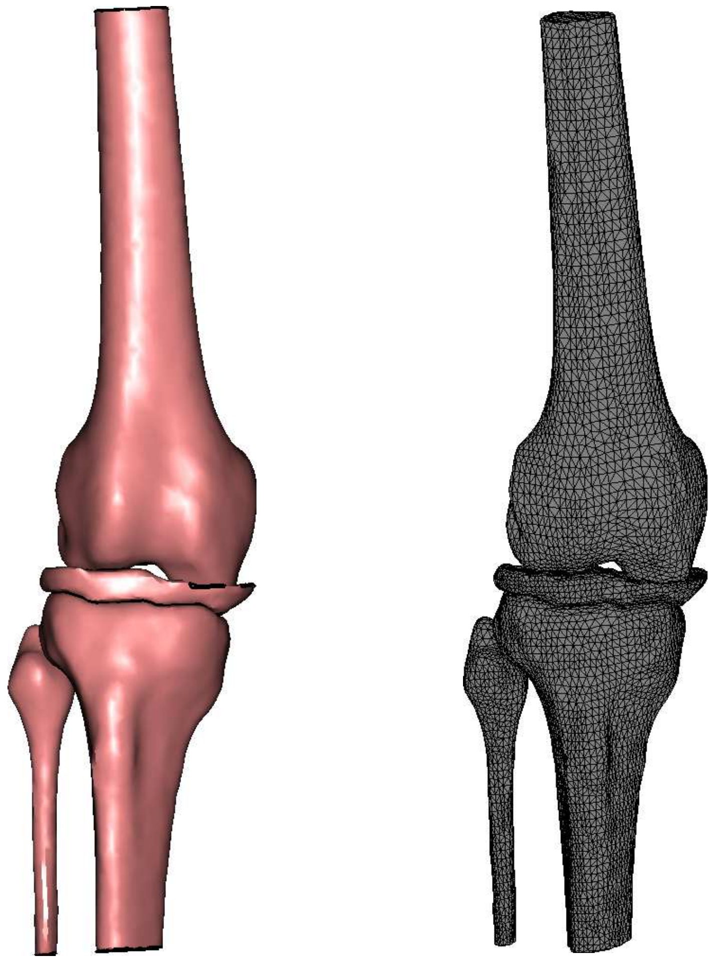
Obrázek 5.27: Hlavní napětí - model 2



Obrázek 5.28: Hlavní napětí - model 3

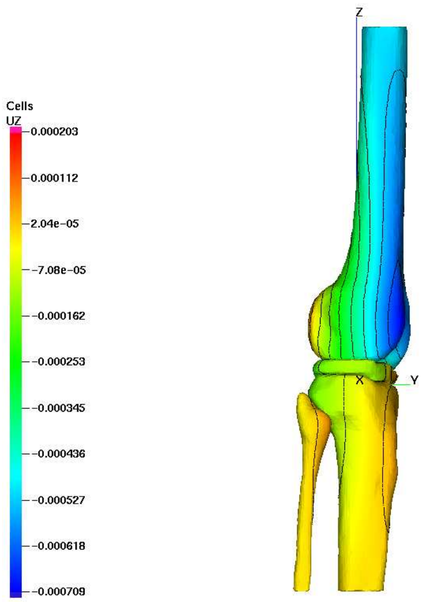
5.1.7 3D model kolenního kloubu

Pokud použijeme třírozměrný model kolenního kloubu, popř. jeho náhrady, získáme detailnější a přesnější informaci o silovém zatížení jednotlivých částí modelu. V práci [83] jsou uvedeny výsledky pro 3D model kolenního kloubu, jehož geometrie (viz obr. 5.29) byla získána postupem zmíněným ve 3. kapitole. V modelu jsou 4 kontaktní plochy (dvě mezi femorálními kondyly a meniskem, jedna mezi meniskem a holenní kostí a jedna mezi holenní a lýtkovou kostí. Dále je předepsán nulový posun na dolních řezech kosti holenní a lýtkové a pro horní část (řez) stehenní kosti

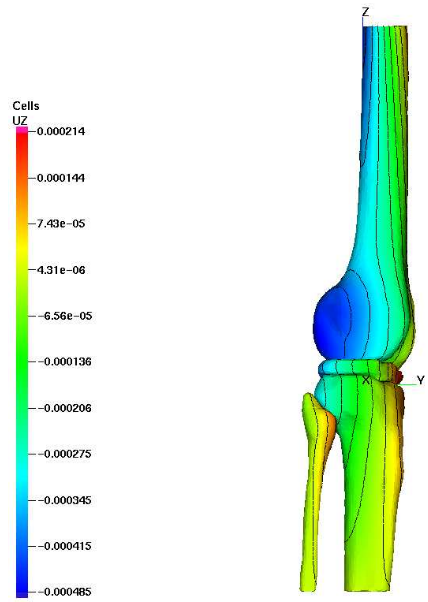


Obrázek 5.29: 3D model kolenního kloubu - geometrie a konečněprvková síť

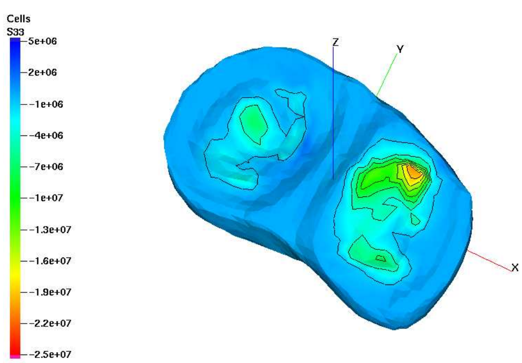
je předepsán posun 0.5 mm ve směru ke středu menisku a současně je předepsána rotace tohoto řezu v sagitální rovině okolo středu menisku o velikosti od $-0.9°$ do $0.9°$. Volbou těchto podmínek simulujeme deformaci zatížené nohy, kterou mírně propínáme a pokrčujeme. Na obr. 5.30 a 5.31 jsou vykresleny hodnoty vertikálních posunů pro natočení o velikosti $0°$ a $0.9°$. Vertikální složka napětí na povrchu menisku je pro stejné hodnoty natočení stehenní kosti znázorněna na obr. 5.32 a 5.33. Informaci o rozložení napětí mezi oba kondyly lze získat ze dvou sagitálních řezů (viz obr. 5.34). Na obr. 5.35 a 5.36 jsou prezentovány hodnoty napětí ve směru osy y (předozadní) pro sagitální řez procházející přes vnější kondyl pro natočení stehenní kosti o velikosti $0°$ a $0.9°$.



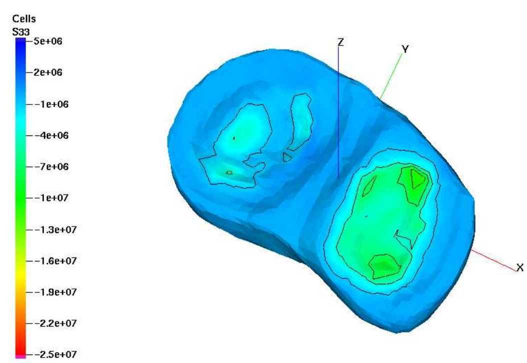
Obrázek 5.30: Vertikální posun - natočení 0°



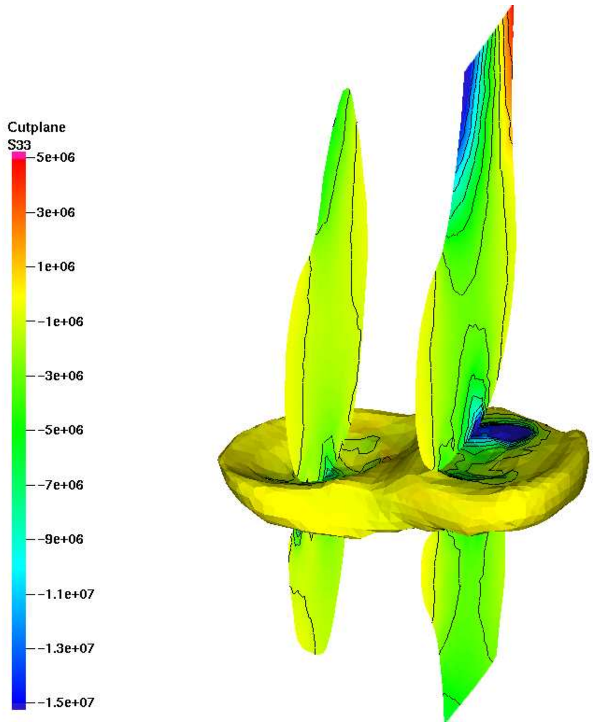
Obrázek 5.31: Vertikální posun - natočení 0.9°



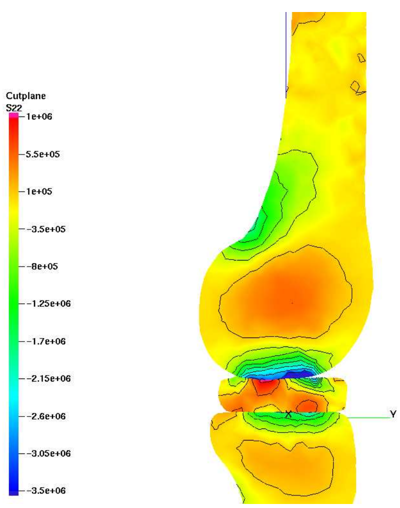
Obrázek 5.32: Vertikální napětí - natočení 0°



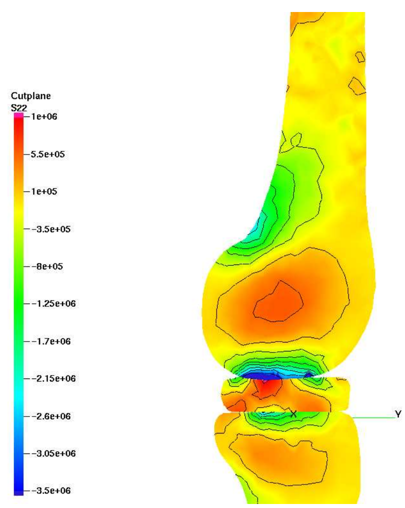
Obrázek 5.33: Vertikální napětí - natočení 0.9°



Obrázek 5.34: Sagitální řezy vedené přes oba kondyly



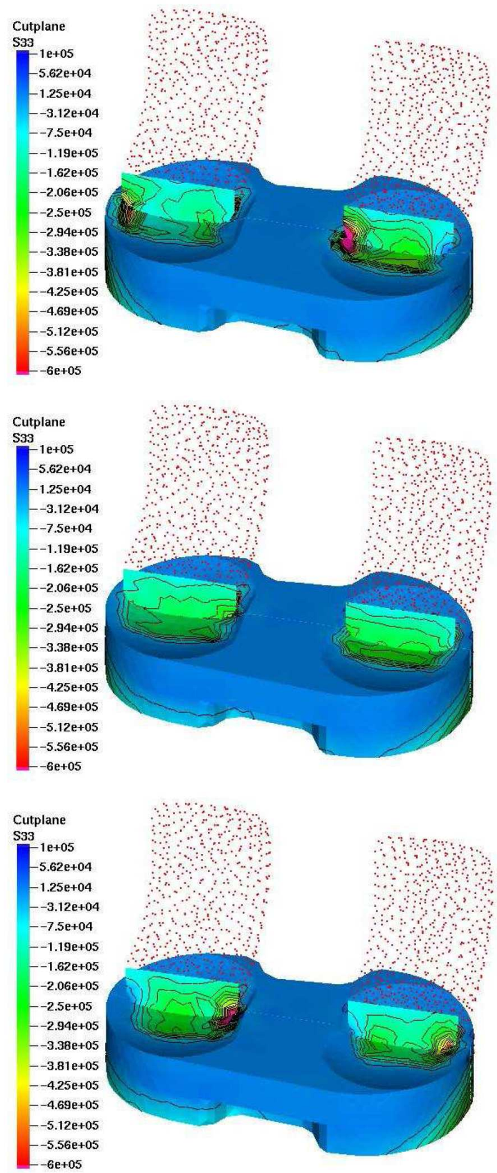
Obrázek 5.35: Napětí ve směru osy y (předozadní) - natočení 0°



Obrázek 5.36: Napětí ve směru osy y (předozadní) - natočení 0.9°

5.1.8 3D model náhrady kolenního kloubu

Numerické výsledky pro 3D model náhrady kolenního kloubu jsou uvedeny v práci [83]. Pro vlastní výpočet byl použit zjednodušený model náhrady, ve kterém byly vypuštěny neadekvátní detaily (viz 3. kapitola, obr. 3.8 a 3.9), a který sestává z kovové femorální komponenty (CoCrMo), tibiální komponenty (Ti6Al4V) a plastové vložky (UHMWPE). Předpokládáme, že plastová vložka je pevně spojena s tibiální částí náhrady, a že ke kontaktu dochází na rozhraní plastové vložky a femorální částí náhrady. Dále předepisujeme nulový posun pro celou spodní plochu tibiální části a femorální část zatěžujeme na její horní horizontální ploše časově proměnou silou působící svisle dolů s odchylkou $\pm 5^\circ$ od svislé osy ve frontální rovině (jedná se o kvazistacionární úlohu). Motivací je modelovat reakci náhrady při zatížení vahou těla, kde různé úhlové odchylky zatěžující síly mohou vystihovat situaci při různém umístění komponent náhrady vzhledem k mechanické ose končetiny. Abychom zabránili rotaci femorální komponenty v sagitální rovině, předepisujeme pro její horní svislou stěnu podmínku oboustranného kontaktu, tj. komponenta se může pohybovat pouze ve frontální rovině a pohyby v předozadním směru umožněny nejsou. Na obr. 5.37 jsou vykresleny hodnoty napětí ve svislém (proximálním) směru pro případy odchylky -4°, 0° a $+4^\circ$ vektoru zatěžující síly od svislé osy ve frontální rovině. Pro názornost je vytečkována dorzální část femorální komponenty.

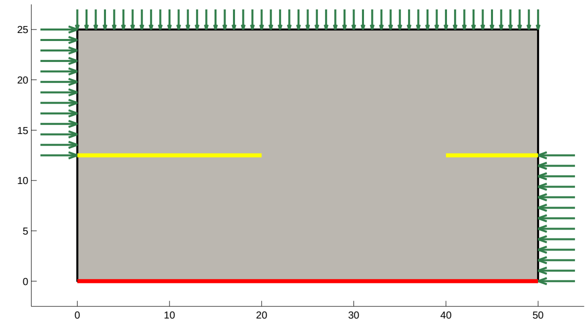


Obrázek 5.37: Vertikální napětí pro řezy femorální komponentou pro odchylku zatěžující síly od svislé osy ve frontální rovině -4°, 0° a $+4^\circ$

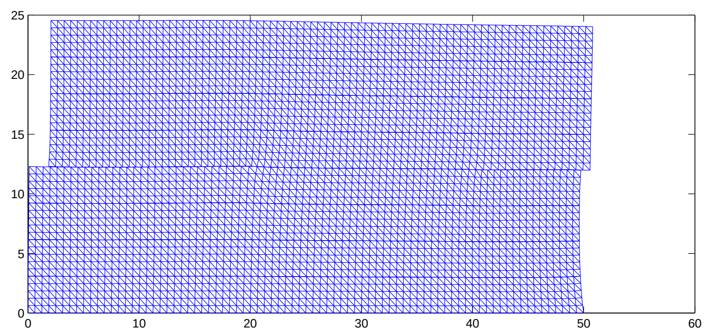
5.2 Geomechanické modely

5.2.1 Model žulového bloku narušeného trhlinou

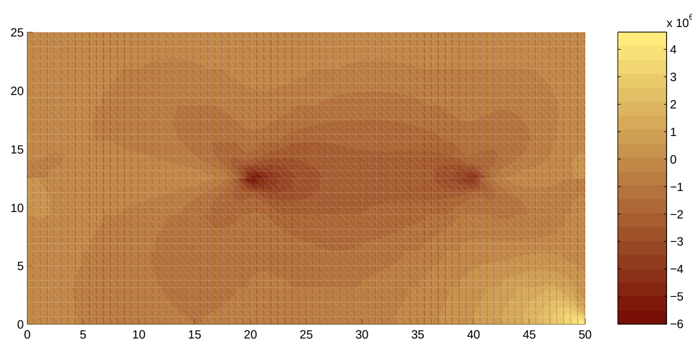
V článku [24] jsou uvedeny numerické výsledky pro model žulového bloku narušeného trhlinou. Geometrie modelu je zobrazena na obr. 5.38. Červeně označenou část hranice fixujeme, žlutě je vyznačena kontaktní hranice a zelené šipky naznačují povrchové zatížení. Na obr. 5.39 je vykreslen blok po deformaci (50-ti násobné zvětšení), na obr. 5.40 smyková napětí a na obr. 5.41 hlavní napětí.



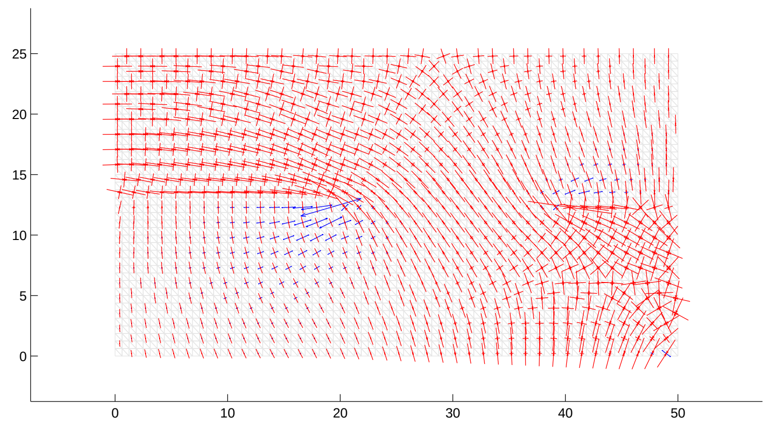
Obrázek 5.38: Model žulového bloku narušeného trhlinou



Obrázek 5.39: Deformace



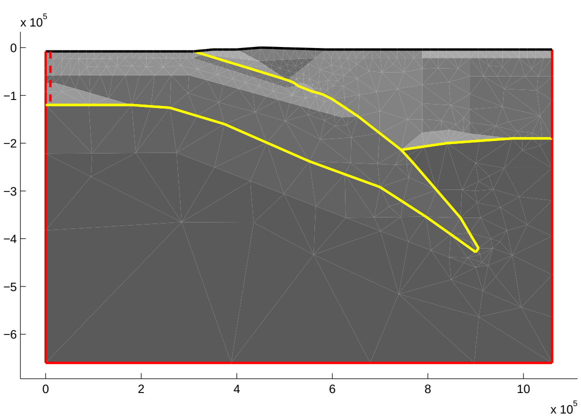
Obrázek 5.40: Smyková napětí



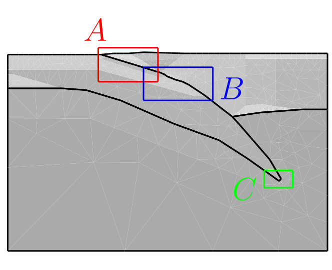
Obrázek 5.41: Hlavní napětí

5.2.2 Model Himálají

Geometrie modelu je zobrazena na obr. 5.42. Uvažujeme celkem 27 podoblastí s různými elastickými koeficienty (na obrázku jsou rozlišeny různou intenzitou šedi). Červenou barvou je vyznačena část hranice s předepsaným posunutím, které je nulové s výjimkou levé horní části, kde je předpsáno posunutí $[3\times10^3\ \mathrm{m}, 0\ \mathrm{m}]$, které odpovídá pohybu litosferické desky v časovém intervalu 6×10^4 let. Žlutou barvou je vyznačena kontaktní hranice.

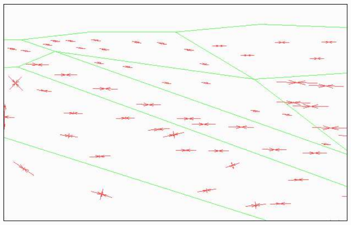


Obrázek 5.42: Model Himálají

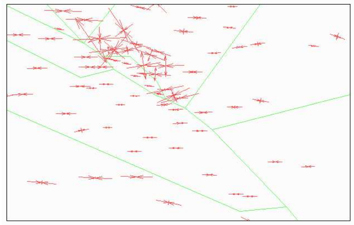


Obrázek 5.43: Označení výřezů

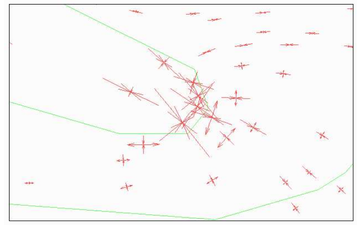
Numerické výsledky pro tento model jsou obsaženy v práci [15]. Představu o rozložení silového zatížení získáme z vykreslených hlavních napětí. Na obr. 5.44, 5.45 a 5.46 jsou vykresleny detaily hlavních napětí pro výřezy zobrazené na obr. 5.43.



Obrázek 5.44: Hlavní napětí - výřez A



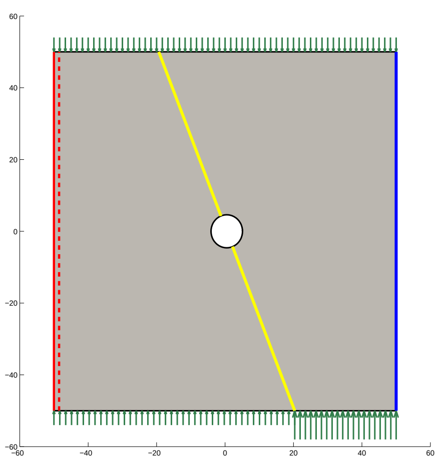
Obrázek 5.45: Hlavní napětí - výřez B



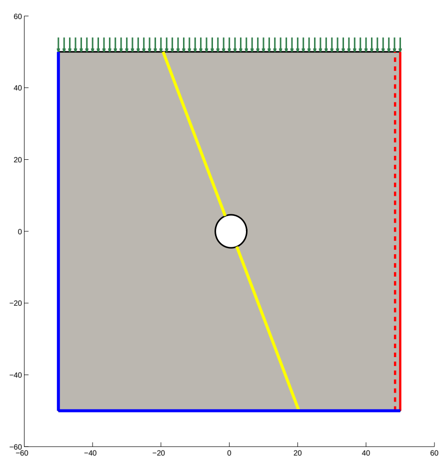
Obrázek 5.46: Hlavní napětí - výřez C

5.2.3 Model tunelu narušeného geologickým zlomem

V článku [68⋆], který je součástí práce, jsou uvedeny numerické výsledky pro model, který popisuje tunel kruhového průřezu ve skalním masívu, kterým prochází geologický zlom. Geometrie je znázorněna na obr. 5.47. Na červeně označené části hranice předepisujeme posun $[2,5 \times 10^{-2}\ \mathrm{m}, 0\ \mathrm{m}]$, žlutou barvou je vykreslena hranice, kde předpokládáme jednostranný kontakt, na modré hranici platí podmínky oboustranného kontaktu a zelené šipky naznačují předepsaná povrchová zatížení $0,5 \times 10^7$ [Pa], resp. 1×10^7 [Pa].

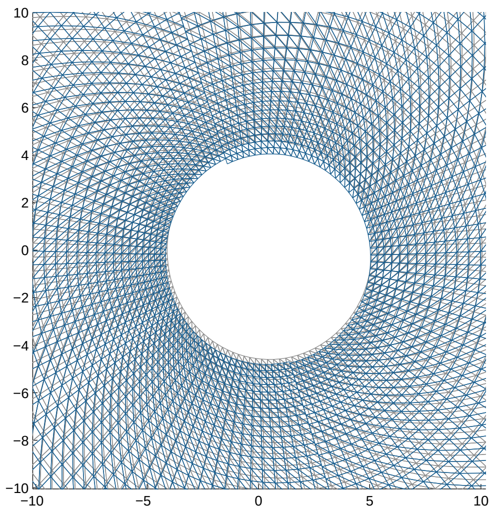


Obrázek 5.47: Model tunelu - 1

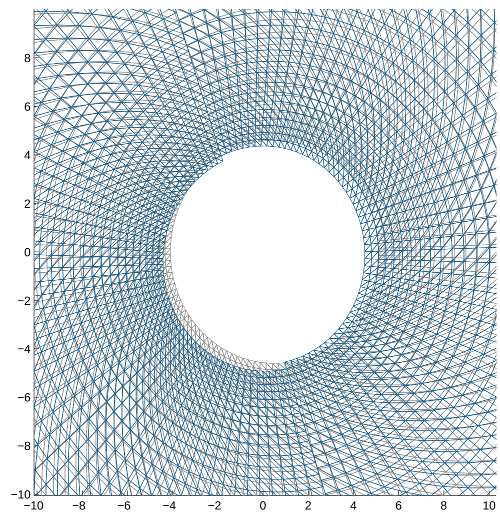


Obrázek 5.48: Model tunelu - 2

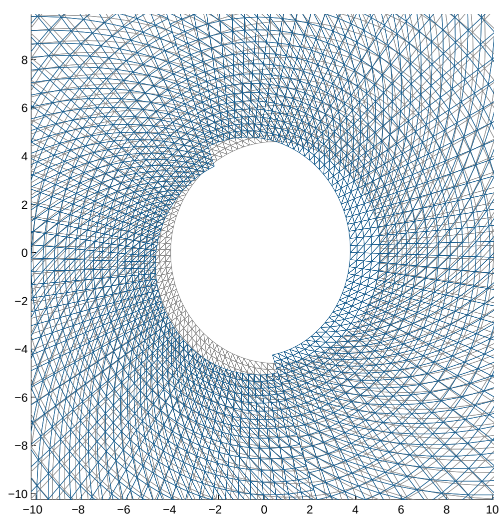
Na obr. 5.48 je model, ve kterém uvažujeme jiné okrajové podmínky. Povrchové zatížení na horní hranici je $0,5 \times 10^7$ [Pa], na modré hranici platí podmínky oboustranného kontaktu a na žluté podmínky jednostranného kontaktu. Hodnotu posunutí na červeně označené hranici předepisujeme ve vodorovném směru ve třech variantách: (I) $-2,5 \times 10^{-2}$ m, (II) -1×10^{-1} m a (III) -2×10^{-1} m. Deformace okolí tunelu jsou pro tyto tři možnosti vykresleny na obr. 5.49, 5.50 a 5.51 (zvětšeno 10×). Posunutí bodů na stěnách tunelu ukazují obr. 5.52, 5.54 a 5.56. Obr. 5.53, 5.55 a 5.57 zachycují hlavní napětí na stěnách tunelu.



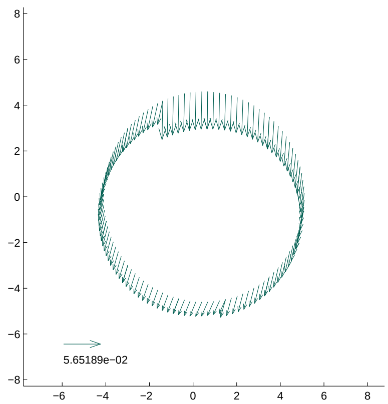
Obrázek 5.49: Deformace - verze (I)



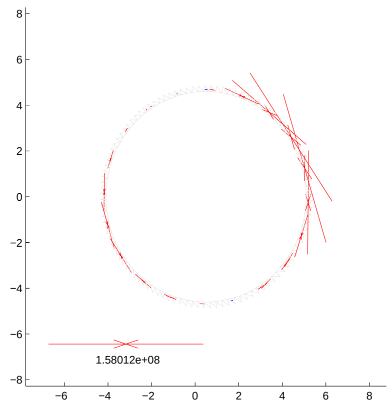
Obrázek 5.50: Deformace - verze (II)



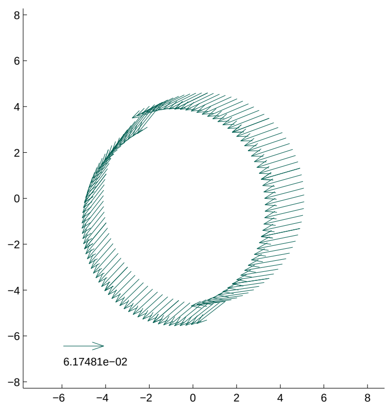
Obrázek 5.51: Deformace - verze (III)



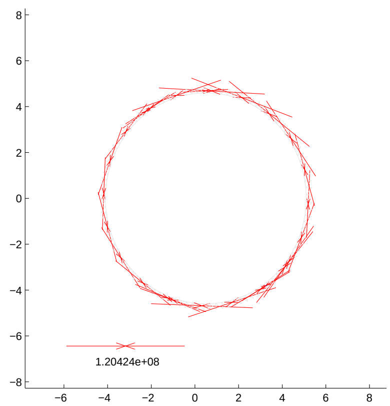
Obrázek 5.52: Posunutí - (I)



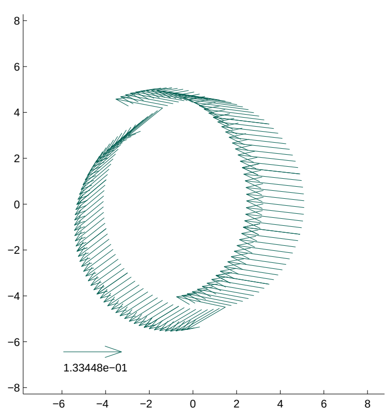
Obrázek 5.53: Hlavní napětí - (I)



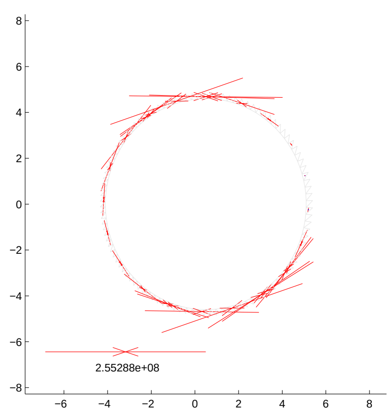
Obrázek 5.54: Posunutí - (II)



Obrázek 5.55: Hlavní napětí - (II)



Obrázek 5.56: Posunutí - (III)



Obrázek 5.57: Hlavní napětí - (III)

6

Shrnutí

Předkládaná habilitační práce, která je souborem původních prací autora doplněným komentářem, je věnována úlohám z oblasti biomechaniky a geomechaniky, které lze formulovat jako kontaktní problém. Práce, které jsou zařazeny do předkládané práce, nepředstavují veškeré výstupy autora. Jedná se o výběr, který charakterizuje určité dílčí přístupy, které lze využít pro efektivní řešení kontaktního problému. V komentáři jsou použity i odkazy na další původní práce autora, které souvisí s danou problematikou. Cílem práce je snaha podat komplexní pohled na řešení problému, od formulace až po vlastní numerické řešení. Z komentáře je patrné, že pro řešení problému jsou důležité poznatky z různých oborů, zejména z matematiky, mechaniky, počítačových věd, medicíny, geologie a dalších. Právě v propojení různých vědních disciplín spatřuji velký význam a potenciál pro další práci.

V článku [17⋆] je popsána metoda rozkladu oblasti bez překrývání vycházející z primární formulace pro posunutí pro řešení kontaktního problému obecně více pružných těles ve dvou dimenzích. Na kontaktu předpokládáme nulové tření. Popsaná metoda je efektivní v případě, že je kontaktní hranice „relativně“ malá. Zadané oblasti rozdělíme na podoblasti tak, že jejich rozhraní neprochází kontaktní hranicí. Vzniklé podoblasti můžeme rozdělit do dvou skupin: a) oblasti, které neosahují kontaktní hranici a b) dvojice oblastí, které jsou v kontaktu. Princip metody vychází z metody Schurova doplňku, tj. hledáme hodnoty posunutí na rozhraních mezi oblastmi. Vzhledem k tomu, že jsou operátory Schurova doplňku pro dvojice podoblastí v kontaktu nelineární, řešíme problém metodou postupných aproximací. Počáteční aproximaci určíme jako řešení linearizovaného problému, kdy je podmínka jednostranného kontaktu nahrazena podmínkou oboustranného kontaktu. Pro řešení linearizovaného problému i pro řešení v jednotlivých krocích metody postupných aproximací používáme metodu sdružených gradientů s předpodmíněním typu Neumann-Neumann. V článku je dále dokázána konvergence metody postupných aproximací a jsou uvedeny výsledky numerického experimentu pro modelovou úlohu.

Metoda rozkladu oblasti bez překrývání pro řešení kontaktního problému pružných těles ve 2D je popsána také v článku [21⋆], kde je na rozdíl od článku [17⋆] použita pro řešení kontaktního problému se zadaným třením. Struktura článku [21⋆] je proto obdobná struktuře článku [17⋆]. V závěru jsou uvedeny numerické výsledky pro geomechanický model tunelu, kterým prochází geologický zlom, a pro biomechanický model náhrady kolenního kloubu.

V článku [19⋆] jsou uvedeny a diskutovány výsledky numerických experimentů pro dvourozměrný model náhrady kolenního kloubu. Umístění femorální komponenty je uvažováno ve třech variantách, pro úhel resekce hlavice stehenní kosti 3°, 5° a 7°. Motivací zkoumání je na základě výsledného rozložení napětí v jednotlivých případech určit nejvhodnější polohu femorální části náhrady. Uvedené výsledky byly předneseny na mezinárodní konferenci The 2004 International Conference on Computational Science and its Applications (ICCSA 2004).

Modelu náhrady kolenního kloubu je věnován také článek [28⋆], kde jsou uvedeny výsledky pro řezy ve frontální a v sagitální rovině. Na základě analýzy rozložení napětí v jednotlivých částech náhrady jsme se snažili potvrdit praktické zkušenosti lékařů a zdůvodnit asymetrické opotřebení tibiálního plata, zejména jeho zadní části. V důsledku přetěžování náhrady dochází k tzv. otěru, což je progresivní úbytek materiálu spojený s uvolňováním otěrových částic, které ve svém důsledku mohou narušit spojení implantátu s živou tkání. Výsledky byly prezentovány na mezinárodní konferenci The 3rd IMACS Conference on Mathematical Modelling and Computational Methods in Applied Sciences and Engineering (MODELLING 2005).

Obsahem článku [68⋆] je metoda nejhoršího scénáře pro kvazi-sdruženou (quasi-coupled) úlohu v termo-pružnosti a výše zmiňovaná metoda rozkladu oblasti. Nejprve je uvedena formulace kontaktního problému v termo-pružnosti, následuje výklad metody rozkladu oblasti pro řešení kontaktního problému pružných těles se zadaným třením a dále je popsána metoda nejhoršího scénáře pro nejistá vstupní data. Jsou zde zavedeny množiny přípustných dat a poté je uvedena věta o existenci a jednoznačnosti slabého řešení kontaktního problému pro každý prvek z množiny přípustných dat. „Spolehlivým řešením“ úlohy definujeme nejhorší mezi všemi možnými řešeními, kde stupeň nespolehlivosti je měřen jistým funkcionálem, kritériem (criterion functional), který závisí na řešení studované úlohy. V článku je ukázáno 6 možností volby tohoto kritéria a je dokázána existence řešení problému nejhoršího scénáře (worst scenario problem).

V článku [33⋆] je formulována metoda mortarových konečných prvků pro řešení lineárních eliptických problémů ve dvou dimenzích. Použití metody mortarových konečných prvků vychází z metody rozkladu oblasti, kde jednotlivé podoblasti jsou dikretizovány (triangulovány) nezávisle, tzn. že sítě na společných hranicích (rozhraních) mezi podoblastmi nesouhlasí - jedná se o nekonformní metodu. Takový přístup je více flexibilní a umožňuje např. lokální zjemňování pouze na některých podoblastech. Spojitost řešení na rozhraních je zajištěna ve slabém smyslu. V článku je dále popsána implementace metody v systému MATLAB a na dvou příkladech jsou ilustrovány vlastnosti metody.

V článku [101⋆] je popsána metoda pro numerické řešení dynamického kontaktního problému viskopružných těles. Prezentovaný model je použit pro popis šíření seismických vln napříč zemskou kůrou. Pro řešení je použito semi-implicitní schéma, metoda mortarových konečných prvků a primárně duální metoda aktivních množin PDAS. Uvedená metoda zatím nebyla imlementována a součástí článku proto nejsou žádné numerické výsledky. Tato práce určuje směr dalšího zkoumání.

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Daněk, J.: *Domain decomposition method for contact problems with small range contact*, J. Mathematics and Computers in Simulation, vol. 61, issue 3-6, pp. 359–373, Elsevier Science Ltd., North-Holland 2003.

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Mathematics and Computers in Simulation 61 (2003) 359–373

MATHEMATICS AND COMPUTERS IN SIMULATION

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Domain decomposition method for contact problems with small range contact

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Abstract

A non-overlapping domain decomposition algorithm of Neumann–Neumann type for solving variational inequalities arising from the elliptic boundary value problems in two dimensions with unilateral boundary condition is presented. We suppose that boundary with inequality condition is 'relatively' small. First, the linear auxiliary problem, where the inequality condition is replaced by the equality condition, is solved. In the second step, the solution of the auxiliary problem is used in a successive approximations method. In these solvers, a preconditioned conjugate gradient method with Neumann–Neumann preconditioner is used for solving the interface problems, while local problems within each subdomain are solved by direct solvers. A convergence of the iterative method is proved and results of computational test are reported.

Keywords: Domain decomposition; Schur complement; Unilateral contact problems; Parallel computing; Preconditioning

1. Equilibrium of a system of bodies in contact

We consider a system of elastic bodies decomposed into subdomains each of which occupies, in reference configuration, a domain Ω_i^M in $\mathbb{R}^2$, $i = 1, \dots, I_M$, $M = 1, \dots, \mathcal{J}$, with boundary $\partial\Omega_i^M$. Suppose that boundary $\bigcup_{M=1}^{\mathcal{J}} \partial\Omega^M$ consists of four disjoint parts Γ_u, Γ_τ, Γ_c and Γ_0 and that the displacements $u_0 : \Gamma_u \to \mathbb{R}^2$ and forces P: $\Gamma_\tau \to \mathbb{R}^2$ are given. The part Γ_c denote the part of boundary that may get into unilateral contact with some other subdomain and the part Γ_0 denote the part of boundary on that the condition of the bilateral contact is prescribed (see Fig. 1).

We shall look for the displacements that satisfy the conditions of equilibrium in the set $K = \{v \in V | v_n^k + v_n^l \leq 0 \quad \text{on}\, \Gamma_c\}$ of all kinematically admissible displacements $v \in V$, $V = \{v \in \mathcal{H}^1(\Omega) | v = u_0 \quad \text{on}\, \Gamma_u, v_n = 0 \quad \text{on}\, \Gamma_0\}$, $\mathcal{H}^1(\Omega) = [H^1(\Omega_1^1)]^2 \times \cdots \times [H^1(\Omega_{I_\mathcal{J}}^\mathcal{J})]^2$ is standard Sobolev space.

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0378-4754/02/$ – see front matter © 2002 Published by Elsevier Science B.V. on behalf of IMACS.
PII: S0378-4754(02)00089-7

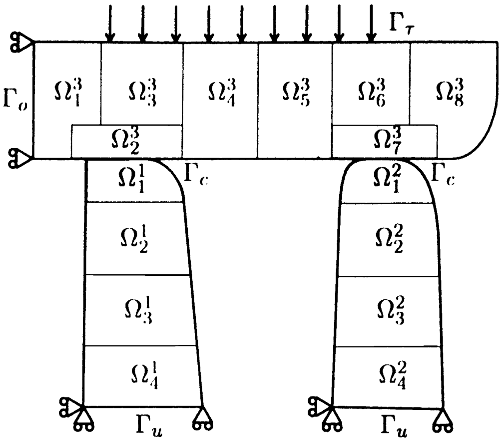

Fig. 1. The contact problem with decomposition.

The displacement $u \in K$ of the system of bodies in equilibrium then minimizes the energy functional $\mathcal{L}(v) = (1/2)a(v, v) - L(v)$:

$$\mathcal{L}(u) \leq \mathcal{L}(v) \quad \text{for any } v \in K. \tag{1}$$

Conditions that guarantee existence and uniqueness of the solution may be expressed in terms of coercivity of $\mathcal{L}$ and may be found, for example, in [1].

We define $\Gamma_i^M = \partial\Omega_i^M / \partial\Omega^M$ and the interface $\Gamma = \bigcup_{M=1}^{\mathcal{J}} \bigcup_{i=1}^{I_M} \Gamma_i^M$. Let us introduce

$$T^M = \{j \in \{1, \dots, I_M\} : \bar{\Gamma}_c \cap \partial\bar{\Omega}_j^M = \emptyset\}, \quad M = 1, \dots, \mathcal{J}, \tag{2}$$

The number of a separate subset Γ_c is P_c, i.e. $\Gamma_c = \bigcup_{j=1}^{P_c} \Gamma_{cj}$. Further, we denote

$$\Omega^{*j} = \left\{ x \in \bigcup_{M=1}^{\mathcal{J}} \bigcup_{i=1}^{I_M} \Omega_i^M : \partial\Omega_i^M \cap \Gamma_{cj} \neq \emptyset \right\}, \quad j = 1, \dots, P_c, \tag{3}$$

$$\vartheta^j = \{[i, M] : \partial\Omega_i^M \cap \Gamma_{cj} \neq \emptyset\}, \quad j = 1, \dots, P_c, \tag{4}$$

i.e.

$$\Omega^{*j} = \bigcup_{[i,M]\in\vartheta^j} \Omega_i^M, \quad j = 1, \dots, P_c.$$

We suppose that

$$\Gamma \cap \Gamma_c = \emptyset, \tag{5}$$

then

$$V_\Gamma = \gamma K|_\Gamma = \gamma V|_\Gamma, \tag{6}$$

for trace operator γ: $[H^1(\Omega_i^M)]^2 \to [L^2(\partial\Omega_i^M)]^2$. We suppose that γ^{-1}: $V_\Gamma \to V$ is arbitrary linear inverse mapping for which

$$\sum_{M=1}^{\mathcal{J}} (\gamma^{-1}\bar{v}^M)_n = 0 \quad \text{on } \Gamma_c \quad \forall \bar{v} \in V_\Gamma. \tag{7}$$

After denoting restrictions $\bar{R}_i^M$: $V_\Gamma \to \Gamma_i^M$, L_i^M: $L^M \to \Omega_i^M$, $a_i^M(.,.)$: $a^M(.,.) \to \Omega_i^M$, $V(\Omega_i^M)$: $V(\Omega^M) \to \Omega_i^M$ and introduction

$$V^0(\Omega_i^M) = \left\{ v \in V | v = 0 \quad \text{on } \overline{\left(\bigcup_{M=1}^{\mathcal{J}} \Omega^M\right) \Big\backslash \Omega_i^M} \right\},$$

we can formulate the Theorem 1.1.

Theorem 1.1. *A function $u \in K$ is the solution of the global problem (1) if and only if the function u satisfies*

1. $$\sum_{M=1}^{\mathcal{J}} \sum_{i=1}^{I_M} (a_i^M(u_i^M(\bar{u}), \gamma^{-1}\bar{w}) - L_i^M(\gamma^{-1}\bar{w})) = 0 \quad \forall \bar{w} \in V_\Gamma, \bar{u} \in V_\Gamma, \tag{8}$$
   for the trace $\bar{u} = \gamma u|_\Gamma$ on the interface Γ.
2. *Its rescriction $u_i^M(\bar{u}) \equiv u|_{\Omega_i^M}$ satisfies following conditions*

   (a)

   $$a_i^M(u_i^M(\bar{u}), \phi_i^M) = L_i^M(\phi_i^M) \quad \forall \phi_i^M \in V^0(\Omega_i^M),$$
   $$u_i^M(\bar{u}) \in V(\Omega_i^M), \gamma u_i^M(\bar{u})|_{\Gamma_i^M} = \bar{R}_i^M \bar{u}, \tag{9}$$

   for $i \in T^M$, $M = 1, \ldots, \mathcal{J}$,

   (b)

   $$\sum_{[i,M]\in\vartheta^j} a_i^M(u_i^M(\bar{u}), \phi_i^M)$$
   $$\geq \sum_{[i,M]\in\vartheta^j} L_i^M(\phi_i^M) \quad \forall \phi \equiv (\phi_i^M, [i, M] \in \vartheta^j), \phi_i^M \in V^0(\Omega_i^M), \tag{10}$$

   such that

   $$u + \phi \in K; \qquad \gamma u_i^M(\bar{u})|_{\Gamma_i^M} = \bar{R}_i^M \bar{u} \quad \text{for } [i, M] \in \vartheta^j,$$

   for $j = 1, \ldots, P_c$.

Proof. See [2,5]. □

2. The local and global Schur complement

We now want to write the interface problem (8) in operator form. For this purpose, we first introduce additional notation. We introduce the local trace spaces

$$V_i^M = \{\gamma v|_{\Gamma_i^M} | v \in K\} = \{\gamma v|_{\Gamma_i^M} | v \in V\}, \tag{11}$$

and the extension $\mathrm{Tr}_{iM}^{-1} : V_i^M \to V(\Omega_i^M)$ defined by

$$\begin{aligned} &\gamma(\mathrm{Tr}_{iM}^{-1}\bar{u}_i^M)|_{\Gamma_i^M} = \bar{u}_i^M, \quad i = 1, \ldots, I_M, \quad M = 1, \ldots, \mathcal{J} \\ &a_i^M(\mathrm{Tr}_{iM}^{-1}\bar{u}_i^M, v_i^M) = 0 \forall v_i^M \in V^0(\Omega_i^M), \mathrm{Tr}_{iM}^{-1}\bar{u}_i^M \in V(\Omega_i^M) \quad \text{for } i \in T^M, \quad M = 1, \ldots, \mathcal{J}. \end{aligned} \tag{12}$$

For subdomains Ω^{*j}, $j = 1, \ldots, P_c$, we completed definition Tr_{iM}^{-1} with boundary condition

$$\sum_{[i,M]\in\vartheta^j} (\mathrm{Tr}_{iM}^{-1}\bar{u}_i^M)_n = 0 \quad \text{on } \Gamma_{cj}, \quad \text{for } j = 1, \ldots, P_c,$$

i.e.

$$\sum_{[i,M]\in\vartheta^j} a_i^M(\mathrm{Tr}_{iM}^{-1}\bar{u}_i^M, v_i^M) = 0 \quad \forall(v_i^M, [i, M] \in \vartheta^j) : v_i^M \in V^0(\Omega_i^M) \quad \text{so that}$$

$$\sum_{[i,M]\in\vartheta^j} (v_i^M)_n = 0 \quad \text{on } \Gamma_{cj}, \ j = 1, \ldots, P_c. \tag{13}$$

Definition 2.1. The local Schur complement, for $i \in T^M$, $M = 1, \ldots, \mathcal{J}$, is operator $S_i^M : V_i^M \to (V_i^M)^*$ defined by

$$\langle S_i^M \bar{u}_i^M, \bar{v}_i^M \rangle = a_i^M(\mathrm{Tr}_{iM}^{-1}\bar{u}_i^M, \mathrm{Tr}_{iM}^{-1}\bar{v}_i^M) \quad \forall \bar{u}_i^M, \bar{v}_i^M \in V_i^M. \tag{14}$$

In matrix form, we have

$$S_i^M \bar{U}_i^M = (\bar{A}_{iM} - B_{iM}^T \mathring{A}_{iM}^{-1} B_{iM})\bar{U}_i^M, \tag{15}$$

where we decompose the degrees of freedom U_i of u_i into internal degrees of freedom $\mathring{U}_i^M$ and interface degrees of freedom $\bar{U}_i^M$:

$$U_i^M = [\mathring{U}_i^M, \bar{U}_i^M]^T.$$

With this decomposition, the matrix representation of $a_i^M(.,.)$ on $H^1(\Omega_i^M)$ takes the form

$$A_{iM} = \begin{bmatrix} \mathring{A}_{iM} & B_{iM} \\ B_{iM}^T & \bar{A}_{iM} \end{bmatrix}. \tag{16}$$

Definition 2.2. The combined local Schur complement, for subdomains Ω^{*j}, $j = 1, \ldots, P_c$, is operator

$$S_{*j} : (V_i^M, [i, M] \in \vartheta^j) \to (V_i^M, [i, M] \in \vartheta^j)^*, \quad j = 1, \ldots, P_c,$$

defined by

$$\begin{aligned}&\langle S_{*j}(\bar{u}_i^M, [i,M] \in \vartheta^j), (\bar{v}_i^M, [i,M] \in \vartheta^j)\rangle \\ &\quad = \sum_{[i,M]\in\vartheta^j} a_i^M(u_i^M(\bar{u}_i^M), \mathrm{Tr}_{iM}^{-1}\bar{v}_i^M) \quad \forall(\bar{v}_i^M, [i,M] \in \vartheta^j) \in (V_i^M, [i,M] \in \vartheta^j),\end{aligned} \tag{17}$$

where $u_i^M(\bar{u}_i^M)$ is the solution of the problem (10) and $\bar{R}_i^M\bar{u} \equiv \bar{u}_i^M, [i,M] \in \vartheta^j$.

Lemma 2.1. *The condition (8) for the function $\bar{u}$ on interface Γ is equivalent to the condition (18):*

$$\begin{aligned}&\sum_{M=1}^{\mathcal{J}}\sum_{i\in T^M}\langle S_i^M\bar{u}_i^M, \bar{w}_i^M\rangle + \sum_{j=1}^{P_c}\langle S_{*j}(\bar{u}_i^M, [i,M] \in \vartheta^j), (\bar{w}_i^M, [i,M] \in \vartheta^j)\rangle \\ &\quad = \sum_{M=1}^{\mathcal{J}}\sum_{i\in I_M} L_i^M(\mathrm{Tr}_{iM}^{-1}\bar{w}_i^M) \quad \forall\bar{w} \in V_\Gamma, \text{ where } \bar{w}_i^M = \bar{R}_i^M\bar{w}, \bar{u}_i^M = \bar{R}_i^M\bar{u},\end{aligned} \tag{18}$$

by using the local Schur complements.

Proof. See [5]. □

We rewrite the condition (18) in the form

$$S_0\bar{U} + S_{\mathrm{KON}}\bar{U} = F, \tag{19}$$

where

$$S_0 = \sum_{M=1}^{\mathcal{J}}\sum_{i\in T^M}(\bar{R}_i^M)^T S_i^M \bar{R}_i^M,$$

$$F = \sum_{M=1}^{\mathcal{J}}\sum_{i\in I_M}(\bar{R}_i^M)^T(\mathrm{Tr}_{iM}^{-1})^T L_i^M,$$

$$S_{\mathrm{KON}} = \sum_{j=1}^{P_c}\bar{R}_{*j}^T S_{*j}\bar{R}_{*j},$$

and

$$\bar{R}_{*j}\bar{u} = (\bar{R}_i^M\bar{u}, [i,M] \in \vartheta^j)^T, \quad \bar{u} \in V_\Gamma \quad \forall j = 1, \dots, P_c.$$

Since the operator S_{KON} is non-linear, we solve the Eq. (19) successive aproximations method. We choose the solution of the auxiliary linear problem as an initial aproximation $\bar{U}^0$. In the auxiliary problem, we replace the set K by

$$K^0 = \left\{ v \in V | \sum_{[i,M]\in\vartheta^j}(v_i^M)_n = 0 \quad \text{on } \Gamma_{cj}, j = 1, \dots, P_c \right\}$$

and we obtain

$$u_0 = \arg\min_{v\in K^0} \mathcal{L}(v),$$

$$\bar{U}^0 = \gamma u^0|_\Gamma.$$

Now, we come back to Eq. (19) and we compute $\bar{U}^k$ as the solution of the linear problem

$$S_0\bar{U}^k = F - S_{\mathrm{KON}}\bar{U}^{k-1}, \quad k = 1, 2, \dots \tag{20}$$

3. The auxiliary problem

We solve the variational equation

$$u^0 \in K^0, \quad D\mathcal{L}(u^0, v) = 0 \quad \forall v \in K^0. \tag{21}$$

There exists a unique solution of the problem (21). For problem (21) we can describe analogy of Theorem 1.1.

Theorem 3.1. *A function $u^0 \in K^0$* is the solution of the auxiliary problem (21) *if and only if the function u^0 satisfies*

1. $$\sum_{M=1}^{\mathcal{J}}\sum_{i=1}^{I_M}\left(a_i^M(u_i^{0M}(\bar{u}^0), \gamma^{-1}\bar{w}) - L_i^M(\gamma^{-1}\bar{w})\right) = 0 \quad \forall \bar{w} \in V_\Gamma, \bar{u}^0 \in V_\Gamma, \tag{22}$$
   for the trace $\bar{u}^0 = \gamma u^0|_\Gamma$ on the interface Γ.
2. *Its restriction $u_i^{0M}(\bar{u}) \equiv u^0|_{\Omega_i^M}$ satisfies following conditions*:
   (a)
   $$\begin{aligned} &a_i^M(u_i^{0M}(\bar{u}^0), \phi_i^M)\\ &\quad = L_i^M(\phi_i^M) \quad \forall \phi_i^M \in V^0(\Omega_i^M), \quad u_i^{0M}(\bar{u}^0) \in V(\Omega_i^M), \gamma u_i^{0M}(\bar{u}^0)|_{\Gamma_i^M} = \bar{R}_i^M \bar{u}^0, \end{aligned} \tag{23}$$
   for $i \in T^M$, $M = 1, \dots, \mathcal{J}$,
   (b)
   $$\begin{aligned} &\sum_{[i,M]\in\vartheta^j} a_i^M(u_i^{0M}(\bar{u}^0), \phi_i^M)\\ &\quad = \sum_{[i,M]\in\vartheta^j} L_i^M(\phi_i^M) \;\; \forall\phi \equiv (\phi_i^M, [i,M] \in \vartheta^j), \phi_i^M \in V^0(\Omega_i^M) \;\text{ such that } \phi \in K^0 \end{aligned} \tag{24}$$
   for $j = 1, \dots, P_c$.

Proof. See [5]. □

Definition 3.1. We define a combined local Schur complements for subdomains Ω^{*j}

$$S_{*j}^0 : (V_i^M, [i,M] \in \vartheta^j) \to (V_i^M, [i,M] \in \vartheta^j)^*, \qquad j = 1, \dots, P_c$$

by

$$\langle S^0_{*j}(\bar{u}^{0M}_i, [i, M] \in \vartheta^j), (\bar{v}^M_i, [i, M] \in \vartheta^j)\rangle = \sum_{[i,M]\in\vartheta^j} a^M_i(\mathrm{Tr}^{-1}_{iM} u^{0M}_i, \mathrm{Tr}^{-1}_{iM}\bar{v}^M_i) \quad \forall(\bar{v}^M_i, [i, M] \in \vartheta^j) \in (V^M_i, [i, M] \in \vartheta^j). \tag{25}$$

Lemma 3.1. *The condition (22) for the function $\bar{u}$ on interface Γ is equivalent to the condition (26):*

$$\sum_{M=1}^{\mathcal{J}} \sum_{i\in T^M} \langle S^M_i \bar{u}^{0M}_i, \bar{w}^M_i\rangle + \sum_{j=1}^{P_c} \langle S^0_{*j}(\bar{u}^{0M}_i, [i, M] \in \vartheta^j), (\bar{w}^M_i, [i, M] \in \vartheta^j)\rangle = \sum_{M=1}^{\mathcal{J}} \sum_{i\in I_M} L^M_i(\mathrm{Tr}^{-1}_{iM}\bar{w}^M_i) \quad \forall\bar{w} \in V_\Gamma, \textit{where } \bar{w}^M_i = \bar{R}^M_i \bar{w}, \bar{u}^{0M}_i = \bar{R}^M_i \bar{u}^0. \tag{26}$$

Proof. See [5]. □

Definition 3.2. We define a global Schur complement:

$$S = \sum_{j=1}^{P_c} \bar{R}^T_{*j} S^0_{*j} \bar{R}_{*j} + \sum_{M=1}^{\mathcal{J}} \sum_{i\in T^M} (\bar{R}^M_i)^T S^M_i \bar{R}^M_i, \tag{27}$$

and the condition (26) on the interface Γ has form

$$S\bar{U} = F, \tag{28}$$

in dual space $(V_\Gamma)^*$.

