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WEAK ASYMPTOTICS OF THE BAYES ESTIMATOR OF THE RELIABILITY FUNCTION IN THE KOZIOL-GREEN MODEL

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Abstract. The Bayesian estimator (assuming conjugate prior density) of the reliability function in the Koziol-Green model with exponential distribution is considered, and the weak convergence of the estimator process to the Gaussian process in $C[0,\infty]$ is proved.

1. Introduction

We are dealing with the Koziol-Green model of random censorship with an exponential distribution, the model, which is described in many reliability and survival data literature. We continue the works [3] and [4], where Bayesian estimators and above all their asymptotic properties and Bayesian risk are studied. [6] deals with the asymptotic properties of the maximum likelihood estimator, in [5] a test of fit with the Koziol-Green model is given. A review of models and methods of estimation is found in [7].

Let us have two independent random samples: $X_1, X_2, \dots, X_n$ (failure times) distributed $\mathrm{Exp}(\lambda)$ $(\lambda > 0)$, the exponential distribution with the reliability function

$$R(t) = \mathrm{e}^{-\lambda t}, \qquad t \geq 0,$$

and $T_1, T_2, \dots, T_n$ (time censors) distributed $\mathrm{Exp}(\lambda\gamma)$ $(\gamma > 0)$. The reliability function of T equals R^γ and satisfies thus the assumption of the Koziol-Green model — to be a power of that of X. Instead of γ we can consider the parameter $p = (1+\gamma)^{-1}$.

Due to the censoring by T_j's, the information available is fully contained in

$$W_j = \min(X_j, T_j) \qquad \text{and} \qquad I_j = \chi_{[X_j \leq T_j]}.$$

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If we denote the total time of an experiment and the number of uncensored items by

$$W = \sum_{j=1}^{n} W_j \qquad \text{and} \qquad I = \sum_{j=1}^{n} I_j,$$

the likelihood function for the parameters of the model, λ and γ, can be written as

$$L(\lambda, \gamma; W, I) = \lambda^n \,\mathrm{e}^{-\lambda(1+\gamma)W}\, \gamma^{n-I}, \qquad \lambda > 0, \gamma > 0.$$

We choose a prior density

$$k_{a,r,s}(\lambda, \gamma) = \frac{a^{r+s}}{\Gamma(r)\Gamma(s)} \lambda^{r+s-1}\, \mathrm{e}^{-\lambda(1+\gamma)}\, \gamma^{s-1}, \qquad \lambda, \gamma > 0,$$

from the natural conjugate system for L, $\mathcal{K} = \{k_{a,r,s},\ a, r, s > 0\}$. Then the Bayesian estimators of λ and $R(t)$ are

$$\widehat{\lambda} = \frac{I + r}{W + a} \qquad \text{and} \qquad \widehat{R(t)} = \left(\frac{W + a}{W + a + t}\right)^{I+r},$$

respectively.

Given λ, γ, the convergences

$$\widehat{\lambda} \xrightarrow[n\to\infty]{\text{a. s.}} \lambda, \qquad \sqrt{n}(\widehat{\lambda} - \lambda) \xrightarrow[n\to\infty]{\mathcal{D}} \mathrm{N}(0, \lambda^2/p)$$

and

$$\sqrt{n}\left(\widehat{R(1)} - R(1)\right) \xrightarrow[n\to\infty]{\mathcal{D}} \mathrm{N}\left(0, (R(1)\ln R(1))^2/p\right). \tag{1}$$

are proved in [4]. In this paper we will consider $\widehat{R(t)}$ and $R(t)$ as the trajectories of continuous processes $\widehat{R}$ and R and extend (1) to the process $\sqrt{n}(\widehat{R} - R)$.