Eq. (28) we solve by a preconditioned conjugate gradient method PCG1:

Choose $\bar{U}^{[0]}$ and $H^{[0]}$ (by Eq. (51)), $\bar{P}^{[0]} = 0$.
Iteration loop on n.
Compute the preconditioned direction of descent $G^{[n]} = \mathcal{M}^{-1} H^{[n]}$,
Compute $P^{[n]} = G^{[n]} + (\langle H^{[n]}, G^{[n]}\rangle / \langle H^{[n-1]}, G^{[n-1]}\rangle) P^{[n-1]}$,
On each subdomain Ω^M_i, $i \in T^M$, $M = 1, \dots, \mathcal{J}$,
solve in parallel Dirichlet problem

$$\mathring{A}_{iM} \mathring{U}^M_i = B_{iM} \bar{R}^M_i P^{[n]},$$

On subdomain Ω^{*j}, $j = 1, \dots, P_c$ compute $S^0_{*j} \bar{R}_{*j} P^{[n]}$,
Compute

$$Z^{[n]} = SP^{[n]} = \sum_{j=1}^{P_c} (\bar{R}_{*j})^T S^0_{*j} \bar{R}_{*j} P^{[n]} + \sum_{M=1}^{\mathcal{J}} \sum_{i\in T^M} (\bar{R}^M_i)^T S^M_i \bar{R}^M_i P^{[n]} = \sum_{j=1}^{P_c} (\bar{R}_{*j})^T S^0_{*j} \bar{R}_{*j} P^{[n]} + \sum_{M=1}^{\mathcal{J}} \sum_{i\in T^M} (\bar{R}^M_i)^T (\bar{A}_{iM} \bar{R}^M_i P^{[n]} - B^T_{iM} \mathring{U}^M_i),$$

$$\alpha^{[n]} = \frac{\langle H^{[n]}, G^{[n]}\rangle}{\langle Z^{[n]}, P^{[n]}\rangle},$$

$$H^{[n+1]} = H^{[n]} - \alpha^{[n]} Z^{[n]},$$

$$\bar{U}^{[n+1]} = \bar{U}^{[n]} + \alpha^{[n]} P^{[n]},$$

End loop on n.

This method does not require the explicit construction of the local Schur complement matrix S_i^M but does require an efficient preconditioner $\mathcal{M}^{-1}$ (see [3,4]). By construction, the Schur complement operator is defined by the sum (27). Its inverse $(S_i^M)^{-1}$, $i \in T^M$, $M = 1, \ldots, \mathcal{J}$ simply consists in associating to the generalized derivative $g \in (V_i^M)^*$ the trace $\gamma\phi_i^M$ on Γ_i^M of the solution ϕ_i^M of the corresponding Neumann problem in variational form

$$a_i^M(\phi_i^M, v) = \langle g, \gamma v|_{\Gamma_i^M}\rangle \quad \forall v \in V(\Omega_i^M),\ \phi_i^M \in V(\Omega_i^M). \tag{29}$$

Then

$$(S_i^M)^{-1} g = \gamma\phi_i^M|_{\Gamma_i^M}. \tag{30}$$

Similarly, we define inverse operator

$$(S_{*j}^0)^{-1} : (V_i^M, [i, M] \in \vartheta^j)^* \to (V_i^M, [i, M] \in \vartheta^j), \quad j = 1, \ldots, P_c.$$

We solve the Neumann problem for $(g_i^M, [i, M] \in \vartheta^j) \in (V_i^M, [i, M] \in \vartheta^j)^*$

$$\sum_{[i,M]\in\vartheta^j} a_i^M(\phi_i^M, v_i^M) = \sum_{[i,M]\in\vartheta^j} \langle g_i^M, \gamma v_i^M|_{\Gamma_i^M}\rangle \quad \forall v, \phi \in \hat{V}_j \tag{31}$$

where $\hat{V}_j = \{(v_i^M, [i, M] \in \vartheta^j) | v_i^M \in V(\Omega_i^M), \sum_{[i,M]\in\vartheta^j} (v_i^M)_n = 0 \text{ on } \Gamma_{cj}\}$.

Then

$$(S_{*j}^0)^{-1}(g_i^M, [i, M] \in \vartheta^j) = (\gamma\phi_i^M|_{\Gamma_i^M}, [i, M] \in \vartheta^j). \tag{32}$$

Definition 3.3. We define an injection

$$\begin{aligned} &D_i^M : V_i^M \to V_\Gamma, \quad i \in T^M, \quad M = 1, \ldots, \mathcal{J}, \\ &D_{*j} : (D_i^M, [i, M] \in \vartheta^j) \to V_\Gamma,\ D_{*j} = (D_i^M, [i, M] \in \vartheta^j), \quad j = 1, \ldots, P_c, \end{aligned} \tag{33}$$

such that on each interface degree of freedom is

$$D_i^M \bar{v}(P_l) = \bar{v}(P_k)\frac{\varrho_i^M}{\varrho_T}, \qquad i = 1, 2, \ldots, I_M, \quad M = 1, \ldots, \mathcal{J}, \tag{34}$$

if the lth degree of freedom of V_Γ corresponds to the kth degree of freedom of V_i^M and

$$D_i^M \bar{v}(P_l) = 0, \qquad \text{if not.} \tag{35}$$

Here, ϱ_i^M is a local measure of the stiffness of subdomain Ω_i^M (for example, an average Young modulus on Ω_i^M) and

$$\varrho_T = \sum_{P_l \in \bar{\Omega}_j^M} \varrho_j^M, \tag{36}$$

is the sum of ϱ_j^M on all subdomains Ω_j^M containing P_l.

We define

$$\mathcal{M}^{-1} = \sum_{M=1}^{\mathcal{J}} \sum_{i \in T^M} D_i^M (S_i^M)^{-1} (D_i^M)^T + \sum_{j=1}^{P_c} D_{*j} (S_{*j}^0)^{-1} D_{*j}^T. \tag{37}$$

In operator form, the action of $\mathcal{M}^{-1}$ on $L \in (V_\Gamma)^*$ is thus given by

$$\mathcal{M}^{-1} L = \sum_{M=1}^{\mathcal{J}} \sum_{i \in T^M} D_i^M \bar{U}_i^M + \sum_{j=1}^{P_c} D_{*j} \bar{U}_{*j},$$

$$\langle S_i^M \bar{U}_i^M, \bar{V}_i^M \rangle = \langle L, D_i^M \bar{V}_i^M \rangle \quad \forall \bar{V}_i^M \in V_i^M, \quad \bar{U}_i^M \in V_i^M, \quad \text{for } i \in T^M, M = 1, \dots, \mathcal{J} \tag{38}$$

and

$$\langle S_{*j}^0 \bar{U}_{*j}, \bar{V}_j \rangle = \langle L, D_{*j} \bar{V}_j \rangle \quad \forall \bar{V}_j \in \hat{V}_j, \quad \bar{U}_{*j} \in \hat{V}_j, \quad \text{for } j = 1, \dots, P_c. \tag{39}$$

The original Neumann–Neumann preconditioner supposes that the solution of each local Neumann problem is uniquely defined, whereas rigid body motions are possible. This weakness can be fixed by replacing $(S_i^M)^{-1}$, $(S_{*j}^0)^{-1}$ by a regularized inverse $(\tilde{S}_i^M)^{-1}$, respectively $(\tilde{S}_{*j}^0)^{-1}$. We introduce on each subdomain Ω_i^M, respectively Ω^{*j} a small local coarse space Z_i^M, respectively Z^{*j} containing all rigid body motions.

The general trick to upgrade the original preconditioner then consists in adding to the initial local contribution ϕ_i^M, respectively ϕ_j a “bad” $z_i^M \in Z_i^M$, respectively $z_j \in Z^{*j}$ which is chosen in order to get the smallest difference $(\mathcal{M}^{-1} - S^{-1})$.

We suppose that L satisfies the invariance property

$$\langle L, D_i^M \gamma z_i^M \rangle = 0 \quad \forall z_i^M \in Z_i^M, \quad i \in T^M, \quad M = 1, \dots, \mathcal{J}, \tag{40}$$

$$\langle L, D_{*j} \gamma z_j \rangle \equiv \sum_{[i,M] \in \vartheta^j} \langle L, D_i^M \gamma z_i^M \rangle = 0 \quad \forall z_j \in Z^{*j}, \quad j = 1, \dots, P_c, \tag{41}$$

We introduce a closed orthogonal complement space $Q(\Omega_i^M)$ of Z_i^M in $V(\Omega_i^M)$ and a closed orthogonal complement space $Q(\Omega^{*j})$ of Z^{*j} in $\hat{V}_j$. Let then $\phi_i^{0M} \in Q(\Omega_i^M)$ be the particular solution of the variational problem (29) defined by

$$a_i^M(\phi_i^{0M}, v_i^M) = \langle L, D_i^M (\gamma v_i^M)|_{\Gamma_i^M} \rangle \qquad \forall v_i^M \in V(\Omega_i^M) \tag{42}$$

and $\phi^0_{*j} = (\phi_i^{0M}, [i, M] \in \vartheta^j) \in Q(\Omega^{*j})$ be the particular solution of the variational problem (31) defined by

$$\sum_{[i,M]\in\vartheta^j} a_i^M(\phi_i^{0M}, v_i^M) = \sum_{[i,M]\in\vartheta^j} \langle L, D_i^M(\gamma v_i^M)|_{\Gamma_i^M}\rangle \quad \forall v_j \in \hat{V}_j. \tag{43}$$

Eqs. (42) and (43) are well posed varitional problems set on $Q(\Omega_i^M)$, $Q(\Omega^{*j})$.

Definition 3.4. We define our new Neumann–Neumann preconditioner $\mathcal{M}^{-1}(z^0)$ by

$$\mathcal{M}^{-1}(z^0)L = \sum_{M=1}^{\mathcal{J}} \sum_{i=1}^{I_M} D_i^M \gamma(\phi_i^{0M} + z_i^{0M})|_{\Gamma_i^M}, \tag{44}$$

with the solution z_i^{0M} of the minimization problem

$$z^0 = \arg\min_{z\in\Pi Z} \underbrace{\langle S(\mathcal{M}^{-1}(z) - S^{-1})L, (\mathcal{M}^{-1}(z) - S^{-1})L\rangle}_{J(z)}, \tag{45}$$

$$\Pi Z \equiv (\times_{i\in T^M, M=1,\dots,\mathcal{J}}(Z_i^M)) \times (\times_{j=1,\dots,P_c}(Z^{*j})).$$

By construction, and since L satisfies (40) and (41), we have

$$J(z) = \left\langle S\sum_{M=1}^{\mathcal{J}}\sum_{i=1}^{I_M} D_i^M\gamma z_i^M, \sum_{M=1}^{\mathcal{J}}\sum_{j=1}^{I_M}(D_j^M\gamma z_j^M + 2D_j^M\gamma\phi_j^{0M})\right\rangle + \text{constant} \tag{46}$$

Its minimum is attained for the function z^0 which cancels its gradient, that is for the solution of the variational coarse equality

$$\left\langle S\sum_{M=1}^{\mathcal{J}}\sum_{j=1}^{I_M} D_j^M\gamma z_j^M, \sum_{M=1}^{\mathcal{J}}\sum_{i=1}^{I_M} D_i^M\gamma z_i^{0M}\right\rangle = -\left\langle S\sum_{M=1}^{\mathcal{J}}\sum_{j=1}^{I_M} D_j^M\gamma z_j^M, \sum_{M=1}^{\mathcal{J}}\sum_{i=1}^{I_M} D_i^M\gamma\phi_i^{0M}\right\rangle \; \forall z \in \Pi Z. \tag{47}$$

The upgraded Neumann–Neumann preconditioner (44) is therefore obtained by first solving the local Neumann problems (42) and (43) and then the variational coarse problem (47) set on the coarse product space ΠZ.

We introduce the coarse trace space

$$V_H = \sum_{M=1}^{\mathcal{J}} \sum_{i\in T^M} D_i^M\gamma Z_i^M + \sum_{j=1}^{P_c} D_{*j}\gamma Z_{*j}, \tag{48}$$

a set $V_H^\perp \subset (V_\Gamma)^*$ given by

$$L \in V_H^\perp \Leftrightarrow \langle L, z\rangle = 0 \quad \forall z \in V_H$$

and the S-orthogonal projection P_Z from V_Γ onto V_H given by

$$\langle Sz, \bar{U} - P_Z\bar{U}\rangle = 0 \quad \forall z \in V_H, \forall\bar{U} \in V_\Gamma, P_Z\bar{U} \in V_H. \tag{49}$$

Under this notation, the coarse problem (47) can be written as

$$\sum_{M=1}^{\mathcal{J}}\sum_{i=1}^{I_M} D_i^M \gamma z_i^{0M} = -P_Z \sum_{M=1}^{\mathcal{J}}\sum_{i=1}^{I_M} D_i^M \gamma \phi_i^{0M},$$

and thus the new Neumann–Neumann preconditioner (44) takes the final form

$$\mathcal{M}^{-1}(z^0)L = (I - P_Z)\sum_{M=1}^{\mathcal{J}}\sum_{i=1}^{I_M} D_i^M \gamma \phi_i^{0M}. \tag{50}$$

Lemma 3.2. *Suppose that* $H^{[0]} \in V_H^{\perp}$, $P^{[0]} = 0$ *in algorithm PCG1using preconditioner* (44). *Then* $H^{[n]} \in V_H^{\perp}, n = 1, 2, \ldots$

Proof. See [5]. □

According to Lemma 3.2, we must only suppose that $H^{[0]} \in V_H^{\perp}$, which is achieved by setting the initial solution $\bar{U}^{[0]} \in V_H^{\perp}$ to the solution of the coarse problem

$$\langle H^{[0]}, z\rangle \equiv \langle F - S\bar{U}^{[0]}, z\rangle = 0 \quad \forall z \in V_H. \tag{51}$$

This coarse problem is identical to (47) within a change of right-hand side and thus the conditions (40) and (41) do not restrict the generality of the proposed preconditioner.

4. The original problem

Now, we solve by the successive approximations method, Eq. (20). We must effectively compute the solution $\bar{U}^k$ of the linear problem

$$S_0\bar{U}^k = b^k, \tag{52}$$

with

$$S_0 = \sum_{M=1}^{\mathcal{J}}\sum_{i\in T^M} (\bar{R}_i^M)^T S_i^M \bar{R}_i^M, \qquad b^k = F - S_{\mathrm{KON}}\bar{U}^{k-1},$$

$$F = \sum_{M=1}^{\mathcal{J}}\sum_{i\in I_M} (\bar{R}_i^M)^T (\mathrm{Tr}_{iM}^{-1})^T L_i^M, \qquad S_{\mathrm{KON}} = \sum_{j=1}^{P_c} \bar{R}_{*j}^T S_{*j} \bar{R}_{*j}.$$

The Eq. (52) we solve by a preconditioned conjugate gradient method PCG2(we costruct the sequence of the iterations $\bar{\omega}^{[n]} \to \bar{U}^k$ for $n \to \infty$):

Choose $\bar{\omega}^{[0]}$ (by the Eq. (64)), $\eta^{[0]} = b^k - S_0\bar{\omega}^{[0]}$, $\pi^{[0]} = 0$.
Iteration loop on n:
Compute the preconditioned direction of descent $\kappa^{[n]} = \mathcal{M}_0^{-1}\eta^{[n]}$
 Compute $\pi^{[n]} = \kappa^{[n]} + (\langle \eta^{[n]}, \kappa^{[n]}\rangle / \langle \eta^{[n-1]}, \kappa^{[n-1]}\rangle)\pi^{[n-1]}$

On each subdomain Ω_i^M, $i \in T^M$, $M = 1, \dots, \mathcal{J}$,
solve in parallel Dirichlet problem

$$\mathring{A}_{iM}\mathring{\Omega}_i^M = B_{iM}\bar{R}_i^M \pi^{[n]},$$

Compute

$$\xi^{[n]} = S_0\pi^{[n]} = \sum_{M=1}^{\mathcal{J}} \sum_{i\in T^M} (\bar{R}_i^M)^T (\bar{A}_{iM}\bar{R}_i^M \pi^{[n]} - B_{iM}^T \mathring{\omega}_i^M),$$

$$\alpha^{[n]} = \frac{\langle \eta^{[n]}, \kappa^{[n]}\rangle}{\langle \xi^{[n]}, \pi^{[n]}\rangle},$$

$$\eta^{[n+1]} = \eta^{[n]} - \alpha^{[n]}\xi^{[n]},$$

$$\bar{\omega}^{[n+1]} = \bar{\omega}^{[n]} + \alpha^{[n]}\pi^{[n]},$$

End loop on n:

Now, we define a preconditioner $\mathcal{M}_0^{-1}$:

$$\mathcal{M}_0^{-1} = \sum_{M=1}^{\mathcal{J}} \sum_{i\in T^M} D_i^M (S_i^M)^{-1} (D_i^M)^T \tag{53}$$

with a new injection D_i^M.

Definition 4.1. We define an injection

$$D_i^M : V_i^M \to V_\Gamma, \quad i \in T^M, \quad M = 1, \dots, \mathcal{J}, \tag{54}$$

such that on each interface degree of freedom is

$$D_i^M \bar{v}(P_l) = \bar{v}(P_k) \quad \text{if } P_k \in \Gamma_i^M \subset \partial\Omega^{*j} \quad \text{for any } j \in \{1, \dots, P_c\}, \tag{55}$$

$$D_i^M \bar{v}(P_l) = \bar{v}(P_k)\frac{\varrho_i^M}{\varrho_T} \quad \text{for } P_k \in \Gamma_i^M \not\subset \partial\Omega^{*j} \quad \forall j = 1, \dots, P_c, \tag{56}$$

if the lth degree of freedom of V_Γ corresponds to the kth degree of freedom of V_i^M and

$$D_i^M \bar{v}(P_l) = 0 \quad \text{if not.} \tag{57}$$

Let $\phi_i^{0M} \in Q(\Omega_i^M)$ be the particular solution of the variational problem (29) defined by (42). Similarly to the auxiliary problem we now define a new preconditioner.

Definition 4.2. We define our new Neumann–Neumann preconditioner $\mathcal{M}_0^{-1}$ by

$$\mathcal{M}_0^{-1}(z^0)L = \sum_{M=1}^{\mathcal{J}} \sum_{i\in T^M} D_i^M \gamma(\phi_i^{0M} + z_i^{0M})|_{\Gamma_i^M}, \tag{58}$$

with the solution z_i^{0M} of the minimization problem

$$z^0 = \arg\min_{z\in\Pi_0 Z} \langle S_0(\mathcal{M}_0^{-1}(z) - S_0^{-1})L, (\mathcal{M}_0^{-1}(z) - S_0^{-1})L\rangle, \tag{59}$$

$$\Pi_0 Z \equiv \times_{i\in T^M, M=1,\dots,\mathcal{J}}(Z_i^M).$$

Its minimum is attained for the function z^0 which cancels its gradient, i.e. for the solution of the variational coarse equality

$$\left\langle S_0 \sum_{M=1}^{\mathcal{J}} \sum_{j\in T^M} D_j^M \gamma z_j^M, \sum_{M=1}^{\mathcal{J}} \sum_{i\in T^M} D_i^M \gamma z_i^{0M} \right\rangle = -\left\langle S_0 \sum_{M=1}^{\mathcal{J}} \sum_{j\in T^M} D_j^M \gamma z_j^M, \sum_{M=1}^{\mathcal{J}} \sum_{i\in T^M} D_i^M \gamma \phi_i^{0M} \right\rangle \quad \forall z \in \Pi_0 Z. \tag{60}$$

We introduce the coarse trace space

$$V_{oH} = \sum_{M=1}^{\mathcal{J}} \sum_{i\in T^M} D_i^M \gamma Z_i^M, \tag{61}$$

a set $V_{oH}^{\perp} \subset (V_\Gamma)^*$ given by

$$L \in V_{oH}^{\perp} \Leftrightarrow \langle L, z\rangle = 0 \quad \forall z \in V_{oH},$$

and the S_0-orthogonal projection P_{oZ} from V_Γ onto V_{oH} given by

$$\langle S_0 z, \bar{U} - P_{oZ}\bar{U}\rangle = 0 \quad \forall z \in V_{oH}, \forall \bar{U} \in V_\Gamma, P_{oZ}\bar{U} \in V_{oH}. \tag{62}$$

Under this notation, the coarse problem (60) can be written as

$$\sum_{M=1}^{\mathcal{J}} \sum_{i\in T^M} D_i^M \gamma z_i^{0M} = -P_{oZ} \sum_{M=1}^{\mathcal{J}} \sum_{i\in T^M} D_i^M \gamma \phi_i^{0M},$$

and thus the new Neumann–Neumann preconditioner (58) takes the final form

$$\mathcal{M}_0^{-1}(z^0)L = (I - P_{oZ}) \sum_{M=1}^{\mathcal{J}} \sum_{i\in T^M} D_i^M \gamma \phi_i^{0M}. \tag{63}$$

Lemma 4.1. *Suppose that* $\eta^{[0]} \in V_{oH}^{\perp}$, $\pi^{[0]} = 0$ *in algorithm PCG2 using by preconditioner* (58), *then* $\eta^{[n]} \in V_{oH}^{\perp}, n = 1, 2, \dots$

According to Lemma 4.1 we must only suppose that $\eta^{[0]} \in V_{oH}^{\perp}$, which is achieved by setting the initial solution $\bar{\omega}^{[0]} \in V_{oH}^{\perp}$ to the solution of the coarse problem

$$\langle \eta^{[0]}, z\rangle \equiv \langle b^k - S_0\bar{\omega}^{[0]}, z\rangle = 0 \quad \forall z \in V_{oH}. \tag{64}$$

A convergence theorem requires to introduce some definitions. Let Θ be an ortogonal complement of V_{oH} in V_Γ. We introduce seminorms

$$|\bar{R}_{*j}\bar{v}|_{a_{*j}} = \sqrt{\sum_{[i,M]\in\vartheta^j} a_i^M(\mathrm{Tr}_{iM}^{-1}\bar{R}_i^M\bar{v}, \mathrm{Tr}_{iM}^{-1}\bar{R}_i^M\bar{v})}, \qquad j = 1, \ldots, P_c.$$

Lemma 4.2. *The expression*

$$\|\bar{u}\|_Q^2 = \langle S_0\bar{u}, \bar{u}\rangle$$

is a norm on Θ *where*

$$Q = \times_{i,M:i\in T^M;M=1,\ldots,\mathcal{J}} Q(\Omega_i^M)$$

Definition 4.3. Let $\mathcal{T}\colon \Theta \to \Theta$ be a mapping defined by

$$\langle S_0(\mathcal{T}\bar{y}), \bar{v}\rangle = \langle F - S_{\mathrm{KON}}(\bar{y}), \bar{v}\rangle \quad \forall \bar{v} \in \Theta. \tag{65}$$

Theorem 4.1. *Assume that there exists a constant* $\lambda < (1/\sqrt{2P_c})$ *such that the following condition hold*:

$$|\bar{R}_{*j}\bar{u}|_{a_{*j}} \le \lambda\|\bar{u}\|_Q \quad \forall \bar{u} \in \Theta, \quad \forall j \in \{1, \ldots, P_c\}. \tag{66}$$

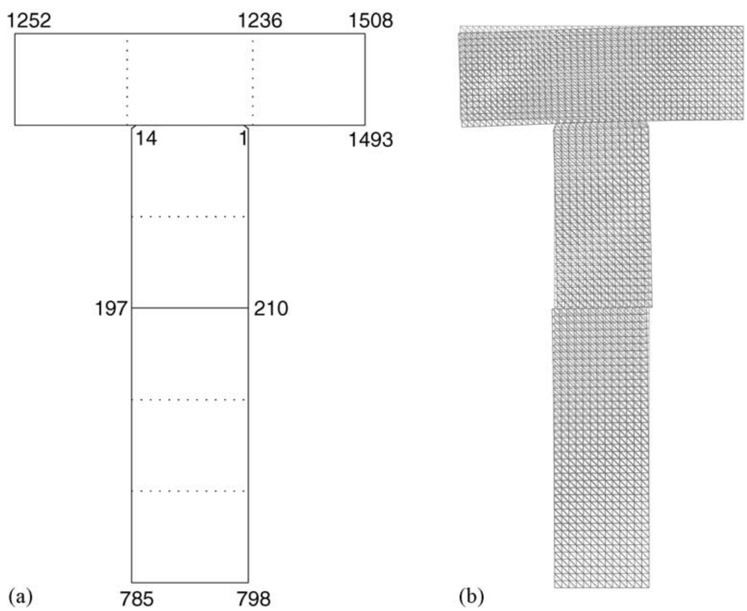

Fig. 2. Model problem—(a) geometry and (b) deformations.

Then the mapping $\mathcal{T}$ is the contraction on Θ. If $\bar{U}^0 \in \Theta$ then the sequence of the iterations $\bar{U}^k$, computed by (52), *are convergent and the limit is a fixed point $\bar{U}$ of the mapping $\mathcal{T}$. The following error estimate holds*

$$\|\bar{U}^k - \bar{U}\|_Q \leq \frac{(2\lambda^2 P_c)^k}{1 - 2\lambda^2 P_c} \|\bar{U}^0 - \mathcal{T}\bar{U}^0\|_Q.$$

Proof. See [5]. □

5. Numerical experiments

In this section, we illustrate the practical behavior of our algorithm on the solution of a model problem. The introduced algorithm has been implemented in the program system MATLAB Version 5.2.1 and in MPI Version 1.2.0 by using FORTRAN 77 compiler. A geometry of the problem is in Fig. 2(a).

Material parameters: Three regions with Young's modulus $E = 10^{10}$ (Pa) and Poisson's ratio $\nu = 0, 3$.

Boundary conditions: Prescribed zero displacement on 785–798. Pressure 0.18×10^7 (Pa) on 1252–1236. Bilateral contact boundary: 1493–1508. Unilateral contact boundary: 14–1, 197–210.

Discretization statistics: Eight subdomains, 1748 nodes, 3040 elements, 3304 unknowns, 28 unilateral contact conditions, 68 interface elements, dimension of the coarse equality for the auxiliary problem is 13, dimension of the coarse equality for the original problem is 7.

Convergence statistics: Nine iterations of the PCG1 algorithm for the auxiliary problem, 17 iterations of the successive approximations method, total 38 iterations of the PCG2 algorithm for the original problem.

Fig. 2(b) represents deformations in model problem.

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Daněk, J., Hlaváček, I., Nedoma, J.: *Domain decomposition for generalized unilateral semi-coercive contact problem with given friction in elasticity*, J. Mathematics and Computers in Simulation 68/3, pp. 271–300, Elsevier Science Ltd., North-Holland 2005.

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Mathematics and Computers in Simulation 68 (2005) 271–300

MATHEMATICS AND COMPUTERS IN SIMULATION

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Domain decomposition for generalized unilateral semi-coercive contact problem with given friction in elasticity

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Received 22 November 2004; received in revised form 20 December 2004; accepted 21 December 2004

Abstract

A non-overlapping domain decomposition is applied to a multibody unilateral contact problem with given friction (Tresca's model). Approximations are proposed on the basis of the primary variational formulation (in terms of displacements) and linear finite elements. For the discretized problem we employ the concept of local Schur complements, grouping every two subdomains which share a contact area. The proposed algorithm of successive approximations can be recommended for "short" contacts only, since the contact areas are not divided by interfaces. The numerical examples show the practical efficiency of the algorithm.

Keywords: Domain decomposition; Unilateral contact; Tresca‘s friction model; Formulation in displacements; Linear finite elements

1. Introduction

In mechanics, geomechanics and biomechanics as well as technological practice there are problems whose investigations lead to solving model problems based on variational formulations. Such problems are described frequently by variational inequalities. Variational inequalities physically describe the principle of virtual work in its inequality form.

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0378-4754/$30.00
doi:10.1016/j.matcom.2004.12.007

Domain decomposition for contact problems has been applied by many authors. An augmented Lagrangian method was used by Dostál et al. [1]. The scalability of the algorithm has been studied by Schöberl [2], Dureisseix and Farhat [3] and Dostál and Horák [4]. Boundary element technique was employed by Kosior et al. [5]. For related papers, we refer to Luo Ping and Liang Guoping [6] and the literature therein.

In the present paper we will deal with numerical solution of a generalized semi-coercive contact problem with the given friction arising in static and quasi-static linear elasticity, for the case that several bodies of arbitrary shapes are in mutual contacts and are loaded by external forces, by using the non-overlapping domain decomposition method. The problem will be formulated as the primary variational inequality problem (see [21]), i.e. in terms of displacements.

We follow the approach proposed by Le Tallec [7] and group every two subdomains which share a contact area into a single "nonlinear" subdomain (see a similar idea used by Barboteu et al. [8]).

Section 2 contains both classical and weak formulation of the problem. Conditions sufficient and necessary for the existence of a weak solution are given. Approximations by linear finite elements on triangulation are proposed in Section 3 together with error estimates. In Section 4 we introduce a nonoverlapping domain decomposition by proving the equivalence of the weak solution on the original domain with that on the interface and subdomains. For the discretized version we employ the concept of local Schur complements. The resulting nonlinear equation on the interface is solved by successive approximations. For the starting approximation we choose the solution of the linear problem, where the unilateral contact conditions are replaced by classical bilateral conditions without friction.

Sections 5 and 6 are devoted to the construction of suitable preconditioning matrices of Neumann–Neumann type. In Section 7 we study the convergence of successive approximations. A sufficient condition and an error estimate is deduced on the basis of contractive mappings theorem. Section 8 contains a numerical test solving a geomechanical model with two domains in contact.

Though the solution of the problem with given friction (Tresca's model) has little physical meaning itself, it can be plugged into an iterative process for the solution of a more realistic Coulomb friction model (see [9–11]).

Since we do not divide the contact areas by interfaces, the proposed algorithm can be recommended for "short" contact only. Such configurations occur e.g. in models of human joints – hips or knees (see [12,13]).

2. Model formulation

Let the investigated part of the elastic body occupy a union Ω of "s" bounded domains Ω^ι, $\iota = 1, \dots, s$ in $\mathbb{R}^2$, with Lipschitz boundaries $\partial\Omega^\iota$. Let the boundary $\partial\Omega = \bigcup_{\iota=1}^{s} \partial\Omega^\iota$ consist of four disjoint parts, i.e. $\partial\Omega = \overline{\Gamma}_u \cup \overline{\Gamma}_\tau \cup \overline{\Gamma}_c \cup \overline{\Gamma}_o$. Let us denote

$$\Gamma_c^{kl} = \partial\Omega^k \cap \partial\Omega^l,\ \ k, l = 1, \dots, s, k \neq l, \Gamma_c = \cup_{k,l}\Gamma_c^{kl},\ \ \Gamma_u = \cup_{\iota=1}^{s}\Gamma_u^\iota,$$

$$\Gamma_u^\iota = \Gamma_u \cap \partial\Omega^\iota, \Gamma_o^\iota = \Gamma_o \cap \partial\Omega^\iota, \Gamma_\tau = \cup_{\iota=1}^{s}\Gamma_\tau^\iota, \Gamma_\tau^\iota = \Gamma_\tau \cap \partial\Omega^\iota.$$

Assume that either

$$\text{meas}\, \Gamma_c^{kl} > 0 \text{ or } \Gamma_c^{kl} = \emptyset$$

and either

$$\text{meas}\,\Gamma_u^\iota > 0 \text{ or } \Gamma_u^\iota = \emptyset.$$

Let body forces $\mathbf{F}$, surface tractions $\mathbf{P}$, boundary displacements $\mathbf{u}_0$ and slip limits g^{kl} be given.

We have the following problem: find the displacements $\mathbf{u}^\iota$ in all Ω^ι such that

$$\frac{\partial}{\partial x_j}\tau_{ij}(\mathbf{u}^\iota) + F_i^\iota = 0 \text{ in } \Omega^\iota,\ \iota = 1, \dots, s,\ i = 1, 2, \tag{2.1}$$

$$\tau_{ij}(\mathbf{u}^\iota) = c_{ijkm}^\iota e_{km}(\mathbf{u}^\iota) \text{ in } \Omega^\iota,\ \mathbf{u}^\iota = \mathbf{u}_0^\iota \text{ on } \Gamma_u^\iota, \tag{2.2}$$

$$u_n^\iota = 0 \text{ and } \boldsymbol{\tau}_t^\iota = 0 \text{ on } \Gamma_o^\iota,\ \tau_{ij}(\mathbf{u}^\iota)n_j^\iota = P_i^\iota \text{ on } \Gamma_\tau^\iota, \tag{2.3}$$

and on every Γ_c^{kl} the following conditions are satisfied:

$$u_n^k - u_n^l \le 0,\ \tau_n^k \le 0,\ (u_n^k - u_n^l)\tau_n^k = 0, \tag{2.4}$$

$$|\boldsymbol{\tau}_t^{kl}| \le g^{kl}, \tag{2.5}$$

$$|\boldsymbol{\tau}_t^{kl}| < g^{kl} \Rightarrow u_t^k - u_t^l = 0, \tag{2.6}$$

$$|\boldsymbol{\tau}_t^{kl}| = g^{kl} \Rightarrow \exists \vartheta \ge 0,\ u_t^k - u_t^l = -\vartheta \boldsymbol{\tau}_t^{kl}. \tag{2.7}$$

We denote the stress tensor by τ_{ij}, $e_{ij}(\mathbf{u}^\iota) = (1/2)((\partial u_i/\partial x_j) + (\partial u_j/\partial x_i))$,

$$u_n^k = u_i^k n_i^k,\ u_n^l = u_i^l n_i^k = -u_i^l n_i^l\ f(\text{nosumover}\ k \text{ or } l),$$

$$u_t^k = u_1^k n_2^k - u_2^k n_1^k,\ u_t^l = u_1^l n_2^k - u_2^l n_1^k,$$

$$\tau_n^k = \tau_{ij}^k n_i^k n_j^k,\ \boldsymbol{\tau}_t^k = (\tau_{ti}^k),\ \tau_{ti}^k = \tau_{ij}^k n_j^k - \tau_n^k n_i^k,\ \boldsymbol{\tau}_t^{kl} \equiv \boldsymbol{\tau}_t^k.$$

In what follows, we introduce

$$W = \prod_{\iota=1}^{s}[H^1(\Omega^\iota)]^2, \quad \|\mathbf{v}\|_W = \left(\sum_{\iota\le s}\sum_{i\le 2}\|v_i^\iota\|_{1,\Omega^\iota}^2\right)^{1/2},$$

$$V_0 = \{\mathbf{v} \in W | \mathbf{v} = 0 \text{ on } \Gamma_u \text{ and } v_n = 0 \text{ on } \Gamma_o\},\ V = \mathbf{u}_0 + V_0,$$

$$K = \{\mathbf{v} \in V | v_n^k - v_n^l \le 0 \text{ on } \cup_{kl}\Gamma_c^{kl}\}.$$

Assume that $u_{0n}^k - u_{0n}^l = 0$ on $\cup_{k,l}\Gamma_c^{kl}$. Let

$$F_i^\iota \in L^2(\Omega^\iota),\ P_i^\iota \in L^2(\Gamma_\tau^\iota),\ c_{ijkl}^\iota \in L^\infty(\Omega^\iota),\ g^{kl} \in L^\infty(\Gamma_c^{kl}),\ \mathbf{u}_0 \in W.$$

Definition 2.1. A function $\mathbf{u}$ is a weak solution of problem $\mathcal{P}_{sg}$, if $\mathbf{u} \in K$ and

$$a(\mathbf{u}, \mathbf{v} - \mathbf{u}) + j(\mathbf{v}) - j(\mathbf{u}) \ge L(\mathbf{v} - \mathbf{u}) \qquad \forall v \in K, \tag{2.8}$$

where

$$a(\mathbf{u},\mathbf{w}) = \sum_{\iota=1}^{s}\int_{\Omega^\iota} c^\iota_{ijkl} e_{ij}(\mathbf{u}^\iota) e_{kl}(\mathbf{w}^\iota)\,\mathrm{d}x,$$

$$j(\mathbf{v}) = \sum_{k,l}\int_{\Gamma_c^{kl}} g^{kl}|v_t^k - v_t^l|\,\mathrm{d}s,$$

$$L(\mathbf{w}) = \sum_{\iota\le s}\left(\int_{\Omega^\iota} F_i^\iota w_i^\iota\,\mathrm{d}x + \int_{\Gamma_\tau^\iota} P_i^\iota w_i^\iota\,\mathrm{d}s\right).$$

Let us denote the sets of rigid displacements and rotations

$$P^\iota = \{\mathbf{v}^\iota = (v_1^\iota, v_2^\iota) | v_1^\iota = a_1^\iota - b^\iota x_2,\ v_2^\iota = a_2^\iota + b^\iota x_1\},\ P = \prod_{\iota=1}^{s} P^\iota,$$

where a_i^ι, $i = 1, 2$ and b^ι are arbitrary real constants. Let

$$P_V = P \cap V_0, \quad P_0 = \left\{\mathbf{v} \in P_V | v_n^k - v_n^l = 0 \text{ on } \bigcup_{k,l}\Gamma_c^{kl}\right\}.$$

Theorem 2.1. *Assume that*

$$P_0 = \{0\},\ P \cap K \ne \{0\}$$

and

$$L(\mathbf{w}) < j(\mathbf{w}) \quad \forall \mathbf{w} \in P \cap K - \{0\}.$$

Then there exists a weak solution. If

$$|L(\mathbf{w})| > j(\mathbf{w}) \quad \forall \mathbf{w} \in P_V - \{0\},$$

the solution is unique. If

$$|L(\mathbf{w})| \le j(\mathbf{w}) \quad \forall \mathbf{w} \in P_V,$$

then for any two solutions $\mathbf{u}, \mathbf{u}^*$

$$\mathbf{u}^* - \mathbf{u} \in P_V \quad \text{and} \quad L(\mathbf{u}^* - \mathbf{u}) = j(\mathbf{u}^*) - j(\mathbf{u})$$

holds.

For the proof see [14, Theorem 4.1].

Remark 2.1. The following condition is necessary for the existence of a weak solution

$$L(\mathbf{w}) \leq j(\mathbf{w}) \quad \forall \mathbf{w} \in P \cap K.$$

For the proof see Lemma 4.4 by Hlaváček and Nedoma [14].

Remark 2.2. A simple example satisfying the conditions of Theorem 2.1 is given in Remark 4.2 of Hlaváček and Nedoma [14].

3. Finite element approximation

Let the domain $\overline{\Omega} = \bigcup_{\iota=1}^{s} \overline{\Omega^{\iota}}$ be approximated by $\overline{\Omega}_h = \bigcup_{\iota=1}^{s} \overline{\Omega_h^{\iota}}$ with polygonal boundary $\partial\Omega_h = \overline{\Gamma}_{uh} \cup \overline{\Gamma}_{\tau h} \cup \overline{\Gamma}_{ch} \cup \overline{\Gamma}_{oh}$, where $\overline{\Gamma}_{uh}, \overline{\Gamma}_{\tau h}, \overline{\Gamma}_{ch}, \overline{\Gamma}_{oh}$ are piecewise linear. Let $\Omega_h = \bigcup_{\iota=1}^{s} \Omega_h^{\iota}$ be triangulated, let q_i be nodes of used triangulation. Let $\mathcal{T}_h^{\iota}, \iota = 1, \ldots, s$, denote triangulations of polygonal domains $\Omega_h^{\iota}, \iota = 1, \ldots, s$, and $\mathcal{T}_h = \{\mathcal{T}^{\iota}_h, \iota = 1, \ldots, s\}$, We assume that $\mathcal{T}^{\iota}_h, \iota = 1, \ldots, s$, are consistent with the respective decompositions of the boundaries $\partial\Omega_h^{\iota}, \iota = 1, \ldots, s$ and let the nodes lie on Γ_c^{kl} belonging to the triangulations corresponding to the neighbouring subdomains Ω^k and Ω^l being in a mutual contact. The triangulation $\mathcal{T}_h$ is said to be regular, if all $\mathcal{T}^{\iota}_h, \iota = 1, \ldots, s$, are regular, h is the maximal side of the triangulation. For every node q_i of the triangulation $\mathcal{T}_h$ on Γ_c^{kl} and Γ_o we define the set of indices $\mathcal{N}_i^{kl} = \{j \in \{1, \ldots, r\} \mid q_i \in \Gamma_{cj}^{kl}\}$ and $\mathcal{N}_i = \{j \in \{1, \ldots, r'\} \mid q_i \in \Gamma_{oj}\}$, where $\Gamma_c^{kl} = \bigcup_{j=1}^{r} \Gamma_{cj}^{kl}, \Gamma_o = \bigcup_{j=1}^{r'} \Gamma_{oj}, \Gamma_{cj}^{kl}, \Gamma_{oj}$ denote segments on Γ_c^{kl}, Γ_o and r, r' the number of segments on Γ_c^{kl} and Γ_o, respectively.

Let us define a finite dimensional space V_h by

$$V_h = \{\mathbf{v}_h \mid \mathbf{v}_h \in [C(\Omega^1)]^2 \times \cdots \times [C(\Omega^s)]^2, \mathbf{v}_{h|T_{hi}} \in [P_1(T_{hi})]^2, \forall T_{hi} \in \mathcal{T}_h;$$
$$v_{hn}(q_i) = 0, q_i \in \Gamma_o; \mathbf{v}_h(q_i) = \mathbf{u}_0(q_i), q_i \in \Gamma_u\}$$

and a finite dimensional set of admissible displacements

$$K_h = \{\mathbf{v}_h | \mathbf{v}_h \in V_h, (v_{hn}^k - v_{hn}^l)(q_i) \leq 0, q_i \in \Gamma_c^{kl}\}.$$

Definition 3.1. Function $\mathbf{u}_h \in K_h$ is a solution of the problem $(\mathcal{P}_{sg})_h$ if

$$a(\mathbf{u}_h, \mathbf{v}_h - \mathbf{u}_h) + j(\mathbf{v}_h) - j(\mathbf{u}_h) \geq L(\mathbf{v}_h - \mathbf{u}_h) \qquad \forall \mathbf{v}_h \in K_h.$$

Note that in a general case $K_h \not\subset K$.

The next theorem gives the connection between the problem $(\mathcal{P}_{sg})$ and the problem $(\mathcal{P}_{sg})_h$ if $h \to 0_+$ under the assumption that the solution of the problem is sufficiently smooth.

Theorem 3.1. *Let $\partial\Omega$ and its parts $\Gamma_u, \Gamma_\tau, \Gamma_o, \Gamma_c$ be piecewise polygonal, $\Gamma_c^{kl} = \bigcup_{j=1}^{r} \Gamma_{cj}^{kl}$. Let the solution of problem $(\mathcal{P}_{sg})$ $\mathbf{u} \in K \cap [H^2(\Omega)]^2$, $\tau_{ij}(\mathbf{u}^{\iota}) \in H^1(\Omega^{\iota}), i, j = 1, 2$ and $\iota = 1, \ldots, s$, $\tau_n^{kl}(\mathbf{u}) \in L^\infty(\Gamma_c^{kl}), u_n^k$, $u_n^l \in H^2(\Gamma_j^{kl}), k, l = 1, \ldots, s$ and $j = 1, \ldots, r$. Let $K_h \subset K$. Let the set of points at which the change of*

$u_n^k - u_n^l < 0$ to $u_n^k - u_n^l = 0$ *occurs, be finite. Then for the semi-coercive case*

$$|\mathbf{u} - \mathbf{u}_h| = O(h),\ where\ |\mathbf{w}| = \left(\sum_{\iota=1}^{s} \int_{\Omega_h^\iota} e_{ij}(\mathbf{w}) e_{ij}(\mathbf{w})\, \mathrm{d}\mathbf{x}\right)^{1/2} \tag{3.1}$$

and for the coercive case

$$\|\mathbf{u} - \mathbf{u}_h\|_W = O(h).$$

For the proof see [14,15].

4. Domain decomposition algorithm

4.1. Introduction

For effective solution of the problem the parallelization technique by using the High Performance Fortran can be employed. The latter techniques can be based on overlapping or non-overlapping domain decomposition methods, respectively. The global problem is splitted in a sequence of local problems of a smaller dimension. In this section we use the non-overlapping domain decomposition method derived from the primal formulation in displacements and group every two subdomains in contact (cf. Barboteu et al. [8], LeTallec [7], Hlaváček [16], Daněk [17], Pavarino, Toselli [18]).

4.2. Domain decomposition

Let every domain $\overline{\Omega}^\iota = \bigcup_{i=1}^{J(\iota)} \overline{\Omega}_i^\iota$, where $J(\iota)$ represents a number of subdomains of Ω^ι. Let us denote $\Gamma_i^\iota = \partial\Omega_i^\iota \backslash \partial\Omega^\iota$, $\iota \in \{1, \ldots, s\}$, $i \in \{1, \ldots, J(\iota)\}$, a part of dividing line and let $\Gamma = \bigcup_{\iota=1}^{s} \bigcup_{i=1}^{J(\iota)} \Gamma_i^\iota$ represent the whole interface boundary. Let us introduce

$$T^\iota = \{j \in \{1, \ldots, J(\iota)\} : \overline{\Gamma}_c \cap \overline{\Omega}_j^\iota = \emptyset\},\ \iota = 1, \ldots, s \tag{4.1}$$

the set of all indices of subdomains of the domain Ω^ι which are not adjacent to a contact, and let

$$\Omega^{*j} = \cup_{[i,\iota]\in\vartheta} \Omega_i^\iota, \tag{4.2}$$

where

$$\vartheta = \{[i, \iota] : \partial\Omega_i^\iota \cap \Gamma_c \neq \emptyset\} \quad \text{(subdomains in unilateral contact).} \tag{4.3}$$

Suppose that

$$\Gamma \cap \Gamma_c = \emptyset. \tag{4.4}$$

Then for the trace operator $\gamma : [H^1(\Omega_i^\iota)]^2 \to [L^2(\partial\Omega_i^\iota)]^2$ we have

$$V_\Gamma = \gamma K|_\Gamma = \gamma V|_\Gamma. \tag{4.5}$$

Let $\gamma^{-1} : V_\Gamma \in V$ be an arbitrary linear inverse mapping satisfying

$$\gamma^{-1}\underline{\mathbf{v}} = 0 \text{ on } \cup_{k,l} \Gamma_c^{kl} \qquad \forall \underline{\mathbf{v}} \in V_\Gamma. \tag{4.6}$$

Further, we introduce restrictions $\overline{R}_i^\iota : V_\Gamma \to \Gamma_i^\iota$; $L_i^\iota : L^\iota \to \Omega_i^\iota$; $j_i^\iota : j^\iota \to S$; $a_i^\iota(.,.) : a^\iota(.,.) \to \Omega_i^\iota$; $V(\Omega_i^\iota) : V \to \Omega_i^\iota$ and let us introduce $V^0(\Omega_i^\iota)$ by

$$V^0(\Omega_i^\iota) = \{\mathbf{v} \in V | \mathbf{v} = 0 \text{ on } \overline{(\cup_{\iota=1}^s \Omega^\iota) \setminus \Omega_i^\iota}\},$$

representing the space of functions with zero traces on Γ_i^ι.

Theorem 4.1. *A function* $\mathbf{u}$ *is a solution of a global problem* $(\mathcal{P}_{sg})$, *if and only if*:
its trace $\underline{\mathbf{u}} = \gamma\mathbf{u}|_\Gamma$ *on the interface* Γ *satisfies the condition*

$$\sum_{\iota=1}^{s} \sum_{i=1}^{J(\iota)} [a_i^\iota(\mathbf{u}_i^\iota(\underline{\mathbf{u}}), \gamma^{-1}\underline{\mathbf{w}}) - L_i^\iota(\gamma^{-1}\underline{\mathbf{w}})] = 0 \quad \forall \underline{\mathbf{w}} \in V_\Gamma, \underline{\mathbf{u}} \in V_\Gamma, \tag{4.7}$$

and its restrictions $\mathbf{u}_i^\iota(\underline{\mathbf{u}}) \equiv \mathbf{u}|_{\Omega_i^\iota}$ *satisfy*

(i) *the condition*

$$a_i^\iota(\mathbf{u}_i^\iota(\underline{\mathbf{u}}), \phi_i^\iota) = L_i^\iota(\phi_i^\iota) \quad \forall \phi_i^\iota \in V^0(\Omega_i^\iota),\ \mathbf{u}_i^\iota(\underline{\mathbf{u}}) \in V(\Omega_i^\iota),\ \gamma\mathbf{u}_i^\iota(\underline{\mathbf{u}})|_{\Gamma_i^\iota} = \overline{R}_i^\iota \underline{\mathbf{u}}, \tag{4.8}$$

for $i \in T^\iota$, $\iota = 1, \ldots, s$,

(ii) *the condition*

$$\sum_{[i,\iota]\in\vartheta} a_i^\iota(\mathbf{u}_i^\iota(\underline{\mathbf{u}}), \phi_i^\iota) + j^\iota(\mathbf{u}_i^\iota(\underline{\mathbf{u}}) + \phi_i^\iota) - j^\iota(\mathbf{u}_i^\iota(\underline{\mathbf{u}})) \sum_{[i,\iota]\in\vartheta} L_i^\iota(\phi_i^\iota) \tag{4.9}$$

for all $\phi \equiv (\phi_i^\iota, [i,\iota] \in \vartheta)$, $\phi_i^\iota \in V^0(\Omega_i^\iota)$, *and such that*

$$\mathbf{u} + \phi \in K, \quad \gamma\mathbf{u}_i^\iota(\underline{\mathbf{u}})|_{\Gamma_i^\iota} = \overline{R}_i^\iota \underline{\mathbf{u}} \text{ for } [i,\iota] \in \vartheta. \tag{4.10}$$

Proof. Let $\mathbf{u}$ be a weak solution of the problem $(\mathcal{P}_{sg})$. Then $\mathbf{u}$ satisfies

$$a(\mathbf{u}, \mathbf{v} - \mathbf{u}) + j(\mathbf{v}) - j(\mathbf{u}) \geq L(\mathbf{v} - \mathbf{u}) \qquad \forall \mathbf{v} \in K. \tag{4.11}$$

Let us put test functions

$$\mathbf{v}^\iota = \mathbf{u}^\iota \pm \phi_i^\iota, \quad \phi_i^\iota \in V^0(\Omega_i^\iota), \quad i \in T^\iota, \iota = 1, \ldots, s, \mathbf{v}^\mu = \mathbf{u}^\mu, \quad \mu \neq \iota, \mu \in \{1, \ldots, s\}$$

into (4.11). Thus we obtain

$$a_i^\iota(\mathbf{u}_i^\iota(\underline{\mathbf{u}}), \phi_i^\iota) - L_i^\iota(\phi_i^\iota) = 0 \tag{4.12}$$

so that (4.8) holds.

Now let us put $\mathbf{v} = \mathbf{u} + \phi$, $\phi = (\phi_i^\iota, [i,\iota] \in \vartheta, \phi_i^\iota \in V^0(\Omega_i^\iota))$, such that $\mathbf{u} + \varphi \in K$, i.e. such that

$$(u_n^k - u_n^l) + (\varphi_{in}^k - \varphi_{jn}^l) \leq 0 \text{ on } \cup_{k,l} \Gamma_c^{kl}.$$

Hence (4.9) follows.

Next, we will derive the condition (4.7) for $\underline{\mathbf{u}} = \gamma \mathbf{u}|_{\Gamma}$. First, we decompose the set of admissible displacements as

$$\begin{aligned} K = \gamma^{-1} V_{\Gamma} \oplus V^0(\Omega_1^1) \oplus V^0(\Omega_2^1) \oplus \cdots \oplus V^0(\Omega_{J(1)}^1) \oplus \cdots \\ \cdots \oplus V^0(\Omega_1^2) \oplus V^0(\Omega_2^2) \oplus \cdots \oplus V^0(\Omega_{J(2)}^2) \oplus \cdots \\ \cdots \oplus V^0(\Omega_1^s) \oplus V^0(\Omega_2^s) \oplus \cdots \oplus V^0(\Omega_{J(s)}^s). \end{aligned} \tag{4.13}$$

Any function $\mathbf{v} \in K$ can be written in the following form

$$\mathbf{v} = \gamma^{-1}(\underline{\mathbf{v}}) + \sum_{\iota=1}^{s} \sum_{i=1}^{J(\iota)} \phi_i^{\iota}(\underline{\mathbf{v}}), \tag{4.14}$$

where

$$\underline{\mathbf{v}} = \gamma \mathbf{v}|_{\Gamma}, \phi_i^{\iota}(\underline{\mathbf{v}}) \in V^0(\Omega_i^{\iota}).$$

According to the definition of the mapping γ^{-1} on Γ_c, given by (4.6), we find

$$(\phi_i^k(\underline{\mathbf{v}}))_n - (\phi_j^l(\underline{\mathbf{v}}))_n \leq 0 \text{ on } \cup_{kl} \Gamma_c^{kl} \tag{4.15}$$

if $[i, k] \in \vartheta$ and $[j, l] \in \vartheta$.

From (4.14) we infer

$$\mathbf{v} - \mathbf{u} = \gamma^{-1}(\underline{\mathbf{v}}) - \gamma^{-1}(\underline{\mathbf{u}}) + \sum_{\iota=1}^{s} \sum_{i=1}^{J(\iota)} (\phi_i^{\iota}(\underline{\mathbf{v}}) - \phi_i^{\iota}(\underline{\mathbf{u}})). \tag{4.16}$$

The inequality (4.11) implies

$$\begin{aligned} 0 \leq \sum_{\iota=1}^{s} [a^{\iota}(\mathbf{u}^{\iota}, \gamma^{-1}(\underline{\mathbf{v}}) - \gamma^{-1}(\underline{\mathbf{u}})) - L^{\iota}(\gamma^{-1}(\underline{\mathbf{v}}) - \gamma^{-1}(\underline{\mathbf{u}}))] + \sum_{\iota=1}^{s} \sum_{i \in T^{\iota}} [a_i^{\iota}(\mathbf{u}_i^{\iota}, \phi_i^{\iota}(\underline{\mathbf{v}}) - \phi_i^{\iota}(\underline{\mathbf{u}})) \\ - L_i^{\iota}(\phi_i^{\iota}(\underline{\mathbf{v}}) - \phi_i^{\iota}(\underline{\mathbf{u}}))] + \sum_{[i,\iota] \in \vartheta} [a_i^{\iota}(\mathbf{u}_i^{\iota}, \phi_i^{\iota}(\underline{\mathbf{v}}) - \phi_i^{\iota}(\underline{\mathbf{u}})) - L_i^{\iota}(\phi_i^{\iota}(\underline{\mathbf{v}}) - \phi_i^{\iota}(\underline{\mathbf{u}}))] + j(\mathbf{v}) - j(\mathbf{u}). \end{aligned}$$

By virtue of (4.8), the second sum vanishes. According to (4.6), we may take

$$\phi_i^{\iota}(\underline{\mathbf{v}}) = \phi_i^{\iota}(\underline{\mathbf{u}}) \quad \forall \iota = 1, \ldots, s \quad \text{and} \quad i \leq J(\iota), \tag{4.17}$$

so that

$$0 \leq \sum_{\iota=1}^{s} [a^{\iota}(\mathbf{u}^{\iota}, \gamma^{-1}(\underline{\mathbf{v}}) - \gamma^{-1}(\underline{\mathbf{u}})) - L^{\iota}(\gamma^{-1}(\underline{\mathbf{v}}) - \gamma^{-1}(\underline{\mathbf{u}}))] + j(\mathbf{v}) - j(\mathbf{u}).$$

Next, we have

$$\gamma^{-1}(\underline{\mathbf{v}}) - \gamma^{-1}(\underline{\mathbf{u}}) = \gamma^{-1}(\underline{\mathbf{v}} - \underline{\mathbf{u}})$$

and we can take

$$\underline{\mathbf{v}} - \underline{\mathbf{u}} = \pm\underline{\mathbf{w}},$$

where $\underline{\mathbf{w}}$ is an arbitrary function from $V|_\Gamma$. Moreover,

$$j(\underline{\mathbf{v}}) - j(\underline{\mathbf{u}}) = j(\gamma^{-1}(\underline{\mathbf{v}}) + \sum_{[i,\iota]\in\vartheta} \phi_i^\iota(\underline{\mathbf{v}})) - j(\gamma^{-1}(\underline{\mathbf{u}}) + \sum_{[i,\iota]\in\vartheta} \phi_i^\iota(\underline{\mathbf{u}})) = 0$$

due to (4.6) and (4.17). As a consequence, (4.7) follows.

On the contrary, let us assume that the conditions (4.7)–(4.9) hold for same function $\mathbf{u} \in K$. For any $\mathbf{v} \in K$ we may write

$$\begin{aligned} I &\equiv a(\mathbf{u}, \mathbf{v} - \mathbf{u}) - L(\mathbf{v} - \mathbf{u}) + j(\mathbf{v}) - j(\mathbf{u}) \\ &= \sum_{\iota=1}^{s}[a^\iota(\mathbf{u}^\iota, \gamma^{-1}(\underline{\mathbf{v}} - \underline{\mathbf{u}})) - L^\iota(\gamma^{-1}(\underline{\mathbf{v}} - \underline{\mathbf{u}}))] + \sum_{\iota=1}^{s}\sum_{i\in T^\iota}[a_i^\iota(\mathbf{u}_i^\iota, \phi_i^\iota(\underline{\mathbf{v}}) - \phi_i^\iota(\underline{\mathbf{u}})) - L_i^\iota(\phi_i^\iota(\underline{\mathbf{v}}) - \phi_i^\iota(\underline{\mathbf{u}}))] \\ &\quad + \sum_{[i,\iota]\in\vartheta}[a_i^\iota(\mathbf{u}_i^\iota, \phi_i^\iota(\underline{\mathbf{v}}) - \phi_i^\iota(\underline{\mathbf{u}})) - L_i^\iota(\phi_i^\iota(\underline{\mathbf{v}}) - \phi_i^\iota(\underline{\mathbf{u}}))] + j(\mathbf{v}) - j(\mathbf{u}). \end{aligned}$$

Due to (4.7) and (4.8), the first and the second sum vanish.

Next, we have for all $\iota \leq s$ and $i \leq J(\iota)$

$$\psi_i^\iota \equiv \phi_i^\iota(\underline{\mathbf{v}}) - \phi_i^\iota(\underline{\mathbf{u}}) \in V^0(\Omega_i^\iota), \tag{4.18}$$

$$\mathbf{u}_i^\iota + \psi_i^\iota = \gamma^{-1}(\underline{\mathbf{u}})|_{\Omega_i^\iota} + \phi_i^\iota(\underline{\mathbf{v}}), \tag{4.19}$$

so that

$$\mathbf{u} + \sum_{\iota=1}^{s}\sum_{i=1}^{J(\iota)} \psi_i^\iota \in K$$

follows from (4.15) and (4.6).

Then (4.9) yields that

$$\sum_{[i,\iota]\in\vartheta}[a_i^\iota(\mathbf{u}_i^\iota, \psi_i^\iota) - L_i^\iota(\psi_i^\iota)] \geq j(\mathbf{u}) - j\left(\mathbf{u} + \sum_{\iota=1}^{s}\sum_{i=1}^{J(\iota)} \psi_i^\iota\right).$$

Altogether, we obtain

$$I \geq j(\mathbf{v}) - j\left(\mathbf{u} + \sum_{\iota=1}^{s}\sum_{i=1}^{J(\iota)} \psi_i^\iota\right) = 0$$

since

$$j(\mathbf{v}) = j(\gamma^{-1}(\underline{\mathbf{v}}) + \sum_{[i,\iota]\in\vartheta} \phi_i^\iota(\underline{\mathbf{v}})) = j\left(\sum_{[i,\iota]\in\vartheta} \phi_i^\iota(\underline{\mathbf{v}})\right)$$

and

$$j\left(\mathbf{u} + \sum_{\iota=1}^{s}\sum_{i=1}^{J(\iota)} \psi_i^\iota\right) = j\left(\gamma^{-1}(\underline{\mathbf{u}}) + \sum_{[i,\iota]\in\vartheta} \phi_i^\iota(\underline{\mathbf{v}})\right) = j\left(\sum_{[i,\iota]\in\vartheta} \phi_i^\iota(\underline{\mathbf{v}})\right)$$

follows from (4.19) and (4.6). As $\mathbf{v} \in K$ was arbitrary, $\mathbf{u}$ is a weak solution of the (global) problem $\mathcal{P}_{sg}$. □

4.3. Local and global operators of the Schur complement

The aim of this subsection is to analyze in detail the condition (4.7) and to employ it for numerical computation of our discretized problem $(\mathcal{P}_{sg})_h$. We will introduce the concept of the *local Schur complement*.

Let us denote

$$V_i^\iota = \{\gamma\mathbf{v}|_{\Gamma_i^\iota} | \mathbf{v} \in K\} = \{\gamma\mathbf{v}|_{\Gamma_i^\iota} | \mathbf{v} \in V\}$$

and define a particular case of the restriction of the inverse mapping $\gamma^{-1}(\cdot)|_{\Omega_i^\iota}$ by

$$\begin{cases} Tr_{i\iota}^{-1} : V_i^\iota \to V(\Omega_i^\iota),\ \gamma(Tr_{i\iota}^{-1}\underline{\mathbf{u}}_i^\iota)|_{\Gamma_i^\iota} = \underline{\mathbf{u}}_i^\iota,\ i = 1,\ldots,J(\iota),\ \iota = 1,\ldots,s, \\ a_i^\iota(Tr_{i\iota}^{-1}\underline{\mathbf{u}}_i^\iota, \mathbf{v}_i^\iota) = 0 \quad \forall \mathbf{v}_i^\iota \in V_0^0(\Omega_i^\iota),\ Tr_{i\iota}^{-1}\underline{\mathbf{u}}_i^\iota \in V(\Omega_i^\iota),\ i \in T^\iota,\ \iota = 1,\ldots,s. \end{cases} \tag{4.20}$$

where

$$V_0^0(\Omega_i^\iota) = \{\mathbf{v} \in V_0 | \mathbf{v} = 0 \text{ on } (\cup\Omega_i^\iota) \setminus \Omega_i^\iota\}. \tag{4.21}$$

For $[i,\iota] \in \vartheta$ we complete the definition by the boundary condition (4.6), i.e.

$$Tr_{i\iota}^{-1}\underline{\mathbf{u}}_i^\iota = 0 \text{ on } \cup_{k,l}\Gamma^{kl}. \tag{4.22}$$

Definition 4.1. By the local Schur complement for $i \in T^\iota$ it is meant the operator $\mathcal{S}_i^\iota : V_i^\iota \to (V_i^\iota)^*$ defined by

$$\langle \mathcal{S}_i^\iota \underline{\mathbf{u}}_i^\iota, \underline{\mathbf{v}}_i^\iota\rangle = a_i^\iota(Tr_{i\iota}^{-1}\underline{\mathbf{u}}_i^\iota, Tr_{i\iota}^{-1}\underline{\mathbf{v}}_i^\iota) \qquad \forall \underline{\mathbf{u}}_i^\iota, \underline{\mathbf{v}}_i^\iota \in V_i^\iota. \tag{4.23}$$

In the matrix form we may write

$$\mathcal{S}_i^\iota \underline{\mathbf{U}}_i^\iota = (\overline{A}_{i\iota} - B_{i\iota}^T \mathring{A}_{i\iota}^{-1} B_{i\iota})\underline{\mathbf{U}}_i^\iota, \tag{4.24}$$

where we use the decomposition of the matrix $A_{i\iota} \equiv a_i^\iota(.,.)$ into blocks of the form

$$A_{i\iota} = \begin{bmatrix} \mathring{A}_{i\iota} & B_{i\iota} \\ B_{i\iota}^T & \overline{A}_{i\iota} \end{bmatrix}, \tag{4.25}$$

which corresponds to the decomposition of the vector of nodal parameters $\mathbf{U}_i^\iota = [\mathring{\mathbf{U}}_i^{\iota T}, \underline{\mathbf{U}}_i^{\iota T}]^T$ of $\overline{\Omega}_i^\iota$ where the nodes of $\underline{\mathbf{U}}_i^\iota$ belong to Γ_i^ι and the internal degrees of freedom are $\mathring{\mathbf{U}}_i^\iota$.

To derive (4.24) we firstly find

$$Tr_{i\iota}^{-1} = \begin{bmatrix} -\mathring{A}_{i\iota}^{-1} B_{i\iota} \\ I \end{bmatrix}, \quad i \in T^\iota, \iota = 1, \dots, s. \tag{4.26}$$

Then from (4.20) for the vector $\mathbf{w} = (\mathring{\mathbf{w}}, \underline{\mathbf{w}})^T = Tr_{i\iota}^{-1}\underline{\mathbf{U}}_i^\iota$ we obtain that $\underline{\mathbf{w}} = \underline{\mathbf{U}}_i^\iota$ and using the decomposition (4.25) we arrive at

$$\mathring{A}_{i\iota}\mathring{\mathbf{w}} + B_{i\iota}\underline{\mathbf{w}} = 0,$$

so that

$$\mathring{\mathbf{w}} = -\mathring{A}_{i\iota}^{-1} B_{i\iota}\underline{\mathbf{U}}_i^\iota.$$

Hence (4.26) follows. Inserting (4.26) into the definition (4.23) with the matrix (4.25) we obtain (4.24).

Next, for subdomains which are in contact we will define a *common local Schur complement* as follows.

Definition 4.2. The common local Schur complement for the union $\Omega_i^k \cup \Omega_j^l$ (where $\Gamma_c^{kl} \subset \Gamma_c$ and $[i, k] \in \vartheta$, $[j, l] \in \vartheta$) is the operator

$$\mathcal{S}^{kl} : (V_i^k \times V_j^l) \to (V_i^k \times V_j^l)^* = (V_i^k)^* \times (V_j^l)^*$$

defined by the relation

$$\langle \mathcal{S}^{kl}(\underline{\mathbf{y}}_i^k, \underline{\mathbf{y}}_j^l), (\underline{\mathbf{v}}_i^k, \underline{\mathbf{v}}_j^l)\rangle = a_i^k(\mathbf{u}_i^k(\underline{\mathbf{y}}_i^k), Tr_{ik}^{-1}\underline{\mathbf{v}}_i^k) + a_j^l(\mathbf{u}_j^l(\underline{\mathbf{y}}_j^l), Tr_{jl}^{-1}\underline{\mathbf{v}}_j^l) \quad \forall (\underline{\mathbf{v}}_i^k, \underline{\mathbf{v}}_j^l) \in V_i^k \times V_j^l. \tag{4.27}$$

Here Tr_{ik}^{-1} and Tr_{jl}^{-1} are defined again by means of (4.20)–(4.22).

The functions $\mathbf{u}_i^k(\underline{y}_i^k)$ and $\mathbf{u}_j^l(\underline{y}_j^l)$ denote the solution of the problem (4.9), i.e.

$$\begin{aligned} &a_i^k(\mathbf{u}_i^k(\underline{\mathbf{y}}_i^k), \phi_i^k) + a_j^l(\mathbf{u}_j^l(\underline{\mathbf{y}}_j^l), \phi_j^l) + j(\mathbf{u}(\underline{\mathbf{y}}) + \phi) - j(\mathbf{u}(\underline{\mathbf{y}})) \geq L_i^k(\phi_i^k) + L_j^l(\phi_j^l) \\ &\forall \phi \equiv (\phi_i^k, \phi_j^l), \phi_i^k \in V^0(\Omega_i^k), \phi_j^l \in V^0(\Omega_j^l), \end{aligned} \tag{4.28}$$

such that

$$\mathbf{u}(\underline{\mathbf{y}}) + \phi \in K, \quad \gamma\mathbf{u}(\underline{\mathbf{y}})|_{\Gamma_i^\iota} = \overline{R}_i^\iota\underline{\mathbf{y}},$$

$$\mathbf{u}(\underline{\mathbf{y}}) \equiv (\mathbf{u}_i^k(\underline{\mathbf{y}}_i^k), \mathbf{u}_j^l(\underline{\mathbf{y}}_j^l)) \in V(\Omega_i^k) \times V(\Omega_j^l), \quad (\mathbf{u}_i^k(\underline{\mathbf{y}}))_n - (\mathbf{u}_j^l(\underline{\mathbf{y}}))_n \leq 0 \text{ on } \Gamma_c^{kl}.$$

The condition (4.7) can be expressed by means of local Schur complements. In fact, we have the following lemma.

Lemma 4.1. *The trace* $\underline{\mathbf{u}} = \gamma \mathbf{u}|_\Gamma$ *of the weak solution satisfies the following condition*

$$\sum_{\iota=1}^{s}\sum_{i\in T^\iota}\langle \mathcal{S}_i^\iota \underline{\mathbf{u}}_i^\iota, \underline{\mathbf{v}}_i^\iota\rangle + \sum_{k,l}\langle \mathcal{S}^{kl}(\underline{\mathbf{u}}_i^k, \underline{\mathbf{u}}_j^l), (\underline{\mathbf{v}}_i^k, \underline{\mathbf{v}}_j^l)\rangle = \sum_{\iota=1}^{s}\sum_{i=1}^{J(\iota)} L_i^\iota(Tr_{i\iota}^{-1}\underline{\mathbf{v}}_i^\iota)$$
$$\forall \underline{\mathbf{v}} \in V_\Gamma, [i,k] \in \vartheta,\ [j,l] \in \vartheta, \Gamma_c^{kl} \subset \Gamma_c, \tag{4.29}$$

where $\underline{\mathbf{v}}_i^\iota = \overline{R}_i^\iota \underline{\mathbf{v}}$, $\underline{\mathbf{u}}_i^\iota = \overline{R}_i^\iota \underline{\mathbf{u}}$.

Proof. By definition (4.20) and (4.23) we have

$$\langle \mathcal{S}_i^\iota \underline{\mathbf{u}}_i^\iota, \underline{\mathbf{v}}_i^\iota\rangle = a_i^\iota(Tr_{i\iota}^{-1}\underline{\mathbf{u}}_i^\iota, Tr_{i\iota}^{-1}\underline{\mathbf{v}}_i^\iota) = a_i^\iota(u_i^\iota(\underline{\mathbf{u}}), Tr_{i\iota}^{-1}\underline{\mathbf{v}}_i^\iota) \tag{4.30}$$

since

$$u_i^\iota(\underline{\mathbf{u}}) - Tr_{i\iota}^{-1}\underline{\mathbf{u}}_i^\iota \in V^0(\Omega_i^\iota).$$

For $[i,k] \in \vartheta$ and $[j,l] \in \vartheta$, $\Gamma_c^{kl} \subset \Gamma_c$ we have (4.27). Inserting (4.30) and (4.27) into the equation (4.7), we obtain (4.29).

Remark 4.1. Conditions (4.7) and/or (4.29), respectively, express the continuity of the stress vector $\tau(\mathbf{u}).\mathbf{n}$ on the interface Γ (see [16]).

In accordance with Lemma 4.1 we will solve Eq. (4.29) on the interface Γ in the dual space $(V_\Gamma)^*$. Let us denote

$$\mathcal{S}_{\mathrm{CON}} = \sum_{k,l} \overline{R}_{kl}^T \mathcal{S}^{kl} \overline{R}_{kl}, \tag{4.31}$$

where

$$\overline{R}_{kl}(\underline{\mathbf{u}}) = (\overline{R}_i^k(\underline{\mathbf{u}}), \overline{R}_j^l(\underline{\mathbf{u}}))^T, \underline{\mathbf{u}} \in V_\Gamma, [i,k] \in \vartheta, [j,l] \in \vartheta, \Gamma_c^{kl} \subset \Gamma_c.$$

By definition, we may write

$$_{(V_i^k\times V_j^l)^*}\langle \mathcal{S}^{kl}(\overline{R}_{kl}(\underline{\mathbf{u}})), \overline{R}_{kl}(\underline{\mathbf{w}})\rangle_{V_i^k\times V_j^l} \equiv {}_{(V_\Gamma)^*}[(\overline{R}_{kl})^T \mathcal{S}^{kl} \overline{R}_{kl}(\underline{\mathbf{u}}), \underline{\mathbf{w}}]_{V_\Gamma}$$
$$_{(V_i^\iota)^*}\langle \mathcal{S}_i^\iota \overline{R}_i^\iota \underline{\mathbf{u}}, \overline{R}_i^\iota \underline{\mathbf{w}}\rangle_{V_i^\iota} \equiv {}_{(V_\Gamma)^*}[(\overline{R}_i^\iota)^T \mathcal{S}_i^\iota \overline{R}_i^\iota \underline{\mathbf{u}}, \underline{\mathbf{w}}]_{V_\Gamma}$$

and condition (4.29) can be expressed in the space V_Γ^* as follows:

$$\mathcal{S}_0\underline{\mathbf{U}} + \mathcal{S}_{\mathrm{CON}}\underline{\mathbf{U}} = \mathbf{F}, \tag{4.32}$$

where

$$\mathcal{S}_0 = \sum_{\iota=1}^{s}\sum_{i\in T^\iota}(\overline{R}_i^\iota)^T \mathcal{S}_i^\iota \overline{R}_i^\iota \tag{4.33}$$

and

$$\mathbf{F} = \sum_{\iota=1}^{s}\sum_{i=1}^{J(\iota)}(\overline{R}_i^\iota)^T (Tr_{i\iota}^{-1})^T L_i^\iota. \tag{4.34}$$

Eq. (4.32) will be solved by *successive approximations*, because the operators $\mathcal{S}^{kl}$ and therefore $\mathcal{S}_{\mathrm{CON}}$ are nonlinear.

We choose a suitable initial approximation $\underline{\mathbf{U}}^0$, for instance the solution of the global primal problem, where the boundary conditions on Γ_c are replaced by the linear "classical" bilateral conditions

$$u_n^k - u_n^l = 0, \quad \boldsymbol{\tau}_t^{kl} = 0 \quad \text{on } \Gamma_{c0} \equiv \cup_{k,l}\Gamma_{c0}^{kl} \tag{4.35}$$

(note that these conditions corresponds with $g^{kl} \equiv 0$ and $j(\mathbf{u}) \equiv 0$), where every Γ_{c0}^{kl} is a part of Γ_c^{kl}, meas $\Gamma_{c0}^{kl} > 0$, chosen a priori ($\Gamma_{c0}^{kl} = \Gamma_c^{kl}$ is allowed, for example). On $\Gamma_c^{kl} \backslash \Gamma_{c0}^{kl}$ we consider homogeneous conditions of zero surface load $P_j^k = P_j^l = 0$, $j = 1, 2$.

Denoting $K^0 = \{\mathbf{v} \in V | v_n^k - v_n^l = 0 \text{ on } \cup_{k,l} \Gamma_{c0}^{kl}\}$,we therefore solve the following problem

$$\mathbf{u}^0 = \arg\min_{\mathbf{v}\in\mathbf{K^0}} \mathcal{L}(\mathbf{v}), \tag{4.36}$$

where $\mathcal{L}(\mathbf{v}) = \frac{1}{2}a(\mathbf{v}, \mathbf{v}) - L(\mathbf{v})$ and set

$$\underline{\mathbf{U}}^0 = \gamma \mathbf{u}^0 \mid_\Gamma .$$

The auxiliary problem (4.36) is a linear elliptic boundary value problem of a system of "s" elastic bodies with bilateral contact. We can solve it by the domain decomposition method again, as we will see later.

Returning to the non-linear Eq. (4.32), we assume that the approximation $\underline{\mathbf{U}}^{k-1}$ is known and define the next approximation $\underline{\mathbf{U}}^k$ as the solution of the following linear problem

$$\mathcal{S}_0 \underline{\mathbf{U}}^k = \mathbf{F} - \mathcal{S}_{\mathrm{CON}} \underline{\mathbf{U}}^{k-1}, \quad k = 1, 2, \ldots \tag{4.37}$$

To the linear operator $\mathcal{S}_0$ a suitable preconditioning of the Neumann–Neumann type will be applied.

4.4. *Solution of the auxiliary problem*

Let us assume that

$$K^0 \cap P = \{0\}. \tag{4.38}$$

Then the Korn's inequality holds for the union $\bigcup_{\iota=1}^s \Omega^\iota$ (see [20]). Consequently, there exists a unique solution $\mathbf{u}^0$ of the auxiliary problem (4.36).

Instead of the variational inequality (3.1) we have the following equation for $\mathbf{u}^0 \in K^0$:

$$D\mathcal{L}(\mathbf{u}^0, \mathbf{v}) = 0 \qquad \forall \mathbf{v} \in K_0^0 = \{\mathbf{v} \in V_0 | v_n^k - v_n^l = 0 \text{ on } \cup_{k,l} \Gamma_{c0}^{kl}\}. \tag{4.39}$$

Thus an analogue of Theorem 4.1 can be derived, where a mapping $\gamma_0^{-1} : V_\Gamma \to V$ plays a role, being linear and satisfying conditions

$$(\gamma_0^{-1}\underline{\mathbf{v}})_n^k - (\gamma_0^{-1}\underline{\mathbf{v}})_n^l = 0 \quad \text{on } \cup_{k,l} \Gamma_{c0}^{kl}. \tag{4.40}$$

Theorem 4.2. *A function $\mathbf{u}^0 \in K^0$ is a solution of the auxiliary problem (4.36), if and only if its trace $\underline{\mathbf{u}}^0 \equiv \gamma \mathbf{u}^0 \mid_\Gamma$ on the interface Γ fulfils the condition*

$$\sum_{\iota=1}^{s} \sum_{i=1}^{J(\iota)} [a_i^\iota(\mathbf{u}_i^{0\iota}(\underline{\mathbf{u}}^0), \gamma_0^{-1}\underline{\mathbf{w}}) - L_i^\iota(\gamma_0^{-1}\underline{\mathbf{w}})] = 0 \qquad \forall \underline{\mathbf{w}} \in V_\Gamma, \underline{\mathbf{u}}_0 \in V_\Gamma \tag{4.41}$$

and its restrictions $\mathbf{u}_i^{0\iota}(\underline{\mathbf{u}}^0) \equiv \mathbf{u}^0|_{\Omega_i^\iota}$ *satisfy the following conditions*

$$\begin{aligned} & a_i^\iota(\mathbf{u}_i^{0\iota}(\underline{\mathbf{u}}^0), \phi_i^\iota) = L_i^\iota(\phi_i^\iota) \qquad \forall \phi_i^\iota \in V^0(\Omega_i^\iota), \\ & \mathbf{u}_i^{0\iota}(\underline{\mathbf{u}}^0) \in V(\Omega_i^\iota),\ \gamma \mathbf{u}_i^{0\iota}(\underline{\mathbf{u}}^0) = \overline{R}_i^\iota \underline{\mathbf{u}}^0 \quad \text{for } i \in T^\iota, \iota \leq s, \end{aligned} \tag{4.42}$$

and

$$\begin{aligned} & \sum_{[i,\iota] \in \vartheta} a_i^\iota(\mathbf{u}_i^{0\iota}(\underline{\mathbf{u}}^0), \phi_i^\iota) = \sum_{[i,\iota] \in \vartheta} L_i^\iota(\phi_i^\iota) \qquad \forall \phi = (\phi_i^\iota, [i,\iota] \in \vartheta), \\ & \phi_i^\iota \in V^0(\Omega_i^\iota) \text{ such that } (\phi_i^k)_n - (\phi_j^l)_n = 0 \text{ on } \textstyle\bigcup_{k,l} \Gamma_{c0}^{kl}, \text{ (i.e. } \phi \in K_0^0). \end{aligned} \tag{4.43}$$

Proof. It is analogous to that of Theorem 4.1. □

Next, we will rewrite condition (4.41) by means of operators of Schur complements. For $i \in T^\iota$, $\iota = 1, \ldots, s$, we define the mappings $Tr_{i\iota}^{-1}$ according to (4.20) and the local Schur complements $\mathcal{S}_i^{0\iota}$ by (4.23), so that (4.24)–(4.26) hold in the matrix form.

Definition 4.3. The common local Schur complement for the union $\Omega_i^k \cup \Omega_j^l$, where $\Gamma_{c0}^{kl} \subset \Gamma_c$ and $[i,k] \in \vartheta$, $[j,l] \in \vartheta$

$$\mathcal{S}^{0kl} : (V_i^k \times V_j^l) \to (V_i^k)^* \times (V_j^l)^*$$

is defined by the following relation

$$\langle \mathcal{S}^{0kl}(\underline{\mathbf{u}}_i^{0k}, \underline{\mathbf{u}}_j^{0l}), (\underline{\mathbf{v}}_i^k, \underline{\mathbf{v}}_j^l) \rangle = a_i^k(\mathbf{u}_i^k(\underline{\mathbf{u}}_i^k), Tr_{ik}^{-1} \underline{\mathbf{v}}_i^k) + a_j^l(\mathbf{u}_j^l(\underline{\mathbf{u}}_j^l), Tr_{jl}^{-1} \underline{\mathbf{v}}_j^l) \forall (\underline{\mathbf{v}}_i^k, \underline{\mathbf{v}}_j^l) \in V_i^k \times V_j^l. \tag{4.44}$$

Here Tr_{ik}^{-1} and Tr_{jl}^{-1} are defined by means of

$$(Tr_{ik}^{-1} \underline{\mathbf{v}}_i^k)_n - (Tr_{jl}^{-1} \underline{\mathbf{v}}_j^l)_n = 0 \text{ on } \Gamma_{c0}^{kl} \tag{4.45}$$

and

$$a_i^k(Tr_{ik}^{-1} \underline{\mathbf{v}}_i^k, \mathbf{w}_i^k) + a_j^l(Tr_{jl}^{-1} \underline{\mathbf{v}}_j^l, \mathbf{w}_j^l) = 0 \quad \forall \mathbf{w}_i^k \in V^0(\Omega_i^k), \mathbf{w}_j^l \in V^0(\Omega_j^l) \tag{4.46}$$

such that

$$(\mathbf{w}_i^k)_n - (\mathbf{w}_j^l)_n = 0 \ \text{ on } \Gamma_{c0}^{kl}.$$

Like in Lemma 4.1 we can rewrite condition (4.41) in terms of Schur complements as follows:

$$\begin{aligned} & \sum_{\iota=1}^{s} \sum_{i \in T^\iota} \langle \mathcal{S}_i^{0\iota} \underline{\mathbf{u}}_i^{0\iota}, \underline{\mathbf{v}}_i^\iota \rangle + \sum_{k,l} \langle \mathcal{S}^{0kl}(\underline{\mathbf{u}}_i^{0k}, \underline{\mathbf{u}}_j^{0l}), (\underline{\mathbf{v}}_i^k, \underline{\mathbf{v}}_j^l) \rangle = \ = \sum_{\iota=1}^{s} \sum_{i \in T^\iota} L_i^\iota(Tr_{i\iota}^{-1} \underline{\mathbf{v}}_i^\iota) \\ & \forall \underline{\mathbf{v}} \in V_\Gamma, \underline{\mathbf{v}}_i^\iota = \overline{R}_i^\iota \underline{\mathbf{v}},\ \underline{\mathbf{u}}_i^{0\iota} = \overline{R}_i^\iota \underline{\mathbf{u}}^0. \end{aligned} \tag{4.47}$$

Definition 4.4. A global Schur complement $\mathcal{S}$ is defined by

$$\mathcal{S} = \mathcal{S}_0 + \sum_{k,l} (\overline{R}_{kl})^T \mathcal{S}^{0kl} \overline{R}_{kl},$$

where $\mathcal{S}_0$ has been defined in (4.33) and $\mathcal{S}^{0kl}$ in Definition 4.3.

Then condition (4.47) on the interface implies the equation

$$\mathcal{S}\underline{\mathbf{U}} = \mathbf{F} \tag{4.48}$$

in the dual space $(V_\Gamma)^*$ (cf. (4.31)–(4.34)).

Let a suitable matrix of preconditioning $\mathcal{M}$ be chosen. In what follows, we will describe a method of so-called Neumann–Neumann preconditioner.

Algorithm PCG1. Choose $\underline{\mathbf{U}}^0$, $\mathbf{H}^0$ and let $\mathbf{P}^0 = 0$:

(i) compute the preconditioned direction of descent

$$\mathbf{G}^m = \mathcal{M}^{-1}\mathbf{H}^m, \quad m = 0, 1, \ldots,$$

(ii) compute

$$\mathbf{P}^m = \mathbf{G}^m + \frac{\langle \mathbf{H}^m, \mathbf{G}^m \rangle}{\langle \mathbf{H}^{m-1}, \mathbf{G}^{m-1} \rangle}\mathbf{P}^{m-1}, \quad m = 1, 2, \ldots,$$

(iii) on each subdomain Ω_i^ι, $i \in T^\iota$, $\iota = 1, \ldots, s$, solve the Dirichlet problem $\mathring{A}_{i\iota}\mathring{\mathbf{U}}_i^\iota = B_{i\iota}\bar{R}_i^\iota\mathbf{P}^m$ (see (4.25) for the definition of the above matrices),

(iv) on every subdomain $\Omega_i^k \cup \Omega_j^l$, (where $\Gamma_{c0}^{kl} \subset \Gamma_c$) solve the following problem:

$$(\mathring{\mathbf{U}}_i^k, \mathring{\mathbf{U}}_i^l) = \operatorname*{arg\,min}_{E^k\mathring{\mathbf{V}}^k + E^l\mathring{\mathbf{V}}^l = 0} \Psi(\mathring{\mathbf{V}}^k, \mathring{\mathbf{V}}^l), \tag{4.49}$$

where

$$\Psi(\mathring{\mathbf{V}}^k, \mathring{\mathbf{V}}^l) = (\mathring{\mathbf{V}}^k)^T \mathring{A}_{ik}\mathring{\mathbf{V}}^k + 2(\bar{R}_i^k\mathbf{P}^m)^T B_{ik}^T\mathring{\mathbf{V}}^k + (\mathring{\mathbf{V}}^l)^T \mathring{A}_{jl}\mathring{\mathbf{V}}^l 2(\bar{R}_j^l\mathbf{P}^m)^T B_{jl}^T\mathring{\mathbf{V}}^l$$

and the condition

$$E^k\mathring{\mathbf{V}}^k + E^l\mathring{\mathbf{V}}^l = 0 \quad \text{is equivalent with } (\mathbf{v}^k)_n - (\mathbf{v}^l)_n = 0 \text{ on } \Gamma_{c0}^{kl}. \tag{4.50}$$

Then

$$(\mathcal{S}^{0kl}\underline{\mathbf{U}})_i^k = B_{ik}^T\mathring{\mathbf{U}}^k + A_{ik}\bar{R}_i^k\mathbf{P}^m, (\mathcal{S}^{0kl}\underline{\mathbf{U}})_j^k = B_{jl}^T\mathring{\mathbf{U}}^l + A_{jl}\bar{R}_j^l\mathbf{P}^m.$$

Here we used the decompositions $\mathbf{U}^\iota = [\mathring{\mathbf{U}}^{\iota T}, \underline{\mathbf{U}}^{\iota T}]^T$, $\iota = k, l$ and inserted $\underline{\mathbf{U}}^k = \bar{R}_i^k\mathbf{P}^m$, $\underline{\mathbf{U}}^l = \bar{R}_j^l\mathbf{P}^m$,

(v) compute

$$\mathcal{Z}^m = \mathcal{S}\mathbf{P}^m = \sum_{k,l}(\bar{R}_{kl})^T\mathcal{S}^{0kl}\bar{R}_{kl}\mathbf{P}^m + \sum_{\iota=1}^{s}\sum_{i\in T^\iota}(\bar{R}_i^\iota)^T(\bar{A}_{i\iota}\bar{R}_i^\iota\mathbf{P}^m - B_{i\iota}^T\mathring{\mathbf{U}}_i^\iota),$$

$$\alpha_m = \langle \mathbf{H}^m, \mathbf{G}^m \rangle / \langle \mathcal{Z}^m, \mathbf{P}^m \rangle,$$

$$\mathbf{H}^{m+1} = \mathbf{H}^m - \alpha_m\mathcal{Z}^m,$$

$$\underline{\mathbf{U}}^{m+1} = \underline{\mathbf{U}}^m + \alpha_m\mathbf{P}^m,$$

goto (i).

Remark 4.2. It is not needed to construct the matrix $\mathcal{S}$ explicitly in step (v).

5. Preconditioner of Neumann–Neumann type

A suitable preconditioning matrix $\mathcal{M}$ can be defined by means of the inverse Schur complements (see e.g. [7,19]). To this end, we realize, that the inverse mappings $(\mathcal{S}_i^\iota)^{-1}$ for $i \in T^\iota$, $\iota = 1, \ldots, s$, map a given surface loading $\mathbf{g} \in (V_i^\iota)^*$ on the trace $\gamma\Phi_i^\iota$ on Γ_i^ι, where Φ_i^ι is the solution of the following "Neumann problem": find $\Phi_i^\iota \in V(\Omega_i^\iota)$ such that

$$a_i^\iota(\phi_i^\iota, \mathbf{v}) = \langle \mathbf{g}, \gamma\mathbf{v} \rangle \quad \forall \mathbf{v} \in V(\Omega_i^\iota). \tag{5.1}$$

Then

$$(\mathcal{S}_i^\iota)^{-1}\mathbf{g} = \gamma\phi_i^\iota|_{\Gamma_i^\iota}. \tag{5.2}$$

Likewise the operator inverse to any common local Schur complement is defined:

$$(\mathcal{S}^{0kl})^{-1} : (V_i^k)^* \times (V_j^l)^* \to V_i^k \times V_j^l.$$

For given data $(\mathbf{g}^k, \mathbf{g}^l) \in (V_i^k)^* \times (V_j^l)^*$ we solve a "Neumann–Neumann" problem: find $\Phi \equiv (\Phi_i^k, \Phi_j^l) \in \hat{V}$ such that

$$a_i^k(\phi_i^k, \mathbf{v}^k) + a_j^l(\phi_j^l, \mathbf{v}^l) = \langle \mathbf{g}^k, \gamma\mathbf{v}^k|_{\Gamma_i^k} \rangle + \langle \mathbf{g}^l, \gamma\mathbf{v}^l|_{\Gamma_j^l} \rangle \quad \forall \mathbf{v} \in \widehat{V} \tag{5.3}$$

where

$$\hat{V} = \{(\mathbf{v}^k, \mathbf{v}^l) \in V(\Omega_i^k) \times V(\Omega_j^l) |\, (\mathbf{v}^k)_n - (\mathbf{v}^l)_n = 0 \text{ on } \Gamma_{c0}^{kl}\}.$$

Then we define

$$(\mathcal{S}^{0kl})^{-1}(\mathbf{g}^k, \mathbf{g}^l) = (\gamma\phi_i^k|_{\Gamma_i^k}, \gamma\phi_j^l|_{\Gamma_j^l}). \tag{5.4}$$

In terms of the degrees of freedom the problem (5.3) leads to a quadratic programming problem with a linear equality constraint (4.50).

Furthermore, we introduce the following injections

$$D_i^\iota : V_i^\iota \to V_\Gamma, \quad i \in T^\iota, \iota = 1, \ldots, s,$$

$$D^{kl} : (V_i^k \times V_j^l) \to V_\Gamma, \quad [i,k] \in \vartheta, [j,l] \in \vartheta, \Gamma_{c0}^{kl} \subset \Gamma_c,$$

by means of the following rule.

For each degree of freedom on the interface

$$D_i^\iota \underline{\mathbf{v}}(P_m) = \underline{\mathbf{v}}(P_n)\varrho_i^\iota/\varrho_T, \quad i = 1, \ldots, J(\iota), \iota = 1, \ldots, s, \tag{5.5}$$

if the mth degree of freedom of V_Γ corresponds with the nth degree of freedom of V_i^ι and

$$D_i^\iota \underline{\mathbf{v}}(P_m) = 0 \tag{5.6}$$

in the opposite case; here ϱ_i^ι denotes the local measure of stiffness of the subdomain Ω_i^ι (e.g. the average of the Young modulus) and

$$\varrho_T = \sum_{P_l \in \bar{\Omega}_j^\iota} \varrho_j^\iota \tag{5.7}$$

is the sum of ϱ_j^ι over all subdomains $\bar{\Omega}_j^\iota$, which contain the point P_l.

Note, that a necessary conditions for a correct choice of the injections D_i^ι is

$$\sum_{\iota=1}^{s} \sum_{i=1}^{J(\iota)} D_i^\iota \bar{R}_i^\iota = I_d \quad \text{on } V_\Gamma \tag{5.8}$$

(where I_d is the identity operator).

The boundary value (Neumann) problems (5.1) and (5.3) are not solvable, in general (for details see [19]). We denote $\mathcal{Z}_i^\iota = \operatorname{Ker} A_{i\iota}$, $\iota = 1, \ldots, s$, $i \in T^\iota$ and $\mathcal{Z}^{kl}$ a common subspace of displacements and rotation of the block $\Omega_i^k \cup \Omega_j^l$. Let $\Phi_i^{0\iota} \in Q(\Omega_i^\iota) = V(\Omega_i^\iota) \ominus \mathcal{Z}_i^\iota$ be a particular solution of the problem (5.1) and let the couple $(\Phi_i^{0k}, \Phi_j^{0l})$ be a particular solution of the problem (5.3) in the subspace $Q(\Omega_i^k \cup \Omega_j^l) = \hat{V} \ominus \mathcal{Z}^{kl}$.

Now, we will define the Neumann–Neumann preconditioner. This preconditioner was described e.g. in Le Tallec [7].

Definition 5.1. A preconditioner $\mathcal{M}^{-1}(\mathbf{z}^0)$ of the Neumann–Neumann type will be defined by the formula

$$\mathcal{M}^{-1}(\mathbf{z}^0)\mathbf{S} = \sum_{\iota=1}^{s} \sum_{i \in T^\iota} D_i^\iota \gamma(\phi_i^{0\iota} + \mathbf{z}_i^{0\iota})|_{\Gamma_i^\iota}, \tag{5.9}$$

where $\mathbf{z}_i^{0\iota}$ will be found from the following global optimization problem

$$\mathbf{z}^0 = \arg\min_{\mathbf{z} \in \Pi \mathcal{Z}_i^\iota} \langle \mathcal{S}(\mathcal{M}^{-1}(\mathbf{z}) - \mathcal{S}^{-1})\mathbf{S}, (\mathcal{M}^{-1}(\mathbf{z}) - \mathcal{S}^{-1})\mathbf{S} \rangle \tag{5.10}$$

where

$$\Pi \mathcal{Z}_i^\iota = \prod_{\iota=1}^{s} \prod_{i \in T^\iota} \mathcal{Z}_i^\iota \times \prod_{k,l} \mathcal{Z}^{kl}.$$

Remark 5.1. We see that the optimal preconditioner $\mathcal{M}^{-1}(\mathbf{z}^0)$ minimizes the difference $\mathcal{M}^{-1}(\mathbf{z}) - \mathcal{S}^{-1}$ in the subset of kernels $\Pi \mathcal{Z}_i^\iota$.

In what follows, we analyze the question, how to satisfy the equilibrium conditions in the course of the Algorithm PCG1. To this end, we introduce the "course space" of traces

$$V_H = \sum_{\iota=1}^{s} \sum_{i \in T^\iota} D_i^\iota \gamma \mathcal{Z}_i^\iota + \sum_{k,l} D^{kl} \gamma \mathcal{Z}^{kl} \tag{5.11}$$

and the set

$$V_H^\perp \subset (V_\Gamma)^*$$

by the relation

$$\mathbf{S} \in V_H^{\perp} \Leftrightarrow \langle \mathbf{S}, \mathbf{z} \rangle = 0 \qquad \forall \mathbf{z} \in V_H.$$

Next, let $P_{\mathcal{Z}}$ be the $\mathcal{S}$ – orthogonal projection of V_Γ on V_H defined by

$$P_{\mathcal{Z}}\underline{\mathbf{U}} \in V_H, \quad \langle \mathcal{S}\mathbf{z}, \underline{\mathbf{U}} - P_{\mathcal{Z}}\underline{\mathbf{U}} \rangle = 0 \quad \forall \mathbf{z} \in V_H, \forall \underline{\mathbf{U}} \in V_\Gamma. \tag{5.12}$$

The Euler necessary condition of a minimum of functional in (5.10) leads to a variational "coarse" equation that can be written as

$$\sum_{i,\iota} D_i^\iota \gamma \mathbf{z}_i^{0\iota} = -P_{\mathcal{Z}} \sum_{i,\iota} D_i^\iota \gamma \phi^{0\iota}, \tag{5.13}$$

so that the preconditioner (5.9) takes the form

$$\mathcal{M}^{-1}\mathbf{S} \equiv \mathcal{M}^{-1}(\mathbf{z}^0)\mathbf{S} = (I - P_{\mathcal{Z}}) \sum_{i,\iota} D_i^\iota \gamma \phi^{0\iota}. \tag{5.14}$$

Lemma 5.1. *Assume that in the Algorithm PCG1 we take $\mathbf{H}^0 \in V_H^{\perp}$. Then $\mathbf{H}^m \in V_H^{\perp}$, $m = 1, 2, \ldots$*

Proof. Let $\mathbf{H}^m \in V_H^{\perp}$ and $\mathcal{S}\mathbf{P}^{m-1} \in V_H^{\perp}$. Using (5.14), we derive that

$$\mathbf{G}^m \equiv \mathcal{M}^{-1}\mathbf{H}^m = (I - P_{\mathcal{Z}})\phi^0$$

holds for the element $\Phi^0 \in V_\Gamma$, calculated on the basis of the functional $\mathbf{H}^m$. By virtue of (5.12) and due to the symmetry we may write

$$\langle \mathcal{S}\mathbf{G}^m, \mathbf{z} \rangle = \langle \mathcal{S}\mathbf{z}, \mathbf{G}^m \rangle = \langle \mathcal{S}\mathbf{z}, (I - P_{\mathcal{Z}})\phi^0 \rangle = 0 \quad \forall \mathbf{z} \in V_H,$$

so that $\mathcal{S}\mathbf{G}^m \in V_H^{\perp}$. In the algorithm, we have

$$\mathcal{Z}^m = \mathcal{S}\mathbf{P}^m = \mathcal{S}\mathbf{G}^m + \beta_m \mathcal{S}\mathbf{P}^{m-1} \in V_H^{\perp},$$

since β_m is a constant and $V_H^{\perp}$ is a linear subspace. Finally, we also have

$$\mathbf{H}^{m+1} = \mathbf{H}^m - \alpha_m \mathcal{Z}^m \in V_H^{\perp}.$$

Since $\mathcal{S}\mathbf{P}^0 = 0 \in V_H^{\perp}$ and $\mathbf{H}^1 = \mathbf{H}^0 - \alpha_0 \mathcal{S}\mathbf{P}^0 = \mathbf{H}^0 \in V_H^{\perp}$, the assertion of the lemma follows by induction.

To satisfy the condition $\mathbf{H}^0 \in V_H^{\perp}$, we calculate the initial approximation $\underline{\mathbf{U}}^0$ in PCG1 such that $\underline{\mathbf{U}}^0 \in V_H^{\perp}$ and

$$\langle \mathbf{H}^0, \mathbf{z} \rangle \equiv \langle \mathbf{F} - \mathcal{S}\underline{\mathbf{U}}^0, \mathbf{z} \rangle = 0 \quad \forall \mathbf{z} \in V_H. \tag{5.15}$$

Note that this problem is the same as the calculation of $\mathbf{z}^0$ from Eq. (5.13), except for another right-hand side. □

6. Solution of the original problem on the interface

Recall that we have to solve the problem (4.32) by successive approximations (see [19]), i.e., by means of a sequence of problems (4.37), i.e.

$$\mathcal{S}_0\underline{\mathbf{U}}^k = \mathbf{b}^k, \quad k = 1, 2, \dots, \tag{6.1}$$

where

$$\mathcal{S}_0 = \sum_{\iota=1}^{s}\sum_{i\in T^\iota}(\bar{R}_i^\iota)^T\mathcal{S}_i^\iota\bar{R}_i^\iota, \tag{6.2}$$

$$\mathbf{b}^k = \mathbf{F} - \mathcal{S}_{\mathrm{CON}}\underline{\mathbf{U}}^{k-1}; \tag{6.3}$$

now $\underline{\mathbf{U}}^0$ is the solution of the auxiliary problem, i.e. $\underline{\mathbf{U}}^0 = \gamma\mathbf{u}^0\,|_\Gamma$, where $\mathbf{u}^0$ is a solution of problem (4.36). To solve problem (6.1), we use the method of preconditioned conjugate gradients, as follows.

Algorithm PCG2. Choose ω_0, $\rho_0 = \mathbf{b}^k - \mathcal{S}_0\omega_0$, $\boldsymbol{\pi}_0 = 0$.

(i) compute the preconditioned direction of descent $\mathbf{g}_m = \mathcal{M}_0^{-1}\rho_m$, where $\mathcal{M}_0^{-1}$ is a “reduced” preconditioner;

(ii) compute

$$\boldsymbol{\pi}_m = \mathbf{g}_m + \frac{\langle\boldsymbol{\rho}_m, \mathbf{g}_m\rangle}{\langle\boldsymbol{\rho}_{m-1}, \mathbf{g}_{m-1}\rangle}\boldsymbol{\pi}_{m-1};$$

(iii) on every subdomain Ω_i^ι, $\iota = 1, \dots, s$ and $i \in T^\iota$ solve (parallel) the system $\mathring{A}_{i\iota}\mathring{\boldsymbol{\omega}}_i^\iota = B_{i\iota}\bar{R}_i^\iota\boldsymbol{\pi}_m$; (Dirichlet problem);

(iv) compute $\boldsymbol{\xi}_m = \mathcal{S}_0 - \boldsymbol{\pi}_m = \sum_{\iota=1}^{s}\sum_{i\in T^\iota}(\bar{R}_i^\iota)^T(\bar{A}_{i\iota}\bar{R}_i^\iota\boldsymbol{\pi}_m - B_{i\iota}^T\mathring{\boldsymbol{\omega}}_i^\iota)$;

(v) compute

$$\alpha_m = \langle\boldsymbol{\rho}_m, \mathbf{g}_m\rangle/\langle\boldsymbol{\xi}_m, \boldsymbol{\pi}_m\rangle,$$

$$\boldsymbol{\rho}_{m+1} = \boldsymbol{\rho}_m - \alpha_m\boldsymbol{\xi}_m,$$

$$\boldsymbol{\omega}_{m+1} = \boldsymbol{\omega}_m + \alpha_m\boldsymbol{\pi}_m,$$

goto (i).

Now, the “injection operators”

$$\mathring{D}_i^\iota : V_i^\iota \to V_\Gamma, \quad \iota = 1, \dots, s \text{ and } i \in T^\iota$$

differ from the previous. Namely, for the nodes on $\Gamma_i^k \cup \Gamma_j^l$ ($\Gamma_c^{kl} \subset \Gamma_c$, $[i, k] \in \vartheta$, $[j, l] \in \vartheta$)

$$\mathring{D}_i^\iota\underline{\mathbf{v}}(P_m) = \underline{\mathbf{v}}(P_n) \text{ if } P_n \in \Gamma_i^k \cup \Gamma_j^l, \tag{6.4}$$

$$\mathring{D}_i^\iota\underline{\mathbf{v}}(P_m) = \underline{\mathbf{v}}(P_n)\varrho_i^\iota/\varrho^T \tag{6.5}$$

if the mth degree of freedom corresponds with the nth degree of V_i^ι and $P_n \notin \Gamma_i^k \cup \Gamma_j^l$ and

$$\mathring{D}_i^\iota \underline{\mathbf{v}}(P_m) = 0 \tag{6.6}$$

in the remaining cases; ϱ_i^ι and ϱ_T have been defined in (5.7).

Now we define new Neumann–Neumann preconditioner (for more details see [19]).

Definition 6.1. The new preconditioner of the Neumann–Neumann type is defined by

$$\mathcal{M}_0^{-1}(\mathbf{z}^0)\mathbf{S} = \sum_{\iota=1}^{s} \sum_{i \in T^\iota} \mathring{D}_i^\iota \gamma(\phi_i^{0\iota} + \mathbf{z}_i^{0\iota}), \tag{6.7}$$

where $\mathbf{z}_i^{0\iota}$ denotes the solution of the following reduced global optimization problem

$$\mathbf{z}^0 = \arg\min_{\mathbf{z} \in \Pi_0 \mathcal{Z}_i^\iota} \langle \mathcal{S}_0(\mathcal{M}_0^{-1}(\mathbf{z}) - \mathcal{S}_0^{-1})\mathbf{S}, (\mathcal{M}_0^{-1}(\mathbf{z}) - \mathcal{S}_0^{-1})\mathbf{S} \rangle, \tag{6.8}$$

where

$$\Pi_0 \mathcal{Z}_i^\iota = \prod_{\iota=1}^{s} \prod_{i \in T^\iota} \mathcal{Z}_i^\iota.$$

Let us define the "coarse" reduced space of traces

$$V_{0H} = \sum_{\iota=1}^{s} \sum_{i \in T^\iota} \mathring{D}_i^\iota \gamma \mathcal{Z}_i^\iota$$

and a linear set $V_{0H}^\perp \in (V_\Gamma)^*$ of functionals by the relation

$$\mathbf{S} \in V_{0H}^\perp \Leftrightarrow \langle \mathbf{S}, \mathbf{z} \rangle = 0 \qquad \forall \mathbf{z} \in V_{0H}.$$

Lemma 6.1. *In the Algorithm* PCG2*let us set* $\rho_0 \in V_{0H}^\perp$, $\boldsymbol{\pi}_0 = 0$. *Then* $\rho_m \in V_{0H}^\perp$, $m = 1, 2, \ldots$

Proof. It is analogous to that of Lemma 5.1. □

7. A convergence theorem for the successive approximations

In the present section we will analyze the convergence of the method of successive approximation (6.1), to the solution of the original problem (4.32) in the space $(V_\Gamma)^*$.

To this end, we introduce a seminorm and a norm.

Definition 7.1. Let $\mathbb{H}_0$ be an orthogonal complement of the subspace V_{0H} *in* V_Γ. Let us introduce a seminorm

$$|\overline{R}_c \underline{\mathbf{v}}|_\vartheta = \left(\sum_{k,l} [a_i^k (Tr_{ik}^{-1} \overline{R}_i^k \underline{\mathbf{v}}, Tr_{ik}^{-1} \overline{R}_i^k \underline{\mathbf{v}}) + a_j^l (Tr_{jl}^{-1} \overline{R}_j^l \underline{\mathbf{v}}, Tr_{jl}^{-1} \overline{R}_j^l \underline{\mathbf{v}})] \right)^{1/2}$$

where $\Gamma_c^{kl} \subset \Gamma_c$, $[i,k] \in \vartheta$ and $[j,l] \in \vartheta$.

Lemma 7.1. *The expression* $\|\underline{\mathbf{u}}\|_Q^2 = \langle \mathcal{S}_0 \underline{\mathbf{u}}, \underline{\mathbf{u}} \rangle$ *defines a norm in* $\mathbb{H}_0$.

For the proof see Lemma 5.1 by Hlaváček [16].

Definition 7.2. Let a mapping $T : \mathbb{H}_0 \to \mathbb{H}_0$ be defined by the relation

$$\langle \mathcal{S}_0(T\underline{\mathbf{y}}), \underline{\mathbf{v}} \rangle = \langle \mathbf{F} - \mathcal{S}_{\mathrm{CON}}(\underline{\mathbf{y}}), \underline{\mathbf{v}} \rangle \qquad \forall \underline{\mathbf{v}} \in \mathbb{H}_0. \tag{7.1}$$

Assumption 7.1. Let a constant β exist such that

$$|\bar{R}_c \underline{\mathbf{u}}|_\vartheta \le \beta \|\underline{\mathbf{u}}\|_Q \quad \forall \underline{\mathbf{u}} \in \mathbb{H}_0. \tag{7.2}$$

Lemma 7.2. *If Assumption 7.1 is satisfied, the mapping T is well-defined, i.e. for all* $\underline{\mathbf{y}} \in \mathbb{H}_0$ *there exists a unique element* $T\underline{\mathbf{y}} \in \mathbb{H}_0$, *satisfying (7.1).*

Proof. By Lemma 7.1 the mapping $\mathcal{S}_0$ is positive definite on $\mathbb{H}_0$. Since we have

$$\begin{aligned} |\langle \mathcal{S}_0 \underline{\mathbf{u}}, \underline{\mathbf{v}} \rangle| &= \left| \sum_{\iota=1}^{s} \sum_{i \in T^\iota} a_i^\iota (Tr_{i\iota}^{-1} \bar{R}_i^\iota \underline{\mathbf{u}}, Tr_{i\iota}^{-1} \bar{R}_i^\iota \underline{\mathbf{v}}) \right| \\ &\le \sum_{\iota,i} [a_i^\iota (Tr_{i\iota}^{-1} \bar{R}_i^\iota \underline{\mathbf{u}}, Tr_{i\iota}^{-1} \bar{R}_i^\iota \underline{\mathbf{u}})]^{1/2} [a_i^\iota (Tr_{i\iota}^{-1} \bar{R}_i^\iota \underline{\mathbf{v}}, Tr_{i\iota}^{-1} \bar{R}_i^\iota \underline{\mathbf{v}})]^{1/2} \le \|\underline{\mathbf{u}}\|_Q \|\underline{\mathbf{v}}\|_Q. \end{aligned}$$

$\mathcal{S}_0$ is continuous. Therefore, it suffices to show that $\mathcal{S}_{\mathrm{CON}}(\underline{\mathbf{y}}) \in (\mathbb{H}_0)^*$ for any $\underline{\mathbf{y}} \in \mathbb{H}_0$. We may write, using the corresponding definitions and the Schwarz inequality,

$$\begin{aligned} \langle \mathcal{S}_{\mathrm{CON}}(\underline{\mathbf{y}}), \underline{\mathbf{v}} \rangle &= \left\langle \sum_{k,l} \mathcal{S}^{kl}(\bar{R}_{kl} \underline{\mathbf{y}}), \bar{R}_{kl} \underline{\mathbf{v}} \right\rangle = \sum_{k,l} (a_i^k(\mathbf{u}_i^k(\underline{\mathbf{y}}_i^k), Tr_{ik}^{-1} \underline{\mathbf{v}}_i^k) + a_j^l(\mathbf{u}_j^l(\underline{\mathbf{y}}_j^l), Tr_{jl}^{-1} \underline{\mathbf{v}}_j^l)) \\ &\le [\mathbf{u}_i^k(\underline{\mathbf{y}}_i^k), \mathbf{u}_j^l(\underline{\mathbf{y}}_j^l)]_a |\bar{R}_c \underline{\mathbf{v}}|_\vartheta \le C(\underline{\mathbf{y}}) \beta \|\underline{\mathbf{v}}\|_Q \end{aligned}$$

where

$$C(\underline{\mathbf{y}}) \equiv [\mathbf{u}_i^k(\underline{\mathbf{y}}_i^k), \mathbf{u}_j^l(\underline{\mathbf{y}}_j^l)]_a = \left(\sum_{k,l} [a_i^k(\mathbf{u}_i^k(\underline{\mathbf{y}}_i^k), \mathbf{u}_i^k(\underline{\mathbf{y}}_i^k)) + a_j^l(\mathbf{u}_j^l(\underline{\mathbf{y}}_j^l), \mathbf{u}_j^l(\underline{\mathbf{y}}_j^l))] \right)^{1/2}.$$

Since $C(\underline{\mathbf{y}}) < \infty$ for any $\underline{\mathbf{y}} \in \mathbb{H}_0$, $\mathcal{S}_{\mathrm{CON}}(\mathbf{y}) \in (\mathbb{H}_0)^*$ follows.