2. Weak convergence of $\widehat{R}$

Lemma. *Let $U(t) = (I + r)\ln(1 + t/(a + W)) - \lambda t$. Then*

$$\sqrt{n}\,\mathrm{e}^{-\lambda t}\, U(t) \xrightarrow[n\to\infty]{\mathcal{D}} Z(t) \quad \text{in } C([0, \infty]), \tag{2}$$

where Z is a zero mean Gaussian process with $\operatorname{cov}(Z(s), Z(t)) = st\,\mathrm{e}^{-\lambda(s+t)}\,\lambda^2/p$.

Proof. We use Taylor expansion of $\ln(1 + x)$ at $x = 0$ to express

$$\ln\left(1 + \frac{t}{W + a}\right) = \frac{t}{W + a} + Q_1(t),$$

where

$$Q_1(t) = -\frac{1}{2(1+\xi)^2}\left(\frac{t}{W + a}\right)^2 \quad \text{for some } \xi(t) \in [0, t/(W + a)]. \tag{3}$$

Since

$$(4)\quad |\sqrt{n}\,\mathrm{e}^{-\lambda t}(I+r)Q_1(t)| \le \frac{1}{2}t^2\,\mathrm{e}^{-\lambda t}\,\frac{(I+r)\sqrt{n}}{(W+a)^2} \le K\frac{(I+r)\sqrt{n}}{(W+a)^2} \xrightarrow[n\to\infty]{} 0 \quad \text{a. s.},$$

where $K > 0$ is a constant depending on λ only, both

$$U'(t) = \sqrt{n}\,\mathrm{e}^{-\lambda t}\left(\frac{I+r}{W+a}t - \lambda t\right) = t\,\mathrm{e}^{-\lambda t}\sqrt{n}\left(\frac{I+r}{W+a} - \lambda\right)$$

and $\sqrt{n}\,\mathrm{e}^{-\lambda t}\,U(t) = U'(t) + \sqrt{n}\,\mathrm{e}^{-\lambda t}(I+r)Q_1(t)$ converge to equally distributed processes.

We know that $\sqrt{n}[(I+r)/(W+a) - \lambda] \to \mathrm{N}(0, \lambda^2/p)$ in distribution as $n \to \infty$ and hence for every $t_1, t_2, \ldots, t_k > 0$ finite-dimensional distributions

$$(U'(t_1), \ldots, U'(t_k)) \xrightarrow[n\to\infty]{\mathcal{D}} \mathrm{N}_k(\mathbf{0}, \mathrm{var}(Z(t_1), \ldots, Z(t_k))).$$

Let $\varepsilon > 0$ be given.

$$\mathrm{P}[\sup_{|s-t|\le\delta} |U'(t) - U'(s)| > \varepsilon] = \mathrm{P}[\sup_{|s-t|\le\delta} |t\,\mathrm{e}^{-\lambda t} - s\,\mathrm{e}^{-\lambda s}|\sqrt{n}\left|\frac{I+r}{W+a} - \lambda\right| > \varepsilon] \le$$

$$\le \mathrm{P}[\delta\sqrt{n}\left|\frac{I+r}{W+a} - \lambda\right| > \varepsilon] \xrightarrow[n\to\infty]{} 2\Phi\left(\frac{-\varepsilon}{\delta\lambda/\sqrt{p}}\right),$$

where Φ is the distribution function of $\mathrm{N}(0,1)$. We can conclude that

$$\lim_{\delta\to 0+} \limsup_{n\to\infty} \mathrm{P}[\sup_{|s-t|\le\delta} |U'(t) - U'(s)| > \varepsilon] = 0$$

(Stone's tightness condition). Thus we have the convergence (2) in $C([0,T])$ for every $T > 0$ (e. g. [8], Theorem III.5.7, or [1], Chapter 8). The extension to $C([0,\infty])$ can be made through the condition on tightness at ∞ (based on [1], Theorem 4.2, an example of application [2], Theorem 6.2.1) which reads

$$\lim_{s\to\infty} \limsup_{n\to\infty} \mathrm{P}[\sup_{t\ge s} |U'(t) - U'(s)| > \varepsilon] = 0 \quad \text{for every } \varepsilon > 0,$$