Theorem 7.1. *Let the Assumption 7.1 hold. Then*

$$\|T(\underline{\mathbf{y}}) - T(\underline{\mathbf{w}})\|_Q \le 2\beta^2 \|\underline{\mathbf{y}} - \underline{\mathbf{w}}\|_Q \qquad \underline{\mathbf{y}}, \underline{\mathbf{w}} \in \mathbb{H}_0.$$

Proof. By definitions, we may write

$$\mathcal{S}_0(T(\underline{\mathbf{y}})) - \mathcal{S}_0(T(\underline{\mathbf{w}})) = \mathcal{S}_0(T(\underline{\mathbf{y}}) - T(\underline{\mathbf{w}})) = \mathcal{S}_{\mathrm{CON}}(\underline{\mathbf{w}}) - \mathcal{S}_{\mathrm{CON}}(\underline{\mathbf{y}}).$$

Let us denote $\underline{\mathbf{v}} = T(\underline{\mathbf{y}}) - T(\underline{\mathbf{w}})$ and

$$a^{kl}(\mathbf{u}, \mathbf{v}) = a_i^k(\mathbf{u}_i^k, \mathbf{v}_i^k) + a_j^l(\mathbf{u}_j^l, \mathbf{v}_j^l) \quad \text{for } \Gamma_{\mathrm{c}}^{\mathrm{kl}} \subset \Gamma_{\mathrm{c}}, \quad \forall \mathbf{u}, \mathbf{v} \in \mathrm{V}(\Omega_{\mathrm{i}}^{\mathrm{k}}) \times \mathrm{V}(\Omega_{\mathrm{j}}^{\mathrm{l}}),$$

$$L^{kl}(\mathbf{v}) = L_i^k(\mathbf{v}_i^k) + L_j^l(\mathbf{v}_j^l), \quad Tr_{kl}^{-1}\underline{\mathbf{v}} = (Tr_{ik}^{-1}\underline{\mathbf{v}}_i^k, Tr_{jl}^{-1}\underline{\mathbf{v}}_j^l).$$

Then we may write

$$\|\underline{\mathbf{v}}\|_Q^2 = \langle \mathcal{S}_0\underline{\mathbf{v}}, \underline{\mathbf{v}}\rangle = \langle \mathcal{S}_{\mathrm{CON}}(\underline{\mathbf{w}}) - \mathcal{S}_{\mathrm{CON}}(\underline{\mathbf{y}}), \underline{\mathbf{v}}\rangle = \sum_{k,l} a^{kl}(\mathbf{u}(\underline{\mathbf{w}}) - \mathbf{u}(\underline{\mathbf{y}}), Tr_{kl}^{-1}\underline{\mathbf{v}}). \tag{7.3}$$

Here $\mathbf{u}(\underline{\mathbf{y}})$ is the solution of the variational inequality (4.28), i.e. $\mathbf{u}(\underline{\mathbf{y}}) \in V(\Omega_i^k) \times V(\Omega_j^l)$, $(\mathbf{u}_i^k(\underline{\mathbf{y}}))_n - (\mathbf{u}_j^l(\underline{\mathbf{y}}))_n \leq 0$ on Γ_c^{kl},

$$a^{kl}(\mathbf{u}(\underline{\mathbf{y}}), \phi) + j(\mathbf{u}(\underline{\mathbf{y}}) + \phi) - j(\mathbf{u}(\underline{\mathbf{y}})) \geq L^{kl}(\phi) \tag{7.4}$$

for all $\varphi \in V^0(\Omega_i^k) \times V^0(\Omega_j^l)$ such that $\mathbf{u}(\underline{\mathbf{y}}) + \varphi$ satisfies the unilateral contact condition on Γ_c^{kl} and $\mathbf{u}(\underline{\mathbf{y}}) + \varphi \in V(\Omega_i^k) \times V(\Omega_j^l)$. We can write $\mathbf{u}(\underline{\mathbf{y}}) = Tr_{kl}^{-1}\underline{\mathbf{y}} + \mathbf{y}^0$, where $\mathbf{y}^0 \in K_{kl}^0 = \{\mathbf{v} \in V^0(\Omega_i^k) \times V^0(\Omega_j^l) | (\mathbf{v}_i^k)_n - (\mathbf{v}_j^l)_n \leq 0 \text{ on } \Gamma_c^{kl}\}$.

Analogous characterization holds for $\mathbf{u}(\underline{\mathbf{w}})$, so that

$$\mathbf{u}(\underline{\mathbf{w}}) = Tr_{kl}^{-1}\underline{\mathbf{w}} + \mathbf{w}^0, \quad \mathbf{w}^0 \in K_{kl}^0.$$

If we denote $\mathbf{v} = \mathbf{u}(\underline{\mathbf{y}}) + \varphi$, then $\phi = \mathbf{v} - (Tr_{kl}^{-1}\underline{\mathbf{y}} + \mathbf{y}^0) = \mathbf{v}^0 - \mathbf{y}^0$. where $\mathbf{v}^0 = \mathbf{v} - Tr_{kl}^{-1}\underline{\mathbf{y}} \in K_{kl}^0$. The inequality (7.4) is equivalent with

$$a^{kl}(\mathbf{u}(\underline{\mathbf{y}}), \mathbf{v}^0 - \mathbf{y}^0) + j(\mathbf{u}(\underline{\mathbf{y}}) + \mathbf{v}^0 - \mathbf{y}^0) - j(\mathbf{u}(\underline{\mathbf{y}})) \geq L^{kl}(\mathbf{v}^0 - \mathbf{y}^0) \tag{7.5}$$

for all $\mathbf{v}^0 \in K_{kl}^0$.

By a parallel argument, we arrive at

$$a^{kl}(\mathbf{u}(\underline{\mathbf{w}}), \mathbf{v}^0 - \mathbf{w}^0) + j(\mathbf{u}(\underline{\mathbf{w}}) + \mathbf{v}^0 - \mathbf{w}^0) - j(\mathbf{u}(\underline{\mathbf{w}})) \geq L^{kl}(\mathbf{v}^0 - \mathbf{w}^0) \tag{7.6}$$

for all $\mathbf{v}^0 \in K_{kl}^0$.

Let us insert $\mathbf{v}^0 := \mathbf{w}^0$ in (7.5) and $\mathbf{v}^0 := \mathbf{y}^0$ in (7.6) and sum up.

Recall that $Tr_{kl}^{-1}\underline{\mathbf{y}} = Tr_{kl}^{-1}\underline{\mathbf{w}} = 0$ on Γ_c^{kl}, (cf. (4.22)) so that

$$j(\mathbf{u}(\underline{\mathbf{y}}) + \mathbf{w}^0 - \mathbf{y}^0) = j(Tr_{kl}^{-1}\underline{\mathbf{y}} + \mathbf{w}^0) = j(\mathbf{w}^0) = j(\mathbf{u}(\underline{\mathbf{w}})),$$

$$j(\mathbf{u}(\underline{\mathbf{w}}) + \mathbf{y}^0 - \mathbf{w}^0) = j(\mathbf{u}(\underline{\mathbf{y}})).$$

As a consequence, the summing gives

$$a^{kl}(\mathbf{u}(\underline{\mathbf{w}}) - \mathbf{u}(\underline{\mathbf{y}}), \mathbf{y}^0 - \mathbf{w}^0) \geq 0. \tag{7.7}$$

Let us denote $\psi = \mathbf{y}^0 - \mathbf{w}^0$ and

$$\|\mathbf{t}\|_{kl} = [a^{kl}(\mathbf{t}, \mathbf{t})]^{1/2} \quad \forall \mathbf{t} \in V(\Omega_i^k) \times V(\Omega_j^l).$$

Then (7.7) and the Schwarz inequality yields the estimate

$$\|\psi\|_{kl}^2 \leq a^{kl}(Tr^{-1}\underline{\mathbf{w}} - Tr^{-1}\underline{\mathbf{y}}, \psi) \leq \|Tr^{-1}(\mathbf{w} - \mathbf{y})\|_{kl}\|\psi\|_{kl},$$

so that

$$\|\psi\|_{kl} \leq \|Tr^{-1}(\mathbf{w} - \mathbf{y})\|_{kl}. \tag{7.8}$$

Making use of (7.3) and (7.8), we may write

$$\begin{aligned}\|\underline{\mathbf{v}}\|_Q^2 &= \sum_{k,l} a^{kl}(Tr^{-1}(\underline{\mathbf{w}} - \underline{\mathbf{y}}) - \psi, Tr^{-1}\underline{\mathbf{v}}) = \sum_{k,l}[a^{kl}(Tr^{-1}(\underline{\mathbf{w}} - \underline{\mathbf{y}}), Tr^{-1}\underline{\mathbf{v}}) - a^{kl}(\psi, Tr^{-1}\underline{\mathbf{v}})] \\ &\leq \sum_{k,l}(\|Tr^{-1}(\underline{\mathbf{w}} - \underline{\mathbf{y}})\|_{kl} + \|\psi\|_{kl})\|Tr^{-1}\underline{\mathbf{v}}\|_{kl} \leq 2|\bar{R}_c(\underline{\mathbf{w}} - \underline{\mathbf{y}})|_\vartheta |\bar{R}_c\underline{\mathbf{v}}|_\vartheta.\end{aligned}$$

Finally, the Assumption 7.1 implies $\|\underline{\mathbf{v}}\|_Q^2 \leq 2\beta^2\|\underline{\mathbf{w}} - \underline{\mathbf{y}}\|_Q\|\underline{\mathbf{v}}\|_Q$ and proof is complete. □

Corollary 7.1. *Let the Assumption 7.1 hold with $\beta < \sqrt{2}/2$ for all $\underline{\mathbf{U}} \in \mathbb{Y}$, where $\mathbb{Y}$ is a subset of $\mathbb{H}_0$ such that*

$$T(\mathbb{Y}) \subset \mathbb{Y} \text{ and } \underline{\mathbf{U}}^0 \in \mathbb{Y}. \tag{7.9}$$

Then the mapping T is contractive on $\mathbb{Y}$. The successive approximations (6.1) converge to the fixed point of the mapping T, which represents a solution of Eq. (4.32). The following error estimate holds

$$\|\underline{\mathbf{U}}^k - \underline{\mathbf{U}}\|_Q \leq (2\beta^2)^k(1 - 2\beta^2)^{-1}\|\underline{\mathbf{U}}^0 - T\underline{\mathbf{U}}^0\|_Q, \quad k = 1, 2, \ldots$$

Proof is classical – see e.g. the book by Nečas and Hlaváček ([20], Section 11.7, Theorem 7.1).

Remark 7.1. To find conditions guaranteeing (7.2) with $\beta < \sqrt{2}/2$ and (7.9) is a difficult task. Having the definitions of the norms in mind, it seems that the subdomains adjacent to the contact boundary should be "small" in comparison with the union of the remaining subdomains. Nevertheless, these conditions are only sufficient for the convergence and the successive approximations may converge in other cases too.

8. Implementation of the algorithm and numerical results

The algorithm which is described in the previous section is based on the nonoverlapping domain decomposition method. The original problem is first decomposed into smaller problems defined on nonoverlapping subdomains. Parallel iterative procedures are then constructed for decoupling the whole domain problem into subdomain problems. During the iterative process, information must be transmitted between subdomains. For passing informations we used Message Passing Interface (MPI).

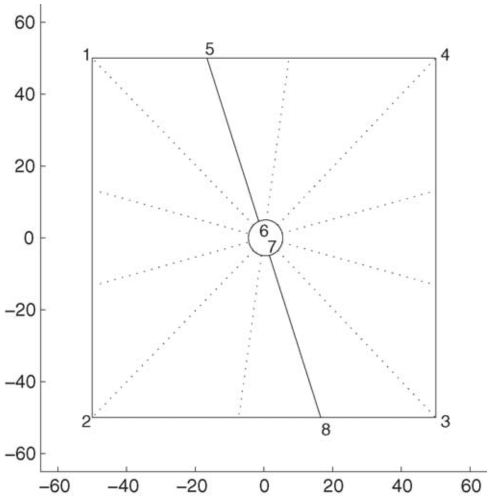

Fig. 1. A geometry of the geomechanical problem.

Hence the introduced algorithm has been implemented in MPI Version 1.2.0 by using FORTRAN 77 compiler. The implementation of the described parallelization is based on model “master-slaves”, where one computer (master) directs all other computers (slaves) which compute partial problems for subdomains without contact and partial problems for couples of subdomains with contact. In this section, we illustrate the practical behavior of our algorithm on the solution of the following problems.

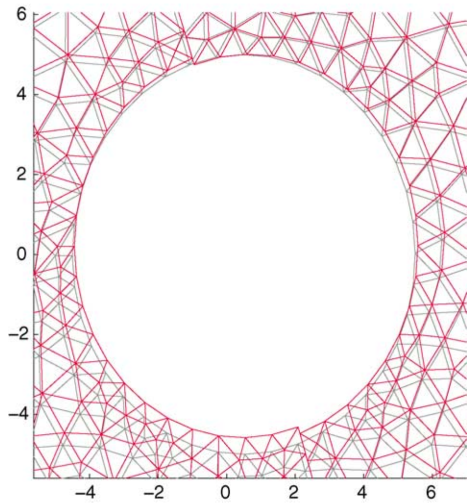

Fig. 2. A detail of deformations (enlarging factor is 10).

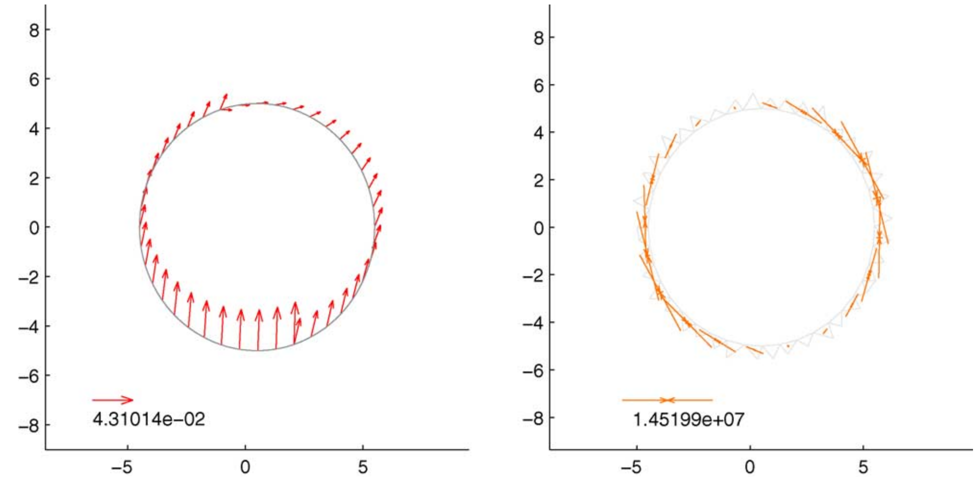

Fig. 3. A detail of displacements and principal stresses in a neighbourhood of the tunnel.

8.1. The geomechanical problem

The geomechanical model problem describes a loaded tunnel which is crossing by a deep fault and based on the geomechanical theory and models having connection with radioactive waste repositories (see Nedoma [15]). A geometry of the problem is in Fig. 1.

8.1.1. Material parameters

Two regions with Young's modulus $E = 5.2 \times 10^9$ Pa and Poisson's ratio $\nu = 0.18$. Specific gravity is 2.45×10^4 Pa/m.

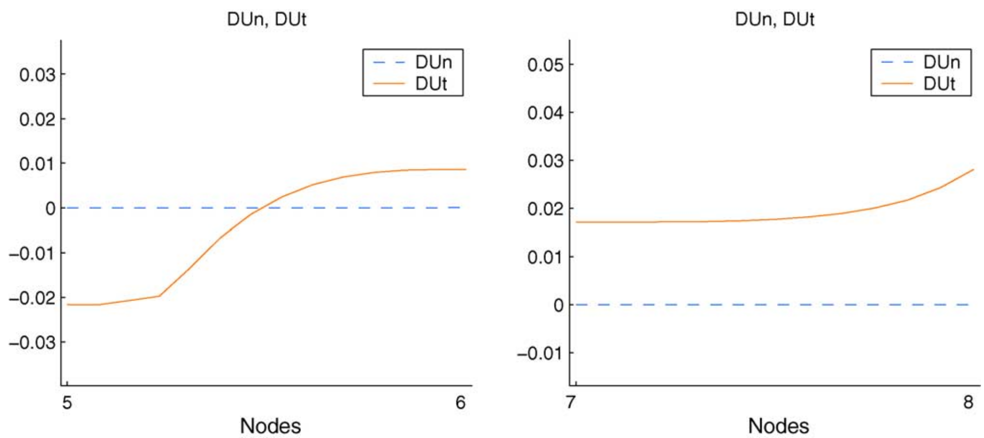

Fig. 4. Normal and tangential components of displacements on contact boundary (parts 5–6 and 7–8).

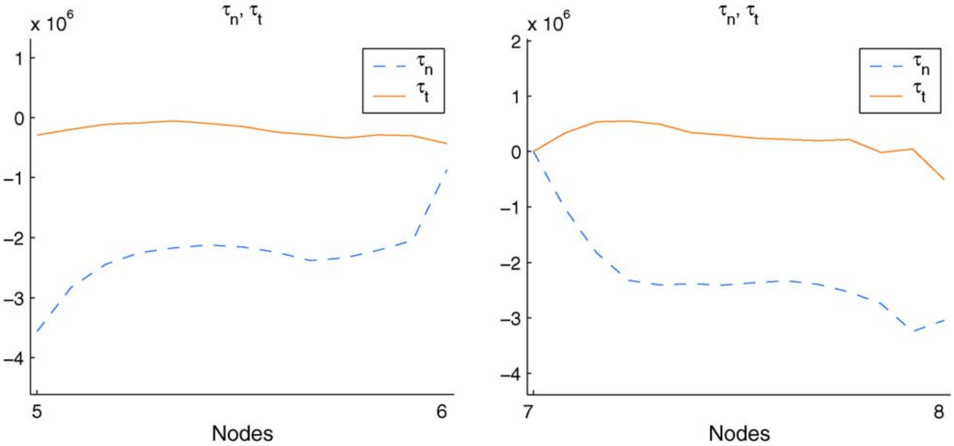

Fig. 5. Normal and tangential components of stress on contact boundary (parts 5–6 and 7–8).

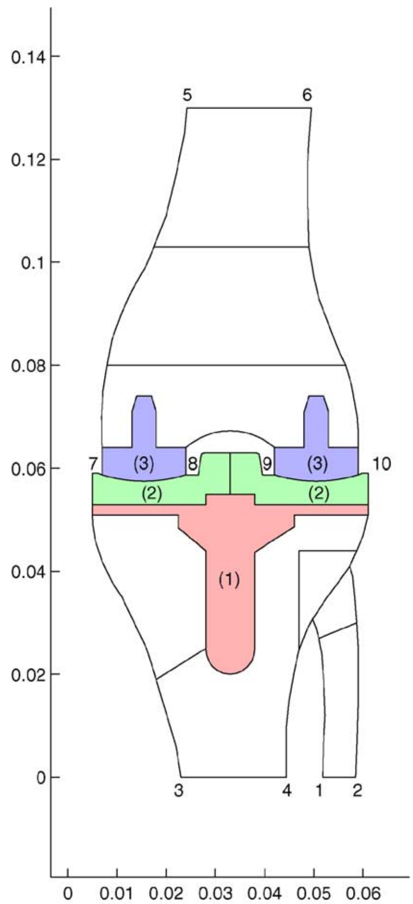

Fig. 6. A geometry of the biomechanical problem.

8.1.2. Boundary conditions

Prescribed displacement $(2.5 \times 10^{-2};\ 0)$ m on 1–2. Pressure 0.5×10^7 Pa on 1–4 and 2–8 and 1×10^7 Pa on 8–3. Bilateral contact boundary on 3–4. Unilateral contact boundary: 5–6 and 7–8. Given slip limit is 10^6 Pa. Zero surface forces on the tunnel wall.

8.1.3. Discretization statistics

Twelve subdomains, 5501 nodes, 9676 elements, 10428 unknowns, 89 unilateral contact conditions, 466 interface elements.

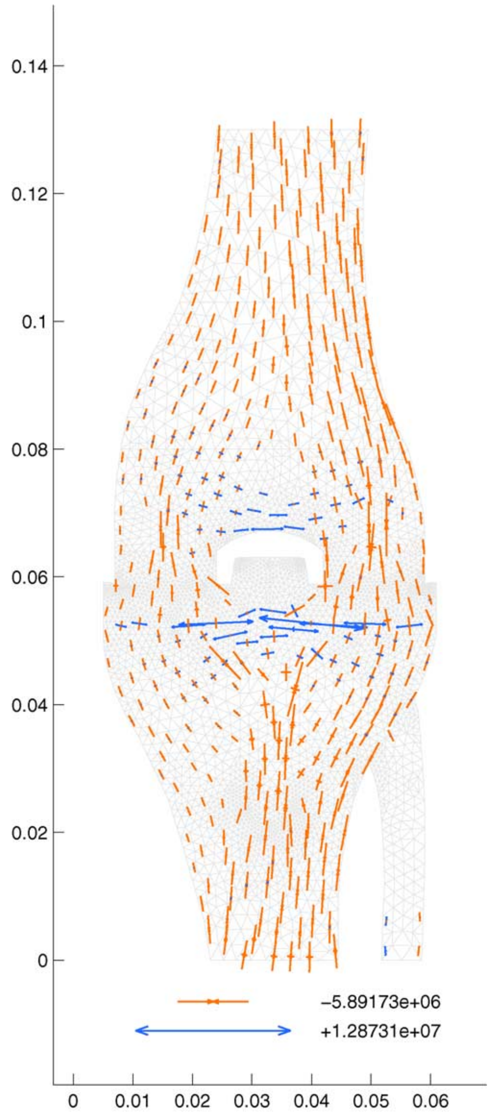

Fig. 7. The principal stresses (→← represents compression and ←→ extension).

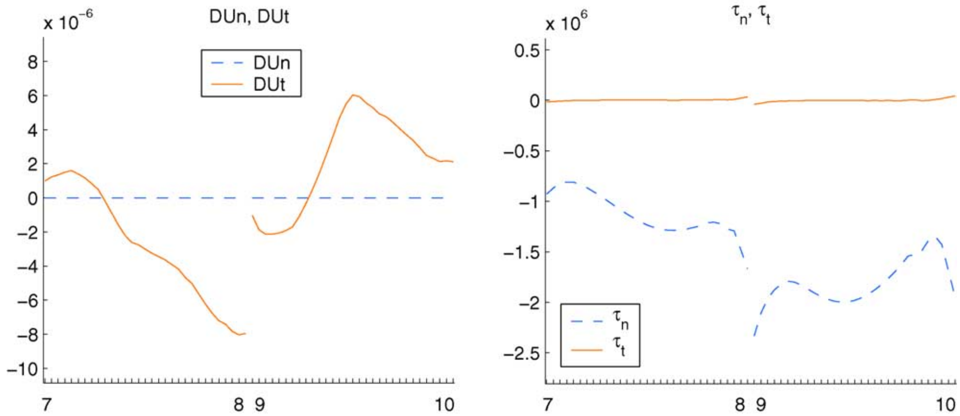

Fig. 8. Normal and tangential components of displacements and stress on contact boundary (parts 7–8 and 9–10).

8.1.4. Convergence statistics

Twenty one iterations of the PCG algorithm for the auxiliary problem, 15 iterations of the successive approximations method for accuracy 10^{-6}, total 39 iterations of the PCG algorithm for the original problem.

Fig. 2 represents detail of deformations and Fig. 3 shows displacements and principal stresses in a neighbourhood of the tunnel.

Fig. 4 shows normal and tangential components of displacements on two parts of the contact boundary, where we denoted DUn $\equiv u_n^k - u_n^l$ and DUt $\equiv u_t^k - u_t^l$. Fig. 5 shows normal and tangential components of stress on two parts contact boundary.

8.2. The biomechanical problem

The biomechanical model problem was derived from the X-ray image and describes total knee replacement. We deal with and simulations of mechanical processes taking place during static loadening. A geometry of the problem is in Fig. 6.

8.2.1. Material parameters

Bone: Young's modulus $E = 1.71 \times 10^{10}$ Pa, Poisson's ratio $\nu = 0.25$, (1) Ti6Al4V: $E = 1.15 \times 10^{11}$ Pa, $\nu = 0.3$, (2) Chirulen: $E = 3.4 \times 10^8$ Pa, $\nu = 0.4$, (3) CoCrMo: $E = 2.08 \times 10^{11}$ Pa, $\nu = 0.3$.

8.2.2. Boundary conditions

The femur is loaded between points 5 and 6 by a loading 0.215×10^7 Pa, the tibia and the fibula are fixed between points 1 and 2 (the tibia) and between 3 and 4 (the fibula) are fixed and the unilateral contact boundary are between points 7 and 8 as well as between 9 and 10. Given slip limit is 10^5 Pa.

8.2.3. Discretization statistics

Thirteen subdomains of domain decomposition, 3800 nodes, 7200 elements, 62 unilateral contact nodes, 350 interface elements. The loadings evoked by muscular forces were neglected.

8.2.4. Convergence statistics

Nineteen iterations of the PCG algorithm for the auxiliary problem, 14 iterations of the successive approximations method for accuracy 10^{-6} and total 36 iterations of the PCG algorithm for the original problem.

In Fig. 7 the principal stresses are presented. Fig. 8 shows normal and tangential components of displacements and stress on two parts of the contact boundary, where we denoted $\mathsf{DUn} \equiv u_n^k - u_n^l$ and $\mathsf{DUt} \equiv u_t^k - u_t^l$.

Acknowledgements

The research was supported partly by the grant COPERNICUS-HIPERGEOS II – KIT 977006, partly by the grants of the Ministry of Education, Youth and Sport of the Czech Republic No. OK-407 and MSM 235200001 and partly by the grant of the Ministry of Industry and Trade of the Czech Republic FT-TA/087.

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DANĚK, J., DENK, F., HLAVÁČEK, I., NEDOMA, J., STEHLÍK, J., VAVŘÍK, P.: *On the stress-strain analysis of the knee replacement*, Lecture Notes in Computer Science 3044, pp. 456–466, Springer Vlg., Berlin, Heidelberg 2004.

On the Stress-Strain Analysis of the Knee Replacement

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Abstract. The paper deals with the stress/strain analysis of an artificial knee joint. Three cases, where femoral part of the knee joint part is cut across under 3, 5 and 7 degrees, are analysed. Finite element method and the nonoverlapping decomposition technique for the contact problem in elasticity are applied. Numerical experiments are presented and discussed.

1 Introduction

The success of artificial replacements of human joints depends on many factors. The mechanical factor is an important one. The idea of a prothesis being a device that transfers the joint loads to the bone allows one to explain the mechanical factor in terms of the load transfer mechanism. A complex relation exists between this mechanism and the magnitude and direction of the loads, the geometry of the bone/joint prothesis configuration, the elastic properties of the materials and the physical connections at the material connections. Authors in [7,12,9, 10] showed that the contact problems in a suitable rheology, and their finite element approximations [2,5] are a very useful tools for analyzing these relations for several types of great human joints and their artificial replacements. The aim of the paper is to analyze the total knee replacement in dependence on the femoral cut and the orientation of the anatomic joint line.

2 The Model

The model of the knee is based on the contact problem in elasticity, and the finite element approximation. The problem leads to solving variational inequalities, which describe physically the principle of virtual work in its inequality form.

A. Laganà et al. (Eds.): ICCSA 2004, LNCS 3044, pp. 456–466, 2004.

In the present paper we deal with mathematical simulations of total knee joint replacements and simulations of mechanical processes taking place during static loadening. The model problem investigated was formulated as the primary semi-coercive contact problem with the given friction and for the numerical solution of the studied problem the nonoverlapping domain decomposition method is used ([1,2,3]).

Let the investigated part of the knee joint occupy a union Ω of bounded domains Ω^ι, $\iota = f, t$ in $\mathbb{R}^N$ $(N = 2)$, denoting separate components of the knee joint - the femur (f) and the tibia together with the fibula (t), with Lipschitz boundaries $\partial\Omega^\iota$. Let the boundary $\partial\Omega = \partial\Omega^f \cup \partial\Omega^t$ consist of three disjoint parts such that $\partial\Omega = \Gamma_\tau \cup \Gamma_u \cup \Gamma_c$. Let $\Gamma_\tau = {}^1\Gamma_\tau \cup {}^2\Gamma_\tau$, where by ${}^1\Gamma_\tau$ we denote the loaded part of the femur and by ${}^2\Gamma_\tau$ the unloaded part of the boundary $\partial\Omega$. By Γ_u we denote the part of the tibia boundary, where we simulate its fixation. The common contact boundary between both joint components Ω^f and Ω^t before deformation we denote by $\Gamma_c = \partial\Omega^f \cap \partial\Omega^t$.

Let body forces $\mathbf{F}$, surface tractions $\mathbf{P}$ and slip limits g^{ft} be given.
We have the following problem: find the displacements $\mathbf{u}^\iota$ in all Ω^ι such that

$$\frac{\partial}{\partial x_j}\tau_{ij}(\mathbf{u}^\iota) + F_i^\iota = 0 \quad \text{in } \Omega^\iota,\ \iota = f, t,\ i = 1, ..., N, \tag{1}$$

where the stress tensor τ_{ij} is defined by

$$\tau_{ij}(\mathbf{u}^\iota) = c_{ijkl}^\iota e_{kl}(\mathbf{u}^\iota) \qquad \text{in } \Omega^\iota,\ \iota = f, t,\ i = 1, ..., N, \tag{2}$$

with boundary conditions

$$\tau_{ij}(\mathbf{u})n_j = P_i \quad \text{on } {}^1\Gamma_\tau,\ i = 1, ..., N, \tag{3}$$

$$\tau_{ij}(\mathbf{u})n_j = 0 \quad \text{on } {}^2\Gamma_\tau,\ i = 1, ..., N, \tag{4}$$

$$\mathbf{u} = \mathbf{u}_0\,(= 0) \qquad \text{on } \Gamma_u, \tag{5}$$

$$u_n^f - u_n^t \le 0,\ \tau_n^f \le 0,\ (u_n^f - u_n^t)\tau_n^f = 0 \quad \text{on } \Gamma_c, \tag{6}$$

$$\begin{aligned} &|\tau_t^{ft}| \le g^{ft} \quad \text{on } \Gamma_c,\\ &|\tau_t^{ft}| < g^{ft} \implies \mathbf{u}_t^f - \mathbf{u}_t^t = 0,\\ &|\tau_t^{ft}| = g^{ft} \implies \text{there exists } \vartheta \ge 0 \text{ such that } \mathbf{u}_t^f - \mathbf{u}_t^t = -\vartheta\tau_t^{ft}. \end{aligned} \tag{7}$$

Here $e_{ij}(\mathbf{u}) = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$ is the small strain tensor, normal and tangential components of displacement vector $\mathbf{u}$ $(\mathbf{u} = (u_i),\ i = 1, 2)$ and stress vector τ $(\tau = (\tau_i))$ $u_n^f = u_i^f n_i^f$, $u_n^t = u_i^t n_i^f$ (no sum over t or f), $\mathbf{u}_t^f = (u_{ti}^f)$, $u_{ti}^f = u_i^f - u_n^f n_i^f$, $\mathbf{u}_t^t = (u_{ti}^t)$, $u_{ti}^t = u_i^t + u_n^t n_i^t$, $i = 1, ..., N$, $\tau_n^f = \tau_{ij}^f n_i^f n_j^f$, $\tau_t^f = (\tau_{ti}^f)$, $\tau_{ti}^f = \tau_{ij}^f n_j^f - \tau_n^f n_i^f$, $\tau_n^t = \tau_{ij}^t n_i^t n_j^t$, $\tau_t^t = (\tau_{ti}^t)$, $\tau_{ti}^t = \tau_{ij}^t n_j^t - \tau_n^t n_i^t$, $\tau_t^{ft} \equiv \tau_t^f$.

Assume that c^ι_{ijkl} are positive definite symmetric matrices such that
$0 < c^\iota_0 \le c^\iota_{ijkl}\xi_{ij}\xi_{kl} \mid \xi \mid^{-2} \le c^\iota_1 < +\infty$ for a.a. $\mathbf{x} \in \Omega^\iota, \xi \in \mathrm{IR}^{N^2}$, $\xi_{ij} = \xi_{ij}$,
where c^ι_0, c^ι_1 are constants independent of $\mathbf{x} \in \Omega^\iota$.

Let us introduce $W = \prod_{\iota=f,t}[H^1(\Omega^\iota)]^N$, $\|\mathbf{v}\|_W = (\sum_{\iota=f,t}\sum_{i\le N}\|v^\iota_i\|^2_{1,\Omega^\iota})^{\frac{1}{2}}$ and the sets of virtual and admissible displacements $V_0 = \{\mathbf{v} \in W \mid \mathbf{v} = 0 \text{ on } \Gamma_u\}$, $V = \mathbf{u}_0 + V_0$, $K = \{\mathbf{v} \in V \mid v^f_n - v^t_n \le 0 \text{ on } \Gamma_c\}$. Assume that $u^f_{0n} - u^t_{0n} = 0$ on Γ_c Let $c^\iota_{ijkl} \in L^\infty(\Omega^\iota)$, $F^\iota_i \in L^2(\Omega^\iota)$, $P_i \in L^2({}^1\Gamma_\tau)$, $\mathbf{u}^\iota_0 \in [H^1(\Omega^\iota)]^N$.

Then we have to solve the following variational problem $(\mathcal{P})$:
find a function $\mathbf{u}$, $\mathbf{u} - \mathbf{u}_0 \in K$, such that

$$a(\mathbf{u}, \mathbf{v} - \mathbf{u}) + j(\mathbf{v}) - j(\mathbf{u}) \ge L(\mathbf{v} - \mathbf{u}) \qquad \forall\, \mathbf{v} \in K \tag{8}$$

holds, where

$$\begin{aligned}
a(\mathbf{u}, \mathbf{v}) &= \textstyle\sum_{\iota=f,t} \int_{\Omega^\iota} c^\iota_{ijkl} e_{ij}(\mathbf{u}^\iota) e_{kl}(\mathbf{v}^\iota)\, d\mathbf{x},\\
j(\mathbf{v}) &= \textstyle\int_{\Gamma_c} g^{ft} \mid \mathbf{v}^f_t - \mathbf{v}^t_t \mid ds,\\
L(\mathbf{v}) &= \textstyle\sum_{\iota=f,t} \int_{\Omega^\iota} F^\iota_i v^\iota_i\, d\mathbf{x} - \sum_{\iota=f,t} \int_{\Gamma^\iota_\tau} P^\iota_i v^\iota_i\, ds.
\end{aligned} \tag{9}$$

Let us define the sets of rigid displacements and rotations
$P = P^f \times P^t$, $P^\iota = \{\mathbf{v}^\iota = (v^\iota_1, v^\iota_2) \mid v^\iota_1 = a^\iota_1 - b^\iota x_2, v^\iota_2 = a^\iota_2 + b^\iota x_1\}$ where $a^\iota_i, i = 1,2$ and b^ι are arbitrary real constants and $\iota = f, t$.

It can be shown that the problem (8) has a unique solution, if (see [5]):

$$\begin{gathered}
L(\mathbf{v}) < j(\mathbf{v}) \quad \forall \mathbf{v} \in P \cap K - \{0\},\\
\{\mathbf{v} \in P \cap V_0, v^f_n - v^t_n = 0 \ \text{ on } \Gamma_c\} \quad \Rightarrow \quad v = 0\\
\text{and} \quad |L(\mathbf{v})| > j(\mathbf{v}) \quad \forall \mathbf{v} \in P \cap V_0 - \{0\}.
\end{gathered}$$

3 Short Description of the Domain Decomposition Algorithm

Let every domain $\overline{\Omega}^\iota = \cup_{i=1}^{J(\iota)} \overline{\Omega}^\iota_i$, where $J(\iota)$ is a number of subdomains of Ω^ι. Let $\Gamma^\iota_i = \partial\Omega^\iota_i \backslash \partial\Omega^\iota$, $\iota = f, t$, $i \in \{1, ..., J(\iota)\}$, be a part of dividing line (boundary line) and let $\Gamma = \cup_{\iota=f,t} \cup_{i=1}^{J(\iota)} \Gamma^\iota_i$ be the whole interface boundary. Let

$$T^\iota = \{j \in \{1, .., J(\iota)\} : \overline{\Gamma}_c \cap \partial\overline{\Omega}^\iota_j = \emptyset\}, \iota = f, t, \tag{10}$$

be the set of all indices of subdomains of the domain Ω^ι which are not adjacent to a contact, and let

$$\Omega^{*j} = \cup_{[i,\iota]\in\vartheta} \Omega^\iota_i, \tag{11}$$

where $\vartheta = \{[i, \iota] : \partial\Omega^\iota_i \cap \Gamma_c \ne \emptyset\}$ represent subdomains in unilateral contact. Suppose that $\Gamma \cap \Gamma_c = \emptyset$. Then for the trace operator $\gamma : [H^1(\Omega^\iota_i)]^N \to [L^2(\partial\Omega^\iota_i)]^N$ we have

$$V_\Gamma = \gamma K|_\Gamma = \gamma V|_\Gamma \tag{12}$$

Let $\gamma^{-1} : V_\Gamma \in V$ be an arbitrary linear inverse mapping satisfying

$$\gamma^{-1}\overline{\mathbf{v}} \;= 0 \quad \text{on } \Gamma_c \quad \forall \overline{\mathbf{v}} \in V_\Gamma. \tag{13}$$

Let us introduce restrictions $\overline{R}_i^\iota : V_\Gamma \to \Gamma_i^\iota$; $L_i^\iota : L^\iota \to \Omega_i^\iota$; $j_i^\iota : j^\iota \to S$; $a_i^\iota(.,.) : a(.,.) \to \Omega_i^\iota$; $V(\Omega_i^\iota) \to \Omega_i^\iota$ and let $V^0(\Omega_i^\iota) = \{\mathbf{v} \in V \mid \mathbf{v} = 0 \text{ on } \overline{(\cup_{\iota=f,t}\Omega^\iota)\backslash\Omega_i^\iota}\}$ be the space of functions with zero traces on Γ_i^ι. The algorithm is based on the next theorem and on the use of local and global Schur complements.

Theorem 3.1: A function $\mathbf{u}$ is a solution of a global problem $(\mathcal{P})$, if and only if its trace $\overline{\mathbf{u}} = \gamma\mathbf{u}|_\Gamma$ on the interface Γ satisfies the condition

$$\sum_{\iota=f,t}\sum_{i=1}^{J(\iota)}[a_i^\iota(\mathbf{u}_i^\iota(\overline{\mathbf{u}}), \gamma^{-1}\overline{\mathbf{w}}) - L_i^\iota(\gamma^{-1}\overline{\mathbf{w}})] = 0 \quad \forall \overline{\mathbf{w}} \in V_\Gamma, \overline{\mathbf{u}} \in V_\Gamma \tag{14}$$

and its restrictions $\mathbf{u}_i^\iota(\mathbf{u}) \equiv \mathbf{u}|_{\Omega_i^\iota}$ satisfy:
(i) the condition

$$\begin{aligned} &a_i^\iota(\mathbf{u}_i^\iota(\overline{\mathbf{u}}), \varphi_i^\iota) - L_i^\iota(\varphi_i^\iota) \quad \forall \varphi_i^\iota \in V^0(\Omega_i^\iota),\ \mathbf{u}_i^\iota(\overline{\mathbf{u}}) \in V(\Omega_i^\iota), \\ &\qquad\qquad\qquad\qquad \gamma\mathbf{u}_i^\iota(\overline{\mathbf{u}})|_{\Gamma_i^\iota} = \overline{R_i^\iota}\overline{\mathbf{u}},\ i \in T^\iota, \iota = f,t, \end{aligned} \tag{15}$$

(ii) the condition

$$\begin{aligned} &\textstyle\sum_{[i,\iota]\in\vartheta} a_i^\iota(\mathbf{u}_i^\iota(\overline{\mathbf{u}}), \varphi_i^\iota) + j^\iota(\mathbf{u}_i^\iota(\overline{\mathbf{u}}) + \varphi_i^\iota) - j^\iota(\mathbf{u}_i^\iota(\overline{\mathbf{u}})) \geq \\ &\qquad \geq \textstyle\sum_{[i,\iota]\in\vartheta} L_i^\iota(\varphi_i^\iota) \quad \forall\varphi \in (\varphi_i^\iota, [i,\iota] \in \vartheta, \varphi_i^\iota \in V^0(\Omega_i^\iota), \end{aligned} \tag{16}$$

and such that

$$\mathbf{u} + \varphi \in K, \quad \gamma\mathbf{u}_i^\iota(\overline{\mathbf{u}})|_{\Gamma_i^\iota} = \overline{R_i^\iota}\overline{\mathbf{u}} \quad \text{for } [i,\iota] \in \vartheta. \tag{17}$$

For the proof see [2].

To analyze the condition (14) the **local and global Schur complements** are introduced. Let

$$V_i^\iota = \{\gamma\mathbf{v}_{|\Gamma_i^\iota} \mid \mathbf{v} \in K\} = \{\gamma\mathbf{v}_{|\Gamma_i^\iota} \mid \mathbf{v} \in V\}$$

and define a particular case of the restriction of the inverse mapping $\gamma^{-1}(.)|_{\Omega_i^\iota}$ by

$$\begin{aligned} &Tr_{i\iota}^{-1} : V_i^\iota \to V(\Omega_i^\iota),\ \gamma(Tr_{i\iota}^{-1}\overline{\mathbf{u}})|_{\Gamma_i^\iota} = \mathbf{u}_i^\iota,\ i = 1,..,J(\iota),\ \iota = f,t \\ &a_i^\iota(Tr_{i\iota}^{-1}\overline{\mathbf{u}}_i^\iota, \mathbf{v}_i^\iota) = 0 \ \ \forall \mathbf{v}_i^\iota \in V_0^0(\Omega_i^\iota), \\ &Tr_{i\iota}^{-1}\overline{\mathbf{u}}_i^\iota \in V(\Omega_i^\iota) \text{ for } i \in T^\iota, \iota = f,t, \end{aligned} \tag{18}$$

where $V_0^0(\Omega_i^\iota) = \{\mathbf{v} \in V_0 \mid \mathbf{v} = 0 \text{ on } (\cup\Omega_i^\iota)\backslash\Omega_i^\iota\}$. For $[i,\iota] \in \vartheta$ we complete the definition by the boundary condition (13), i.e.

$$Tr_{i\iota}^{-1}\overline{\mathbf{u}}_i^\iota = 0 \text{ on } \Gamma_c. \tag{19}$$

The local Schur complement for $i \in T^\iota$ is the operator $\mathcal{S}_i^\iota : V_i^\iota \to (V_i^\iota)^*$ defined by

$$\langle \mathcal{S}_i^\iota \overline{\mathbf{u}}_i^\iota, \overline{\mathbf{v}}_i^\iota \rangle = a_i^\iota (Tr_{i\iota}^{-1} \overline{\mathbf{u}}_i^\iota, Tr_{i\iota}^{-1} \overline{\mathbf{v}}_i^\iota) \quad \forall \overline{\mathbf{u}}_i^\iota, \overline{\mathbf{v}}_i^\iota \in V_i^\iota. \tag{20}$$

For subdomains which are in contact we define a **common local Schur complement** for the union $\Omega_i^f \cup \Omega_j^t$ (where $[i,f] \in \vartheta, [j,t] \in \vartheta$) as the operator $\mathcal{S}^{ft} : (V_i^f \times V_j^t) \to (V_i^f \times V_j^t)^* = (V_i^f)^* \times (V_j^t)^*$ defined by

$$\begin{aligned} \Big\langle \mathcal{S}^{ft}(\overline{\mathbf{y}}_i^f, \overline{\mathbf{y}}_j^t), (\overline{\mathbf{v}}_i^f, \overline{\mathbf{v}}_j^t) \Big\rangle = a_i^f (\mathbf{u}_i^f(\overline{\mathbf{y}}_i^f), Tr_{if}^{-1} \overline{\mathbf{v}}_i^f) + a_j^t (\mathbf{u}_j^t(\overline{\mathbf{y}}_j^t), Tr_{jt}^{-1} \overline{\mathbf{v}}_j^t) \\ \forall (\overline{\mathbf{v}}_i^f, \overline{\mathbf{v}}_j^t) \in V_i^f \times V_j^t, \end{aligned} \tag{21}$$

where Tr_{if}^{-1} and Tr_{jt}^{-1} are defined by means of (18) and (19).

The condition (14) can be expressed by means of local Schur complements in the form

$$\begin{aligned} \textstyle\sum_{\iota=f,t} \sum_{i \in T^\iota} \langle \mathcal{S}_i^\iota \overline{\mathbf{u}}_i^\iota, \overline{\mathbf{v}}_j^\iota \rangle + \sum_{\iota=f,t} \Big\langle \mathcal{S}^{ft}(\overline{\mathbf{u}}_i^f, \overline{\mathbf{u}}_j^t), (\overline{\mathbf{v}}_i^f, \overline{\mathbf{v}}_j^t) \Big\rangle = \\ = \textstyle\sum_{\iota=f,t} \sum_{i=1}^{J(\iota)} L_i^\iota (Tr_{i\iota}^{-1} \overline{\mathbf{v}}_i^\iota) \quad \forall \overline{\mathbf{v}} \in V_\Gamma, [i,f] \in \vartheta, [j,t] \in \vartheta, \end{aligned} \tag{22}$$

where $\overline{\mathbf{u}} = \gamma \mathbf{u}_{|\Gamma}$, $\overline{\mathbf{v}}_i^\iota = \overline{R}_i^\iota \overline{\mathbf{v}}, \overline{\mathbf{u}}_i^\iota = \overline{R}_i^\iota \overline{\mathbf{u}}$. Then we will solve the equation (22) on the interface Γ in the dual space $(V_\Gamma)^*$. We rewrite (22) into the following form

$$\mathcal{S}_0 \overline{\mathbf{U}} + \mathcal{S}_{CON} \overline{\mathbf{U}} = \mathbf{F}, \tag{23}$$

where

$$\begin{aligned} \mathcal{S}_0 = \textstyle\sum_{\iota=f,t} \sum_{i \in T^\iota} (\overline{R}_i^\iota)^T \mathcal{S}_i^\iota \overline{R}_i^\iota, \quad \mathcal{S}_{CON} = \sum_{\iota=f,t} \overline{R}_{ft}^T \mathcal{S}^{ft} \overline{R}_{ft}, \\ \mathbf{F} = \textstyle\sum_{\iota=f,t} \sum_{i=1}^{J(\iota)} (\overline{R}_i^\iota)^T (Tr_{i\iota}^{-1})^T \mathcal{S}_i^\iota, \end{aligned} \tag{24}$$

and $\overline{R}_{ft}(\overline{\mathbf{u}}) = (\overline{R}_i^f(\overline{\mathbf{u}}), \overline{R}_j^t(\overline{\mathbf{u}}))^T, \overline{\mathbf{u}} \in V_\Gamma, [i,f] \in \vartheta, [j,t] \in \vartheta$. Equation (23) will be solved by **successive approximations**, because the operators $\mathcal{S}^{ft}$ and therefore $\mathcal{S}_{CON}$ are nonlinear. As a initial approximation $\overline{\mathbf{U}}^0$ we choose the solution of the global primal problem, where the boundary conditions on Γ_c are replaced by the linear bilateral conditions with $g^{ft} \equiv 0$ (i.e. $j(\mathbf{u}) \equiv 0$)

$$u_n^f - u_n^t = 0, \ \tau_t^{ft} = 0 \ \text{ on } \Gamma_{c0}. \tag{25}$$

On $\Gamma_c \backslash \Gamma_{c0}$ we consider homogeneous conditions of zero surface load $P_j^f = P_j^t = 0$, $j = 1, 2$.

Then we replace the set K by $K^0 = \{\mathbf{v} \in V \mid v_n^f - v_n^t = 0 \text{ on } \Gamma_{c0}\}$ and therefore, we solve the following problem

$$\mathbf{u}^0 = \arg\min_{\mathbf{v} \in K^0} \mathcal{L}(\mathbf{v}) \tag{26}$$

where $\mathcal{L}(\mathbf{v}) = \frac{1}{2} a(\mathbf{v}, \mathbf{v}) - L(\mathbf{v})$ and we set $\overline{\mathbf{U}}^0 = \gamma \mathbf{u}^0|_\Gamma$. The auxiliary problem (26) represents a linear elliptic boundary value problem with bilateral contact and it can be solved by the domain decomposition method again.

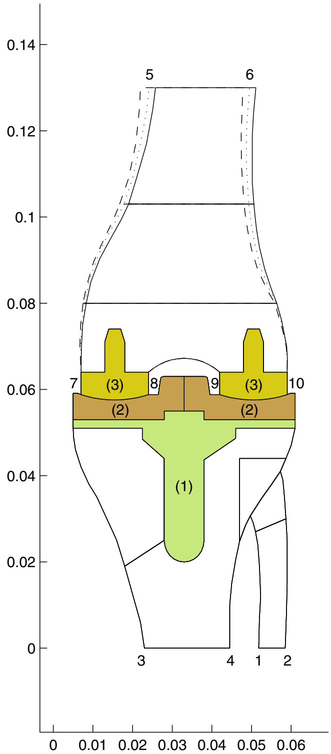


Fig. 1. The models.

The non-linear equation (23) will be solved by successive approximations. We will assume that the approximation $\overline{\mathbf{U}}^{k-1}$ is known and the next approximation $\overline{\mathbf{U}}^{k}$we find as the solution of the following linear problem

$$\mathcal{S}_0\overline{\mathbf{U}}^k = \mathbf{F} - \mathcal{S}_{CON}\overline{\mathbf{U}}^{k-1}, k = 1, 2, ... \tag{27}$$

In [2] the convergence of the method of successive approximation (27) to the solution of the original problem (23) in the space $(V_\Gamma)^*$ is proved.

Numerically (26) and (27) are solved by the finite element method.

4 Discussion of Numerical Results

The model of the knee joint replacement was derived from the X-ray image after application the total knee prothesis under the resulting femoral cuts 3, 5 and 7 degrees.

In the model the material parameters are as follows: Bone: Young's modulus $E = 1.71 \times 10^{10}$ $[Pa]$, Poisson's ratio $\nu = 0.25$, (1) $Ti6Al4V$: $E = 1.15 \times 10^{11}$ $[Pa]$, $\nu = 0.3$, (2) Chirulen: $E = 3.4 \times 10^8$ $[Pa]$, $\nu = 0.4$, (3) $CoCrMo$:

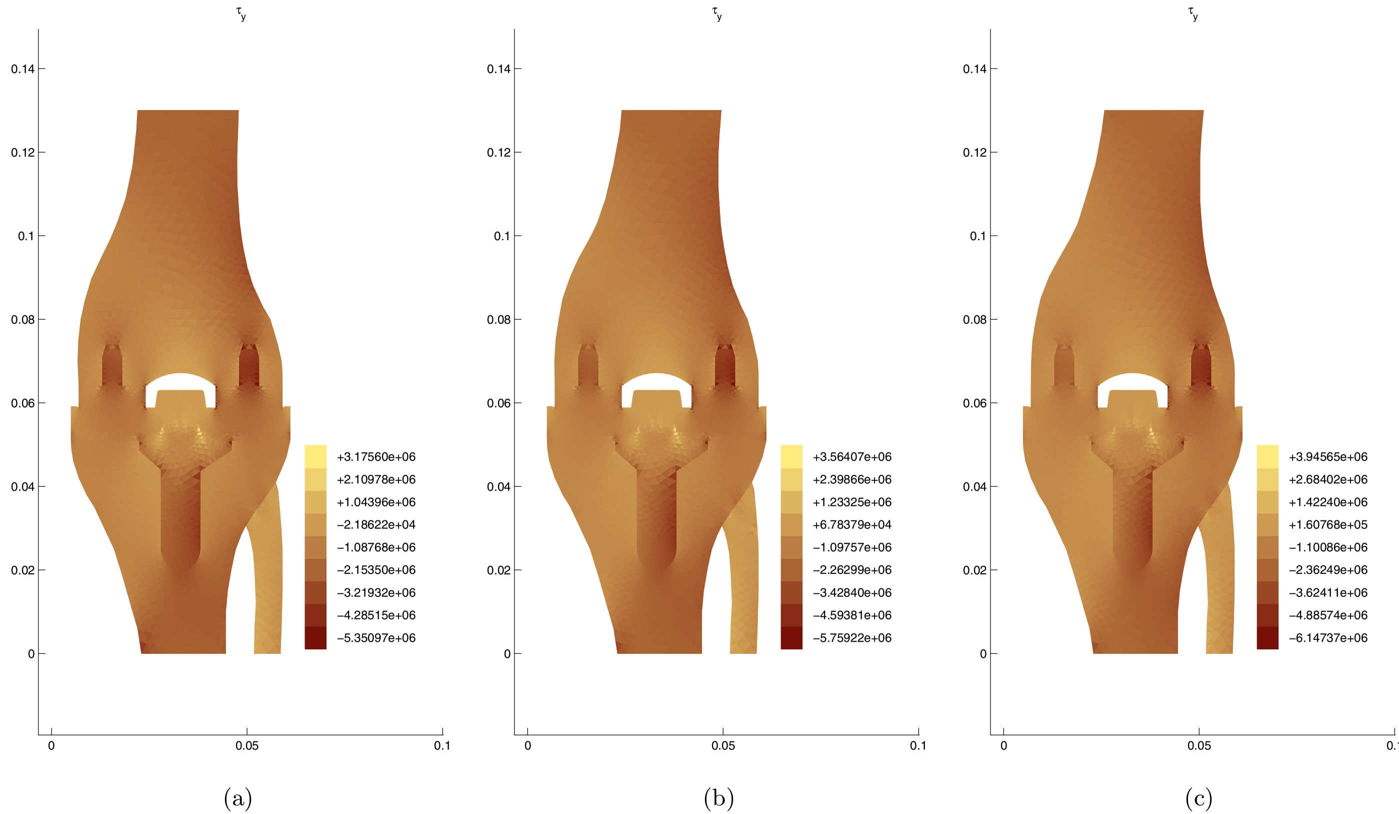


Fig. 2. The vertical stress tensor component - the frontal cross-section of the *CoCrMo* prothesis (a) the femoral cut under 3 degree, (b) the femoral cut under 5 degree, (c) the femoral cut under 7 degree.

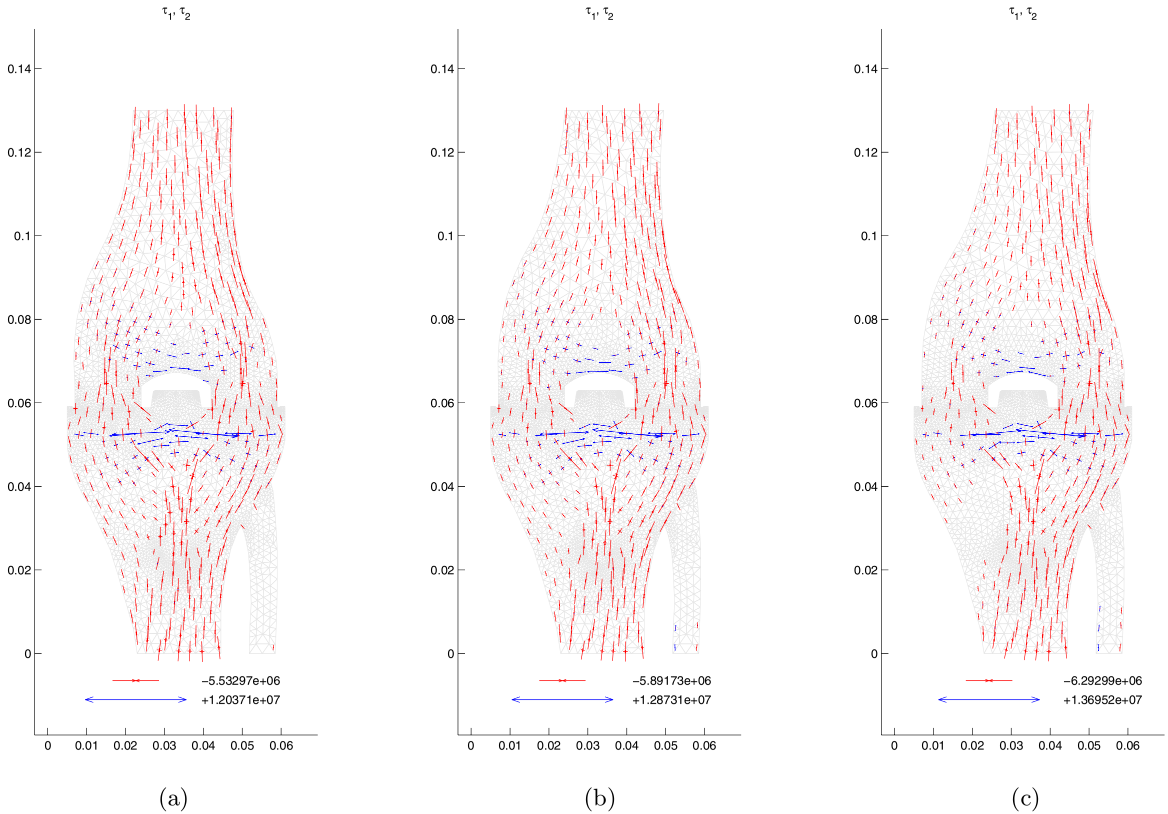
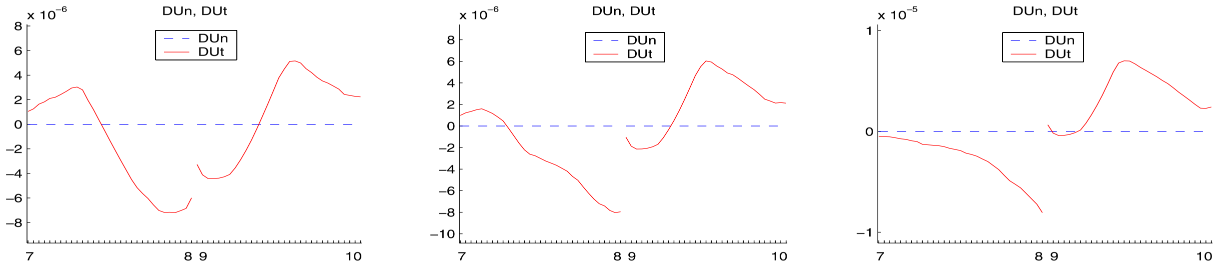
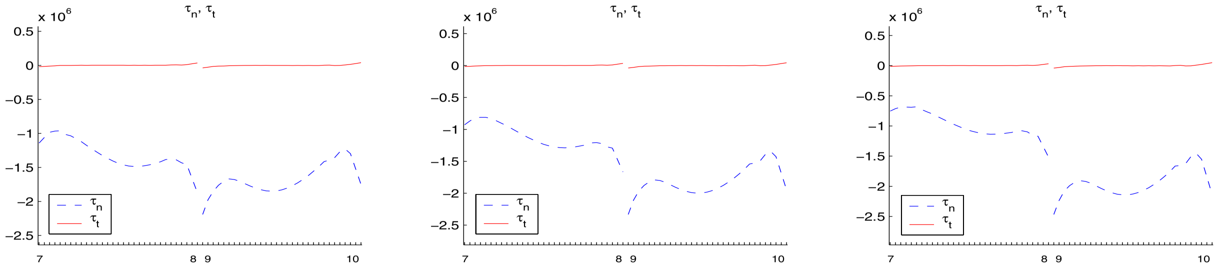


Fig. 3. The principal stresses - the frontal cross-section $CoCrMo$ prothesis ($\rightarrow\leftarrow$ represents compression and $\leftarrow\rightarrow$ extension) (a) the femoral cut under 3 degree, (b) the femoral cut under 5 degree, (c) the femoral cut under 7 degree.



(a) (b) (c)

Fig. 4. Normal and tangential components of displacement - the frontal cross-section $CoCrMo$ prothesis with the cut under (a) 3 degree, (b) 5 degree, (c) 7 degree, where the normal and tangential components of displacement vector are denoted by $u_n \equiv DUn$, $\mathbf{u}_t \equiv DUt$.



(a) (b) (c)

Fig. 5. Normal and tangential components of stress vectors - the frontal cross-section $CoCrMo$ prothesis with the cut under (a) 3 degree, (b) 5 degree, (c) 7 degree.

$E = 2.08 \times 10^{11}$ $[Pa]$, $\nu = 0.3$. The femur is loaded between points 5 and 6 by a loading $0.215 \times 10^7 [Pa]$, the tibia and the fibula are fixed between points 1 and 2 (the tibia) and between 3 and 4 (the fibula) are fixed and the unilateral contact boundary are between points 7 and 8 as well as between 9 and 10. On the contact boundary we suppose that $g^{ft} = 0$. Discretization statistics are characterized by 13 subdomains of domain decoposition, 3800 nodes, 7200 elements, 62 unilateral contact nodes, 350 interface elements. The loadings evoked by muscular forces were neglected. The paper presents three models - the frontal cross-section prothesis with the cut under 3 degree - model (a), 5 degree - model (b) and 7 degree - model (c). All models are presented in Fig.1. In Fig.2 a,b,c the vertical stress tensor components for the

frontal cross-section are presented, while in Figs 3 a,b,c the principal stresses are presented. The presented graphical results represent distribution of stresses in the femur, in the total protheses and in the tibia as well as in the fibula. On Figs 4 a,b,c the normal and tangential components of displacement and on Figs 5 a,b,c stress vectors on the contact boundaries of both condyles (i.e. between points $7 - 8$ and $9 - 10$) are presented. We see that both parts of the prothesis are in a contact and that they mutually move in the tangential direction.

The obtained numerical results, in their graphical forms, corespond to the observed distribution of stress field in the bones and in the knee prothesis, and therefore, they are in a good agreement with the orthopaedic practice. The presented models facilitate to compare the protheses made from the different materials like the $CoCrMo$ alloy, the Al_2O_3 and ZrO_2 ceramics, respectively. The aim of the mathematical modelling of the knee prothesis is to determine the best version of the knee prothesis.

Acknowledgements. The authors have been partly supported by the Ministry of Education, Youth and Sport of the Czech Republic, MSM 235200001.

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Daněk, J., Nedoma, J., Hlaváček, I., Vavřík, P., Denk, F.: *Numerical Modelling of the Weight-Bearing Total Knee Joint Replacement and Usage in Practice*, J. Mathematics and Computers in Simulation 76/1 (2007), pp. 49–56, Elsevier Science Ltd., North-Holland.

Available online at www.sciencedirect.com
ScienceDirect
Mathematics and Computers in Simulation 76 (2007) 49–56
MATHEMATICS AND COMPUTERS IN SIMULATION
www.elsevier.com/locate/matcom

Numerical modelling of the weight-bearing total knee joint replacement and usage in practice

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Available online 23 January 2007

Abstract

In the contribution a weight-bearing total knee joint replacement will be based on numerical results on a non-linear contact problem with Coulombian friction in elasticity. The non-overlapping domain decomposition algorithm will be used. The main goal of the contribution represents an application of mathematical modelling and obtained numerical results to the practice—the total knee joint replacement.

Keywords: Total knee joint replacement; Contact problem; Non-overlapping domain decomposition method

1. Introduction

The first attempt of artificial replacement of human knee joint was made by Gluck in 1890. The technology of artificial joint replacement has developed namely during the last 50 years. The fundamental idea of Charnley, the so-called low friction arthroplasty, changed the development of all types of joint prostheses and is used up to the present. During recent years, considerable progress has been made in theoretical investigation of total joint replacements and their successful application in the practice. Let us mention here [1,2,7,11].

In the contribution a weight-bearing total knee replacement will be discussed and analysed. The model was derived from the X-ray radiograph of the knee joint after the implantation of artificial knee joint prosthesis. The stress–strain analysis is based on the theory of contact problems in elasticity and the non-overlapping domain decomposition method.

In our presentation, we would like to present our numerical results and their application in practice. Therefore, we will also present the information about the knee joint replacements WALTER UNIVERSAL and WALTER MODULAR.

The intention of total knee replacement is to replace damaged articular surfaces and to restore the mechanical axis of the extremity in such a way to change the non-functional knee joint by the complete functional one.

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doi:10.1016/j.matcom.2007.01.013

The model concerned with the above-discussed problem makes it possible to test total knee prosthesis from the aspect of biomechanics during statically loading of the knee prosthesis as well as tribology and the distribution of stress–strain field in the natural knee and in its replacement. The best surgical procedure for a given patient is then determined to preserve the condition of an approximately equal distribution of the stress–strain field in the knee joint and its replacement (see [6]). Since the implanted artificial knee joint, as well as the natural one, is a physically defined static system, where the shape corresponds to the function, then its impairment leads to a mechanical failure of the knee prosthesis, i.e. to a loosening of the knee prosthesis.

2. The Model

For the geometry of the model the X-ray radiograph was used. The mathematical model is based on the theory of contact problem in elasticity and the finite element approximation. The algorithm used for our computation is based on the non-overlapping domain decomposition method. On the contact boundary between both collided parts of the knee the Coulombian type of friction acts.

We will assume that the investigated knee joint occupies the domain $\Omega = \cup_{\iota=1}^{3} \Omega^{\iota}$, where Ω^1 is occupied by the femoral part of the knee joint, Ω^2 by the tibial part of the knee joint and Ω^3 by the fibula. The boundary $\partial\Omega$ is assumed to be sufficiently smooth and consists of three disjoint parts such that $\partial\Omega = \Gamma_\tau \cup \Gamma_u \cup \Gamma_c$, where by Γ_τ we denote the loaded and unloaded part of the boundary $\partial\Omega$. By Γ_u we denote the parts of the tibial and fibula's boundaries, where the tibia and the fibula are fixed. By $\Gamma_c = (\partial\Omega^1 \cap \partial\Omega^2) \cup (\partial\Omega^2 \cap \partial\Omega^3)$ we denote the common contact boundary between both knee joint components Ω^1 and Ω^2 and the connection between the tibia and the fibula. The knee joint is assumed to be loaded by the loading parallel with the femur's axis, corresponding to the weight of the human body. Let the body forces F, the surface forces P and the slip limits g_c be given. Then we have to solve the following problem:

Problem ($\mathcal{P}$). Find the displacements $\mathbf{u}^{\iota}$ in all Ω^{ι} such that:

$$\frac{\partial \tau_{ij}(\mathbf{u}^{\iota})}{\partial x_j} + F_i^{\iota} = 0, \quad i, j = 1, 2 \text{ in } \Omega^{\iota}, \quad \iota = 1, 2, 3, \tag{1}$$

$$\tau_{ij} n_j = P_i, \quad i, j = 1, 2 \text{ on } \Gamma_\tau, \tag{2}$$

$$u_i = u_{0i}, \quad i = 1, 2 \text{ on } \Gamma_u, \tag{3}$$

$$\left.\begin{array}{l} u_n^k - u_n^l \le 0,\ \tau_n^{kl} \le 0,\ (u_n^k - u_n^l)\tau_n^{kl} = 0 \\ |\boldsymbol{\tau}_t^{kl}(\mathbf{u})| \le F_c^{kl} |\tau_n^{kl}(\mathbf{u})| \equiv g_c^{kl} \\ |\boldsymbol{\tau}_t^{kl}(\mathbf{u})| < g_c^{kl} \Rightarrow \mathbf{u}_t^k - \mathbf{u}_t^l = 0 \\ |\boldsymbol{\tau}_t^{kl}(\mathbf{u})| = g_c^{kl} \Rightarrow \text{there exists } \vartheta \ge 0 \\ \text{such that } \mathbf{u}_t^k - \mathbf{u}_t^l = -\vartheta \boldsymbol{\tau}_t^{kl}(\mathbf{u}) \end{array}\right\} \text{ on } \Gamma_c^{kl}, \tag{4}$$

where $\Gamma_c^{kl} = \partial\Omega^k \cap \partial\Omega^l, k \neq l, k \in \{1, 2\}, l = k + 1$ and where the normal and tangential components of displacement vector $\mathbf{u} = (u_i)$, $i = 1, 2$, and stress vector $\boldsymbol{\tau} = (\tau_i)$, $i = 1, 2$, are defined as follows: $u_n = u_i n_i$, $\mathbf{u}_t = \mathbf{u} - u_n \mathbf{n}$, $\tau_n = \tau_{ij} n_j n_i$, $\boldsymbol{\tau}_t = \boldsymbol{\tau} - \tau_n \mathbf{n}$, where $\mathbf{n}$ denotes the outerward unit normal to the boundary $\partial\Omega$, and therefore, $\tau_n^k = \tau_n^l \equiv \tau_n^{kl}$, $\boldsymbol{\tau}_t^k(\mathbf{u}) = \boldsymbol{\tau}_t^l(\mathbf{u}) \equiv \boldsymbol{\tau}_t^{kl}(\mathbf{u})$. Moreover, the elastic coefficients c_{ijkl} satisfy the conditions of symmetry $c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij}$ and the condition $0 < c_0^{\iota} \le c_{ijkl}^{\iota} \xi_{ij} \xi_{kl} |\xi|^{-2} \le c_1^{\iota} < +\infty$, for a.a. $\mathbf{x} \in \boldsymbol{\Omega}^{\iota}$, $\xi \in \mathbb{R}^4$, $\xi_{ij} = \xi_{ji}$, $c_0^{\iota}, c_1^{\iota} = \text{constant} > 0$ independent of $\mathbf{x} \in \Omega^{\iota}$, $\iota \in \{1, 2, 3\}$.

Let $W = \sqcap_{\iota=1}^{3} [H^1(\Omega^{\iota})]^2$ be the Sobolev space in the usual sense, let $\|\mathbf{v}\|_W = (\sum_{\iota} \sum_i \|v_i\|_{1,\Omega^{\iota}}^2)^{1/2}$. Let us introduce the sets of virtual and admissible displacements:

$$V_0 = \{\mathbf{v} \in W | \mathbf{v} = 0 \text{ on } \Gamma_u\}, \qquad V = \mathbf{u}_0 + V_0, \qquad K = \{\mathbf{v} \in V | v_n^k - v_n^l \le 0 \text{ on } \Gamma_c\}$$

and let V_h, $K_h = V_h \cap K$ be their finite element approximations. Assume that $u_{0n}^k - u_{0n}^l = 0$ on Γ_c^{kl}. Let $c_{ijkl}^{\iota} \in L^\infty(\Omega^{\iota})$, $F_i^{\iota} \in L^2(\Omega^{\iota})$, $P_i \in L^2(\Gamma_\tau)$, $\mathbf{u}_0^{\iota} \in [H^1(\Omega^{\iota})]^2$. Then we have to solve the following variational problem.

Problem ($\mathcal{P}_v$). Find a function $\mathbf{u} \in K$, such that:

$$a(\mathbf{u}, \mathbf{v} - \mathbf{u}) + j(\mathbf{v}) - j(\mathbf{u}) \geq L(\mathbf{v} - \mathbf{u}), \quad \forall \mathbf{v} \in K, \tag{5}$$

where

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) &= \sum_{\iota} \int_{\Omega^\iota} c^\iota_{ijkl} e_{ij}(\mathbf{u}^\iota) e_{kl}(\mathbf{v}^\iota)\, \mathrm{d}\mathbf{x}, \\ j(\mathbf{v}) &= \int_{\cup_{k,l} \Gamma^{kl}_c} g^{kl}_c |\mathbf{v}^k_t - \mathbf{v}^l_t|\, \mathrm{d}s, \\ L(\mathbf{v}) &= \sum_{\iota} \int_{\Omega^\iota} F^\iota_i\, \mathrm{d}\mathbf{x} - \sum_{\iota} \int_{{}^1\Gamma_\tau \cap \partial\Omega^\iota} P^\iota_i v^\iota_i\, \mathrm{d}s. \end{aligned} \tag{6}$$

Let us introduce the sets of rigid displacements and rotations $R = \cup_\iota R^\iota$, $R^\iota = \{\mathbf{v}^\iota = (v^\iota_1, v^\iota_2) | v^\iota_1 = a^\iota_1 - b^\iota x_2, v^\iota_2 = a^\iota_2 + b^\iota x_1\}$, where $a^\iota_1, a^\iota_2, b^\iota, \iota = 1, 2, 3$, are arbitrary real constants. If:

$$\begin{aligned} & L(\mathbf{v}) < j(\mathbf{v}), \quad \forall \mathbf{v} \in R \cap K - \{0\}, \\ & \{\mathbf{v} \in R \cap V_0, v^k_n - v^l_n = 0 \text{ on } \Gamma^{kl}_c\} = \{0\} \text{ and} \\ & |L(\mathbf{v})| > j(\mathbf{v}), \quad \forall \mathbf{v} \in R \cap V_0 - \{0\}, \end{aligned} \tag{7}$$

then the problem (5) has a unique solution. The finite element approximation leads to solve the following problem.

Problem ($\mathcal{P}_h$). Find a function $\mathbf{u}_h \in K_h$, such that:

$$a(\mathbf{u}_h, \mathbf{v}_h - \mathbf{u}_h) + j(\mathbf{v}_h) - j(\mathbf{u}_h) \geq L(\mathbf{v}_h - \mathbf{u}_h), \quad \forall \mathbf{v}_h \in K_h. \tag{8}$$

For the numerical solution the non-overlapping domain decomposition method [3–5,9,10] was used.

3. Numerical results

For the geometry of the model the X-ray radiograph was used. We will study the knee joint replacement in two cases—the cut in the frontal plane (Fig. 1 a) and the cut in the sagital plane (Fig. 1b). The numerical results were used for verifications of the real existing total knee joint replacements WALTER-UNIVERSAL and WALTER-MODULAR, described below in Section 4.

The material parameters are as follows (E, Young's modulus; ν, Poisson's ratio): bone: $E = 1.71 \times 10^{10}$ Pa, $\nu = 0.25$; [1] Ti_6Al_4V: $E = 1.15 \times 10^{11}$ Pa, $\nu = 0.3$; [2] UHMWPE: $E = 3.4 \times 10^8$ Pa, $\nu = 0.4$; [3] CoCrMo: $E = 2.08 \times 10^{11}$ Pa, $\nu = 0.3$.

The femoral part of the knee joint was loaded between points 5 and 6 by a loading 0.215×10^7 Pa, the tibia and the fibula are fixed between points 1 and 2, 3 and 4, and the contact between the femoral and tibial parts of the knee joint replacement is between 7 and 8, 9 and 10, and between the tibial and fibula's parts 11 and 12. Figs. 2 and 3 represent the vertical τ_y and shear τ_{xy} components of stress tensor and Fig. 4 the principal stresses in the knee joint replacement.

Ideal distribution of loading on the tibial component of the total knee joint replacement represents the main requirement for its high quality. Stresses in the soft tissues (ligaments, joint capsule) in the neighbourhood of the total knee replacement determine the pressure relations in the joint. Observations of the reoperated knee joints show that certain asymmetric overloading of medial or lateral compartment, as well as overloading of the hinder part of the tibial plate increases to a wear of the UHMWPE insert or even starts its deformation.

Our numerical results show that for horizontal stresses predominate relatively small tractions in the region around inner medial and outer lateral parts of the contact area and in the region between condyles, otherwise the pressures are indicated in the femoral and tibial parts of the knee joint. Moreover, analyses of our numerical results indicate that the gradients of stresses in the joint are counterpoised in the epiphysis and then the metaphysis is strained uniformly. For vertical stress components in the whole area are indicated pressures, the area between condyles and the medial periphery are lightened and in the lateral direction the stresses increase. The similar situation is also in the tibia. Our numerical results indicate that pressures are transferred across outward lateral and inner medial contact area. From the vertical and shear stress components as well as from the principal stresses, it is shown, that greater parts of the

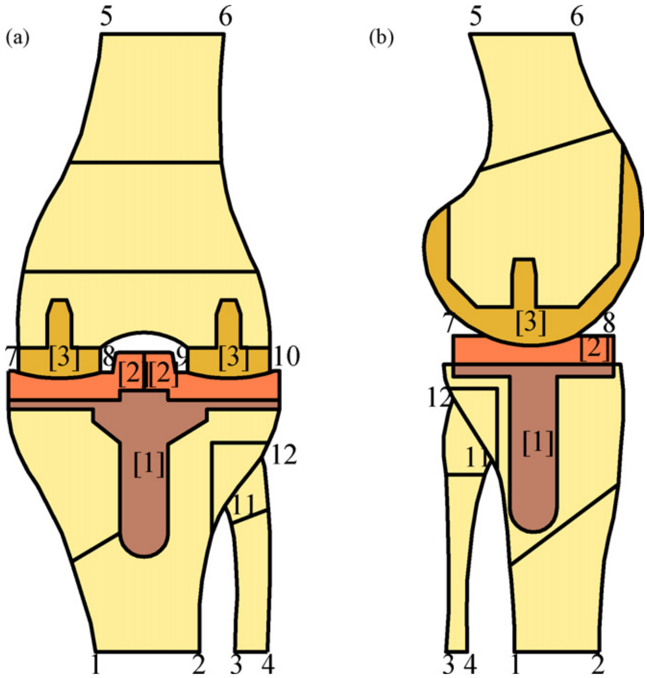

Fig. 1. The geometry of the model (a) in the frontal plane and (b) in the sagital plane.

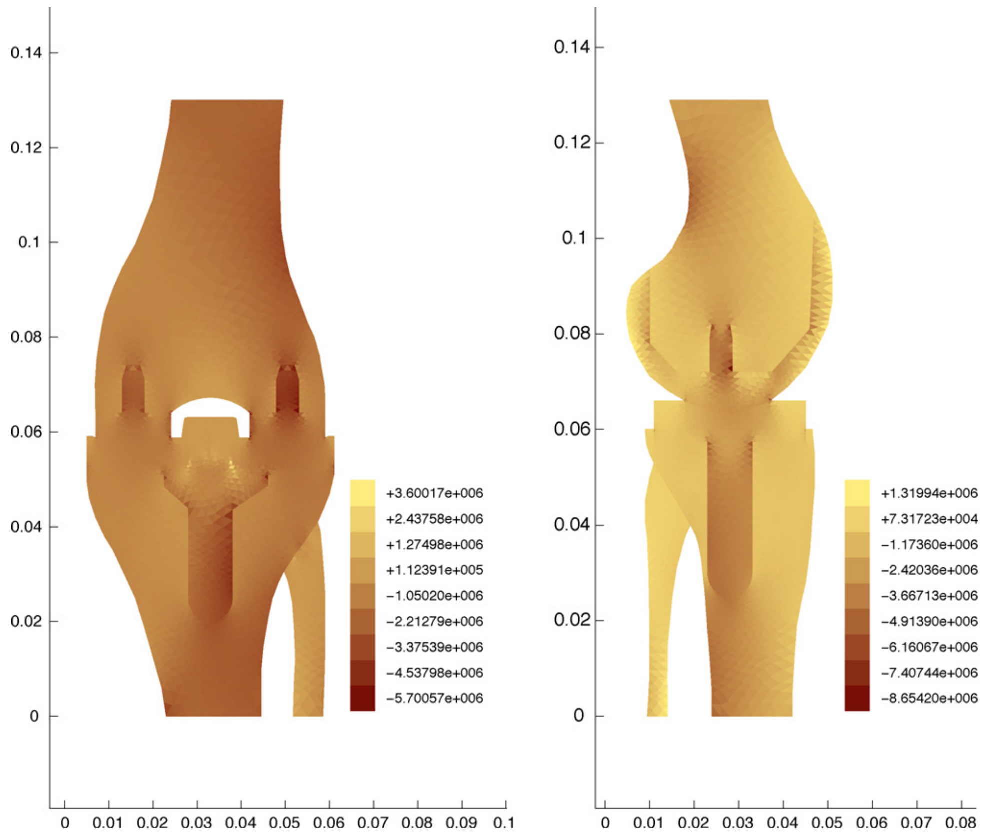

Fig. 2. The vertical component τ_y of stress tensor.

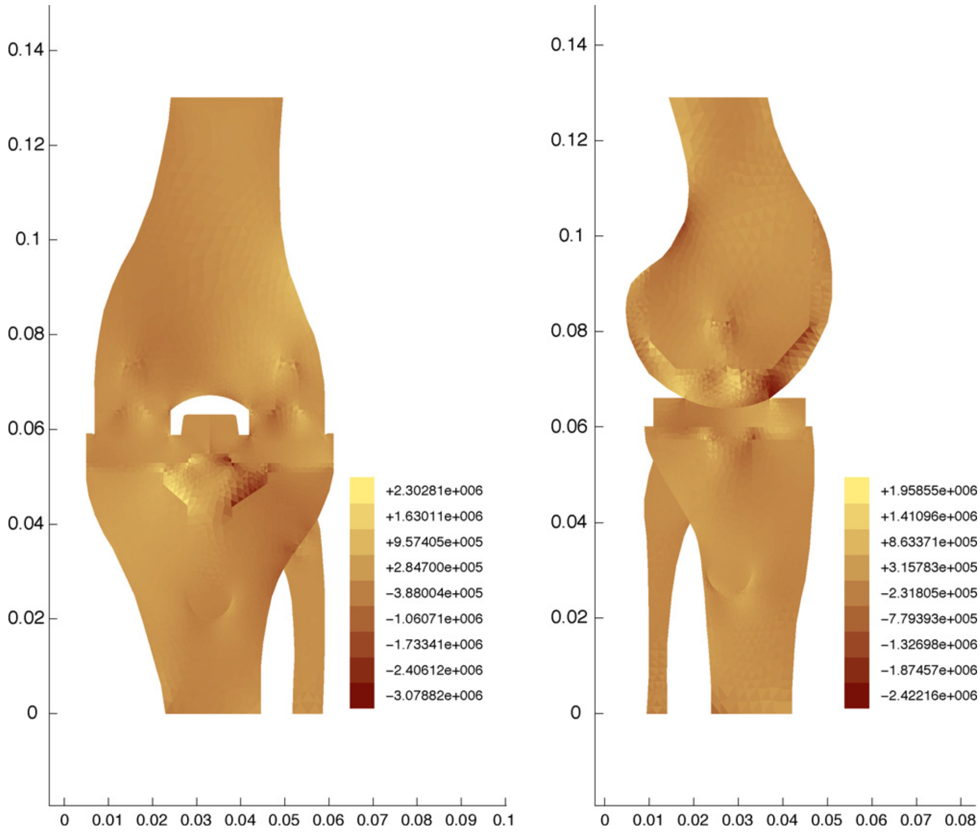

Fig. 3. The shear component τ_{xy} of stress tensor.

concentration of pressures are situated into the areas of outward condyles of the femur and the tibia, the smaller part across the inner condyles and tractions are situated in the area between the condyles.

4. Applications

Analyses of our numerical results were used for verification of function of the knee joint replacements WALTER UNIVERSAL and WALTER MODULAR (see [8]). Both are indicated for severe destruction of the knee joint in rheumatoid arthritis or osteoarthritis, and for some post-traumatic cases, as well as for some systemic diseases. The implants are intended for the primary replacement of the knee joint.

WALTER UNIVERSAL (WU) (see Fig. 5) is the older design which has used since 1984 and has implanted successfully in more than 12,000 cases. Femoral component of WU is designed as a symmetrical one and is manufactured of the CoCrMo (ISO 5832-4) alloy. Its basic type is intended for the application with bone cement. The femoral components are supplied in four sizes and can be provided with distal or dorsal inserts for the filling of local bone defects. Tibial component of WU is designed as a symmetrical one. The total tibial plateau is composed of two inseparable parts of (i) symmetric anchoring plate with a stem manufactured of the CoCrMo (ISO 5832-4) alloy and (ii) an insert produced of UHMWPE (ISO 5834-2), which forms the contact areas. It is supplied in four sizes and each of four thicknesses.

WALTER MODULAR (WM) (see Fig. 6) is based on the experiences with the previous model WU. The femoral component is designed as an asymmetrical one—right and left versions, and is produced of CoCrMo (ISO 5832-4) alloy. In its basic version is used with the bone cement. Femoral component is supplied in four sizes and it can be also provided with distal or dorsal inserts for the treatment of the local bone defects. Tibial component is designed as a modular one, in right and left versions. It is formed by an anchoring plate with a stem, and with an interchangeable

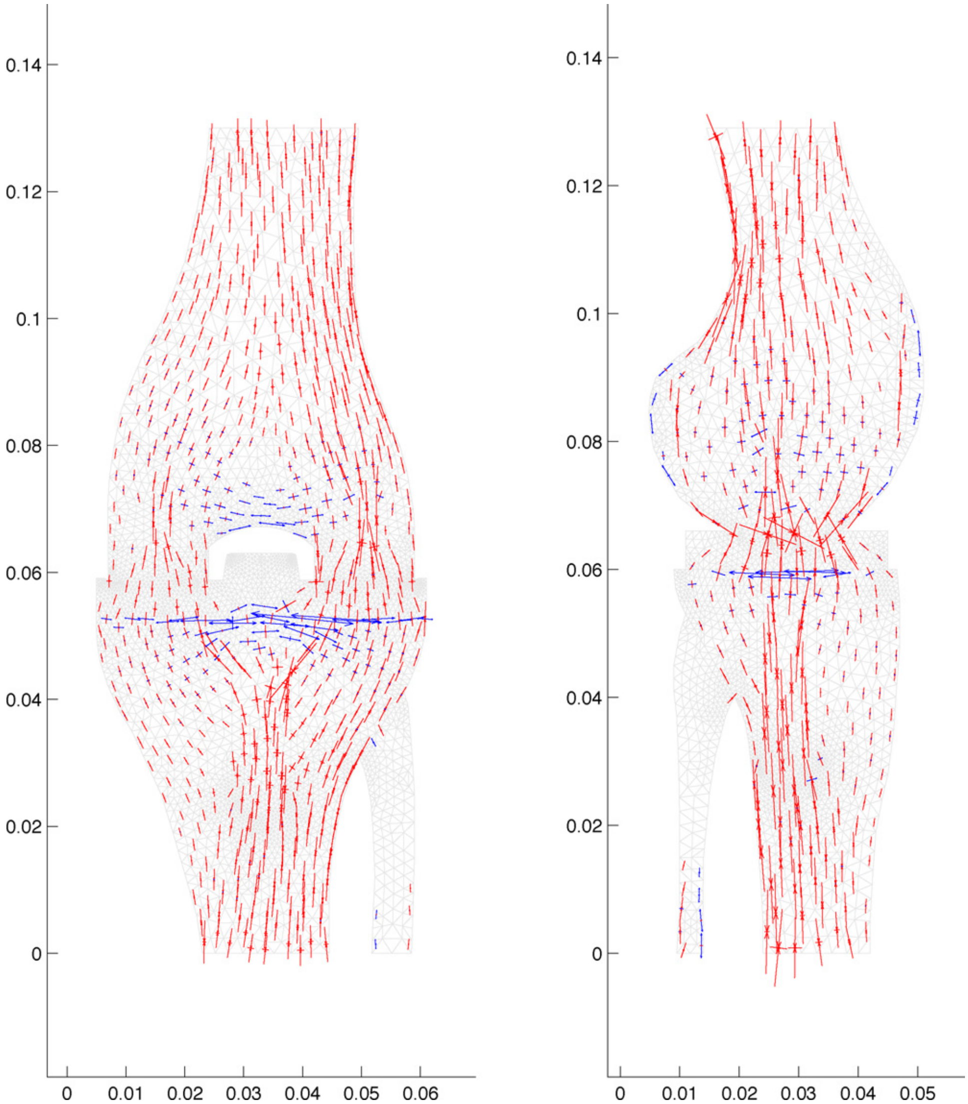

Fig. 4. The principal stresses.

Fig. 5. The knee joint replacement—WALTER UNIVERSAL.

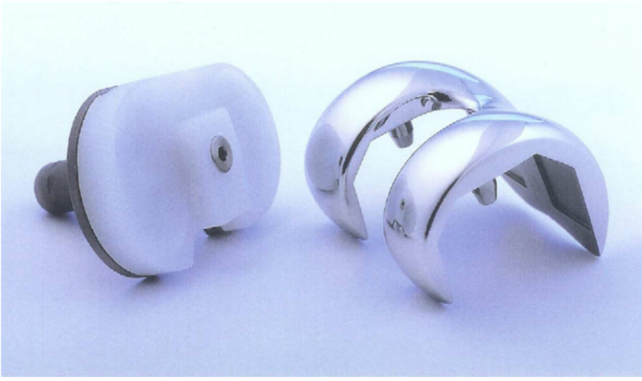

Fig. 6. The knee joint replacement—WALTER MODULAR.

insert with corresponding contact areas. The asymmetrical anchoring plate of tibial component shape corresponds to the different shapes of medial and lateral tibial condyles. The slots on plate provide space to preserve the posterior cruciate ligament. For fixation, the anchoring plate is provided with couple of anti-rotating ribs, short stem and two holes for bone screws. Basic version is supplied with a standard stem, which is provided with closing bolt at the end. All parts together with the anchoring plate are manufactured of titanium Ti_6Al_4V (ISO 5832-3) alloy. The articular insert is manufactured of the UHMWPE (ISO 5834-2) material in the right and left versions. The insert is fixed in the anchoring plate by several bosses and one central screw. The insert is delivered in four basic and three intermediate sizes to compensate possible differences between the femoral and tibial components.

Fig. 7 represents the X-ray images of WALTER UNIVERSAL in the frontal and sagital plane.

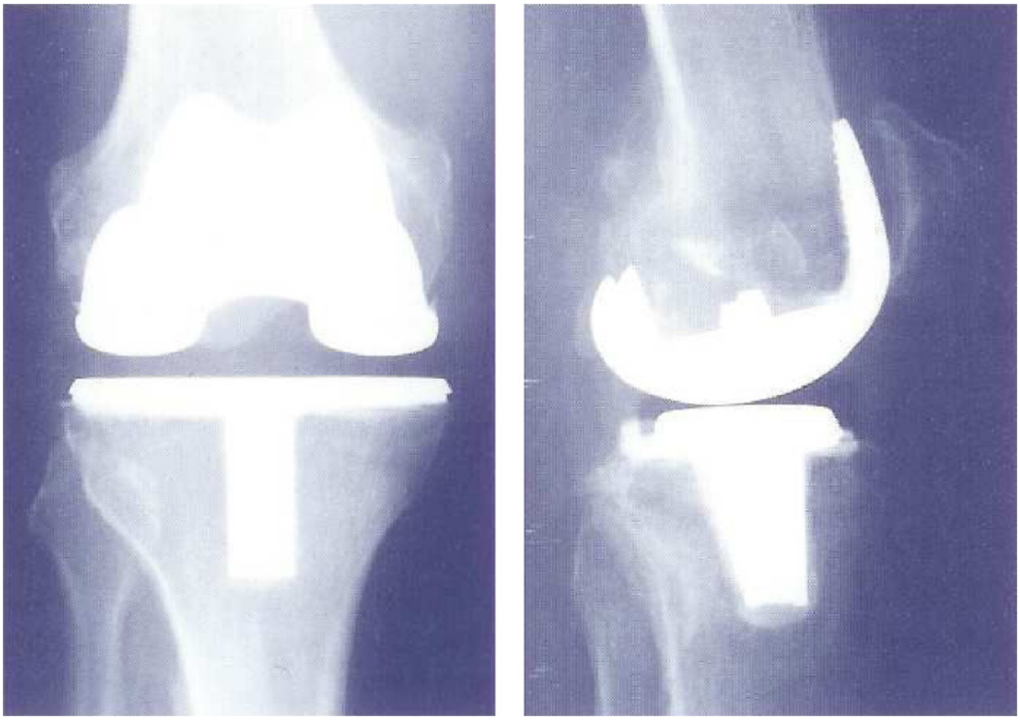

Fig. 7. The X-ray images of WALTER UNIVERSAL.

Acknowledgements

The authors have been partly supported by the Ministry of Education, Youth and Sport of the Czech Republic, MSM4977751301 and by the Ministry of Industry and Trade of the Czech Republic, FT-TA/087.

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Hlaváček, I., Nedoma, J., Daněk, J.: *Worst Scenario and Domain Decomposition Methods in Geomechanics*, Future Generation Computer Systems 22/4, Special section: Numerical modelling in geomechanics and geodynamics, pp. 468–483, Elsevier Science Ltd., North-Holland 2006.

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Future Generation Computer Systems 22 (2006) 468–483

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Worst scenario and domain decomposition methods in geomechanics

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Available online 29 June 2005

Abstract

In geomechanics, there are problems whose investigations lead to solving model problems based on variational formulations. Such problems are frequently formulated by variational inequalities as they physically describe the principle of virtual work in its inequality form. In the first part of the contribution, the algorithm for the numerical solution of the discussed variational inequality problem will be investigated. The used parallel algorithm is based on a nonoverlapping domain decomposition method for unilateral contact problem with the given friction and the finite element approach. The conditions of solvability will be presented. In the second part of the contribution, a unilateral contact problem with friction and with uncertain input data in quasi-coupled thermo-elasticity is analysed. Method of worst scenario will be applied to find the most "dangerous" admissible input data. The solvability of the corresponding worst scenario (antioptimization) problem will be shortly discussed. Numerical experiments, e.g. a tunnel crossing by an active fault will be presented.

1. Introduction

In this paper, we will deal with semi-coercive contact problem with friction and uncertain input data in linear quasi-coupled thermo-elasticity. The problem represents extension of problems solved in [1,2] for their application in geomechanics of high level radioactive waste repositories. Such problems are frequently formulated by variational inequalities as they physically describe the principle of virtual work in its inequality form.

The first part of the contribution will deal with numerical solution of a geomechanical problem based on the generalized semi-coercive contact problem with the given friction in quasi-coupled thermo-elasticity for the case that "*s*" bodies of arbitrary shapes are in mutual contacts and are loaded by external forces. The problem will be formulated as the primary variational inequality problem. The corresponding algorithm, employing

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0167-739X/$ – see front matter
doi:10.1016/j.future.2005.04.004

properties of modern parallel computers with greater number of processors, will be based on nonoverlapping domain decomposition method.

In the second part of the contribution we will assume that the input data will be also uncertain. By uncertain data we mean input data (physical coefficients, right-hand sides, boundary values, friction, etc.), which cannot be determined uniquely but only in some intervals determined by their measurement errors. The notation *reliable solution* denotes the *worst case* among a set of possible solutions, where possibility is given by uncertain input data, and the degree of badness is measured by a criterion-functional [3–5]. The main goal of our investigation will be to find maximal values of this functional depending on the solution of the problem to be solved. Therefore, we will formulate and analyze a corresponding maximization (worst scenario) problem.

2. Formulation of the thermo-elastic contact problem

Let us consider a union Ω of bounded domains Ω^ι, $\iota = 1, \dots, s$, with Lipschitz boundaries $\partial\Omega^\iota$, occupied by elastic bodies such that $\Omega = \cup_{\iota=1}^s \Omega^\iota \subset \mathbb{R}^N$, $N \in \{2, 3\}$. Let the boundary $\partial\Omega = \cup_{\iota=1}^s \partial\Omega^\iota$ consist of three disjoint parts Γ_τ, Γ_u and Γ_c, such that $\partial\Omega = \overline{\Gamma}_\tau \cup \overline{\Gamma}_u \cup \overline{\Gamma}_c$.

Assume that $(N-1)$-dimensional measures of Γ_τ, Γ_u and Γ_c are positive, where $\Gamma_c = \cup_{k,l} \Gamma^{kl}$, $\Gamma^{kl} = \partial\Omega^k \cap \partial\Omega^l$, $1 \le k, l \le s$, $k \ne l$, and $\overline{\Gamma}_\tau$, $\overline{\Gamma}_u$, $\overline{\Gamma}_c$ denote the closures in $\partial\Omega$.

We will deal with the following quasi-coupled problem of thermo-elasticity, which consists of a pair of boundary value and contact problems to be solved gradually.

2.1. Problem of stationary heat conduction—problem $\mathcal{P}_1$

Let W^ι and T_1 be given functions. Problem $\mathcal{P}_1$ is to find a function of temperature $T = (T^1, \dots, T^s)$ such that

$$\frac{\partial}{\partial x_i}\left(\kappa_{ij}^\iota \frac{\partial T^\iota}{\partial x_j}\right) + W^\iota = 0$$

$$\text{in } \Omega^\iota,\ 1 \le \iota \le s,\ i, j = 1, \dots, N, \tag{2.1}$$

$$\kappa_{ij} \frac{\partial T}{\partial x_j} n_i = 0 \quad \text{on } \Gamma_u, \tag{2.2}$$

$$T = T_1 \quad \text{on } \Gamma_\tau, \tag{2.3}$$

$$T^k = T^l,\ \left(\kappa_{ij}\frac{\partial T}{\partial x_j} n_i\right)^k + \left(\kappa_{ij}\frac{\partial T}{\partial x_j} n_i\right)^l = 0$$

$$\text{on } \cup_{k,l} \Gamma^{kl},\ 1 \le k, l \le s. \tag{2.4}$$

Throughout the paper we use the summation convention, i.e. a repeated index implies summation from 1 to N. Furthermore, $\mathbf{n}^k = (n_i^k)$, $i = 1, \dots, N$, $1 \le k \le s$, denotes the unit normal with respect to $\partial\Omega^k$, $\mathbf{n}^k = -\mathbf{n}^l$ on Γ^{kl}; (κ_{ij}^ι) is the matrix of thermal conductivities. Assume that κ^ι are positive definite symmetric matrices,

$$0 < \kappa_0^\iota \le \kappa_{ij}^\iota \zeta_i \zeta_j |\zeta|^{-2} \le \kappa_1^\iota < +\infty$$

$$\text{for a.a. } \mathbf{x} \in \Omega^\iota,\ \zeta \in \mathbb{R}^N,$$

where κ_0^ι, κ_1^ι are constants independent of $\mathbf{x} \in \Omega^\iota$. Let $\kappa_{ij}^\iota \in L^\infty(\Omega^\iota)$, $W^\iota \in L^2(\Omega^\iota)$, $T_1 \in H^1(\Omega^\iota)$, $T_1^k = T_1^l$ on $\cup_{k,l}\Gamma^{kl}$.

Definition 2.1. We say that a function T is a weak solution of problem $\mathcal{P}_1$, if $T - T_1 \in V_1$ and

$$b(T, z) = s(z) \quad \forall\, z \in V_1, \tag{2.5}$$

where

$$b(T, z) = \sum_{\iota=1}^s \int_{\Omega^\iota} \kappa_{ij}^\iota \frac{\partial T^\iota}{\partial x_i} \frac{\partial z^\iota}{\partial x_j}\, \mathrm{d}\mathbf{x},$$

$$s(z) = \sum_{\iota=1}^s \int_{\Omega^\iota} W^\iota z^\iota\, \mathrm{d}\mathbf{x},$$

$$V_1 = \{z \in W_1 = \sqcap_{\iota=1}^s H^1(\Omega^\iota) | z = 0 \quad \text{on } \Gamma_\tau,$$
$$z^k = z^l \quad \text{on } \cup_{k,l} \Gamma^{kl}\}.$$

The formulation (2.5) can be obtained by multiplying Eq. (2.1) by a test function, integrating by parts over the domain Ω^ι and using the boundary conditions.

2.2. Problem of unilateral contact problem with friction—problem $\mathcal{P}_2$

Let the body forces $\mathbf{F}$, the surface tractions $\mathbf{P}$, boundary displacements $\mathbf{u}_0$, elastic coefficients c^ι_{ijkl}, coefficients of thermal expansion β^ι_{ij} and slip limits g^{kl}_c, the temperature T^ι and the reference temperature $T^\iota_0 = T^\iota_0(\mathbf{x})$ be given.

We will deal with the following problem:

Problem $\mathcal{P}_2$: Find the displacement field $\mathbf{u} = (u_i)$, $i = 1, \ldots, N$ in Ω, such that

$$\frac{\partial}{\partial x_j}\tau_{ij}(\mathbf{u}^\iota, T^\iota) + F^\iota_i = 0 \quad \text{in } \Omega^\iota,$$
$$1 \le \iota \le s,\ i = 1, \ldots, N, \tag{2.6}$$

$$\tau_{ij}(\mathbf{u}^\iota, T^\iota) = c^\iota_{ijkl} e_{kl}(\mathbf{u}^\iota) - \beta^\iota_{ij}(T^\iota - T^\iota_0) \quad \text{in } \Omega^\iota,\ 1 \le \iota \le s,\ i = 1, \ldots, N, \tag{2.7}$$

$$\mathbf{u} = \mathbf{u}_0 \quad \text{on } \Gamma_u, \tag{2.8}$$

$$\tau_{ij}(\mathbf{u}, T)n_j = P_i \quad \text{on } \Gamma_\tau,\ i = 1, \ldots, N, \tag{2.9}$$

$$u^k_n - u^l_n \le 0,\ \tau^k_n \le 0,\ (u^k_n - u^l_n)\tau^k_n = 0$$
$$\text{on } \cup_{k,l}\Gamma^{kl},\ 1 \le k, l \le s, \tag{2.10}$$

$$|\tau^{kl}_t| \le g^{kl} \quad \text{on } \cup_{k,l}\Gamma^{kl},\ 1 \le k,\ l \le s, \tag{2.11}$$

$$|\tau^{kl}_t| < g^{kl} \Longrightarrow \mathbf{u}^k_t - \mathbf{u}^l_t = 0, \tag{2.12}$$

$$|\tau^{kl}_t| = g^{kl} \Longrightarrow \text{there exists } \vartheta \ge 0 \text{ such that } \mathbf{u}^k_t - \mathbf{u}^l_t = -\vartheta\tau^{kl}_t. \tag{2.13}$$

Here, $e_{ij}(\mathbf{u}) = \frac{1}{2}((\partial u_i/\partial x_j) + (\partial u_j/\partial x_i))$, $u^k_n = u^k_i n^k_i$, $u^l_n = u^l_i n^k_i$ (no sum over k or l), $\mathbf{u}^k_t = (u^k_{ti})$, $u^k_{ti} = u^k_i - u^k_n n^k_i$, $\mathbf{u}^l_t = (u^l_{ti})$, $u^l_{ti} = u^l_i - u^l_n n^l_i$, $i = 1, \ldots, N$, $\tau^k_n = \tau^k_{ij} n^k_i n^k_j$, $\tau^k_t = (\tau^k_{ti})$, $\tau^k_{ti} = \tau^k_{ij} n^k_j - \tau^k_n n^k_i$, $\tau^l_n = \tau^l_{ij} n^l_i n^l_j$, $\tau^l_t = (\tau^l_{ti})$, $\tau^l_{ti} = \tau^l_{ij} n^l_j - \tau^l_n n^l_i$, $\tau^{kl}_t \equiv \tau^k_t$.

Assume that c^ι_{ijkl} are positive definite symmetric matrices such that

$$0 < c^\iota_0 \le c^\iota_{ijkl}\xi_{ij}\xi_{kl}|\xi|^{-2} \le c^\iota_1 < +\infty$$
$$\text{for a.a. } \mathbf{x} \in \Omega^\iota,\ \xi \in \mathbb{R}^{N^2},\ \xi_{ji} = \xi_{ij},$$

where c^ι_0, c^ι_1 are constants independent of $\mathbf{x} \in$ **Omega**$^\iota$. Let $c^\iota_{ijkl} \in L^\infty(\Omega^\iota)$, $F^\iota_i \in L^2(\Omega^\iota)$, $P_i \in L^2(\Gamma_\tau)$, $\beta^\iota_{ij} \in L^\infty(\Omega^\iota)$, $\mathbf{u}^\iota_0 \in [H^1(\Omega^\iota)]^N$. Let coefficients of thermal expansion β_{ij} be such that $\beta_{ij} = \beta_{ji}$.

To simplify the formulation of stress–strain relations, the entries of any symmetric $(N \times N)$ matrix $\{\tau_{ij}\}$ will be denoted by *the vector notation* $\{\tau_j\}$, $j = 1, \ldots, j_N$, where $j_N = N(N+1)/2$, as follows:

$$\tau_i = \tau_{ii} \quad \text{for } 1 \le i \le N, \qquad \tau_3 = \tau_{12} \quad \text{for } N = 2,$$

$$\tau_4 = \tau_{23}, \qquad \tau_5 = \tau_{31}, \qquad \tau_6 = \tau_{12} \quad \text{for } N = 3.$$

Likewise, we replace the symmetric matrices $(e_{ij}(\mathbf{u}))$, (β_{ij}) by vectors $\{e_j(\mathbf{u})\}$, $\{\beta_j\}$. Then the stress–strain relation (2.7) can be rewritten as

$$\tau_i(\mathbf{u}^\iota, T^\iota) = \sum_{j=1}^{j_N} \mathcal{A}^\iota_{ij} e_j(\mathbf{u}^\iota) - \beta^\iota_i(T^\iota - T^\iota_0),$$
$$1 \le i,\ j \le j_N,\ 1 \le \iota \le s, \tag{2.7'}$$

where $\mathcal{A}^\iota$ is a symmetric $(j_N \times j_N)$ matrix, $\mathcal{A}^\iota_{ik} \in L^\infty(\Omega^\iota)$, $\iota = 1, \ldots, s$.

It is readily seen that

$$\tau : e \equiv \tau_{ij}e_{ij} = \sum_{i=1}^{N} \tau_i e_i + 2\sum_{i=N+1}^{j_N} \tau_i e_i.$$

Therefore, we can write

$$c^\iota_{ijkl} e_{ij} e_{kl} = \sum_{i,j=1}^{j_N} B^\iota_{ij} e_i e_j,$$

where B^ι is a symmetric $(j_N \times j_N)$ matrix such that

$$B^\iota_{ij} = \mathcal{A}^\iota_{ij} \quad \text{for } 1 \le i,\ j \le N,$$
$$B^\iota_{ij} = 0 \quad \text{for } 1 \le i \le N,\ N+1 \le j \le j_N,$$
$$B^\iota_{ij} = 2\mathcal{A}^\iota_{ij} \quad \text{for } N+1 \le i,\ j \le j_N.$$

Let us denote

$$W_1 = \sqcap^s_{\iota=1} H^1(\Omega^\iota), \quad \|w\|_{W_1} = \left(\sum_{\iota \le s} \|w^\iota\|^2_{1,\Omega^\iota}\right)^{\frac{1}{2}},$$

$$W = \sqcap_{\iota=1}^{s}[H^1(\Omega^\iota)]^N,$$

$$\|\mathbf{v}\|_W = \left(\sum_{\iota\le s}\sum_{i\le N}\|v_i^\iota\|_{1,\Omega^\iota}^2\right)^{\frac{1}{2}}.$$

Assume that the matrices B^ι are positive definite, so that

$$0 < a_0^\iota \le \sum_{i,j=1}^{j_N} B_{ij}^\iota \xi_i \xi_j |\xi|^{-2} \le a_1^\iota < +\infty$$

$$\text{for a.a. } \mathbf{x} \in \Omega^\iota, \xi \in \mathbb{R}^{j_N},$$

where the constants a_0^ι, a_1^ι are independent of $\mathbf{x} \in \Omega^\iota$.

Finally, let us assume that

$$\text{meas}_{N-1}(\Gamma_u \cap \partial\Omega^\iota) > 0 \quad \text{and}$$

$$\text{meas}_{N-1}(\Gamma_\tau \cap \partial\Omega^\iota) > 0 \qquad \text{for all } \iota = 1, \dots, s,$$

and let $\mathbf{u}_0 \in W$, $T_0 \in W_1$, $g_c^{kl} \in L^\infty(\Gamma^{kl})$, $\beta_j^\iota \in L^\infty(\Omega^\iota)$. Let us introduce the space of virtual displacements

$$V = \{\mathbf{v} \in W | \mathbf{v} = 0 \quad \text{on } \Gamma_u\}$$

and the set of admissible displacements

$$K = \{\mathbf{v} \in V | v_n^k - v_n^l \le 0 \quad \text{on } \cup_{k,l} \Gamma^{kl}\}.$$

We shall define a weak solution of the problem $\mathcal{P}_2$, which is motivated by the standard procedure: multiply equations (2.6) by a test vector function, integrate by parts over the domain Ω, use the boundary conditions and assume that $\mathbf{u}_0$ satisfies conditions $u_{0n}^k - u_{0n}^l = 0$ on $\cup_{k,l}\Gamma^{kl}$.

Definition 2.2. We say that the function $\mathbf{u}$ is a weak solution of problem $\mathcal{P}_2$, if $\mathbf{u} - \mathbf{u}_0 \in K$ and

$$a(\mathbf{u}, \mathbf{v} - \mathbf{u}) + j_g(\mathbf{v}) - j_g(\mathbf{u}) \ge S(\mathbf{v} - \mathbf{u}, T) \quad \forall\, \mathbf{v} \in \mathbf{u}_0 + K, \tag{2.13}$$

where

$$a(\mathbf{u}, \mathbf{v}) = \sum_{\iota=1}^{s} \int_{\Omega^\iota} \sum_{i,j=1}^{3} B_{ij}^\iota e_i(\mathbf{u}^\iota) e_j(\mathbf{v}^\iota)\, \mathrm{d}\mathbf{x}, \tag{2.14}$$

$$j_g(\mathbf{v}) = \sum_{k,l} \int_{\Gamma^{kl}} g^{kl} |\mathbf{v}_t^k - \mathbf{v}_t^l|\, \mathrm{d}s, \tag{2.15}$$

$$S(\mathbf{v}, T) = \sum_{\iota=1}^{s} \int_{\Omega^\iota} (F_i^\iota v_i^\iota + (T^\iota - T_0^\iota)\beta^\iota : e(\mathbf{v}^\iota))\, \mathrm{d}\mathbf{x} + \int_{\Gamma_\tau} P_i v_i\, \mathrm{d}s, \tag{2.16}$$

where the weak solution T of the problem $\mathcal{P}_1$ in $S(\mathbf{v}, T)$ is inserted.

3. Numerical solution and domain decomposition algoritm

In this section, we deal with the elastic part of problem only, as the domain decomposition algorithm for the thermal part of the problem is the standard problem solved in the literature.

3.1. Formulation of the problem

We follow the approach proposed by Le Tallec [6] and group every two subdomains which share a contact area Γ^{kl} into a single "nonlinear" subdomain. We use discretization by linear finite elements and the concept of local Schur complements. The resulting nonlinear equation on the interface is solved by successive approximations. For the starting approximation we choose the solution of the linear problem, where the unilateral contact conditions are replaced by the classical bilateral contact conditions without friction.

Let every domain $\bar{\Omega}^\iota$ be divided into $J(\iota)$ subdomains Ω_i^ι, $i \le J(\iota)$. Let us denote $\Gamma_i^\iota = \partial\Omega_i^\iota \backslash \partial\Omega^\iota$, $\iota \in \{1, \dots, s\}$, $i \in \{1, \dots, J(\iota)\}$, a part of dividing line and let $\Gamma = \cup_{\iota=1}^{s} \cup_{i=1}^{J(\iota)} \Gamma_i^\iota$ represent the whole interface boundary. Let us introduce

$$T^\iota = \{j \in \{1, \dots, J(\iota)\} : \bar{\Gamma}_c \cap \bar{\Omega}_j^\iota = \emptyset\}, \iota = 1, \dots, s \tag{3.1}$$

the set of all indices of subdomains of the domain Ω^ι which are not adjacent to a contact, and let

$$\Omega^{*j} = \cup_{[i,\iota]\in\vartheta} \Omega_i^\iota, \tag{3.2}$$

where $\vartheta = \{[i, \iota] : \partial\Omega_i^\iota \cap \Gamma_c \ne \emptyset\}$, represent subdomains in unilateral contact. Suppose that $\Gamma \cap \Gamma_c = \emptyset$.

Then for the trace operator $\gamma : [H^1(\Omega_i^\iota)]^N \to [L^2(\partial\Omega_i^\iota)]^N$ we have

$$V_\Gamma = \gamma K|_\Gamma = \gamma V|_\Gamma. \quad (3.3)$$

Let $\gamma^{-1} : V_\Gamma \in V$ be an arbitrary linear inverse mapping satisfying

$$\gamma^{-1}\overline{\mathbf{v}} = 0 \quad \text{on } \cup_{k,l} \Gamma^{kl} \quad \forall\, \overline{\mathbf{v}} \in V_\Gamma. \quad (3.4)$$

Let us introduce restrictions $\overline{R}_i^\iota : V_\Gamma \to \Gamma_i^\iota$; $L_i^\iota : L^\iota \to \Omega_i^\iota$; $j_{gi}^\iota : j_g^\iota \to \Gamma^{kl}$; $a_i^\iota(\cdot,\cdot) : a_i^\iota(\cdot,\cdot) \to \Omega_i^\iota$; $V(\Omega_i^\iota) \to \Omega_i^\iota$ and let

$$V^0(\Omega_i^\iota) = \{\mathbf{v} \in V | \mathbf{v} = 0 \quad \text{on } \overline{(\cup_{\iota=1}^s \Omega^\iota)\backslash\Omega_i^\iota}\}$$

be the space of functions with zero traces on Γ_i^ι.

Theorem 3.1. *A function* $\mathbf{u}$ *is a solution of a global problem* $\mathcal{P}_2$*, if and only if: its trace* $\underline{\mathbf{u}} = \gamma\mathbf{u}|_\Gamma$ *on the interface* Γ *satisfies the condition*

$$\sum_{\iota=1}^{s}\sum_{i=1}^{J(\iota)}[a_i^\iota(\mathbf{u}_i^\iota(\underline{\mathbf{u}}), \gamma^{-1}\underline{\mathbf{w}}) - S_i^\iota(\gamma^{-1}\underline{\mathbf{w}})] = 0$$
$$\forall\, \underline{\mathbf{w}} \in V_\Gamma,\ \underline{\mathbf{u}} \in V_\Gamma, \quad (3.5)$$

and its restrictions $\mathbf{u}_i^\iota(\mathbf{u}) \equiv \mathbf{u}|_{\Omega_i^\iota}$ *satisfy*

(i) *the condition*

$$a_i^\iota(\mathbf{u}_i^\iota(\underline{\mathbf{u}}), \boldsymbol{\varphi}_i^\iota) = S_i^\iota(\boldsymbol{\varphi}_i^\iota) \quad \forall\, \boldsymbol{\varphi}_i^\iota \in V^0(\Omega_i^\iota),$$
$$\mathbf{u}_i^\iota(\underline{\mathbf{u}}) \in V(\Omega_i^\iota),\ \gamma\mathbf{u}_i^\iota(\underline{\mathbf{u}})|_{\Gamma_i^\iota} = \overline{R}_i^\iota\underline{\mathbf{u}}, \quad (3.6)$$

for $i \in T^\iota$, $\iota = 1, \ldots, s$, *and*

(ii) *the condition*

$$\sum_{[i,\iota]\in\vartheta} a_i^\iota(\mathbf{u}_i^\iota(\underline{\mathbf{u}}), \boldsymbol{\varphi}_i^\iota) + j_g^\iota(\mathbf{u}_i^\iota(\underline{\mathbf{u}}) + \boldsymbol{\varphi}_i^\iota) - j_g^\iota(\mathbf{u}_i^\iota(\underline{\mathbf{u}}))$$
$$\geq \sum_{[i,\iota]\in\vartheta} S_i^\iota(\boldsymbol{\varphi}_i^\iota) \quad (3.7)$$

for all $\boldsymbol{\varphi} \equiv (\boldsymbol{\varphi}_i^\iota, [i,\iota] \in \vartheta)$, $\boldsymbol{\varphi}_i^\iota \in V^0(\Omega_i^\iota)$, *and such that*

$$\mathbf{u} + \boldsymbol{\varphi} \in K,$$
$$\gamma\mathbf{u}_i^\iota(\underline{\mathbf{u}})|_{\Gamma_i^\iota} = \overline{R}_i^\iota\underline{\mathbf{u}} \quad \text{for } [i,\iota] \in \vartheta. \quad (3.8)$$

For the proof see [7].

3.2. The Schur complements and the linearized problem

The aim of this subsection is to analyze in detail the condition (3.5) and to employ it for numerical computation of problem $\mathcal{P}_2$. We will introduce the concept of the *local Schur complement*.

Let us denote $V_i^\iota = \{\gamma\mathbf{v}|_{\Gamma_i^\iota} | \mathbf{v} \in K\} = \{\gamma\mathbf{v}|_{\Gamma_i^\iota} | \mathbf{v} \in V\}$ and define a particular case of the restriction of the inverse mapping $\gamma^{-1}(\cdot)|_{\Omega_i^\iota}$ by

$$\begin{cases} Tr_{i\iota}^{-1} : V_i^\iota \to V(\Omega_i^\iota), \\ \quad \gamma(Tr_{i\iota}^{-1}\underline{\mathbf{u}}_i^\iota)|_{\Gamma_i^\iota} = \underline{\mathbf{u}}_i^\iota, \qquad i = 1, \ldots, J(\iota), \\ \qquad\qquad\qquad\qquad\qquad \iota = 1, \ldots, s, \\ a_i^\iota(Tr_{i\iota}^{-1}\underline{\mathbf{u}}_i^\iota, \mathbf{v}_i^\iota) = 0 \\ \quad \forall\, \mathbf{v}_i^\iota \in V^0(\Omega_i^\iota), \\ \quad Tr_{i\iota}^{-1}\underline{\mathbf{u}}_i^\iota \in V(\Omega_i^\iota), \qquad \text{for } i \in T^\iota, \\ \qquad\qquad\qquad\qquad\qquad \iota = 1, \ldots, s. \end{cases} \quad (3.9)$$

For $[i,\iota] \in \vartheta$ we complete the definition by the boundary condition (3.4), i.e.

$$Tr_{i\iota}^{-1}\underline{\mathbf{u}}_i^\iota = 0 \quad \text{on } \cup_{k,l} \Gamma^{kl}. \quad (3.10)$$

Definition 3.1. By the local Schur complement for $i \in T^\iota$ it is meant the operator $\mathcal{S}_i^\iota : V_i^\iota \to (V_i^\iota)^*$ defined by

$$\langle \mathcal{S}_i^\iota \overline{\mathbf{u}}_i^\iota, \overline{\mathbf{v}}_i^\iota \rangle = a_i^\iota(Tr_{i\iota}^{-1}\overline{\mathbf{u}}_i^\iota, Tr_{i\iota}^{-1}\overline{\mathbf{v}}_i^\iota) \quad \forall\, \overline{\mathbf{u}}_i^\iota, \overline{\mathbf{v}}_i^\iota \in V_i^\iota \quad (3.11)$$

and in the matrix form by

$$\mathcal{S}_i^\iota \overline{\mathbf{U}}_i^\iota = (\overline{A}_{i\iota} - \overline{B}_{i\iota}^{\mathrm{T}} \mathring{A}_{i\iota}^{-1} B_{i\iota})\overline{\mathbf{U}}_i^\iota, \quad (3.12)$$

where

$$A_{i\iota} = \begin{pmatrix} \mathring{A}_{i\iota} & B_{i\iota} \\ B_{i\iota}^{\mathrm{T}} & \overline{A}_{i\iota} \end{pmatrix}, \quad \mathbf{U}_i^\iota = \begin{pmatrix} \mathring{\mathbf{U}}_i^\iota \\ \overline{\mathbf{U}}_i^\iota \end{pmatrix}, \quad (3.13)$$

where the nodes of $\overline{\mathbf{U}}_i^\iota$ belong to Γ_i^ι and the internal degrees of freedom are $\mathring{\mathbf{U}}_i^\iota$.

For subdomains which are in contact we will define a *common local Schur complement* as follows:

Definition 3.2. The common local Schur complement for the union $\Omega_i^k \cup \Omega_j^l$ (where $\Gamma_c^{kl} \subset \Gamma_c$ and $[i,k] \in \vartheta$, $[j,l] \in \vartheta$) is the operator

$$\mathcal{S}^{kl} : (V_i^k \times V_j^l) \to (V_i^k \times V_j^l)^* = (V_i^k)^* \times (V_j^l)^*$$

defined by the relation

$$\langle \mathcal{S}^{kl}(\overline{\mathbf{y}}_i^k, \overline{\mathbf{y}}_j^l), (\overline{\mathbf{v}}_i^k, \overline{\mathbf{v}}_j^l)\rangle = a_i^k(\mathbf{u}_i^k(\overline{\mathbf{y}}_i^k), Tr_{ik}^{-1}\overline{\mathbf{v}}_i^k) + a_j^l(\mathbf{u}_j^l(\overline{\mathbf{y}}_j^l), Tr_{jl}^{-1}\overline{\mathbf{v}}_j^l) \quad \forall (\mathbf{v}_i^k, \overline{\mathbf{v}}_j^l) \in V_i^k \times V_j^l. \tag{3.14}$$

where Tr_{ik}^{-1} and Tr_{jl}^{-1} are defined by means of (3.9) and (3.10) and $\mathbf{u}_i^k(\overline{\mathbf{y}}_i^k)$, $\mathbf{u}_j^l(\overline{\mathbf{y}}_j^l)$ denote the solution of the problem (3.7).

The condition (3.5) can be expressed by means of local Schur complements. Then we have

Lemma 3.2. *The trace $\overline{\mathbf{u}} = \gamma\mathbf{u}|_\Gamma$ of the weak solution satisfies the following condition*

$$\sum_{\iota=1}^{s}\sum_{i\in T^\iota}\langle \mathcal{S}_i^\iota\overline{\mathbf{u}}_i^\iota, \overline{\mathbf{v}}_j^\iota\rangle + \sum_{k,l}\langle \mathcal{S}^{kl}(\overline{\mathbf{u}}_i^k, \overline{\mathbf{u}}_j^l), (\overline{\mathbf{v}}_i^k, \overline{\mathbf{v}}_j^l)\rangle = \sum_{\iota=1}^{s}\sum_{i=1}^{J(\iota)} L_i^\iota(Tr_{i\iota}^{-1}\overline{\mathbf{v}}_i^\iota) \quad \forall\, \overline{\mathbf{v}} \in V_\Gamma,\ [i,k] \in \vartheta, \ [j,l] \in \vartheta,\ \Gamma^{kl} \subset \Gamma_c, \tag{3.15}$$

where $\overline{\mathbf{v}}_i^\iota = \overline{R}_i^\iota\overline{\mathbf{v}}, \overline{\mathbf{u}}_i^\iota = \overline{R}_i^\iota\overline{\mathbf{u}}.$

Then we will solve the Eq. (3.15) on the interface Γ in the dual space $(V_\Gamma)^*$. We rewrite (3.15) into the following form

$$\mathcal{S}_0\overline{\mathbf{U}} + \mathcal{S}_{\mathrm{CON}}\overline{\mathbf{U}} = \mathcal{F}, \tag{3.16}$$

where

$$\mathcal{S}_0 = \sum_{\iota=1}^{s}\sum_{i\in T^\iota}(\overline{R}_i^\iota)^{\mathrm{T}}\mathcal{S}_i^\iota\overline{R}_i^\iota, \qquad \mathcal{S}_{\mathrm{CON}} = \sum_{k,l}\overline{R}_{kl}^{\mathrm{T}}\mathcal{S}^{kl}\overline{R}_{kl},$$

$$\mathcal{F} = \sum_{\iota=1}^{s}\sum_{i=1}^{J(\iota)}(\overline{R}_i^\iota)^{\mathrm{T}}(Tr_{i\iota}^{-1})^{\mathrm{T}}\mathcal{S}_i^\iota \tag{3.17}$$

and $\overline{R}_{kl}(\overline{\mathbf{u}}) = (\overline{R}_i^k(\overline{\mathbf{u}}),\ \overline{R}_j^l(\overline{\mathbf{u}}))^{\mathrm{T}},\ \overline{\mathbf{u}} \in V_\Gamma, [i,k] \in \vartheta,$ $[j,l] \in \vartheta,\ \Gamma^{kl} \subset \Gamma_c.$

Eq. (3.16) will be solved by *successive approximations*, because the operators $\mathcal{S}^{kl}$ and therefore $\mathcal{S}_{\mathrm{CON}}$ are nonlinear. We choose a suitable initial approximation $\overline{\mathbf{U}}^0$, for instance the solution of the global primal problem, where the boundary conditions on Γ_c are replaced by the linear "classical" bilateral conditions (which correspond with $g^{kl} \equiv 0$ and $j_g(\mathbf{u}) \equiv 0$)

$$u_n^k - u_n^l = 0, \tau_t^{kl} = 0 \quad \text{on } \Gamma_{c0} \equiv \cup_{k,l}\Gamma_0^{kl} \tag{3.18}$$

where Γ_0^{kl} are parts of Γ^{kl}, $textrmmeas\Gamma_0^{kl} > 0$, chosen a priori (for example, $\Gamma_0^{kl} = \Gamma^{kl}$). On $\Gamma^{kl}\backslash\Gamma_0^{kl}$ we consider homogeneous conditions of zero surface load $P_j^k = P_j^l = 0, j = 1, \dots, N$.

Then we replace the set K by $K^0 = \{\mathbf{v} \in V | v_n^k - v_n^l = 0 \quad \text{on } \cup_{k,l}\,\Gamma_0^{kl}\}$ and therefore, we will solve the following problem

$$\mathbf{u}^0 = \arg\min_{\mathbf{v}\in K^0}\left(\frac{1}{2}a(\mathbf{v},\mathbf{v}) - S(\mathbf{v})\right) \tag{3.19}$$

and we set $\overline{\mathbf{U}}^0 = \gamma\mathbf{u}^0|_\Gamma$. The auxiliary problem (3.19) represents a linear elliptic boundary value problem of a system of "s" elastic bodies with bilateral contact and it can be solved by the domain decomposition method again.

3.3. Solution of the auxiliary problem

Instead of (2.13) we will solve the variational equation for $\mathbf{u}^0 \in K^0$:

$$a(\mathbf{u}^0, \mathbf{v}) = S(\mathbf{v}) \quad \forall\, \mathbf{v} \in K^0. \tag{3.20}$$

Thus, an analogue of Theorem 3.1 can be derived, where the condition (3.7) is replaced by the corresponding variational equality and where a mapping γ_0^{-1} : $V_\Gamma \to V$ satisfies conditions $(\gamma_0^{-1}\overline{\mathbf{v}})_n^k - (\gamma_0^{-1}\overline{\mathbf{v}})_n^l = 0$ on $\cup_{k,l}\Gamma_0^{kl}$.

We introduce operators of Schur complements. For $i \in T^\iota, \iota = 1,\dots,s$, we define the mappings $Tr_{i\iota}^{-1}$ according to (3.9) and the local Schur complemenents $\mathcal{S}_i^{0\iota}$ by (3.11).

Definition 3.3. The common local Schur complement for the union $\Omega_i^k \cup \Omega_j^l$, where $\Gamma_0^{kl} \subset \Gamma_c$ and $[i,k] \in \vartheta, [j,l] \in \vartheta,$

$$\mathcal{S}^{0kl} : (V_i^k \times V_j^l) \to (V_i^k)^* \times (V_j^l)^*$$

is defined by the following relation

$$\langle \mathcal{S}^{0kl}(\overline{\mathbf{u}}_i^{0k}, \mathbf{u}_j^{0l}), (\overline{\mathbf{v}}_i^k, \overline{\mathbf{v}}_j^l)\rangle = a_i^k(\mathbf{u}_i^k(\overline{\mathbf{u}}_i^k), Tr_{ik}^{-1}\overline{\mathbf{v}}_i^k) + a_j^l(\mathbf{u}_j^l(\overline{\mathbf{u}}_j^l), Tr_{jl}^{-1}\mathbf{v}_j^l) \quad \forall (\overline{\mathbf{v}}_i^k, \overline{\mathbf{v}}_j^l) \in V_i^k \times V_j^l, \tag{3.21}$$

where Tr_{ik}^{-1} and Tr_{jl}^{-1} are defined by means of $(Tr_{ik}^{-1}\overline{\mathbf{v}}_i^k)_n - (Tr_{jl}^{-1}\overline{\mathbf{v}}_j^k)_n = 0$ on Γ_0^{kl} and

$$a_i^k(Tr_{ik}^{-1}\overline{\mathbf{v}}_i^k, \mathbf{w}_i^k) + a_j^l(Tr_{jl}^{-1}\overline{\mathbf{v}}_j^l, \mathbf{w}_j^l) = 0$$

$$\forall\, \mathbf{w}_i^k \in V^0(\Omega_i^k),\ \ \mathbf{w}_j^l \in V^0(\Omega_j^l) \tag{3.22}$$

such that $(\mathbf{w}_i^k)_n - (\mathbf{w}_j^l)_n = 0$ on Γ_0^{kl}.

A global Schur complement $\mathcal{S}$ is defined by

$$\mathcal{S} = \mathcal{S}_0 + \sum_{k,l} (\bar{R}_{kl})^{\mathrm{T}} \mathcal{S}^{0kl} \bar{R}_{kl}, \tag{3.23}$$

where $\mathcal{S}_0$ is defined in (3.17). and $\mathcal{S}^{0kl}$ by (3.21) and (3.22).

Then the condition coresponding to (3.15) of the auxiliary problem on the interface implies the equation

$$\mathcal{S}\bar{\mathbf{U}} = \mathcal{F} \quad \text{in the dual space } (V_\Gamma)^*. \tag{3.24}$$

To solve problem (3.24) the method of preconditioned conjugate gradients can be used. In [7], the so-called Neumann–Neumann preconditioner is derived.

3.4. Successive approximation method and its convergence

Recall that we have to solve the problem (3.16) by successive approximations. Now $\bar{\mathbf{U}}^0$ is the solution of the auxiliary problem, i.e. $\bar{\mathbf{U}}^0 = \gamma \mathbf{u}^0|_\Gamma$, where $\mathbf{u}^0$ is a solution of problem (3.19). The next approximations $\bar{\mathbf{U}}^k$, $k = 1, 2, \ldots$, we find as the solution of the following linear problem

$$\mathcal{S}_0 \bar{\mathbf{U}}^k = \mathcal{F} - \mathcal{S}_{CON} \bar{\mathbf{U}}^{k-1}, \quad k = 1, 2, \ldots. \tag{3.25}$$

To solve problem (3.25), we use again the method of preconditioned conjugate gradients with new "reduced" preconditioner of the Neumann–Neumann type (see [7]).

Definition 3.4. We define "injection operators"

$$\mathring{D}_i^\iota : V_i^\iota \to V_\Gamma, \quad \iota = 1, \ldots, s \text{ and } i \in T^\iota$$

by the following relation. For the nodes on $\Gamma_i^k \cup \Gamma_j^l$ $(\Gamma^{kl} \subset \Gamma_c, [i,k] \in \vartheta, [j,l] \in \vartheta)$

$$\mathring{D}_i^\iota \underline{\mathbf{v}}(P_m) = \underline{\mathbf{v}}(P_n) \quad \text{if } P_n \in \Gamma_i^k \cup \Gamma_j^l, \tag{3.26}$$

$$\mathring{D}_i^\iota \underline{\mathbf{v}}(P_m) = \frac{\underline{\mathbf{v}}(P_n)\rho_i^\iota}{\rho^{\mathrm{T}}} \tag{3.27}$$

if the mth degree of freedom corresponds with the nth degree of V_i^ι and $P_n \notin \Gamma_i^k \cup \Gamma_j^l$ and

$$\mathring{D}_i^\iota \underline{\mathbf{v}}(P_m) = 0 \quad \text{in the remaining cases.} \tag{3.28}$$

Here, ρ_i^ι denotes the local measure of stiffness of the subdomain Ω_i^ι (e.g. the average of the Young modulus) and

$$\varrho_T = \sum_{P_l \in \bar{\Omega}_j^\iota} \varrho_j^\iota$$

is the sum of ρ_j^ι over all subdomains $\bar{\Omega}_j^\iota$, which contain the point P_l.

Let us realize that the kernel

$$\mathcal{Z}_i^\iota = \operatorname{Ker} A_{i\iota}, \quad \iota = 1, \ldots, s,\ \ i \in T^\iota \tag{3.29}$$

may contain nonzero elements, i.e. displacements of a rigid body Ω_i^ι. Therefore, we introduce the orthogonal complement of the kernel $\mathcal{Z}_i^\iota$ in the space $V(\Omega_i^\iota)$, so that

$$Q(\Omega_i^\iota) \oplus \operatorname{Ker} A_{i\iota} = V(\Omega_i^\iota). \tag{3.30}$$

Let us define the "coarse" reduced space of traces

$$V_{0H} = \sum_{\iota=1}^{s} \sum_{i \in T^\iota} \mathring{D}_i^\iota \gamma \mathcal{Z}_i^\iota \tag{3.31}$$

and a linear set $V_{0H}^\perp \in (V_\Gamma)^*$ of functionals by the relation

$$\mathbf{S} \in V_{0H}^\perp \Leftrightarrow \langle \mathbf{S}, \mathbf{z} \rangle = 0 \qquad \forall\, \mathbf{z} \in V_{0H}. \tag{3.32}$$

The set $V_{0H}^\perp$ will be used for starting values of the preconditioned conjugate gradients algorithm. Now we will analyze the convergence of the method of sucessive approximation (3.25), to the solution of the original problem (3.16) in the space $(V_\Gamma)^*$.

To this end, we introduce a seminorm and a norm.

Definition 3.5. Let $\mathbb{H}_0$ be an orthogonal complement of the subspace V_{0H} in V_Γ. Let us introduce a seminorm

$$|\overline{R}_c\underline{\mathbf{v}}|_\vartheta = \left(\sum_{k,l}[a_i^k(Tr_{ik}^{-1}\overline{R}_i^k\underline{\mathbf{v}}, Tr_{ik}^{-1}\overline{R}_i^k\underline{\mathbf{v}}) + a_j^l(Tr_{jl}^{-1}\overline{R}_j^l\underline{\mathbf{v}}, Tr_{jl}^{-1}\overline{R}_j^l\underline{\mathbf{v}})]\right)^{\frac{1}{2}} \quad (3.33)$$

where $\Gamma^{kl} \subset \Gamma_c$, $[i,k] \in \vartheta$ and $[j,l] \in \vartheta$.

Lemma 3.3. *The expression*

$$\|\underline{\mathbf{u}}\|_Q^2 = \langle \mathcal{S}_0\underline{\mathbf{u}}, \underline{\mathbf{u}}\rangle \quad (3.34)$$

defines a norm in $\mathbb{H}_0$.

Definition 3.6. Let a mapping $T : \mathbb{H}_0 \to \mathbb{H}_0$ be defined by the relation

$$\langle \mathcal{S}_0(T\underline{\mathbf{y}}), \underline{\mathbf{v}}\rangle = \langle \mathcal{F} - \mathcal{S}_{CON}(\underline{\mathbf{y}}), \underline{\mathbf{v}}\rangle \qquad \forall\, \underline{\mathbf{v}} \in \mathbb{H}_0. \quad (3.35)$$

Assumption 3.4. Let a constant β exist such that

$$|\overline{R}_c\underline{\mathbf{u}}|_\vartheta \le \beta\,\|\underline{\mathbf{u}}\|_Q \quad \forall\, \underline{\mathbf{u}} \in \mathbb{H}_0. \quad (3.36)$$

Lemma 3.5. *If Assumption 3.4 is satisfied, the mapping T is well-defined, i.e. for all* $\underline{\mathbf{y}} \in \mathbb{H}_0$ *there exists a unique element* $T\underline{\mathbf{y}} \in \mathbb{H}_0$, *satisfying (3.35).*

For the proof see [7].

Theorem 3.6. *Let the Assumption 3.4 hold. Then*

$$\|T(\underline{\mathbf{y}}) - T(\underline{\mathbf{w}})\|_Q \le 2\beta^2\|\underline{\mathbf{y}} - \underline{\mathbf{w}}\|_Q \qquad \underline{\mathbf{y}}, \underline{\mathbf{w}} \in \mathbb{H}_0. \quad (3.37)$$

For the proof see [7].

Corollary 3.7. *Let the Assumption 3.4 hold with* $\beta < \sqrt{2}/2$. *Then the mapping T is contractive on* $\mathbb{H}_0$. *The successive approximations (3.25) converge to a fixed point of the mapping T, which represents a solution* $\underline{\mathbf{U}}$ *of the Eq. (3.16). The following error estimate holds*

$$\|\underline{\mathbf{U}}^k - \underline{\mathbf{U}}\|_Q \le (2\beta^2)^k(1-2\beta^2)^{-1}\|\underline{\mathbf{U}}^0 - T\underline{\mathbf{U}}^0\|_Q, \qquad k = 1, 2, \dots \quad (3.38)$$

for any $\underline{\mathbf{U}}^0 \in \mathbb{H}_0$.

For the proof see [8, §11.7].

Remark 3.1. The Assumption 3.4 with $\beta < \sqrt{2}/2$ is fulfilled if the union $\cup_{[i,\iota]\in\vartheta}\Omega_i^\iota$ of subdomains, adjacent to the contact boundary Γ_c, is "small" with regard to the union of remaining subdomains and if the triangulation of every Ω_i^ι, $[i,\iota] \in \vartheta$ is sufficiently fine near Γ_i^ι.

4. Worst scenario problem for uncertain input data

4.1. Sets of uncertain input data

Let us assume that the input data

$$A = \{B^\iota, \kappa^\iota, W^\iota, T_1, \mathbf{F}^\iota, \beta^\iota, \mathbf{P}, \mathbf{u}_0, g^{kl}, \iota = 1, \dots, s, \forall k, l\}$$

are uncertain. Let the only available information about them be that they belong to some sets of admissible data, i.e.

$$A \in U_{\rm ad} \Leftrightarrow B^\iota \in U_{\rm ad}^{B^\iota},\ \kappa^\iota \in U_{\rm ad}^{\kappa^\iota},\ W^\iota \in U_{\rm ad}^{W^\iota},\ T_1 \in U_{\rm ad}^{T_1},\ \mathbf{F}^\iota \in U_{\rm ad}^{F^\iota},\ \beta^\iota \in U_{\rm ad}^{\beta^\iota},\ \mathbf{P} \in U_{\rm ad}^{P},\ \mathbf{u}_0 \in U_{\rm ad}^{u_0},\ g^{kl} \in U_{\rm ad}^{g^{kl}}.$$

Assume that all the bodies Ω^ι are piecewise homogeneous, so that partitions of $\overline{\Omega}^\iota$ exist such that

$$\overline{\Omega}^\iota = \cup_{j=1}^{\overline{j}_\iota}\overline{\Omega}_j^\iota, \quad \Omega_j^\iota \cap \Omega_k^\iota = \emptyset \quad \text{for } j \ne k, 1 \le \iota \le s, \quad (4.1)$$

$$\Gamma^{kl} = \cup_{q=1}^{Q_{kl}}\overline{\Gamma}_q^{kl}, \quad \Gamma_q^{kl} \cap \Gamma_p^{kl} = \emptyset \quad \text{for } q \ne p, \forall\, k, l \quad (4.2)$$

and let the data $B^\iota, \kappa^\iota, \mathbf{F}^\iota, W^\iota, \beta^\iota$ be *piecewise constant* with respect to the partition (4.1).

Let us denote

$$\Gamma_u \cap \partial\Omega^\iota = \Gamma_u^\iota, \quad \iota = 1, \dots, s, \quad (4.3)$$

$$\Gamma_\tau \cap \partial\Omega^\iota = \Gamma_\tau^\iota, \quad \iota \le s. \quad (4.4)$$

We define the sets of admissible matrices:

$$U_{\text{ad}}^{B^\iota} = \{(j_N \times j_N)\,\text{symmetric matrices}\, B^\iota : \underline{B}^\iota_{ik}(j) \le B_{ik}|_{\Omega^\iota_j} = \text{const.} \le \overline{B}^\iota_{ik}(j),\ j \le \overline{j}_\iota,\ i,k = 1,\dots,j_N\} \tag{4.5}$$

where $\underline{B}^\iota(j)$ and $\overline{B}^\iota(j)$ are given $(j_N \times j_N)$ symmetric matrices, $\iota = 1,\dots,s$. Assume that positive constants $c^\iota_B(j)$ exist such that

$$\lambda_{\min}(\tfrac{1}{2}(\underline{B}^\iota(j) + \overline{B}^\iota(j))) - \rho(\tfrac{1}{2}(\overline{B}^\iota(j) - \underline{B}^\iota(j))) \equiv c^\iota_B(j) \quad \text{for } j = 1,\dots,\overline{j}_\iota,\ \iota = 1,\dots,s, \tag{4.6}$$

where $\lambda_{\min}$ and ρ denotes the minimal eigenvalue and the spectral radius, respectively. Next, we define the set of admissible matrices

$$U_{\text{ad}}^{\kappa^\iota} = \{(N \times N) - \text{symmetric matrices}\, \kappa^\iota : \underline{\kappa}^\iota_{ik}(j) \le \kappa^\iota_{ik}|_{\Omega^\iota_j} = \text{const.} \le \overline{\kappa}^\iota_{ik}(j),\ j \le \overline{j}_\iota,\ i,k \le N\} \tag{4.7}$$

where $\underline{\kappa}^\iota(j)$ and $\overline{\kappa}^\iota(j)$ are given $(N \times N)$ symmetric matrices, $j = 1,\dots,\overline{j}_\iota$, $\iota = 1,\dots,s$. Assume that positive constants $c^\iota_\kappa(j)$ exist such that

$$\lambda_{\min}(\tfrac{1}{2}(\underline{\kappa}^\iota(j) + \overline{\kappa}^\iota(j))) - \rho(\tfrac{1}{2}(\overline{\kappa}^\iota(j) - \underline{\kappa}^\iota(j))) \equiv c^\iota_\kappa(j) \quad \text{for } j \le \overline{j}_\iota,\ \iota \le s, \tag{4.8}$$

where $\lambda_{\min}$ and ρ denotes the minimal eigenvalue and the spectral radius, respectively. Then the matrices $\kappa^\iota(j) = \kappa^\iota|_{\Omega^\iota_j}$ are positive definite for any $\kappa^\iota \in U_{\text{ad}}^{\kappa^\iota}$, $\iota \le s$, $j \le \overline{j}_\iota$.

Now, let us introduce

$$U_{\text{ad}}^{F^\iota_i} = \{f \in L^\infty(\Omega) : \underline{F}^\iota_i(j) \le f|_{\Omega^\iota_j} = \text{const.} \le \overline{F}^\iota_i(j),\ j \le \overline{j}_\iota\}, \tag{4.9}$$

for $i \le N$, $\iota \le s$, where $\underline{F}^\iota_i(j)$ and $\overline{F}^\iota_i(j)$ are given constants;

$$U_{\text{ad}}^{W^\iota} = \{w \in L^\infty(\Omega) : \underline{W}^\iota(j) \le w|_{\Omega^\iota_j} = \text{const.} \le \overline{W}^\iota(j),\ j \le \overline{j}_\iota\}, \tag{4.10}$$

for $\iota \le s$, where $\underline{W}^\iota(j)$ and $\overline{W}^\iota(j)$ are given constants;

$$U_{\text{ad}}^{T_1} = \{\mathcal{T} \in L^\infty(\Gamma_\tau) : \underline{T}_1(\iota) \le \mathcal{T}|_{\Gamma^\iota_{\tau *}} = \text{const.} \le \overline{T}_1(\iota),\ \iota \le s\}, \tag{4.11}$$

where $\underline{T}_1(\iota)$ and $\overline{T}_1(\iota)$ are given constants;

$$U_{\text{ad}}^{u_{0i}} = \{u \in L^\infty(\Gamma_u) : \underline{u}_{0i}(\iota) \le u|_{\Gamma^\iota_u} = \text{const.} \le \overline{u}_{0i}(\iota),\ \iota \le s\}, \tag{4.12}$$

where $\underline{u}_{0i}(\iota)$ and $\overline{u}_{0i}(\iota)$, $i = 1,\dots,N$, are given constants;

$$U_{\text{ad}}^{P_i} = \{p \in L^\infty(\Gamma_\tau) : \underline{P}_i(\iota) \le p|_{\Gamma^\iota_\tau} = \text{const.} \le \overline{P}_i(\iota),\ \iota \le s\}, \tag{4.13}$$

where $\underline{P}_i(\iota)$ and $\overline{P}_i(\iota)$, $i = 1,\dots,N$ are given constants;

$$U_{\text{ad}}^{\beta^\iota_i} = \{b \in L^\infty(\Omega) : \underline{\beta}^\iota_i(j) \le b|_{\Omega^\iota_j} = \text{const.} \le \overline{\beta}^\iota(j),\ j \le \overline{j}_\iota\}, \tag{4.14}$$

for $i \le j_N$, $\iota \le s$, where $\underline{\beta}^\iota_i(j)$ and $\overline{\beta}^\iota_i(j)$ are given constants;

$$U_{\text{ad}}^{g^{kl}} = \left\{ g \in L^\infty(\Gamma^{kl}) : g|_{\overline{\Gamma}^{kl}_q} \in C^{(0),1}(\overline{\Gamma}^{kl}_q);\ 0 \le g(s) \le \overline{g}^{kl}_q,\ \left|\frac{\mathrm{d}g}{\mathrm{d}s}\right| \le C^{kl}_g \text{ a.e. in } \Gamma^{kl}_q,\ q \le Q_{kl} \right\}, \tag{4.15}$$

for all pairs k, l under consideration, where $\overline{g}^{kl}_q$ and C^{kl}_g are given positive constants. Here, $C^{(0),1}$ denotes the space of Lipschitz-continuous functions.

Finally, we introduce the set of admissible data, as follows:

$$U_{\text{ad}} = \sqcap_{\iota \le s} U_{\text{ad}}^{B^\iota} \times \sqcap_{\iota \le s} U_{\text{ad}}^{\kappa^\iota} \times \sqcap_{\iota \le s, j \le N} U_{\text{ad}}^{F^\iota_i} \times \sqcap_{\iota \le s} U_{\text{ad}}^{W^\iota} \times U_{\text{ad}}^{T_1} \times \sqcap_{\iota \le s, i \le N} U_{\text{ad}}^{\beta^\iota} \times \sqcap_{i \le N} U_{\text{ad}}^{P_i} \times \sqcap_{i \le N} U_{\text{ad}}^{u_{0i}} \times \sqcap_{k,l} U_{\text{ad}}^{g^{kl}}. \tag{4.16}$$

To obtain $T_1 \in W_1$, we have to extend the boundary values $T_1 \in U_{\text{ad}}^{T_1}$ into the domains Ω^ι properly, i.e. satisfying the conditions $T_1^k = T_1^l$ on all Γ^{kl}. As a consequence at some intersections $\Gamma^{kl} \cap \overline{\Gamma}_\tau$ (if any), additional continuity conditions are necessary in the definition of $U_{\mathrm{ad}}^{T_1}$. An analogous remark holds for the data $u_{0i} \in U_{\mathrm{ad}}^{u_{0i}}$ and $\Gamma^{kl} \cap \overline{\Gamma}_u$.

Definition 4.1. Instead of the bilinear forms and functionals $b(T,z)$, $a(\mathbf{u},\mathbf{v})$, $j_g(\mathbf{v})$, $s(z)$, $S(\mathbf{v},T)$ introduced in Definitions 2.1 and 2.2, we will write $b(A;T,z)$, $a(A;\mathbf{u},\mathbf{v})$, $j_g(A;\mathbf{v})$, $s(A;z)$, $S(A;\mathbf{v},T)$ for any $A \in U_{\mathrm{ad}}$.

Lemma 4.1. *There exist positive constants* c_i, $i = 0,1,\ldots,6$ *independent of* $A \in U_{\mathrm{ad}}$, *such that*

$$b(A;z,z) \geq C_0\|z\|^2_{W^1} \quad \forall z \in V_1, \tag{4.17}$$

$$|b(A;z,y)| \leq C_1\|z\|_{W^1}\|y\|_{W^1} \quad \forall z,y \in W_1, \tag{4.18}$$

$$a(A;\mathbf{v},\mathbf{v}) \geq C_2\|\mathbf{v}\|^2_W \quad \forall \mathbf{v} \in V, \tag{4.19}$$

$$|a(A;\mathbf{v},\mathbf{w})| \leq C_3\|\mathbf{v}\|_W\|\mathbf{w}\|_W \quad \forall \mathbf{v},\mathbf{w} \in W, \tag{4.20}$$

$$|s(A;z)| \leq C_4\|z\|_{0,\Omega} \quad \forall z \in V_1, \tag{4.21}$$

$$|S(A;\mathbf{v},T)| \leq C_5(\|\mathbf{v}\|_{0,\Omega} + \|\mathbf{v}\|_{0,\Gamma_\tau} + \|T - T_0\|_{0,\Omega}\|\mathbf{v}\|_W) \quad \forall \mathbf{v},\mathbf{w} \in W, \tag{4.22}$$

$$|j_g(A;\mathbf{u}) - j_g(A;\mathbf{v})| \leq C_6\sum_{\iota\leq s}\|\mathbf{u}^\iota - \mathbf{v}^\iota\|_{0,\partial\Omega^\iota} \quad \forall \mathbf{u},\mathbf{v} \in W. \tag{4.23}$$

Proof. By Theorem 5 in [9], we have

$$\lambda_{\min}(\kappa^\iota(j)) \geq c_\kappa^\iota(j) \forall \kappa^\iota \in U_{\mathrm{ad}}^{\kappa^\iota}, \quad \iota \leq s,\ j \leq \bar{j}_\iota.$$

As a consequence, we obtain

$$b(A;z,z) \geq (\min_{\iota\leq s, j\leq \bar{j}_\iota} c_\kappa^\iota(j))\sum_{\iota\leq s}\int_{\Omega^\iota}|\mathrm{grad}\, z^\iota|^2\,\mathrm{d}\mathbf{x}. \tag{4.24}$$

Then we have

$$\int_{\Omega^\iota}|\mathrm{grad}\, z^\iota|^2\,\mathrm{d}\mathbf{x} \geq C_1^\iota\|z^\iota\|^2_{1,\Omega^\iota} \tag{4.25}$$

for any restriction z^ι of $z \in V_1$. Combining (4.24) and (4.25), we arrive at (4.17).

The inequality (4.18) follows from the definitions of $U_{\mathrm{ad}}^{\kappa^\iota}$ immediately.

Arguing as in (4.24), we may write

$$a(A;\mathbf{v},\mathbf{v}) \geq (\min_{\iota\leq s, j\leq \bar{j}_\iota} c_B^\iota(j))\sum_{\iota\leq s}\int_{\Omega^\iota}\sum_{k=1}^{j_N} e_k^2(\mathbf{v}^\iota)\,\mathrm{d}\mathbf{x}. \tag{4.26}$$

The Korn's inequality

$$\int_{\Omega^\iota} e(\mathbf{v}^\iota) : e(\mathbf{v}^\iota)\,\mathrm{d}\mathbf{x} \geq C_2^\iota\|\mathbf{v}^\iota\|^2_{1,\Omega^\iota} \tag{4.27}$$

holds for any restriction $\mathbf{v}^\iota$ of $\mathbf{v} \in V$. Since

$$\frac{1}{2}e : e \leq \sum_{k=1}^{j_N} e_k^2, \tag{4.28}$$

recall the formula (2.13), combining (4.26)–(4.28), we obtain the inequality (4.19). The inequality (4.20) is an easy consequence of the definitions of $U_{\mathrm{ad}}^{B^\iota}$.

Thus, we may write

$$|s(A;z)| \leq \sum_{\iota\leq s}(\max_{j\leq \bar{j}_\iota}|\overline{W}^\iota(j)|)\int_{\Omega^\iota}|z^\iota|\,\mathrm{d}\mathbf{x} \leq C_4\|z\|_{0,\Omega}$$

Next, we have

$$\begin{aligned}
&|S(A;\mathbf{v},T)| \\
&\leq \sum_{\iota\leq s}(N^{1/2}(\max\{|\overline{F}_i^\iota(j)|, |\underline{F}_i^\iota(j)|\})\|\mathbf{v}^\iota\|_{0,\Omega^\iota} \\
&\quad + C(\max\overline{\beta}^\iota(j))\int_{\Omega^\iota} 2|T^\iota - T_0^\iota|\,\|e(\mathbf{v}^\iota)\|\,\mathrm{d}\mathbf{x} \\
&\quad + N^{1/2}(\max\{|\overline{P}_i(\iota)|, |\underline{P}_i(\iota)|\})\|\mathbf{v}^\iota\|_{0,\Gamma_\tau^\iota}) \\
&\leq C_5(\|\mathbf{v}\|_{0,\Omega} + \|\mathbf{v}\|_{0,\Gamma_\tau} + \|T - T_0\|_{0,\Omega}\|\mathbf{v}\|_W).
\end{aligned}$$

Finally, we may write

$$\begin{aligned}
&|j_g(A;\mathbf{u}) - j_g(A;\mathbf{v})| \\
&\leq \sum_{k,l}\int_{\Gamma^{kl}} g^{kl}|(\mathbf{u}_t^k - \mathbf{v}_t^k) - (\mathbf{u}_t^l - \mathbf{v}_t^l)|\,\mathrm{d}s \\
&\leq \sum_{k,l}\max_{q\leq Q_{kl}} g_q^{kl}\int_{\Gamma^{kl}}|(\mathbf{u}_t^k - \mathbf{v}_t^k) - (\mathbf{u}_t^l - \mathbf{v}_t^l)|\,\mathrm{d}s \\
&\leq C\sum_{k,l}\sum_{i\leq N}(\|u_i^k - v_i^k\|_{0,\Gamma^{kl}} + \|u_i^l - v_i^l\|_{0,\Gamma^{kl}}) \\
&\leq C_6\sum_{\iota\leq s}\|\mathbf{u}^\iota - \mathbf{v}^\iota\|_{0,\partial\Omega^\iota}. \qquad \square
\end{aligned}$$

Proposition 4.2. *There exists a unique weak solution $T(A)$ of the problem $\mathcal{P}_1$ for any $A \in U_{\mathrm{ad}}$ and $\mathbf{u}(A)$ of problem $\mathcal{P}_2$ for any $A \in U_{\mathrm{ad}}$.*

For the proof see [10].

4.2. Criteria of worst scenario

To find the "worst", i.e. the most "dangerous" input data A in the set U_{ad}, we need a criterion, i.e. a functional, which depends on the solution $T(A)$ or $\mathbf{u}(A)$ of problem $\mathcal{P}_1$ or $\mathcal{P}_2$, respectively.

Next, we present several examples of such criteria.

Let $G_r \subset \Omega$, $r = 1, \dots, \bar{r}$, be (small) subdomains, adjacent to the boundaries $\partial\Omega^\iota$, for example. We can define

$$\Phi_1(T) = \max_{r \leq \bar{r}} \varphi_r(T) \tag{4.29}$$

where $\varphi_r(T) = (\mathrm{meas}_N G_r)^{-1} \int_{G_r} T \,\mathrm{d}\mathbf{x}$; let $G'_r \subset \Gamma_u$, $r \leq \bar{r}$ and

$$\Phi_2(T) = \max_{r \leq \bar{r}} \psi_r(T) \tag{4.30}$$

where $\psi_r(T) = (\mathrm{meas}_{N-1} G'_r)^{-1} \int_{G'_r} T \,\mathrm{d}s$.

Next, we define

$$\Phi_3(\mathbf{u}) = \max_{r \leq \bar{r}} \chi_r(\mathbf{u}) \tag{4.31}$$

where $\chi_r(\mathbf{u}) = (\mathrm{meas}_N G_r)^{-1} \int_{G_r} u_i n_i(X_r) \,\mathrm{d}\mathbf{x}$; where $\mathbf{n}(X_r)$ is the unit outward normal at a fixed point $X_r \in \partial\Omega^\iota \cap \partial G_r$ (if $G_r \subset \Omega^\iota$) to the boundary $\partial\Omega^\iota$;

$$\Phi_4(\mathbf{u}) = \max_{r \leq \bar{r}} \chi'_r(\mathbf{u}) \tag{4.32}$$

where $\chi'_r(\mathbf{u}) = (\mathrm{meas}_{N-1} G'_r)^{-1} \int_{G'_r} u_i n_i(X_r) \,\mathrm{d}s$; $G'_r \subset \cup_{\iota \leq s} \partial\Omega^\iota \setminus \Gamma_u$.

Since the weak solution $\mathbf{u}(A)$ of our problem (2.13) depends on $T(A)$, then $\mathbf{u}(A) = \mathbf{u}(A; T(A))$ and instead of $\Phi_i(\mathbf{u})$ we write $\Phi_i(A; \mathbf{u}, T)$. Another choice is

$$\Phi_5(A; \mathbf{u}, T) = \max_{r \leq \bar{r}} \omega_r(A; \mathbf{u}, T) \tag{4.33}$$

where $\omega_r(A; \mathbf{u}, T) = (\mathrm{meas}_N G_r)^{-1} \int_{G_r} I_2^2(\tau(A; \mathbf{u}, T)) \,\mathrm{d}\mathbf{x}$.

Here, $I_2(\tau)$ denotes the intensity of shear stress (see, e.g. [8]), i.e. the second fundamental invariant of the stress tensor deviator τ^D, i.e.

$$I_2^2(\tau) = \sum_{i,j=1}^{3} \tau_{ij}^D \tau_{ij}^D, \qquad \tau_{ij}^D = \tau_{ij} - \frac{1}{3}\tau_{kk}\delta_{ij};$$

$$I_2^2 = \tfrac{2}{3}[\tau_{11}^2 + \tau_{22}^2 + \tau_{33}^2 - (\tau_{11}\tau_{22} + \tau_{11}\tau_{33} + \tau_{22}\tau_{33}) + 3(\tau_{12}^2 + \tau_{13}^2 + \tau_{23}^2)] \quad \text{for } N = 3.$$

In (4.33), $\tau(A; \mathbf{u}, T)$ is defined by the formula (2.7). For orthotropic material and plane strain, we have to insert $\tau_{13} = \tau_{23} = 0$.

If the friction can be neglected (as in [11,8,2]), we set $g_c^{kl} \equiv 0$ and define, e.g.

$$\Phi_6(A; \mathbf{u}, T) = \max_{r \leq \bar{r}} \mu_r(A; \mathbf{u}, T); \tag{4.34}$$

$\mu_r(A; \mathbf{u}, T) = (\mathrm{meas}_N G_r)^{-1} \int_{G_r} (-\tau_n(A; \mathbf{u}, T)) \,\mathrm{d}\mathbf{x}$; and G_r is a small subdomain adjacent to Γ_c.

Now we formulate the *worst scenario problems* as follows:

find

$$A^{0i} = \arg \max_{A \in U_{\mathrm{ad}}} \Phi_i(T(A)), \quad i = 1, 2, \tag{4.35}$$

and

$$A^{0i} = \arg \max_{A \in U_{\mathrm{ad}}} \Phi_i(\mathbf{u}(A), T(A)), \quad i = 3, 4, 5, 6, \tag{4.36}$$

where $T(A)$ are $\mathbf{u}(A)$ are weak solutions of the problem $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively.

Remark 4.1. Since the weak solution $\mathbf{u}(A)$ of problem $\mathcal{P}_2$ depends on $T(A)$, $\mathbf{u}(A) \equiv \mathbf{u}(A; T(A))$ and we write $\Phi_i(\mathbf{u}(A), T(A))$, instead of $\Phi_i(\mathbf{u}(A))$ for $i = 3, 4$ in (4.36).

4.3. Stability of weak solutions

To analyze the solvability of worst scenario problems (4.35) and (4.36) we have to study the mapping $A \mapsto T(A)$ and $A \mapsto \mathbf{u}(A, T(A))$. First, we introduce the following decomposition of $A \in U_{\mathrm{ad}}$: $A = \{A', A''\}$, where

$$A' = \{\sqcap_{\iota\le s}\sqcap_{j\le\bar{j}_\iota}\kappa^\iota(j), \sqcap_{\iota\le s}\sqcap_{j\le\bar{j}_\iota}W^\iota(j), \sqcap_{\iota\le s}T_1^\iota\},\quad A'\in\mathbb{R}^{p_1},$$

$$p_1 = (j_N+1)\sum_{\iota\le s}\bar{j}_\iota + s$$

and

$$A'' = \{\sqcap_{\iota\le s}\sqcap_{j\le\bar{j}_\iota}B^\iota(j), \sqcap_{\iota\le s}\sqcap_{j\le\bar{j}_\iota}\mathbf{F}^\iota(j), \sqcap_{\iota\le s}\mathbf{P}^\iota, \sqcap_{\iota\le s}\mathbf{u}_0^\iota, \sqcap_{\iota\le s}\sqcap_{j\le\bar{j}_\iota}\beta^\iota(j), \sqcap_{k,l}\sqcap_{q\le Q_{kl}}g^{kl}(q)\},$$

$$A''\in\mathbb{R}^{p_2}\times\sqcap_{k,l}\sqcap_{q\le Q_{kl}}C(\bar{\Gamma}_q^{kl}),$$

$$p_2 = \left(\sum_{\iota\le s}\bar{j}_\iota\right)[(3+j_N)j_N/2 + N(1+2s)].$$

We are going to show the continuity of the mappings $A'\mapsto T(A')$ for $A'\in U'_{\text{ad}} = \sqcap_{\iota\le s}U_{\text{ad}}^{\kappa^\iota}\times\sqcap_{\iota\le s}U_{\text{ad}}^{W^\iota}\times U_{\text{ad}}^{T_1^\iota}$ and $A\mapsto\mathbf{u}(A,T(A'))$ for $A''\in U''_{\text{ad}} = \sqcap_{\iota\le s}U_{\text{ad}}^{B^\iota}\times\sqcap_{\iota\le s,j\le N}U_{\text{ad}}^{F^\iota}\times\sqcap_{\iota\le s,i\le N}U_{\text{ad}}^{\beta^\iota}\times\sqcap_{i\le N}U_{\text{ad}}^{P_i}\times\sqcap_{i\le N}U_{\text{ad}}^{u_{0i}}$, respectively. Since the problem discussed is quasi-coupled, we have the following theorem and lemma:

Lemma 4.3. *If $A_n\in U_{\text{ad}}$, $A_n\to A$ in U, where $U=\mathbb{R}^{p_1+p_2}\times\sqcap_{k,l}\sqcap_{q\le Q_{kl}}C(\bar{\Gamma}_q^{kl})$, and $\mathbf{u}_n\rightharpoonup\mathbf{u}$ weakly in W then*

$$a(A_n;\mathbf{u}_n,\mathbf{v})\to a(A;\mathbf{u},\mathbf{v})\quad\forall\,\mathbf{v}\in W,\tag{4.37}$$

$$S(A_n;\mathbf{u}_n,T)\to S(A;\mathbf{u},T)\quad\forall\,T\in W_1,\tag{4.38}$$

$$j_g(A_n;\mathbf{u})\to j_g(A;\mathbf{u}).\tag{4.39}$$

For the proof see [10].

Theorem 4.4. *Let $A'\in U'_{\text{ad}}$, $A'_n\to A'$ in $\mathbb{R}^{p_1}$ as $n\to\infty$. Then*

$$T(A'_n)\to T(A')\quad\text{in } W_1.$$

Let $A_n\in U_{\text{ad}}$, $A_n\to A$ in $U\equiv\mathbb{R}^{p_1+p_2}\times\sqcap_{k,l}\sqcap_{q\le Q_{kl}}C(\bar{\Gamma}_q^{kl})$. Then

$$\mathbf{u}(A_n)\to\mathbf{u}(A)\quad\text{in } W.$$

For the proof see [10].

4.4. Existence of a solution of the worst scenario problem

To prove the existence of a solution of the worst scenario problem, we will use the following lemma.

Lemma 4.5. *Let $\Phi_i(T)$, $i=1,2$, be defined by (4.29) and (4.30) and let $T_n\to T$ in W_1, as $n\to\infty$. Then*

$$\lim_{n\to\infty}\Phi_i(T_n)=\Phi_i(T),\quad i=1,2.$$

Let $\Phi_i(\mathbf{u})$, $i=3,4$, be defined by (4.31) and (4.32) and let $\mathbf{u}_n\to\mathbf{u}$ in W, as $n\to\infty$. Then

$$\lim_{n\to\infty}\Phi_i(\mathbf{u}_n)=\Phi_i(\mathbf{u}),\quad i=3,4.$$

Let $\Phi_i(A;\mathbf{u},\mathbf{T})$, $i=5,6$, be defined by (4.33) and (4.34). Let $A_n\to A$ in U, $A_n\in U_{\text{ad}}$, $\mathbf{u}_n\to\mathbf{u}$ in W and $T_n\to T$ in $L^2(\Omega)$. Then

$$\lim_{n\to\infty}\Phi_i(A_n,\mathbf{u}_n,T_n)=\Phi_i(A,\mathbf{u},T),\quad i=5,6.$$

The main result gives the next theorem:

Theorem 4.6. *There exists at least one solution of the worst scenario problems (4.35) and (4.36), $i=1,\ldots,6$.*

Proof. Let us denote

$$J_i(A)=\Phi_i(T(A)),\quad i=1,2.$$

If $A_n\in U_{\text{ad}}$, $A_n\to A$ in U as $n\to\infty$, then $A'_n\to A$ in $\mathbb{R}^{p_1}$ and $T(A_n)\to T(A)$ in W_1 by virtue of Theorem 4.4. Using Lemma 4.5, we obtain

$$J_i(A_n)\to J_i(A),$$

so that J_i is continuous on the set U_{ad}.

It is easy to show that U_{ad} is compact subset of U, if we employ Arzela–Ascoli Theorem for $U_{\text{ad}}^{g^{kl}}$.

As a consequence, J_i attains its maximum on U_{ad}.

The same argument can be applied to

$$J_i(A)=\Phi_i(A;\mathbf{u}(A)T(A)),\quad i=3,4,5,6.$$

Here, we employ Theorem 4.4 and Lemma 4.5 to verify the continuity of J_i on the set U_{ad}. □

5. Numerical experiments

For approximations of the problem we can employ the finite element method and the algorithm of Section 3 based on the nonoverlapping domain decomposition approach developed in [7].

The geomechanical model problem describing a loaded tunnel which is crossing by a deep fault and based on the geomechanical theory and models having connection with radioactive waste repositories [2]. A geometry of the problem is in Fig. 1.

5.1. Material parameters

Two regions with Young's modulus $E = 5.2 \times 10^9$ Pa and Poisson's ratio $\nu = 0.18$. Specific gravity is 2.45×10^4 Pa/m.

5.2. Boundary conditions

Prescribed displacement $(2.5 \times 10^{-2};\ 0)$ [m] on 1–2. Pressure 0.5×10^7 Pa on 1–4 and 2–8 and $1 \times$

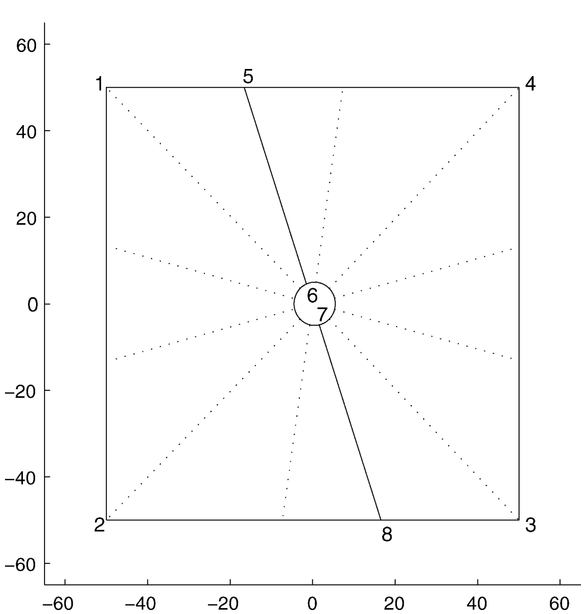

Fig. 1. Geometry of the problem.

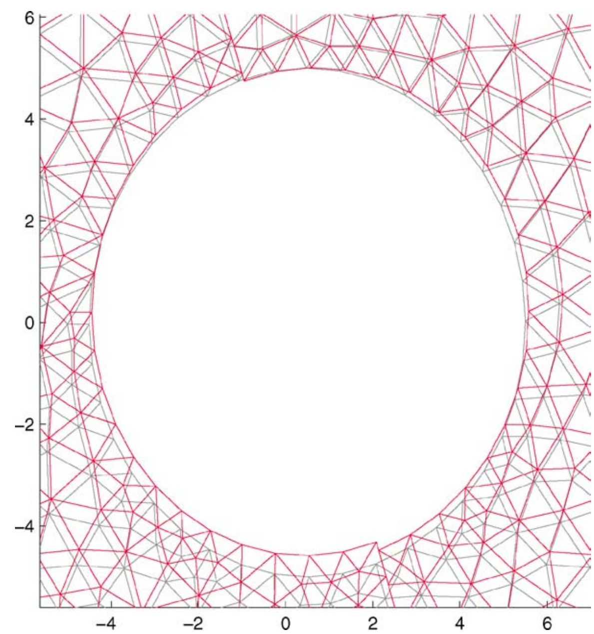

Fig. 2. Detail of deformations (enlarging factor is 10).

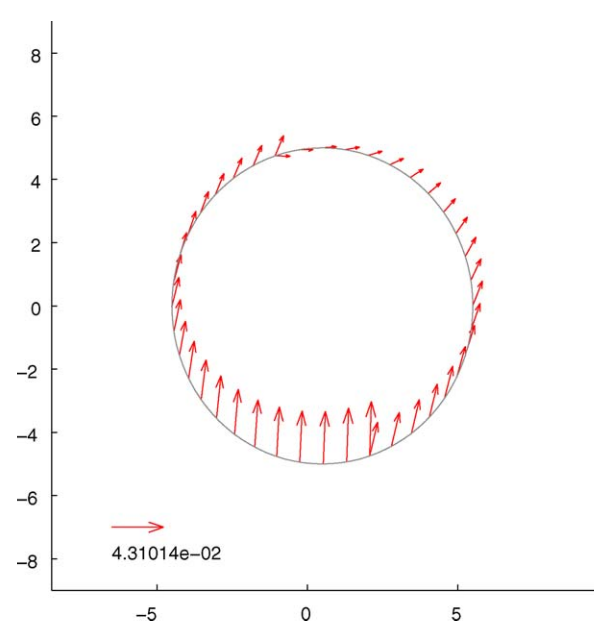

Fig. 3. Detail of displacements on the tunnel wall.

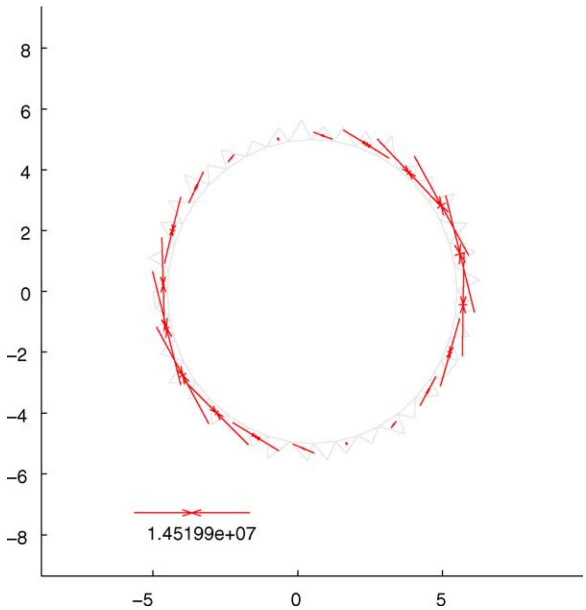

Fig. 4. Detail of principal stresses on the tunnel wall.

10^7 Pa on 8–3. Bilateral contact boundary on 3–4. Unilateral contact boundary: 5–6 and 7–8. Given slip limit is 10^6 Pa. Zero surface forces on the tunnel wall.

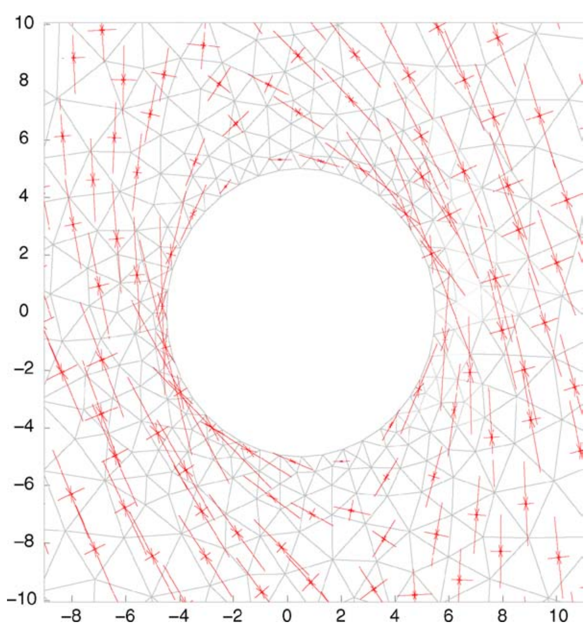

Fig. 5. Detail of principal stresses in a neighbourhood of the tunnel.

5.3. Discretization statistics

Twelve subdomains, 5501 nodes, 9676 elements, 10428 unknowns, 89 unilateral contact conditions, 466 interface elements.

5.4. Convergence statistics

Twenty-one iterations of the PCG algorithm for the auxiliary problem, 15 iterations of the successive approximations method for accuracy 10^{-6}, total 39 iterations of the PCG algorithm for the original problem.

Fig. 2 represents detail of deformations and Fig. 3 shows displacements in a neighbourhood of the tunnel. On Figs. 4 and 5 details of principal stresses are displayed in a neighbourhood of the tunnel.

6. Conclusions

The theory presented in this paper represents extension of geomechanical problems solved in [1,2] for the case if input data, i.e. thermal conductivity and elastic coefficients, body and surface forces, thermal sources, body and surface forces, coefficients of thermal expansion, boundary values, coefficient of friction on contact boundaries, etc. are uncertain. Since the theory is an extension of problems solved in [2] it can be used for mathematical models connected with the safety of construction and of operation of the radioactive waste repositories. The models involve input data (as thermal conductivity and elastic coefficients, body and surface forces, thermal sources, coefficients of thermal expansion, boundary values, coefficient of friction on contact boundaries, etc.) which cannot be determined uniquely, but only in some intervals, given by the accuracy of measurements and the approximate solutions of identification problems. The “reliable solution” denotes the worst case among a set of possible solutions where the degree of badness is measured by a criterion functional. For the safety of the high level radioactive waste repositories and other structures under critical conditions we seek the maximal value of this functional, which depends on the solution of the mathematical model. Then for the computations of such problems (some mean values of temperatures, displacements, intensity of shear stresses, principal stresses, stress tensor components, normal and tangential components of the displacement or stress vector on the contact boundaries, etc.) we have to formulate a corresponding maximization (worst scenario) problem. Then methods and algorithms known from “optimal design” can be used.

To construct a model of structures under the influence of critical conditions the influence of global tectonics onto a local area, where the critical structure is built as well as the influence of the resulting local geomechanical processes on a critical structure must be taken into account [2]. Problems of this kind with uncertain input data are problems with high level radioactive waste repositories. In the case of the high level radioactive waste repositories the effects of geodynamical processes in the sense of plate tectonics must be taken into consideration, namely in regions near tectonic areas (e.g. the Japan island arc, the Central and South Europe, etc.), but also in the platform regions (as in Sweden, Canada, etc.). But in geomechanics and geodynamics our information about input data are very questionable as we obtain input data with very small accuracy. Therefore, the presented method as well as algorithms used give a worst scenario (anti-optimal) solution of the problem studied. In the practice it represents a tolerance solution corresponding to the structures in critical situations and, in fact, the obtained solutions facilitate to ensure the high security of constructions and operations of structures under critical situation (e.g. high level radioactive waste repositories). Another example is represented by modelling an interaction between a tunnel wall and a rock massif in the radioactive waste repository tunnels or by modelling of a tunnel crossing by an active deep fault(s), respectively.

The parallel algorithm presented in this paper is based on the nonoverlapping domain decomposition method developed in [7,12]. The algorithm is derived from the primal formulation in displacement, uses grouping every two subdomains which share a contact area into a single “nonlinear” subdomain and follows the approach proposed by Le Tallec [6] for linear problems. Other possible variants are to consider mixed formulation involving both displacements and stresses or dual formulation eliminating the displacement unknowns from mixed problem.

Acknowledgements

The research was supported by the grant COPERNICUS-HIPERGEOS II - KIT 977006 and grants of the Ministry of Education, Youth and Sport of the Czech Republic No. OK-407 and MSM 235200001.

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Daněk, J., Kutáková, H.: *The mortar finite element method in 2D: Implementation in Matlab*, In 16th Annual Conference Proceedings of Technical Computing Prague 2008, Praha, Česká republika 2008.

THE MORTAR FINITE ELEMENT METHOD IN 2D: IMPLEMENTATION IN MATLAB

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Abstract

The paper is focused on the mortar finite element method for solving linear elliptic problems in 2D. The mortar finite element method is a nonconforming domain decomposition technique tailored to handle problems posed on domains that are partitioned into independently triangulated subdomains. In the contribution we explained the principle and properties of the method. A significant part of the paper is dedicated to the implementation of the mortar method in the Matlab system. The numerical results are showing both the principle and the possibility of practical use of the method.

1 Introduction

The finite element method is applicable to a wide range of physical and engineering problems which can be described by means of partial differential equations. The mortar finite element discretization is a discontinuous Galerkin approximation. The functions in the approximation subspaces have jumps across subdomain interfaces and are standard finite element functions when restricted to the subdomains. The jumps across subdomains interfaces are constrained by conditions associated with one of the two neighboring meshes so that a weak continuity condition must be fulfilled. Because of the discontinuity on the interface we classify the mortar finite element method as a nonconforming finite element method (see [6]).

Mortar finite elements were first introduced in 1994 by Christine Bernardi, Yvon Maday and Anthony T. Patera in [1]. Our paper is focused on the mortar finite element method in two dimensions and its implementation in the Matlab system. Most of the literature describing the mortar finite element method deal with the geometrically conforming partition, which is easier. Therefore, this paper is focused on the nonconforming case.

2 Mortar Finite Element Method

In this section we briefly describe the mortar finite element method in two dimensions. Let $\Omega \subset \mathbb{R}^2$ be a polygonal computational domain. We decompose the domain into P nonoverlapping polygonal subdomains

$$\overline{\Omega} = \bigcup_{i=1}^{P} \overline{\Omega}_i, \qquad \Omega_j \cap \Omega_k = \emptyset \quad \text{for } j \neq k, \quad j,k = 1,\dots,P.$$

The partition can be:

- *Geometrically conforming* – The intersection between two closure of any two subdomains $\overline{\Omega}_i \cap \overline{\Omega}_j$, $i \neq j$ is either an entire edge, a vertex or empty. Figure 1 shows an example of the partition.
- *Geometrically nonconforming* – All the other cases. See example of the nonconforming partition in Figure 2.

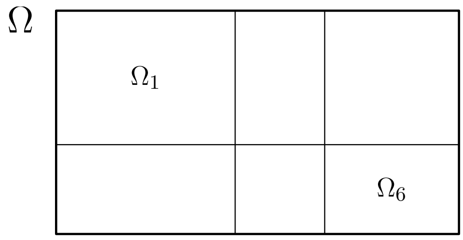


Figure 1: Example of geometrically conforming partition of Ω into subdomains Ω_i.

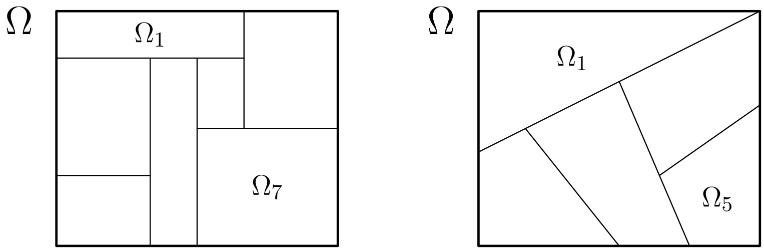


Figure 2: Example of geometrically nonconforming partition of Ω into subdomains Ω_i.

The subdomains Ω_i form together a coarse mesh of the whole domain Ω. We discretize each subdomain by triangular elements known from the finite element method. The size of the triangles can be chosen with regard to the problem. We use finer mesh in subdomains, where big changes in behaviour of the solution are expected, etc. The resulting triangulation can be nonmatching across the interfaces of the subdomains, as you can see in Figure 4.

We define the interface Γ between subdomains Ω_i as a closure of the union of the parts of the boundaries of $\partial\Omega_i$, $i = 1, \dots P$, that are interior to Ω

$$\Gamma = \overline{\cup_{i=1}^{P}(\partial\Omega_i \setminus \partial\Omega)}. \tag{1}$$

The interface can be considered also as a set of nodes, that belong to the boundary of at least two subdomains.

We denote

$$V^h(\Omega) = V_1^h(\Omega_1) \times V_2^h(\Omega_2) \times \cdots \times V_P^h(\Omega_P) \tag{2}$$

the space of mortar finite elements defined on Ω. We consider the low order basis functions, for example the piecewise linear basis functions. $V_i^h(\Omega_i)$ is a finite element space in each subdomain Ω_i. $V_i^h(S)$ is a restriction of functions from V^h to a set S.

For further analysis, we introduce some more notation. A main edge will represent a side of a planar n-agle, see Figure 3. A square has four main edges, n-agle has n main edges. Let Γ_{ij} be an open common edge or a part of an edge of two adjacent subdomains Ω_i and Ω_j

$$\overline{\Gamma}_{ij} = \overline{\Omega}_i \cap \overline{\Omega}_j. \tag{3}$$

The edge Γ_{ij} is a part of a main edge (or is a main edge) both of the subdomain Ω_i and Ω_j. We choose the main edge of one subdomain as a master (mortar) and the main edge of the second subdomain as a slave (nonmortar). We use a new notation:

- *Mortar edge* γ – If it belongs to a particular main edge of a boundary $\partial\Omega_i$, we denote it γ_i.
- *Nonmortar edge* δ – If it belongs to a particular main edge of a boundary $\partial\Omega_j$, we denote it δ_j.

Figures 3 and 4 show examples of situations, that can occur on the interface. If we consider geometrically conforming partition of Ω, it is obvious, that for two subdomains Ω_i and Ω_j with a commom edge holds an equality $\gamma_i = \Gamma_{ij} = \delta_j$, or $\gamma_j = \Gamma_{ij} = \delta_i$ (it's important which edge is chosen as a mortar), see Figure 4. The situation is more complicated in the nonconforming case. There is arising a question, how to choose mortar and nonmortar edges. An example of such a choice is in Figure 5. It can be proven that the partition always exists (see [3] or [6]).

In term of a new notation we can write:

$$\Gamma = \bigcup_{k=1}^{K} \overline{\gamma_k}, \quad \gamma_m \cap \gamma_n = \emptyset, \quad \text{if } m \neq n, \quad m, n = 1, \dots, K, \tag{4}$$

where K is the number of all mortar edges. Obviously also

$$\Gamma = \bigcup_{l=1}^{L} \overline{\delta_l}, \quad \delta_m \cap \delta_n = \emptyset, \quad \text{if } m \neq n, \quad m, n = 1, \dots, L, \tag{5}$$

where L is the number of all nonmortar edges.

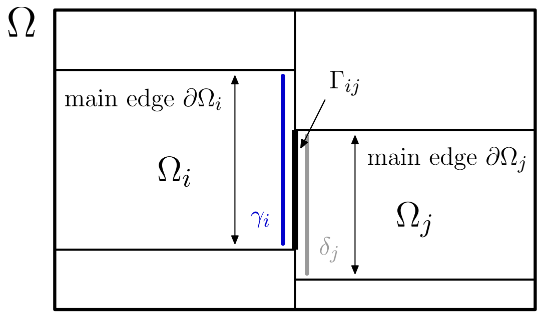


Figure 3: Interface Γ_{ij} of two subdomains Ω_i and Ω_j and signification of edges as mortar and nonmortar.

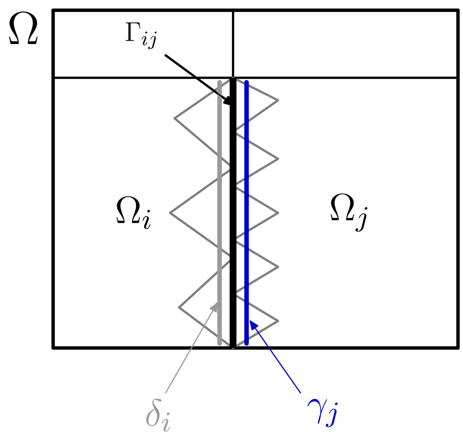


Figure 4: Mortar edge γ and nonmortar edge δ on the interface Γ_{ij} of two subdomains Ω_i and Ω_j.

It remains to solve the most important point of the method - the situation on the interface Γ. It's necessary to join somehow the values of the searched function (we call it mortar function)

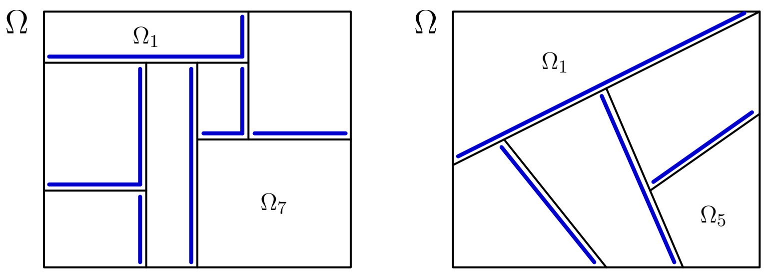


Figure 5: Example of choice of mortar edges in geometrically nonconforming case (mortar edges are blue).

on the interface. Instead of a pointwise continuity we require fulfilment of a weak continuity condition. The exact formulation will follow after some necessary definitions.

Each nonmortar edge δ_l belongs to exactly one subdomain, we denote it Ω_l. Let Γ_l be the union of q parts of the mortar edges $\gamma_{l,i}$ which correspond geometrically to nonmortar edge δ_l

$$\Gamma_l = \bigcup_{i=1}^{q} (\overline{\gamma_{l,i}} \cap \overline{\delta_l}). \tag{6}$$

For each nonmortar edge δ_l we choose a space of test functions $\Psi^h(\delta_l)$, which is a subspace of the space $V_l^h(\delta_l)$, that is a restriction of functions from $V_l^h(\Omega_l)$ to δ_l. So if we choose piecewise linear basis functions, $V_l^h(\Omega_l)$ will be a space of piecewise linear functions. $\Psi^h(\delta_l)$ will be a restriction of functions from $V_l^h(\Omega_l)$ to δ_l with requirement that these continuous, piecewise linear functions are constant in the first and last mesh intervals of δ_l (see Figure 6). Other possibility how to establish the space of test functions is described in [7].

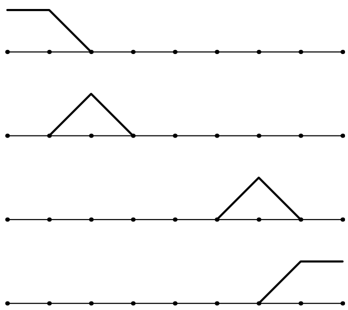


Figure 6: Test functions on δ_l.

We define the mortar projection on δ_l as $\pi_{q_1,q_2}(u_l) : L^2(\Gamma_l) \to V_l^h(\delta_l)$. For two arbitrary values q_1 and q_2 and a function $u_l \in L^2(\Gamma_l)$ it satisfies

$$\int_{\delta_l} (u_l - \pi_{q_1,q_2}(u_l))\psi \, ds = 0 \quad \forall \psi \in \Psi^h(\delta_l). \tag{7}$$

The condition means, that the jump of the mortar function across each nonmortar edge must be orthogonal (in $L^2(\Gamma_l)$) to the space of test functions defined on δ_l. The condition is called the weak continuity condition or the mortar condition.

3 Formulation of the Mortar Problem

3.1 Variational (Weak) Formulation of the Mortar Problem

Let us remind a variational (weak) formulation of a Poisson problem in two dimensions. We need to find a solution $u \in W_2^1(\Omega)$ of the Poisson equation

$$-\triangle u = f \quad \text{in } \Omega \subset \mathbb{R}^2, \tag{8}$$

$$u = g_1 \quad \text{on } \partial\Omega_D, \qquad \frac{\partial u}{\partial n} = g_2 \quad \text{on } \partial\Omega_N, \tag{9}$$

where $f \in L_2(\Omega)$, $g_1 \in L_2(\partial\Omega_D)$, $g_2 \in L_2(\partial\Omega_N)$. The solution $u \in W_2^1(\Omega)$ must be from the set of admissible functions

$$V_g = \{u \in W_2^1(\Omega) : u = g_1 \text{ na } \partial\Omega_D \text{ in the sense of traces}\} \tag{10}$$

and for every test function v must satisfy

$$a(u,v) = F(v) \quad \forall v \in V, \tag{11}$$

where

$$V = \{v \in W_2^1(\Omega) : v = 0 \text{ on } \partial\Omega_D \text{ in the sense of traces}\}. \tag{12}$$

$$a(u,v) = \int_\Omega [\operatorname{grad} u \ \operatorname{grad} v] \, \mathrm{d}\Omega, \tag{13}$$

$$F(v) = \int_\Omega f v \, \mathrm{d}\Omega + \int_{\partial\Omega_N} g_2 v \, \mathrm{d}S. \tag{14}$$

We can formulate the problem (8), (9) in terms of a decomposition of the domain Ω into subdomains Ω_i, $i = 1, 2, \dots, P$ and by using an appropriate mortar finite element space V^h. We appear from the weak formulation (11) of the problem (8), (9) and we rewrite it to the discrete form.

We know, that $V_i^h(\Omega_i) \subset W_2^1(\Omega_i)$, $i = 1, \dots, P$. Thus, we can write

$$a^\Gamma(u,v) = F^\Gamma(v) \quad \forall v \in W_2^{1,0}(\Omega). \tag{15}$$

$a^\Gamma(\cdot,\cdot)$ is a bilinear form, which is defined as a sum of contributions from the particular subdomains

$$a^\Gamma(u,v) = \sum_{i=1}^{P} \int_{\Omega_i} [\operatorname{grad} u \ \operatorname{grad} v] \, \mathrm{d}\Omega_i \tag{16}$$

and

$$F^\Gamma(v) = F(v) = \int_\Omega f v \, \mathrm{d}\Omega + \int_{\partial\Omega_N} g_2 v \, \mathrm{d}S. \tag{17}$$

A variational (weak) formulation of the mortar problem represents such a task: Find $u_h \in V^h$ which satisfies

$$a^\Gamma(u_h, v_h) = F^\Gamma(v_h) \quad \forall v_h \in V^h. \tag{18}$$

The existence and uniqueness of the solution of (18) follows i.a. from the Lax-Milgram lemma.

3.2 Mixed Formulation of the Mortar Problem

First, we present some findings relating to dual spaces. We know, that $V_i^h(\Omega_i) \subset W_2^1(\Omega_i)$. A restriction of a mortar function u to a nonmortar edge δ_l belongs to the space $W_2^{1/2}(\delta_l)$. The space of test functions $\Psi^h(\delta_l)$ can then be a subset of the dual space of space $W_2^{1/2}(\delta_l)$ with respect to the L^2 inner product, thus $\Psi^h(\delta_l) \subset W_2^{-1/2}(\delta_l)$.

For introduction of a mixed formulation we use the mortar condition (7), whose satisfaction is demanded on the interface. We denote $[u_l]$ the jump of $u_h \in V^h$ across δ_l. The test functions from the mortar condition can be considered as Lagrange multipliers. A function u belongs to the space V^h if and only if for all the nonmortar edges δ_l and for all the Lagrange multipliers μ_l, which form a basis of $\Psi^h(\delta_l)$, holds

$$\int_{\delta_l} [u_l]\mu_l \, \mathrm{d}s = 0. \tag{19}$$

From the first paragraph of this subsection results, that $[u_l] \in W_2^{1/2}(\delta_l)$. Lagrange multipliers μ_l must then be from the dual space $W_2^{-1/2}(\delta_l)$. Let $M^h = \prod_l \Psi^h(\delta_l) \subset \prod_l W_2^{-1/2}(\delta_l)$ and $\mu_h \in M^h$, where $\mu_h = (\mu_l)_{l=1:L}$, L is the number of nonmortar edges. We define a bilinear form

$$b(u_h, \mu_h) = \sum_{l=1}^{L} \int_{\delta_l} [u_h]\mu_l \mathrm{d}s. \tag{20}$$

A function u_h is a mortar function if and only if

$$b(u_h, \mu_h) = 0 \quad \forall \mu_h \in M^h. \tag{21}$$

We can rewrite the discrete problem (18) to the mixed formulation: Find a couple $(u_h, \lambda_h) \in V^h \times M^h$ which satisfies

$$\begin{array}{lcll} a^\Gamma(u_h, v_h) \; + \; b(v_h, \lambda_h) & = & F^\Gamma(v_h) & \forall v_h \in V^h, \\ b(u_h, \mu_h) & = & 0 & \forall \mu_h \in M^h. \end{array} \tag{22}$$

As well as in other mixed formulations, it is important to satisfy the Brezzi-Babuška condition (see [7]) also in formulation (22). This is important for the existence of the solution and for the error estimate.

4 Implementation of the Mortar Finite Element Method

We describe briefly the key points of the implementation of the mortar finite element method in the Matlab system. We started from the program [4], which deals with the conforming partition of the computational domain into squares and rectangles, whose sides are parallel to the coordinate axes. The generalization of the program to the nonconforming case is not a trivial task. Our program solves the Poisson or Laplace equation in two dimensions with Dirichlet boundary condition and can deal with polygonal computational domains, that can be divided into general polygonal convex subdomains. Since the conforming partition is a specific case of the nonconforming partition, it's obvious, that the program can handle with conforming case also.

The input data are the geometry of the computational domain, the right hand side and the Dirichlet boundary condition. The geometrical data contain informations about the subdomains - each subdomain and its triangulations is inscribed with a triplet of matrices **p**, **e**, **t** representing matrices of points, edges and triangles. It's necessary to save the information on the mutual relationships of the edges and subdomains. We must distinguish the edges on the boundary because of the fulfilment of the boundary condition and we need to know the adjacent edges and subdomains of each edge and subdomain.

4.1 Mutual Relationship of the Edges

So let us consider a polygonal computational domain Ω. We divide the domain into polygonal subdomains Ω_i. There are, in fact, two types of nonconforming partitions, see Figure 2. In the first case, the subdomains are squares and rectangles, whose sides are parallel to the coordinate axes. The second case is more general – the subdomains are general n-agles or rectangles, whose sides are not parallel to the coordinate axes. Both cases can be combined.

In the following text we will use the terms edge and main edge, see Section 2 for the explanation. The edges and their belongings to the main edges are described in matrices **e**. The comparison of the mutual relationships in the input geometry data is realized through the edges, not the main edges. We look for the parallelism of the edges. The parallelism is indicated by the identical multiples of x- and y- component of the directional vectors. If an edge is parallel to a coordinate axis, we can profit from the fact, that one component of its directional vector is equal to zero. We must identify the edges, that are overlapping. The necessary condition of the overlapping is, that the edges lie on the same line (a special case of parallelism). Two edges can overlap with parts of the edges, one edge can be inside the second one or they can coincide. The information on the mutual relationships of the edges and subdomains is stored in a matrix by a numerical value. On the basis of described information we can do the partition into mortar and nonmortar edges.

4.2 Division of the Edges into Mortar and Nonmortar

We divide the main edges, that are not lying on the outer boundary, into mortars and nonmortars. We have already mentioned, that the partition is always possible. But there is no universal instruction, how to choose the mortars. We introduce some possibilities. First of all, we focus on the conforming case with rectangular subdomains. Each main edge is either on the outer boundary or has one adjacent main edge with which it coincides. The choice of mortar is without any problems. Let's show two possibilities:

- Neumann-Dirichlet partition – Each subdomain has either all the main edges mortar or nonmortar.
- We sign the first two main edges of the subdomain as mortar, the second two as nonmortar. We omit the main edges on the boundary.

The situation is more complicated in the nonconforming case. Neither of the above-mentioned methods can be used. With regard to a lack of information in a literature we use our own way of a choice of the mortar edges.

We will do the following. We go through the particular subdomains in the same sequence as they were entered. If a main edge is unsigned yet and if it is possible, we sign the main edge as mortar and its adjacent main edge (or edges) as nonmortar. So we go through all the main edges of all the subdomains. If a main edge is once called mortar (or nonmortar), we don't change it again. The process of signing of the main edges is shown in Figure 7.

The information on the adjacent edges is known from the previous research and is stored in a matrix. On the basis of a detection of an overlap there is assigned a type of the appropriate edge and of the adjacent edge. It's necessary to verify if one of these two edges already has a value (mortar or nonmortar). In this case, we must keep the value and the type is assigned only to the second edge. The edges, that belong to the same main edge, are of the same type. The type (mortar, nonmortar, outer boundary) is stored as a numerical parameter added in the matrices $\mathbf{e}$ (0 = outer boundary, 1 = nonmortar edge, 2 = mortar edge), see Figure 7.

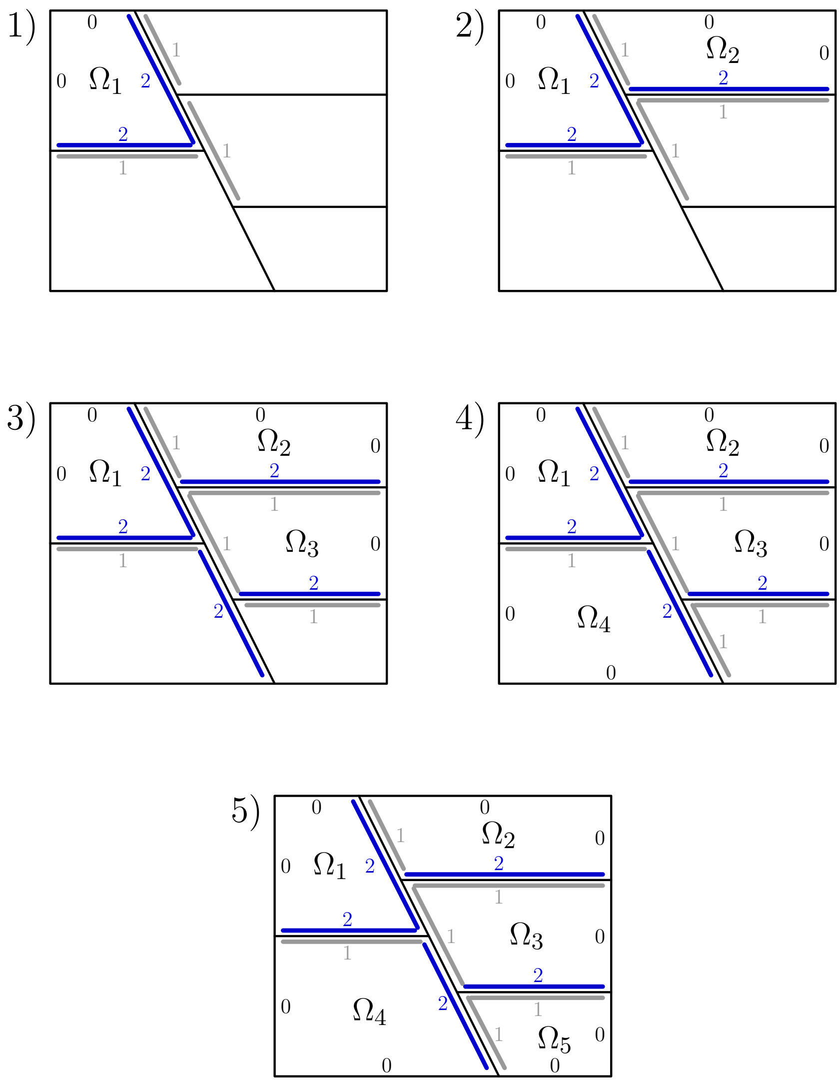


Figure 7: Process of signing of main edges (mortar edges are blue (2), nonmortar edges are grey (1) and 0 is for edges of outer boundary).

4.3 Assembling

We briefly describe the assembling of the stiffness matrix and the right hand side. We add in the assembling the requierement of the fulfilment of the mortar condition (see Section 2) also. First, we need to do some auxiliary steps. Let us consider all the nodes of all the subdomains. For further computation we divide the nodes into several groups - in the local meaning (within the subdomains) and also in global meaning (in terms of the whole computational domain). We divide the nodes into active and inactive, into interior and boundary, etc. We mention closely the global division of the nodes. We introduce a set of global active interior nodes, that contains interior nodes of all the subdomains, interior nodes of the mortar edges and all the corner nodes, that don't lie on the outer boundary. All the nodes lying on the outer boundary are called global active boundary nodes, etc. The sense of these sets will be clear later.

For each mortar (master) edge we assemble the so-called master matrix and for each nonmortar (slave) edge the so-called slave matrix, that introduce the right and left hand side matrix in applying the mortar condition. In the computation of the master matrix we deal with the test functions on the nonmortar edge described in Section 2. Since the test functions have zero values at the end points of the nonmortar edges, it's necessary to have the nonmortar edges with at least three nodes (including the end nodes).

Each subdomain is first assembled in a usual way as in the case of the finite element method. The values on the subdomain edges are re-counted according to whether the appropriate main edge is mortar or nonmortar.

4.4 Solving the Resulting System of Equations

For solving the resulting system of equations, we use the conjugate gradient method with preconditioning, that is implemented in Matlab. This method is highly suitable for solving system of equations with symmetric positive definite and sparse matrix. The method is convenient for large matrices because of the iterative character of the method. The preconditioning accelerates the computation and improves stability.

The system of equations is solved only for the global active interior nodes. After the computation it's necessary to compute the solution on the outer boundary, where the fulfilment of the Dirichlet boundary condition is required, and the solution in the inactive (nonmortar) nodes, which is computed by the mortar projection.

5 Numerical Results

In this section, we present two practical examples showing both the principle and the possibility of practical use of the method. The examples are solved by our program for solving linear elliptic partial differential equations in 2D by the mortar finite element method.

As first example, we consider a Poisson problem on a general computational domain Ω (see Figure 8):

$$\begin{aligned} -\triangle u &= 30 \quad \text{on } \Omega, \\ u &= 0 \quad \text{on } \Gamma_0, \\ u &= 1 \quad \text{on } \Gamma_1. \end{aligned} \tag{23}$$

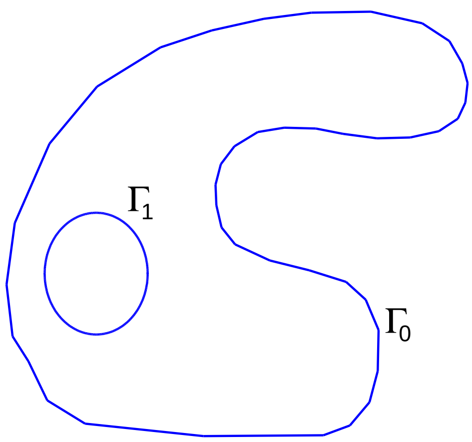


Figure 8: Computational domain Ω for problem (23).

The partition of the computational domain with the appropriate nonmatching mesh is on Figure 9 and the solution of the problem (23) is displayed in Figures 10 and 11. This example serves as an illustration of the principle of the mortar finite element method. There is shown the commonness of the usage of the method.

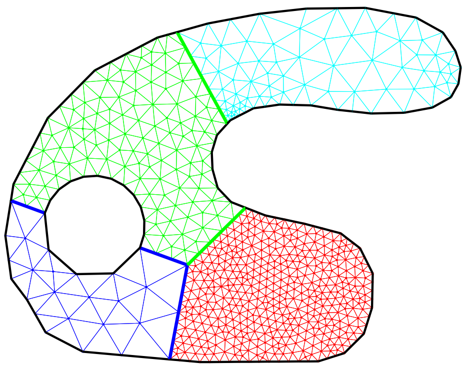


Figure 9: Partition of Ω for problem (23).

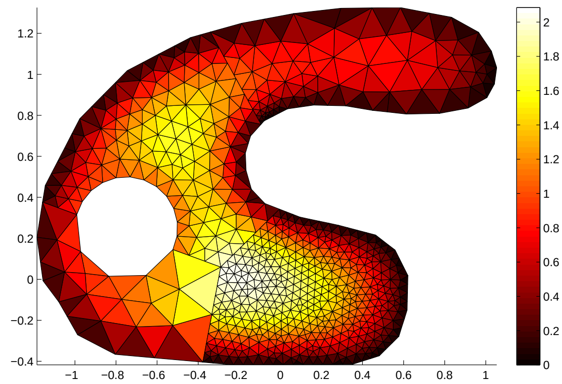


Figure 10: Solution of (23).

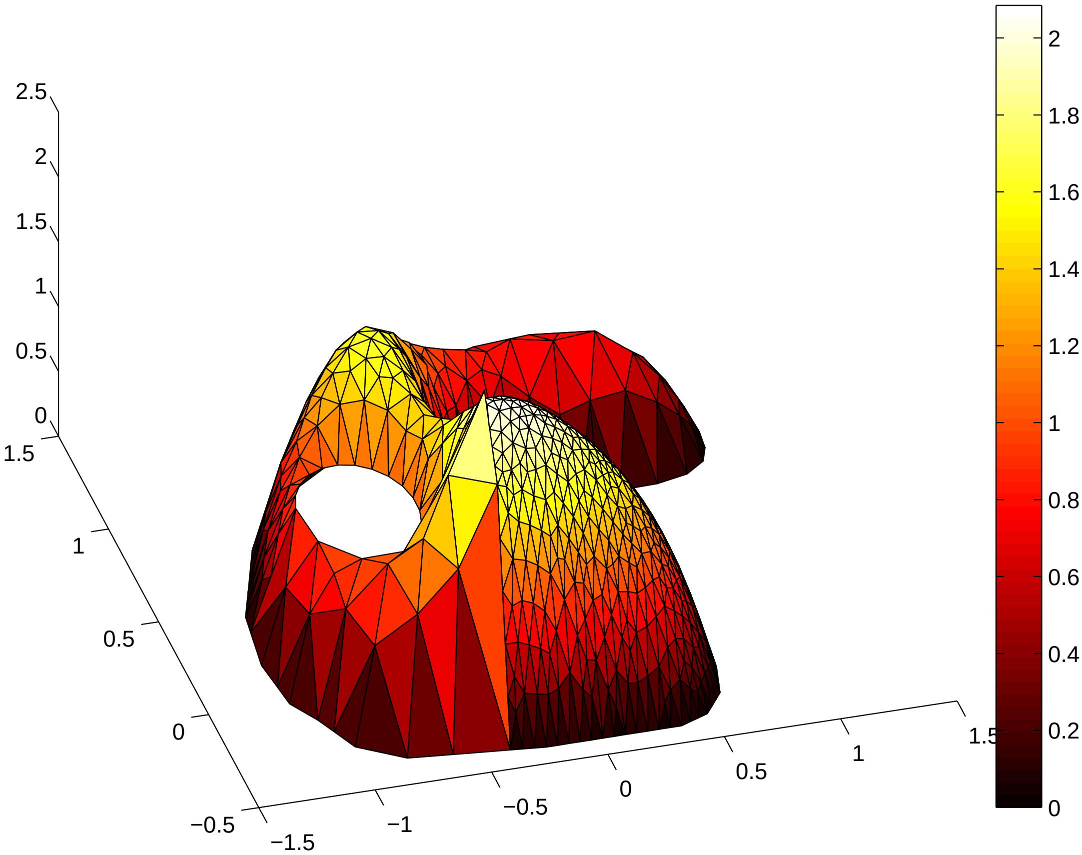
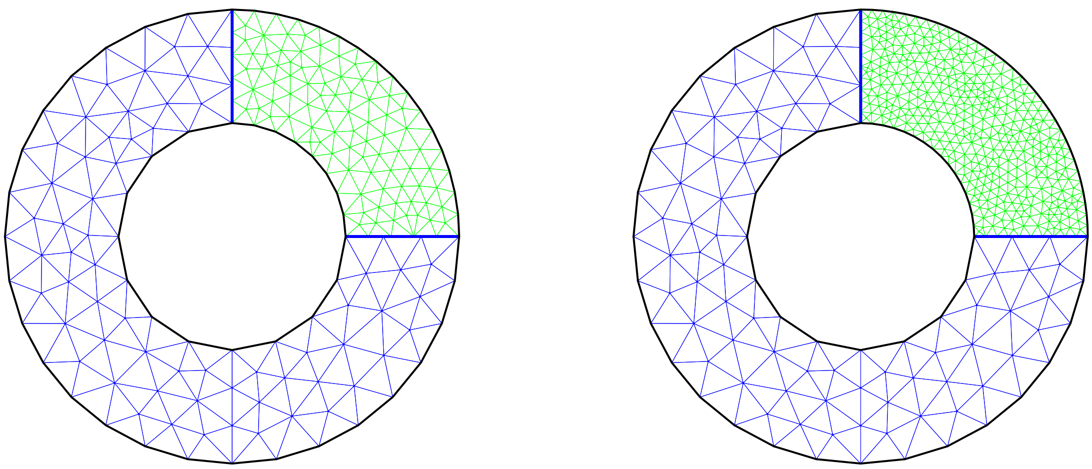


Figure 11: Solution of (23).

As the second example, we consider a Laplace problem:

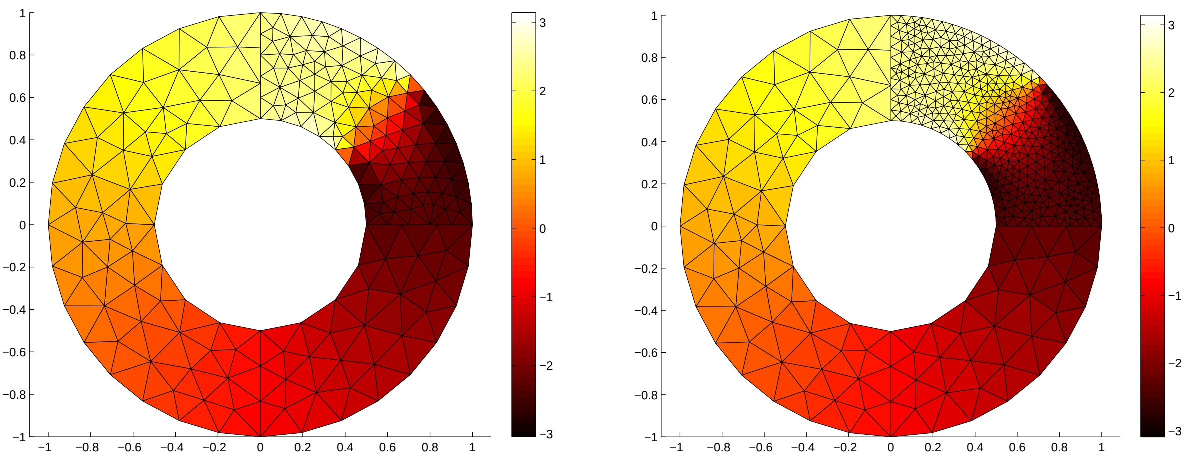
$$
\begin{aligned}
-\triangle u = 0 \quad \text{on } \Omega = \{(x,y) \in \mathbb{R}^2 :\ \frac{1}{4} < x^2 + y^2 < 1\},& \\
u(\cos\varphi, \sin\varphi) \quad = \quad \frac{5\pi}{4} - \varphi, \quad \varphi \in (\frac{\pi}{4}, \frac{9\pi}{4}),& \\
u(\frac{1}{2}\cos\varphi, \frac{1}{2}\sin\varphi) \quad = \quad \frac{5\pi}{4} - \varphi, \quad \varphi \in (\frac{\pi}{4}, \frac{9\pi}{4}).&
\end{aligned}
\tag{24}
$$

Figures 12(a),(b) show the partition of the computational domain with two versions of the appropriate mesh. The solution of the problem (24) is displayed in Figures 13(a),(b) and 14(a),(b). The usage of the mortar finite element method enables a choice of a mesh with respect to the discontinuity of the boundary condition. The subdomain with the jump can be meshed much finer than the other one.



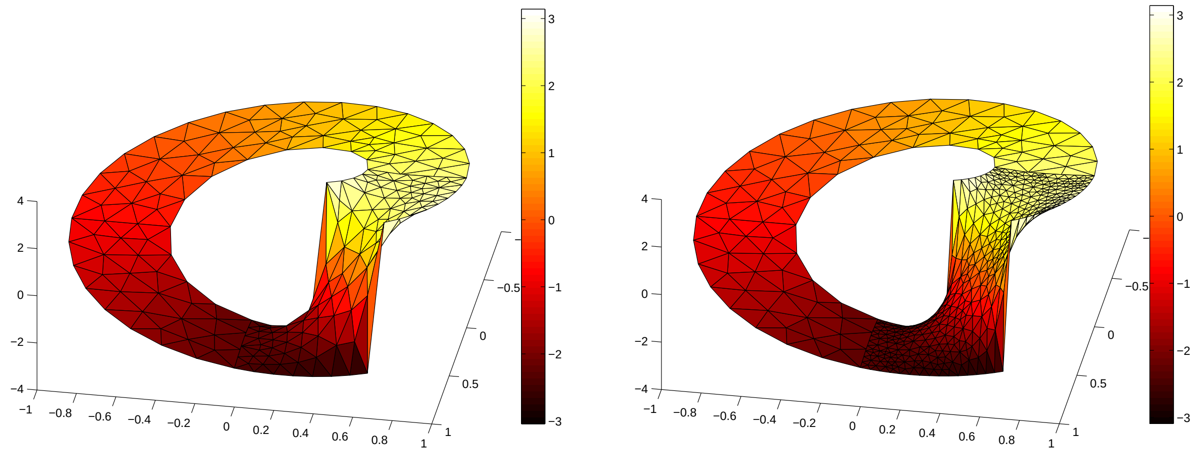
(a) (b)

Figure 12: Partition of Ω and two versions of the appropriate mesh for problem (24).



(a) (b)

Figure 13: Solution of (24) for two versions of the appropriate mesh.



(a) (b)

Figure 14: Solution of (24) for two versions of the appropriate mesh.

Acknowledgements

The first author was supported by the Research Plan MSM 4977751301.

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Článek [101⋆]

NEDOMA, J., DANĚK, J.: *Time-space FE-PDAS method for dynamic unilateral contact problem in viscoelasticity*, Lecture Notes in Computer Science 5073, pp. 707–719, Springer Vlg., Berlin, Heidelberg 2008.

Time-Space FE-PDAS Method for Dynamic Unilateral Contact Problem in Viscoelasticity

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Abstract. The paper deals with numerical analysis of a class of multibody dynamic unilateral contact problems. The presented problem describes the seismological model problem, representing a new approach in this branch and of studying propagation of seismic waves through broken up upper parts of the Earth. The semi-implicit finite element and the primal-dual active set strategy methods will be discussed.

Keywords: multibody unilateral dynamic contact problem, time-space solution of hyperbolic equation, semi-implicit scheme, FEM, seismology.

1 Introduction

This paper deals with a computational efficient method for multibody dynamic visco-elastic bodies being in contact, describing propagation of seismic waves through the elastic and/or viscoelastic media. The dynamic contact problems with given (Tresca) friction involving linearly (thermo−)elastic and visco-elastic bodies were analyzed in Duvaut, Lions (1976), Hilber et al. (1977), Jarušek (1996), Rivera, Racke (1998), Nedoma (1998), Nedoma (2000), Nedoma (2001), Eck et al. (2005), Nedoma, Daněk (2007), etc.

Analyses of seismic wave propagation play an important role in geodynamic investigations. The propagation of elastic as well as seismic waves through the elastic and the visco-elastic media or the Earth body, characterized by inhomogeneous with elastic or visco-elastic parameters are functions of place were studied theoretically by means of simplified models in several papers and books by use of different techniquies, like the ray method, the wave method, the method of propagators, the finite difference and finite element methods, etc. Theoretical seismograms, constructed on this basis of results obtained by various methods mentioned above, do not yield a complete answer to the question of the structure of the Earths crust, mantle as well as core. Their physical structure is so complicated that the present-day methods cannot give the actual information about their structure. The present-day methods analyse the interior of the Earth under the presumption that the Earths interior can be described by continuous inhomogeneous media. But in the real Earth the crust and the upper mantle are broken up into a great number of 3D plates and geological blocks which are in

O. Gervasi et al. (Eds.): ICCSA 2008, Part II, LNCS 5073, pp. 707–719, 2008.

common contacts. Such problems are up-to-date unsolved. Therefore, the main goal of this paper is to give the new methodology and the optimal algorithm which facilitate to investigate such type of seismological problems. The methodology discussed in the paper can be applied also in the other branches like the building industries, the structural mechanics, etc. These type of problems lead to study the dynamic multibody contact problem in linear viscoelasticity, where the damping term and the Coulombian type of friction acting on the contact boundaries between lithosperic plates or geological blocks being in common contact are also assumed.

2 Formulation of the Dynamic Contact Problem

We introduce a viscoelastic bodies of a Kelvin-Voigt type being in a mutual contact which initially occupy a bounded domains of arbitrary shapes Ω^ι of $\mathbb{R}^2$, such that $\Omega = \cup_{\iota=1}^r \Omega^\iota$. Let several neighboring bodies, say Ω^s and Ω^m, be in a mutual contact and let Γ_c^{sm} be the common contact boundary between them before their deformation. Let the boundary $\partial\Omega = \Gamma_\tau \cup \Gamma_u \cup \Gamma_c$, where $\Gamma_c = \cup_{s,m}\Gamma_c^{sm}$, $\Gamma_c^{sm} = \partial\Omega^s \cap \partial\Omega^m$, $s \neq m$, $s, m \in \{1, \ldots, r\}$ and where Γ_j, $j = u, \tau, c$ are open subsets in $\partial\Omega$. Let $I = (0, t_p)$ be a time interval and let $\Omega_t = I \times \Omega$ denote the time-space domain and $\Gamma_\tau(t) = I \times \Gamma_\tau$, $\Gamma_u(t) = I \times \Gamma_u$, $\Gamma_c(t) = I \times \Gamma_c$ denote the parts of its boundary $\partial\Omega_t = I \times \partial\Omega$.

We assume that in the region Ω will act the damping $\alpha(\mathbf{x}) \geq 0$, defined by such a way that the waves propagated through the areas of a certain thickness D (in which $\alpha(\mathbf{x}) > 0$) near the boundary Γ_u are completely absorbed (for more details see Nedoma (1998)). In the body different nonzero densities ρ are assumed. The body is subjected to volume forces $\mathbf{F}$ and as a result of the effect of damping $\alpha(\mathbf{x})$ zero displacements are imposed on Γ_u and on Γ_τ zero or nonzero tractions $\mathbf{P}$ are imposed. The point sources of seismic waves can be of different types and can be simulated by nonzero volume forces (in remaning points of Ω the volume forces are assumed to be zero), see e.g. Nedoma (1998).

Let $\mathbf{n}$ denote the outer normal vector of the boundary $\partial\Omega$, $u_n = u_i n_i$, $\mathbf{u}_t = \mathbf{u} - u_n\mathbf{n}$, $\tau_n = \tau_{ij} n_j n_i$, $\boldsymbol{\tau}_t = \boldsymbol{\tau} - \tau_n \mathbf{n}$ be the normal and tangential components of the displacement and stress vectors $\mathbf{u} = (u_i)$, $\boldsymbol{\tau} = (\tau_i)$, $\tau_i = \tau_{ij} n_j$, $i, j = 1, 2$. On Γ_c^{sm} the positive direction of the outer normal vector $\mathbf{n}$ is assumed with respect to Ω^s. The respective time derivatives are denoted by " $'$ ".

The stress strain relation in every Ω^ι will be defined by the Hooke's law

$$\begin{aligned} \tau_{ij}^\iota = \tau_{ij}^\iota(\mathbf{u}^\iota, \mathbf{u}^{\iota\prime}) = c_{ijkl}^{(0)\iota}(\mathbf{x}) e_{kl}(\mathbf{u}^\iota) + c_{ijkl}^{(1)\iota}(\mathbf{x}) e_{kl}(\mathbf{u}^{\iota\prime}), \\ i, j, k, l = 1, 2,\ \iota = 1, \ldots, r, \end{aligned} \tag{1}$$

where $c_{ijkl}^{(n)\iota}(\mathbf{x})$, $n = 0, 1$, are elastic and viscous coefficients and $e_{ij}(\mathbf{u}) = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$ are components of the small strain tensor. For the tensors $c_{ijkl}^{(n)\iota}(\mathbf{x})$, $n = 0, 1$, we assume

$$\begin{aligned} c_{ijkl}^{(n)\iota} \in L^\infty(\Omega^\iota), \quad n = 0, 1, \quad c_{ijkl}^{(n)\iota} = c_{jikl}^{(n)\iota} = c_{klij}^{(n)\iota} = c_{ijlk}^{(n)\iota}, \\ c_{ijkl}^{(n)\iota} e_{ij} e_{kl} \geq c_0^{(n)\iota} e_{ij} e_{ij} \quad \forall e_{ij},\ e_{ij} = e_{ji} \text{ and a.e. } \mathbf{x} \in \Omega^\iota, c_0^{(n)\iota} > 0. \end{aligned} \tag{2}$$

A repeated index implies summation from 1 to 2.

The problem to be solved has the following classical formulation:

Problem 1. Let $s \geq 2$. Find a vector function $\mathbf{u} : I \times \overline{\Omega} \to \mathbb{R}^2$, satisfying

$$\rho^{\iota} \frac{\partial^2 u_i^{\iota}}{\partial t^2} + \alpha^{\iota} \frac{\partial u_i^{\iota}}{\partial t} = \frac{\partial \tau_{ij}^{\iota} (\mathbf{u}^{\iota}, \mathbf{u}^{\iota\prime})}{\partial x_j} + F_i^{\iota}, \qquad i, j = 1, 2, \ \iota = 1, \ldots, r, \ (t, \mathbf{x}) \in \Omega_t^{\iota} = I \times \Omega^{\iota}, \tag{3}$$

$$\tau_{ij} n_j = P_i, \ i, j = 1, 2, \ (t, \mathbf{x}) \in \Gamma_\tau(t) = I \times \cup_{\iota=1}^{r} (\Gamma_\tau \cap \partial \Omega^{\iota}), \tag{4}$$

$$u_i = 0, \ i = 1, 2, \ \text{on } \Gamma_u(t) = I \times \cup_{\iota=1}^{r} (\Gamma_u \cap \partial \Omega^{\iota}), \tag{5}$$

$$\begin{cases} u_n^s - u_n^m \leq d^{sm}, \ \tau_n^s = \tau_n^m \equiv \tau_n^{sm} \leq 0, \ (u_n^s - u_n^m - d^{sm}) \tau_n^{sm} = 0, \\ \mathbf{u}_t^{\prime s} - \mathbf{u}_t^{\prime m} = 0 \ \Rightarrow \ |\boldsymbol{\tau}_t^{sm}| \leq \mathcal{F}_c^{sm}(0) |\tau_n^{sm}|, \\ \mathbf{u}_t^{\prime s} - \mathbf{u}_t^{\prime m} \neq 0 \ \Rightarrow \ \boldsymbol{\tau}_t^{sm} = -\mathcal{F}_c^{sm} (\mathbf{u}_t^{\prime s} - \mathbf{u}_t^{\prime m}) |\tau_n^{sm}| \frac{\mathbf{u}_t^{\prime s} - \mathbf{u}_t^{\prime m}}{|\mathbf{u}_t^{\prime s} - \mathbf{u}_t^{\prime m}|}, \\ \qquad (t, \mathbf{x}) \in \Gamma_c(t) = I \times \cup_{s,m} \Gamma_c^{sm}, \end{cases} \tag{6}$$

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \quad \mathbf{u}'(\mathbf{x}, 0) = \mathbf{u}_1(\mathbf{x}), \quad \mathbf{x} \in \Omega, \tag{7}$$

where $\mathbf{u}_0$, $\mathbf{u}_1$ are given functions.

The coefficient of friction $\mathcal{F}_c^{sm} \equiv \mathcal{F}_c^{sm}(\mathbf{x}, \mathbf{u}')$ is globally bounded, non-negative and satisfies the Carathéodory property and has a compact support $\mathcal{SF}_c$, which depends on the space variable $\mathbf{x}$ and since we model also the difference between the coefficients of friction and of stick as well as slip, it depends also on the tangential displacement rate component $\mathbf{u}_t'$. Let us denote by ${}^a\Gamma_c^{sm} \subset \Gamma_c^{sm}$ the actual contact set, i.e. for which $u_n^s - u_n^m = d^{sm}$ on ${}^a\Gamma_c^{sm}$ and $u_n^s - u_n^m < d^{sm}$ on ${}^c\Gamma_c^{sm} = \Gamma_c^{sm} \backslash {}^a\Gamma_c^{sm}$. For the continuous displacement $\mathbf{u}$ the actual contact zone ${}^a\Gamma_c^{sm}$ is well-defined and closed subset of $\cup_{s,m} {}^a\Gamma_c^{sm}$.

In the next, the Hilbert and the Sobolev spaces will be used as usual, see e.g. (see Adams (1975)).

The dynamic contact problem with Coulombian friction where the Signorini conditions are formulated in displacements is up-to-date unsolved. In the next, for the numerical solution of the problem discussed, we will assume that the Coulombian friction in every time level depends on its value g_c^{sm} from the previous time levels, where g_c^{sm} is non-negative function and has a meaning of a given friction limit and $-g_c^{sm}$ represents a given frictional force.

The contact problem (3)-(7) has a weak formulation in terms of a variational inequality. Let us introduce the set of virtual displacements and the set of admissible displacements by

$$\mathcal{V} = \{\mathbf{v} \mid \mathbf{v} \in L^2(I; \sqcap_{\iota=1}^{r} H^{1,2}(\Omega^{\iota})), \mathbf{v} = 0 \text{ on } \Gamma_u(t), \quad v_n^s - v_n^m = d^{sm} \ \text{ a.e. on } \ I \times \cup_{s,m} \Gamma_c^{sm}\},$$

$$\mathcal{K} = \{\mathbf{v} \mid \mathbf{v} \in L^2(I; \sqcap_{\iota=1}^{s} H^{1,2}(\Omega^{\iota})), \mathbf{v} = \mathbf{0} \text{ on } \Gamma_u(t), \quad v_n^s - v_n^m \leq d^{sm} \ \text{ a.e. on } \ I \times \cup_{s,m} \Gamma_c^{sm}\}.$$

Then we have the following problem:

Problem 2. A weak solution of (3)-(7) is a function $\mathbf{u} \in B(I; H^{1,2}(\Omega))$ with $\mathbf{u}(t,.) \in \mathcal{K}$ for a.e. $t \in I$, $\mathbf{u}' \in L^2(I; H^{1,2}(\Omega)) \cap L^\infty(I; L^{2,2}(\Omega))$, $\mathbf{u}'(t_p,.) \in L^{2,2}(\Omega)$ such that for all $\mathbf{v} \in H^{1,2}(\Omega(t))$ with $\mathbf{v}(t,.) \in \mathcal{K}$ a.e. in I the following inequality holds

$$\begin{aligned}
&\textstyle\int_I \{(\mathbf{u}''(t), \mathbf{v} - \mathbf{u}(t)) + (\alpha \mathbf{u}'(t), \mathbf{v} - \mathbf{u}(t)) + a^{(0)}(\mathbf{u}(t), \mathbf{v} - \mathbf{u}(t)) + \\
&\quad + a^{(1)}(\mathbf{u}'(\mathbf{t}), \mathbf{v} - \mathbf{u}(t)) + j(\mathbf{v}) - j(\mathbf{u}(t))\}\, dt \geq \textstyle\int_I \left(\overline{\mathbf{f}}(t), \mathbf{v} - \mathbf{u}(t)\right) dt, \\
&\qquad\qquad \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}),\ \mathbf{u}'(\mathbf{x}, 0) = \mathbf{u}_1(\mathbf{x}).
\end{aligned} \tag{8}$$

where we assume that the initial data $\mathbf{u}_0$, $\mathbf{u}_1$ satisfy the static contact multibody linear elastic problem (see Nedoma (1983), Nedoma (1987), Hlaváček, Nedoma (2002)) and where

$$\begin{aligned}
&(\mathbf{u}'', \mathbf{v}) = \textstyle\sum_{\iota=1}^r (\mathbf{u}''^{\iota}, \mathbf{v}^\iota) = \int_\Omega \rho u_i'' v_i \, d\mathbf{x}, \\
&(\alpha \mathbf{u}', \mathbf{v}) = \textstyle\sum_{\iota=1}^r (\alpha^\iota \mathbf{u}'^{\iota}, \mathbf{v}^\iota) = \int_\Omega \alpha u_i' v_i \, d\mathbf{x}, \\
&a^{(n)}(\mathbf{u}, \mathbf{v}) = \textstyle\sum_{\iota=1}^r a^{(n)\iota}(\mathbf{u}^\iota, \mathbf{v}^\iota) = \int_\Omega c_{ijkl}^{(n)} e_{kl}(\mathbf{u}) e_{ij}(\mathbf{v}) \, d\mathbf{x}, \\
&\left(\overline{\mathbf{f}}, \mathbf{v}\right) = \textstyle\sum_{\iota=1}^r \left(\overline{\mathbf{f}}^\iota, \mathbf{v}^\iota\right) = \int_\Omega F_i v_i \, d\mathbf{x} \ + \ \int_{\Gamma_\tau} P_i v_i \, ds, \\
&j(\mathbf{v}) = \textstyle\sum_{\iota=1}^r j^\iota(\mathbf{v}^\iota) = \int_{\cup_{s,m} \Gamma_c^{sm}} g_c^{sm} \left|\mathbf{v}_t^s - \mathbf{v}_t^m\right| ds \equiv \langle g_c^{sm} \left|\mathbf{v}_t^s - \mathbf{v}_t^m\right| \rangle_{\Gamma_c},
\end{aligned}$$

where the bilinear forms $a^{(n)}(\mathbf{u}, \mathbf{v})$, $n = 0, 1$, are symmetric in $\mathbf{u}, \mathbf{v}$ and satisfy $a^{(n)}(\mathbf{u}, \mathbf{u}) \geq c_0^{(n)} \|\mathbf{u}\|_{1,2}^2$, $c_0^{(n)} = const. > 0$, $\left|a^{(n)}(\mathbf{u}, \mathbf{v})\right| \leq c_1^{(n)} \|\mathbf{u}\|_{1,2} \|\mathbf{v}\|_{1,2}$, $c_1^{(n)} = const. > 0$, $\mathbf{u}, \mathbf{v} \in V$.

To prove the existence of the solution the decomposion $\mathbf{v} - \mathbf{u} = \mathbf{v} - \mathbf{u} + \mathbf{u}' - \mathbf{u}' = \mathbf{w} - \mathbf{u}'$ can be used. Then the proof is similar of that of Eck et al. (2005), Chap. 4. For the proof the technique of penalization and regularization will be used.

3 Numerical Solution and the Algorithm

Let Ω be approximated by a polygon Ω_h with the boundary $\partial\Omega_h = \Gamma_{\tau h} \cup \Gamma_{uh} \cup \Gamma_{ch}$. Let $I = (0, t_p)$, $t_p > 0$, let $m > 0$ be an integer, then $\triangle t = t_p/m$, $t_i = i\triangle t$, $i = 0, \ldots, m$. Let $\{\mathcal{T}_{h,\Omega_h}\}$ be a regular family of finite element partitions $\mathcal{T}_h$ of $\overline{\Omega}_h$ compatible to the boundary subsets $\overline{\Gamma}_{\tau h}$, $\overline{\Gamma}_{uh}$ and $\overline{\Gamma}_{ch}$. Let $V_h \subset V$ be the finite element space of linear elements corresponding to the partition $\mathcal{T}_h$. Then $K_h = V_h \cap K$ is the set of continuous piecewise linear functions that vanish at the nodes of $\overline{\Gamma}_{uh}$ and whose normal components are non-positive at the nodes on $\cup_{s,m} \Gamma_c^{sm}$. It is evident that K_h is a nonempty, closed, convex subset of $V_h \subset V$. Let $\mathbf{u}_{0h} \in K_h$, $\mathbf{u}_{1h} \in K_h$ be an approximation of $\mathbf{u}_0$ or $\mathbf{u}_1$. Further, we assume that the end points $\overline{\Gamma}_{\tau h} \cup \overline{\Gamma}_{uh}$, $\overline{\Gamma}_{uh} \cup \overline{\Gamma}_{ch}$, $\overline{\Gamma}_{\tau h} \cup \overline{\Gamma}_{ch}$ coincide with the vertices of T_{hi}. Moreover, we will assume that the frictional term is approximated by its value in the previous time levels, so that the frictional term is approximated by a given friction limit. Then in every time level we have the following discrete problem:

Problem 3. Find a displacement field $\mathbf{u}_h : I \rightarrow V_h$ with $\mathbf{u}_h(0) = \mathbf{u}_{0h}, \mathbf{u}'_h(0) = \mathbf{u}_{1h}$, such that for a.e. $t \in I$, $\mathbf{u}_h(t) \in K_h$

$$\begin{aligned}&(\mathbf{u}''_h(t), \mathbf{v}_h - \mathbf{u}_h(t)) + (\alpha \mathbf{u}'_h(t), \mathbf{v}_h - \mathbf{u}_h(t)) + a^{(0)}(\mathbf{u}_h(t), \mathbf{v}_h - \mathbf{u}_h(t)) + \\ &+ a^{(1)}(\mathbf{u}'_h(t), \mathbf{v}_h - \mathbf{u}_h(t)) + j(\mathbf{v}_h) - j(\mathbf{u}_h(t)) \geq (\overline{\mathbf{f}}(t), \mathbf{v}_h - \mathbf{u}_h(t)) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall \mathbf{v}_h \in K_h, \text{ a.e. } t \in I,\end{aligned} \tag{9}$$

where

$$\begin{aligned}&(\mathbf{u}''_h, \mathbf{v}_h) = \textstyle\sum_{\iota=1}^{r} (\mathbf{u}''^{\iota}_h, \mathbf{v}^{\iota}_h) = \int_\Omega \rho u''_{hi} v_{hi}\, d\mathbf{x}, \\ &(\alpha \mathbf{u}'_h, \mathbf{v}_h) = \textstyle\sum_{\iota=1}^{r} (\alpha^{\iota} \mathbf{u}'^{\iota}_h, \mathbf{v}^{\iota}_h) = \int_\Omega \alpha u'_{hi} v_{hi}\, d\mathbf{x}, \\ &a^{(n)}(\mathbf{u}_h, \mathbf{v}_h) = \textstyle\sum_{\iota=1}^{r} a^{(n)\iota}(\mathbf{u}^{\iota}_h, \mathbf{v}^{\iota}_h) = \int_\Omega c^{(n)}_{ijkl} e_{kl}(\mathbf{u}_h) e_{ij}(\mathbf{v}_h)\, d\mathbf{x},\ n = 0, 1, \\ &(\overline{\mathbf{f}}, \mathbf{v}_h) = \textstyle\sum_{\iota=1}^{r} (\overline{\mathbf{f}}^{\iota}, \mathbf{v}^{\iota}_h) = \int_\Omega F_i v_{hi}\, d\mathbf{x} + \int_{\Gamma_\tau} P_i v_{hi}\, ds, \\ &j(\mathbf{v}_h) = \textstyle\sum_{\iota=1}^{r} j^{\iota}(\mathbf{v}^{\iota}_h) = \int_{\cup_{s,m}\Gamma^{sm}_c} g^{sm}_c |\mathbf{v}^s_t - \mathbf{v}^m_t|\, ds \equiv \langle g^{sm}_c, |\mathbf{v}^s_t - \mathbf{v}^m_t| \rangle_{\Gamma_c},\end{aligned}$$

where we also assume that the frictional term is approximated by the given friction functional in the previous time levels.

To prove the existence of discrete solution $\mathbf{u}_h$ the technique similar of that as in the continuous case based on the decomposition $\mathbf{v}_h - \mathbf{u}_h = \mathbf{v}_h - \mathbf{u}_h + \mathbf{u}'_h - \mathbf{u}'_h = \mathbf{w}_h - \mathbf{u}'_h$, the penalty and regularization technique (Eck et al. (2005)) is used. In the next the test function $\mathbf{v}$ will correspond with $\mathbf{v} + \mathbf{u}' - \mathbf{u}$ in (8) and $\mathbf{v}_h$ with $\mathbf{v}_h + \mathbf{u}'_h - \mathbf{u}_h$ in (9) after used decomposition.

3.1 Algorithm

For the algorithm the semi-implicit scheme in time and the finite elements in space will be used. Let $m > 0$ be integer, then $\triangle t = t_p/m$, $t_i = i\triangle t$, $i = 0, 1, \ldots, m$. Then, the derivatives are approximated by the differences, i.e. $\mathbf{u}''_h = \frac{\mathbf{u}^{i+1}_h - 2\mathbf{u}^i_h + \mathbf{u}^{i-1}_h}{\triangle t^2}$, $\mathbf{u}'_h = \frac{\mathbf{u}^{i+1}_h - \mathbf{u}^{i-1}_h}{\triangle t}$, and after some algebra, in every time level $t = t_{i+1}$ we have to solve the following problem:

$$\begin{aligned}&(\mathbf{u}_h, \mathbf{v}_h - \mathbf{u}_h) + \triangle t\, (\alpha \mathbf{u}_h, \mathbf{v}_h - \mathbf{u}_h) + \\ &\quad + \triangle t^2 a^{(0)}(\mathbf{u}_h, \mathbf{v}_h - \mathbf{u}_h) + \triangle t a^{(1)}(\mathbf{u}_h, \mathbf{v}_h - \mathbf{u}_h) + \\ &\quad + \triangle t^2 \int_{\cup_{s,m}\Gamma^{sm}_c} \tilde{g}^{sm}_{ch} (|\mathbf{v}^s_{th} - \mathbf{v}^m_{th}| - |\mathbf{u}^s_{th} - \mathbf{u}^m_{th}|)\, ds \geq \\ &\quad \geq (\mathbf{f}_h, \mathbf{v}_h - \mathbf{u}_h),\ t = t_{i+1} \in I, \\ &\mathbf{u}_h(0) = \mathbf{u}_{0h},\ \mathbf{u}'_h(0) = \mathbf{u}_{1h},\end{aligned} \tag{10}$$

where we set $\mathbf{u}^i_h = \mathbf{u}_h(t_i)$, $\triangle \mathbf{u}^i_h = \mathbf{u}_h(t_i) - \mathbf{u}_h(t_{i-1})$, $\mathbf{u}^{i+1}_h \equiv \mathbf{u}_h$, $\tilde{g}^{sm}_{ch} = g^{sm}_{ch}(t_i) = \mathcal{F}^{sm}_c\left(\triangle t^{-1}\left(\triangle \mathbf{u}^{si}_h - \triangle \mathbf{u}^{mi}_h\right)\right) |\tau^{sm}_n(\mathbf{u}^i_h, \frac{\triangle \mathbf{u}^i_h}{\triangle t})|$, $(\mathbf{F}(t_{i+1}), \mathbf{v}_h) = \triangle t^2(\overline{\mathbf{f}}_h(t_{i+1}), \mathbf{v}_h) + (2\mathbf{u}^i_h - \mathbf{u}^{i-1}_h, \mathbf{v}_h) + \triangle t(\alpha \mathbf{u}^i_h, \mathbf{v}_h) + \triangle t a^{(1)}_h(\mathbf{u}^i_h, \mathbf{v}_h)$, $\mathbf{F}(t_{i+1}) \equiv \mathbf{f}_h$.

Let us put

$$\begin{aligned}&A(\mathbf{u}_h, \mathbf{v}_h) = (\mathbf{u}_h, \mathbf{v}_h) + \triangle t\, (\alpha \mathbf{u}_h, \mathbf{v}_h) + \triangle t^2 a^{(0)}(\mathbf{u}_h, \mathbf{v}_h) + \triangle t a^{(1)}(\mathbf{u}_h, \mathbf{v}_h), \\ &j(\mathbf{v}_h) = \triangle t^2 \int_{\cup_{s,m}\Gamma^{sm}_c} \tilde{g}^{sm}_{ch} |\mathbf{v}^s_{th} - \mathbf{v}^m_{th}|\, ds,\end{aligned}$$

where $\tilde{g}^{sm}_{ch}$ is the approximate given frictional limit.

Since we assumed that $\rho \geq \rho_0 > 0$, $\alpha \geq \alpha_0 > 0$ and since bilinear forms $a_h^{(n)}(\mathbf{u}_h, \mathbf{v}_h)$, $n = 0, 1$, are symmetric in $\mathbf{u}_h$ and $\mathbf{v}_h$ and satisfy $a^{(n)}(\mathbf{u}_h, \mathbf{u}_h) \geq c_0^{(n)} \|\mathbf{u}_h\|_{1,2}^2$, $c_0^{(n)} = const. > 0$, $\left|a^{(n)}(\mathbf{u}_h, \mathbf{v}_h)\right| \leq c_0^{(n)} \|\mathbf{u}_h\|_{1,2} \|\mathbf{v}_h\|_{1,2}$, $c_1^{(n)} = const. > 0$, $n = 0, 1$, $\mathbf{u}_h, \mathbf{v}_h \in V_h$, then the bilinear form $A(\mathbf{u}_h, \mathbf{v}_h)$ is also symmetric in $\mathbf{u}_h$ and $\mathbf{v}_h$ and

$$\begin{aligned} & A(\mathbf{u}_h, \mathbf{v}_h) \geq a_0 \|\mathbf{u}_h\|_{1,2}^2, \;\; a_0^{(n)} = const. > 0, \\ & |A(\mathbf{u}_h, \mathbf{v}_h)| \leq a_1 \|\mathbf{u}_h\|_{1,2} \|\mathbf{v}_h\|_{1,2}, \;\; a_1 = const. > 0, \;\; \mathbf{u}_h, \mathbf{v}_h \in V_h, \end{aligned}$$

hold.

Then we have to solve in every time level the equivalent problem:

Problem 4. Find $\mathbf{u}_h \in K_h$, a.e. $t = t_{i+1} \in I$, such that

$$A(\mathbf{u}_h, \mathbf{v}_h - \mathbf{u}_h) + j(\mathbf{v}_h) - j(\mathbf{u}_h) \geq (\mathbf{f}_h, \mathbf{v}_h - \mathbf{u}_h), \;\; \forall \mathbf{v}_h \in K_h, \;\; t = t_{i+1} \in I, \tag{11}$$

Then Problem 4 is equivalent with the following problem:

Problem 5. Find $\mathbf{u}_h \in K_h$, a.e. $t = t_{i+1} \in I$, such that

$$\mathcal{L}(\mathbf{u}_h) \leq \mathcal{L}(\mathbf{v}_h), \quad \forall \mathbf{v}_h \in K_h, \; t = t_{i+1} \in I, \tag{12}$$

where the Lagrangian $\mathcal{L}$ is defined as $\mathcal{L}(\mathbf{v}_h) = \mathcal{L}_0(\mathbf{v}_h) + j(\mathbf{v}_h)$, $\mathcal{L}_0(\mathbf{v}_h) = \frac{1}{2} A(\mathbf{v}_h, \mathbf{v}_h) - (\mathbf{f}_h, \mathbf{v}_h)$.

The Mortar Approach - The Non-matching Case. For every polygonal domain Ω^ι, $\iota = 1, \ldots, r$, we introduce triangulations $\mathcal{T}_{h,\Omega^\iota}$ in such a way that on the contact boundaries Γ_c^{sm} the points of Γ_c^s and Γ_c^m are not identical, and therefore, the meshsizes $h_s \neq h_m$.

To give a saddle point formulation, we introduce a Lagrange multiplier space M. We introduce the trace space $W = H^{\frac{1}{2}}(\cup_{s,m} \Gamma_c^{sm})$, being the trace space of V^s restricted to $\cup_{s,m} \Gamma_c^{sm}$, and its dual $W' = H^{-\frac{1}{2}}(\cup_{s,m} \Gamma_c^{sm})$. We introduce the Lagrange multiplier space $M = M_n \times M_t$, where

$$\begin{aligned} & M_n = \{\mu_n \in W'; \; \mu_n \geq 0\}, \\ & M_t = \left\{\mu_t \in L^2(\cup_{s,m} \Gamma_c^{sm}) \mid \|\mu_t\| \leq 1 \;\text{ a.e., } \mu_t = 0 \;\text{ on } \cup_{s,m} \Gamma_c^{sm} \setminus \operatorname{supp} g_c^{sm}\right\}. \end{aligned}$$

We define the bilinear form

$$b(\boldsymbol{\mu}, \mathbf{v}) = \langle \mu_n, [\mathbf{v}.\mathbf{n}]^s - d^{sm} \rangle_{\Gamma_c^s} + \int_{\cup_s \Gamma_c^s} \tilde{g}_{ch}^{sm} \mu_t . [\mathbf{v}_t]^s \, ds,$$

where $[\mathbf{v}.\mathbf{n}]^s = v_n^s(\mathbf{x}) - v_n^m(\mathcal{R}^{sm}(\mathbf{x}))$, $[\mathbf{v}_t]^s = \mathbf{v}_t^s(\mathbf{x}) - \mathbf{v}_t^m(\mathcal{R}^{sm}(\mathbf{x}))$, where $\mathcal{R}^{sm} : \Gamma_c^s(t) \mapsto \Gamma_c^m(t)$, at $t \in I$, is a bijective map satisfying $\Gamma_c^m(t) \subset \mathcal{R}^{sm}(\Gamma_c^s(t))$, $t \in I$ and where $\langle ., . \rangle_{\Gamma_c^s}$ denotes the duality pairing between W and $M = M_n \times M_t$.

We introduce the discrete approximation of the Lagrange multiplier space $M_{hH} = M_{hn} \times M_{Ht}$, where

$$\begin{aligned} & W_{hH}(\Gamma_c^s) = W_{hn}(\Gamma_c^s) \times W_{Ht}(\Gamma_c^s) = \{\boldsymbol{\mu}_{hH}|_{\triangle_r} \in [P_0(\triangle_r)]^2, 0 \leq r \leq n(h^s)\} \\ & M_{hn} = \{\mu_{hn} \in W_{hn}(\Gamma_c^s), \; \mu_{hn} \geq 0 \;\text{ a.e. on every } \Gamma_c^s\}, \\ & M_{Ht} = \{\mu_{Ht} \in W_{Ht}(\Gamma_c^s), \int_{\Gamma_c^s} \mu_{Ht} \psi_H \, ds - \int_{\Gamma_c^s} \tilde{g}_{ch}^{sm} |\psi_H| \, ds \leq 0, \forall \psi_H \in W_{Ht}(\Gamma_c^s)\}, \end{aligned}$$

where $\Gamma_c^s = \cup_{p=1}^{\overline{p}} \Gamma_c^{sp}$, $\Gamma_c^{sp} = \cup_r \triangle_r$, $\triangle_r = (q_r, q_r + h_p)$, $r = 0, 1, \ldots, m(h_p) - 1$, where h_p is a step of mesh in the $p - th$ segment of Γ_c^s.

The problem then leads to the following problem:
In every time level find $(\boldsymbol{\lambda}_{hH}, \mathbf{u}_h) \in M_{hH} \times V_h$ satisfying

$$\begin{aligned} &A(\mathbf{u}_h, \mathbf{v}_h) + b(\boldsymbol{\lambda}_{hH}, \mathbf{v}_h) = (\mathbf{f}_h, \mathbf{v}_h) \quad \mathbf{v}_h \in V_h = \sqcap_{\iota=1}^r V_h^\iota, \\ &b(\boldsymbol{\mu}_{hH} - \boldsymbol{\lambda}_{hH}, \mathbf{v}_h) \leq \langle d^{sm}, \mu_{hn} - \lambda_{hn} \rangle_{\cup \Gamma_c^s} \end{aligned} \tag{13}$$

The existence and uniqueness of $(\boldsymbol{\lambda}_{hH}, \mathbf{u}_h)$ of this saddle-point problem has been stated e.g. in Haslinger et al. (1996), and the discrete Babuška-Brezzi "inf-sup" condition must be satisfied. Thus

Proposition 1. *Let $-\tau_n(\mathbf{u}) \in M_{hn}$. Then the problem (13) has a unique solution $(\boldsymbol{\lambda}_{hH}, \mathbf{u}_h) \in M_{hH} \times V_h$, a.e. $t \in I$. Moreover, we have*

$$\lambda_{hn}^s = -\tau_n^s(\mathbf{u}_h) \quad \textit{and} \quad g_c^s \lambda_{Ht}^s = -\tau_t^s(\mathbf{u}_h),$$

where $\mathbf{u}_h$ is the solution of the discrete primal problem and $g_c^s \equiv \tilde{g}_{ch}^{sm}$.

The above problem is equivalent to that of the Lagrangian formulation:
Find in every time level a pair $(\boldsymbol{\lambda}_{hH}, \mathbf{u}_h) \in (M_{hn} \times M_{Ht}) \times V_h$, such that

$$\begin{aligned} \mathcal{H}(\boldsymbol{\mu}_{hH}, \mathbf{u}_h) \leq \mathcal{H}(\boldsymbol{\lambda}_{hH}, \mathbf{u}) \leq \mathcal{H}(\boldsymbol{\lambda}_{hH}, \mathbf{v}_h) \\ \forall (\boldsymbol{\mu}_{hH}, \mathbf{v}_h) \in (M_{hn} \times M_{Ht}) \times V_h, \; t \in I, \end{aligned} \tag{14}$$

where $\mathcal{H}(\boldsymbol{\mu}_{hH}, \mathbf{v}_h) = \frac{1}{2} A(\mathbf{v}_h, \mathbf{v}_h) + b(\boldsymbol{\mu}_{hH}, \mathbf{v}_h) - (\mathbf{f}_h, \mathbf{v}_h)$.

Matrix Formulation and the Primal-Dual Active Set Strategy Method (PDAS) for the Frictionless Case. Since in the frictionless case the tangential stress component on the contact boundary $\cup \Gamma_c^{sm}$ is zero, then $\lambda_{Ht} = 0$, and therefore, $M_{hH} = M_{hn}$. Wolhmuth and Krause (2003) show that the standard basis of the space $V_h(\Omega^\iota, \mathcal{T}_{h,\Omega^\iota})$ is not a good choice for these type of problems and they introduced the space with a weak condition on $\cup \Gamma_c^{sm}$

$$\hat{V}_h = \{\mathbf{v} \in \sqcap_{\iota=1}^r V_h(\Omega^\iota, \mathcal{T}_{h,\Omega^\iota}) \mid \int_{\cup \Gamma_c^{sm}} [\mathbf{v}.\mathbf{e}_i]^s \psi \, ds = 0, \quad 1 \leq i \leq 2, \psi \in M_{hn}\},$$

where $\mathbf{e}_i$ denotes the i-th unit vector. The definition of $\hat{V}_h$ yields

$$\int_{\cup \Gamma_c^s} [\mathbf{v}.\mathbf{n}_p]^s \psi_s \, ds \leq d_p^s$$

for all vertices on the slide (non-mortar) side. Then the strong form of the non-penetration condition $[\mathbf{u}.\mathbf{n}]^s \leq d^{sm}$ will be replaced by its weak discrete form

$$\int_{\cup \Gamma_c^s} [\mathbf{u}.\mathbf{n}]^s \psi_p \, ds \leq \int_{\cup \Gamma_c^s} d_p^s \psi_p \, ds, \quad p \in S, \tag{15}$$

where S is the set of all vertices in the potential contact part on the slave side. The constraints (15) give a coupling between the vertices on the slave side and the master side. We introduce now a basis transformation of the basis of V_h in such a way that the weak non-penetration condition (15) in the new basis (the so-called dual basis) only deals with the vertices on the slave side. Thus the non-penetration condition is satisfied for all elements in $\hat{V}_h$, and $\hat{V}_h$ is a subspace of K_h.

The Lagrange multipliers λ_{hn} are the approximations of the contact forces $-\tau(\mathbf{u}_h).\mathbf{n} = -(n_j n_k \tau_{jk}(\mathbf{u}_h))$ which are necessary to adjust the contact displacements on the contact boundary $\cup\Gamma_c^{sm}$.

Let the space $V_h = span\{\varphi_p,\ p = 1,\ldots,n_V\}$, n_V denotes the number of degrees of freedom of the space V_h and $M_{hn} = span\{\psi_q, q = 1,\ldots,n_{cn}\}$, where ψ_q denotes the basis function associated with the vertex p and n_{cn} denotes the number of degrees of freedom of the space M_{hn} in each component. φ_p and ψ_q are the scalar basis functions associated with the node p resp. the node q, and satisfying the biorthogonality relation

$$\int_{\cup\Gamma_c^s} \varphi_p \psi_q \, ds = \delta_{pq} \int_{\cup\Gamma_c^s} \varphi_p \, ds, \tag{16}$$

with the Kronecker symbol δ_{pq}, where

$$\delta_{pq} = \langle \begin{matrix} 1 & \text{where the structure node } p \text{ coincides with potential node } q, \\ 0 & \text{otherwise.} \end{matrix}$$

The discrete mortar formulation of the saddle point problem (13) for every time level is defined as follows:

Problem 6. In every time level find $\mathbf{u}_h \in V_h, \lambda_{hn} \in M_{hn}$, a.e. $t \in I$, satisfying

$$\begin{aligned} &A(\mathbf{u}_h, \mathbf{v}_h) + b(\lambda_{hn}, \mathbf{v}_h) = (\mathbf{f}_h, \mathbf{v}_h) \qquad \mathbf{v}_h \in V_h,\ t \in I, \\ &b(\mu_{hn} - \lambda_{hn}, \mathbf{v}_h) \le \langle d^{sm}, (\mu_{hn} - \lambda_{hn}) \rangle_{\cup\Gamma_c^{sm}} \qquad \forall \mu_{hn} \in M_{hn},\ t \in I, \end{aligned} \tag{17}$$

where

$$M_{hn} = \{\mu_{hn} \in M_n | \langle \mu_{hn}, v_h \rangle_{\cup\Gamma_c^{sm}} \ge 0 \ \forall v_h \in \{v_h \in W_h | v_h.\mathbf{n}^s \ge 0\}\},$$
where W_h is the discrete approximation of W

and

$$A(\mathbf{u}_h, \mathbf{v}_h) = (\rho \mathbf{u}_h, \mathbf{v}_h) + \triangle t(\alpha \mathbf{u}_h, \mathbf{v}_h) + \triangle t^2 a^{(0)}(\alpha \mathbf{u}_h, \mathbf{v}_h) + \triangle t a^{(1)}(\alpha \mathbf{u}_h, \mathbf{v}_h),$$

$$b(\mu_{hn}, \mathbf{v}_h) = \int_{\cup\Gamma_c^{sm}} \mu_{hn} [\mathbf{v}_h.\mathbf{n}]^s \, ds = \langle \mu_{hn}, [\mathbf{v}_h.\mathbf{n}]^s \rangle_{\cup\Gamma_c^{sm}} \quad \forall \mathbf{v}_h \in V_h, \mu_{hn} \in M_{hn}.$$

The condition (17b) is equivalent with the following conditions

$$\begin{aligned} &b(\lambda_{hn}, \mathbf{u}_h) = \langle d^{sm}, \lambda_{hn} \rangle_{\cup\Gamma_c^{sm}}, \\ &b(\mu_{hn}, \mathbf{u}_h) \le \langle d^{sm}, \mu_{hn} \rangle_{\cup\Gamma_c^{sm}} \quad \forall \mu_{hn} \in M_{hn}. \end{aligned} \tag{18}$$

Since the space M_{hn} is the convex cone, then putting $\mu_{hn} = 0$ and $\mu_{hn} = 2\lambda_{hn}$ in (17b), we have $b(\lambda_{hn}, \mathbf{u}_h) \geq \langle d^{sm}, \lambda_{hn}\rangle_{\cup\Gamma_c^{sm}}$ and $b(\lambda_{hn}, \mathbf{u}_h) \leq \langle d^{sm}, \lambda_{hn}\rangle_{\cup\Gamma_c^{sm}}$, therefore (18$a$) follows immediately. From (17b) using (18a) the condition (18b) follows and it represents the weak nonpenetration condition.

Let us denote the nodal parameters $\mathbf{u}_h$ by $\mathbf{U}$, of λ_{hn} by $\boldsymbol{\lambda}_{hn}$, and since for the frictionless case the tangential stress component on the contact boundary $\cup\Gamma_c^{sm}$ is zero, then $\lambda_{Ht} = 0$, and thus $\boldsymbol{\lambda}_{Ht} = 0$.

The Eq. (17a) for every $t \in I$ in the matrix form is of the form

$$\mathbb{A}_h\mathbf{U} + \mathbb{B}_h\boldsymbol{\lambda}_h = \mathbb{F}_h. \tag{19}$$

To examine the structure of $\mathbb{B}_h$, we introduce three sets of indices $\mathcal{N}$, $\mathcal{M}$, $\mathcal{S}$. We decompose the set of all vertices in every time level into three disjoint parts $\mathcal{N}$, $\mathcal{M}$ and $\mathcal{S}$, where by $\mathcal{S}$ (dim n_V) we denote all vertices on the possible contact part on the slave side, by $\mathcal{M}$ (dim n_{cn}) all vertices of the possible contact part of the master side, and by $\mathcal{N}$ all the other one. Then the strong formulation of the non-penetration condition (i.e. $[\mathbf{u}.\mathbf{n}]^{sm} \leq d^{sm}$ on $\cup\Gamma_c^{sm}$) will be replaced by its weak discrete form

$$\int_{\cup\Gamma_c^{sm}} [\mathbf{U}.\mathbf{n}]^s\,\psi_p\,ds \leq \int_{\cup\Gamma_c^{sm}} d^{sm}\psi_p\,ds, \quad p \in \mathcal{S}. \tag{20}$$

This condition connects points of sets $\mathcal{S}$ and $\mathcal{M}$. We introduce a basis in a new transformation of the basis of the space V_h in such a way that the weak non-penetration condition (20) in the new basis only deals with the vertices on the slave side (see Hüeber, Wolhmuth (2005), Wolhmuth, Krause (2003)).

Let us introduce the transformation $\varphi = (\varphi_\mathcal{N}, \varphi_\mathcal{M}, \varphi_\mathcal{S})^T$. The matrices and vectors in (19) can be decomposed in the sense of decomposition of the set of all vertices into three disjoint parts $\mathcal{N}$, $\mathcal{M}$ and $\mathcal{S}$, i.e.

$$\begin{bmatrix} \mathbb{A}_{\mathcal{N}\mathcal{N}} & \mathbb{A}_{\mathcal{N}\mathcal{M}} & \mathbb{A}_{\mathcal{N}\mathcal{S}} & \mathbb{O} \\ \mathbb{A}_{\mathcal{M}\mathcal{N}} & \mathbb{A}_{\mathcal{M}\mathcal{M}} & \mathbb{A}_{\mathcal{M}\mathcal{S}} & -\mathbb{M}^T \\ \mathbb{A}_{\mathcal{S}\mathcal{N}} & \mathbb{A}_{\mathcal{S}\mathcal{M}} & \mathbb{A}_{\mathcal{S}\mathcal{S}} & \mathbb{D} \end{bmatrix} \begin{bmatrix} \mathbb{U}_\mathcal{N} \\ \mathbb{U}_\mathcal{M} \\ \mathbb{U}_\mathcal{S} \\ \boldsymbol{\lambda}_\mathcal{S} \end{bmatrix} = \begin{bmatrix} \mathbb{F}_\mathcal{N} \\ \mathbb{F}_\mathcal{M} \\ \mathbb{F}_\mathcal{S} \end{bmatrix}, \tag{21}$$

where elements of $\mathbb{B}_h$ are defined by $\mathbb{B}_h[p,q] = \int_{\cup\Gamma_c^{sm}} \varphi_p\psi_q\,ds\,\mathbb{I}_2 = \delta_{pq}\int_{\cup\Gamma_c^{s}} \varphi_p\,ds,\ p = 1,\ldots,n_V,\ q = 1,\ldots,n_{cn}$, and where $\mathbb{I}_2$ denotes the identity matrix in $\mathbb{R}^{2\times 2}$. The biorthogonality of the basis functions satisfies (16).

Further, the matrix $\mathbb{M}$ represents the coupling matrix the trace of the finite element shape functions on the master side "m" and the shape functions for the Lagrange multiplier on the slave side "s", defined by

$$\mathbb{M}[p,q] = \int_{\cup\Gamma_c^{sm}} \varphi_p\psi_q\,ds\,\mathbb{I}_2, \quad p \in \mathcal{S}, q \in \mathcal{M}. \tag{22}$$

The matrix $\mathbb{M}$ is the block matrix and $\mathbb{D}$ is the block diagonal matrix with

$$\mathbb{D}[p,q] = \delta_{pq}\,\mathbb{I}_2.\int_{\cup\Gamma_c^{sm}} \varphi_p\,ds, \quad p = q \in \mathcal{S}. \tag{23}$$

The structure of the matrix $\mathbb{B}_h$ in every time level, using an approximate node numbering, is thus of the form $\mathbb{B}_h = \left(\mathbb{O}, -\mathbb{M}^T, \mathbb{D}\right)^T$. The block matrices associated with the basis functions of the free structure nodes (i.e. $\mathcal{N}$), the potential contact nodes of the master side (i.e. $\mathcal{M}$) and the potential contact nodes on the slide side (i.e. $\mathcal{S}$) are denoted by $\mathbb{A}_{k,l},\ k,l \in \{\mathcal{N}, \mathcal{M}, \mathcal{S}\}$. The entries of vectors $\mathbb{U}$ and $\mathbb{F}$ for $k \in \{\mathcal{N}, \mathcal{M}, \mathcal{S}\}$ are denoted by $\mathbb{U}_k$ and $\mathbb{F}_k$, respectively.

We introduce a new modified basis $\Phi = (\Phi_{\mathcal{N}}, \Phi_{\mathcal{M}}, \Phi_{\mathcal{S}})^T$ instead of the basis $\varphi = (\varphi_{\mathcal{N}}, \varphi_{\mathcal{M}}, \varphi_{\mathcal{S}})^T$ and the matrix $\widehat{\mathbb{M}} = \mathbb{D}^{-1}\mathbb{M}$. Then

$$\Phi = (\Phi_{\mathcal{N}}, \Phi_{\mathcal{M}}, \Phi_{\mathcal{S}})^T = \begin{bmatrix} \mathbb{I}_2 & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{I}_2 & \widehat{\mathbb{M}}^T \\ \mathbb{O} & \mathbb{O} & \mathbb{I}_2 \end{bmatrix} \begin{bmatrix} \varphi_{\mathcal{N}} \\ \varphi_{\mathcal{M}} \\ \varphi_{\mathcal{S}} \end{bmatrix} = Q\varphi. \tag{24}$$

Hence

$$\mathbb{U} = Q^T \widehat{\mathbb{U}},$$

where $\widehat{\mathbb{U}}$ is the vector of coefficients with respect to the transformed basis Φ.

The modified stiffness matrix $\widehat{\mathbb{A}}_h$ associated with the transformed basis Φ is of the form

$$\widehat{\mathbb{A}}_h = \begin{bmatrix} \mathbb{I}_2 & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{I}_2 & \widehat{\mathbb{M}}^T \\ \mathbb{O} & \mathbb{O} & \mathbb{I}_2 \end{bmatrix} \begin{bmatrix} \mathbb{A}_{\mathcal{N}\mathcal{N}} & \mathbb{A}_{\mathcal{N}\mathcal{M}} & \mathbb{A}_{\mathcal{N}\mathcal{S}} \\ \mathbb{A}_{\mathcal{M}\mathcal{N}} & \mathbb{A}_{\mathcal{M}\mathcal{M}} & \mathbb{A}_{\mathcal{M}\mathcal{S}} \\ \mathbb{A}_{\mathcal{S}\mathcal{N}} & \mathbb{A}_{\mathcal{S}\mathcal{M}} & \mathbb{A}_{\mathcal{S}\mathcal{S}} \end{bmatrix} \begin{bmatrix} \mathbb{I}_2 & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{I}_2 & \mathbb{O} \\ \mathbb{O} & \widehat{\mathbb{M}} & \mathbb{I}_2 \end{bmatrix} = Q\mathbb{A}_h Q^T =$$

$$= \begin{bmatrix} \mathbb{A}_{\mathcal{N}\mathcal{N}} & \mathbb{A}_{\mathcal{N}\mathcal{M}} + \mathbb{A}_{\mathcal{N}\mathcal{S}}\widehat{\mathbb{M}} & \mathbb{A}_{\mathcal{N}\mathcal{S}} \\ \mathbb{A}_{\mathcal{M}\mathcal{N}} + \widehat{\mathbb{M}}^T\mathbb{A}_{\mathcal{S}\mathcal{N}} & \mathbb{A}_{\mathcal{M}\mathcal{M}} + \mathbb{A}_{\mathcal{M}\mathcal{S}}\widehat{\mathbb{M}} + \widehat{\mathbb{M}}^T\mathbb{A}_{\mathcal{S}\mathcal{M}} + \widehat{\mathbb{M}}^T\mathbb{A}_{\mathcal{S}\mathcal{S}}\widehat{\mathbb{M}} & \mathbb{A}_{\mathcal{M}\mathcal{S}} + \widehat{\mathbb{M}}^T\mathbb{A}_{\mathcal{S}\mathcal{S}} \\ \mathbb{A}_{\mathcal{S}\mathcal{N}} & \mathbb{A}_{\mathcal{S}\mathcal{M}} + \mathbb{A}_{\mathcal{S}\mathcal{S}}\widehat{\mathbb{M}} & \mathbb{A}_{\mathcal{S}\mathcal{S}} \end{bmatrix}$$

and the modified vector of the right hand side $\widehat{\mathbb{F}}_h$ is the following

$$\widehat{\mathbb{F}}_h = Q\mathbb{F}_h = \begin{bmatrix} \mathbb{F}_{\mathcal{N}} \\ \mathbb{F}_{\mathcal{M}} + \widehat{\mathbb{M}}^T\mathbb{F}_{\mathcal{S}} \\ \mathbb{F}_{\mathcal{S}} \end{bmatrix}.$$

The algebraic representation of the weak non-penetration condition associated with the transformed basis Φ is the following:

$$\begin{aligned} &[\mathbb{U}_n]^s = \textstyle\sum_{p\in\mathcal{S}} \widehat{\mathbb{U}}_p\, [\Phi_{pn}]^s + \sum_{q\in\mathcal{M}} \widehat{\mathbb{U}}_q\, [\Phi_{qn}]^s = \\ &= (\textstyle\sum_{p\in\mathcal{S}} \widehat{\mathbb{U}}_p\varphi_p).\mathbf{n}_p^s + \sum_{q\in\mathcal{M}} \widehat{\mathbb{U}}_q(-\varphi_q + \sum_{p'\in\mathcal{S}} \widehat{\mathbb{M}}\,[p',q]\,\varphi_{p'}).\mathbf{n}_p^s. \end{aligned}$$

By multiplying this equation with $\psi_p,\ p \in \mathcal{S}$ and integrating the resulting equation over $\cup_s \Gamma_c^s$, we have due to the biorthogonality condition (16) and the definition of the matrix $\widehat{\mathbb{M}}$ and (24)

$$\widehat{\mathbb{U}}_{n,p} \equiv \left(\mathbf{n}_p^s\right)^T \mathbb{D}\,[p,p]\,\widehat{\mathbb{U}}_p \le d_p^{sm} \qquad \forall p \in \mathcal{S}, \tag{25}$$

where $d_p^{sm} = \int_{\cup_s \Gamma_c^s} d_h^{sm}\psi_p\, ds,\ p \in \mathcal{S}$, as the coefficients at $\widehat{\mathbb{U}}_q, q \in \mathcal{M}$, are nullified. This basis transformation glues the vertices of the slave (non-master) side and on the master side together. The displacements of the glued vertices are given by $\widehat{\mathbb{U}}_{\mathcal{M}}$ and the relative displacements between the vertices on the slave side and on the master side are given by $\widehat{\mathbb{U}}_{\mathcal{S}}$.

The modified matrix $\widehat{\mathbb{B}}_h$ associated with the transformed basis Φ has the form

$$\widehat{\mathbb{B}}_h = Q\mathbb{B}_h = \begin{bmatrix} \mathbb{I}_2 & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{I}_2 & \mathbb{M}^T \\ \mathbb{O} & \mathbb{O} & \mathbb{I}_2 \end{bmatrix} \begin{bmatrix} \mathbb{O} \\ -\mathbb{M}^T \\ \mathbb{D} \end{bmatrix} = (\mathbb{O}, \mathbb{O}, \mathbb{D})^T .$$

Then, in every time level, we will solve the following problem

$$\begin{aligned} &\widehat{\mathbb{A}}_h\widehat{\mathbb{U}} + \widehat{\mathbb{B}}_h\boldsymbol{\lambda}_{hH} = \widehat{\mathbb{F}}_h, \\ &\widehat{\mathbb{U}}_{n,p} \le d_p^{sm}, \quad \boldsymbol{\lambda}_{hn,p} \ge 0, \quad (\widehat{\mathbb{U}}_{n,p} - d_p^{sm})\boldsymbol{\Lambda}_{hn,p} = 0, \quad \forall p \in \mathcal{S}, \\ &\boldsymbol{\Lambda}_{Ht,p} = 0, \quad \forall p \in \mathcal{S}, \end{aligned} \tag{26}$$

where in (26) the second line represents the Karush-Kuhn-Tucker conditions of a constrained optimization problem for inequality constraints,

$$\begin{aligned} &\boldsymbol{\Lambda}_{hn,p} = \mathbf{n}_p^{sT}\widehat{\mathbb{D}}\,[p,p]\,\boldsymbol{\Lambda}_{hn}\,(p)\,,\ \boldsymbol{\Lambda}_{hn}\,(p) \in \mathbb{R}^2, \\ &\boldsymbol{\Lambda}_{Ht,p} = \boldsymbol{\Lambda}_{hn}\,(p) - \left(\boldsymbol{\Lambda}_{hn}\,(p)\,.\mathbf{n}_p^s\right)\mathbf{n}_p^s = \left(\boldsymbol{\Lambda}_{hn}\,(p)\,.\mathbf{t}_p^s\right)\mathbf{t}_p^s. \end{aligned}$$

We decompose the set $\mathcal{S}$ as $\mathcal{S} = \mathcal{A} \cup \mathcal{I}$, where $\mathcal{A}$ is the active set and $\mathcal{I}$ is the inactive set. Then the above problem in every time level leads to **the primal-dual active set algorithm**, which is of the form:

Algorithm PDAS:

STEP 1. Initiate the sets $\mathcal{A}_1$ (active set) and $\mathcal{I}_1$ (inactive set), such that $\mathcal{S} = \mathcal{A}_1 \cup \mathcal{I}_1$ and $\mathcal{A}_1 \cap \mathcal{I}_1 = \emptyset$, put the initial value $(\widehat{\mathbb{U}}^0, \boldsymbol{\Lambda}_{hn}^0)$, $c_1 \in \left(10^3, 10^4\right)$ and set $k = 1$.

STEP 2. If $(\widehat{\mathbb{U}}^{k-1},\ \boldsymbol{\Lambda}_{hn}^{k-1})$ is known, find the primal-dual pair $(\widehat{\mathbb{U}}^k, \boldsymbol{\Lambda}_{hn}^k)$ such that

$$\begin{aligned} &\widehat{\mathbb{A}}_h\widehat{\mathbb{U}}^k + \widehat{\mathbb{B}}_h\boldsymbol{\Lambda}_{hn}^k = \widehat{\mathbb{F}}_h, \\ &\widehat{\mathbb{U}}_{n,p}^k = d_p^{sm} \quad \text{for all } p \in \mathcal{A}_k, \\ &\boldsymbol{\Lambda}_{hn,p}^k = 0 \quad \text{for all } p \in \mathcal{I}_k, \\ &\boldsymbol{\Lambda}_{Ht,p}^k = 0 \quad \text{for all } p \in \mathcal{S}. \end{aligned} \tag{27}$$

STEP 3. Set $\mathcal{A}_{k+1}$ and $\mathcal{I}_{k+1}$ to

$$\begin{aligned} \mathcal{A}_{k+1} &= \{p \in \mathcal{S} : \boldsymbol{\Lambda}_{hn,p}^k + c_1(\widehat{\mathbb{U}}_{n,p}^k - d_p^{sm}) > 0\}, \\ \mathcal{I}_{k+1} &:= \{p \in \mathcal{S} : \boldsymbol{\Lambda}_{hn,p}^k + c_1(\widehat{\mathbb{U}}_{n,p}^k - d_p^{sm}) \le 0\}. \end{aligned}$$

STEP 4. If $\mathcal{A}_{k+1} = \mathcal{A}_k$ and $\mathcal{I}_{k+1} = \mathcal{I}_k$, then STOP else $k = k+1$; goto STEP 2, where the active $(\mathcal{A}_k)$ and inactive $(\mathcal{I}_k)$ sets are associated with the transformed basis Φ.

Remark 1. The system (27) can be rewritten if we decompose the set of vertices $\mathcal{S}$ on the slave side in each step k of the PDAS algorithm into the disjoint active and inactive sets $\mathcal{S} = \mathcal{A}_k \cup \mathcal{I}_k$.

4 Conclusion

With the development of computers, methods of numerical analysis of solving partial differential equations were evolved and were applied to the solution of the propagation of seismic waves in the stratified homogeneous and nonhomegeneous media in the time domain. Such models allow the simulation of propagations of seismic waves inside the Earth closed to the real rheology of the investigated upper parts of the Earth. All these methods were evolved for the continuous Earths crust and the mantle. Since the upper parts of the Earth are broken up into a great number of plates and blocks the up-to-date evolved methods cannot be used and we are forced urgently to investigate new methods and new algorithms permit to simulate and study propagation of seismic ans elastic waves through the broken up media. The different problem is to approximate optimally the contact conditions between geological blocks being in common contact. In our paper the primal-dual active set strategy technique for the 2D seismological model is used. The real Earths body is connected from a great number of 3D bodies being in common contact, and therefore, the methods as well as algorithms must be extended and investigated also in the three dimensions, and moreover, for very complicated contact boundaries between geological blocks, as well as for complicated cracks problems which accompanied the propagation seismic waves through elastic media. It is evident that these problems are enormous difficult and will be problems for very long future time period.

Acknowledgments

The paper was partially supported by the Institutional Research Plan AV0Z10300504 and by the Research Plan MSM 4977751301.

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