and this is satisfied here since

$$\mathrm{P}[\sup_{t\ge s} |t\,\mathrm{e}^{-\lambda t} - s\,\mathrm{e}^{-\lambda s}|\sqrt{n}\left|\frac{I+r}{W+a} - \lambda\right| > \varepsilon] \le$$

$$\le \mathrm{P}[s\,\mathrm{e}^{-\lambda s}\sqrt{n}\left|\frac{I+r}{W+a} - \lambda\right| > \varepsilon] \xrightarrow[n\to\infty]{} 2\Phi\left(\frac{-\varepsilon}{s\,\mathrm{e}^{-\lambda s}\lambda/\sqrt{p}}\right),$$

assuming $s > 1/\lambda$. □

Proposition. *For the Bayesian estimator* $\widehat{R}(t) = (\frac{W+a}{W+a+t})^{I+r}$ *of the reliability function* $R(t) = \mathrm{e}^{-\lambda t}$ *we have*

$$\sqrt{n}(\widehat{R} - R) \xrightarrow[n\to\infty]{\mathcal{D}} Z \quad \text{in } C([0,\infty]),$$

where Z is a zero mean Gaussian process with $\mathrm{cov}(Z(s), Z(t)) = st\,\mathrm{e}^{-\lambda(s+t)}\,\lambda^2/p$.

Proof. Using Taylor expansion of e^x at $x = 0$ we obtain

$$\sqrt{n}(\widehat{R}(t) - R(t)) = \sqrt{n}\,\mathrm{e}^{-\lambda t}(\mathrm{e}^{-U(t)} - 1) = \sqrt{n}\,\mathrm{e}^{-\lambda t}(1 - U(t) + Q_2(t) - 1), \tag{5}$$

where

$$Q_2(t) = \mathrm{e}^{\xi(t)}\, U^2(t)/2 \quad \text{for some } \xi(t) \text{ between } 0 \text{ and } -U(t).$$

Let $\delta(t) = 1$ for $0 \le t \le 1$ and $\delta(t) = t$ for $t \ge 1$. Denoting

$$A(t) = \delta^4(t)\,\mathrm{e}^{-\lambda t + \xi(t)}, \quad B(t) = n^{1/4}\delta^{-2}(t)|U(t)|/\sqrt{2}$$

we can write for $\varepsilon > 0$

$$\mathrm{P}[\sup_{t\in[0,\infty]} \sqrt{n}\,\mathrm{e}^{-\lambda t}\,|Q_2(t)| > \varepsilon] \le \mathrm{P}[\sup_{t\in[0,\infty]} A(t) \cdot \sup_{t\in[0,\infty]} B^2(t) > \varepsilon]. \tag{6}$$

If $-U(t) < 0$ then $A(t) \le \delta^4(t)\,\mathrm{e}^{-\lambda t} \le K_1$, for some $K_1 > 0$ independent of t and n. If $-U(t) > 0$ then $A(t) \le \delta^4(t)\,\mathrm{e}^{-\lambda t - U(t)} = \delta^4(t)(1 + t/(W + a))^{-(I+r)}$, which is bounded by 1 on $[0, 1]$; function $t^4(1 + t/(W + a))^{-(I+r)}$, $t > 0$, achieves (assuming $I + r > 4$) its maximum at $t = 4(W + a)/(I + r - 4)$ acquiring the value

$$\left(4\frac{W + a}{I + r - 4}\right)^4 \left(1 + \frac{4}{I + r - 4}\right)^{-(I+r)} \xrightarrow[n\to\infty]{\text{a. s.}} \frac{4^4}{\lambda^4}\,\mathrm{e}^{-4}.$$

Thus we have proved existence of constant $K > \max(K_1, 1, 4^4(\lambda\,\mathrm{e})^{-4})$ which satisfies $\mathrm{P}[\sup_{t\in[0,\infty]} A(t) > K] \to 0$ as $n \to \infty$.

The process $B(t)$ can be treated similarly as in (3), (4). We have

$$\begin{aligned} |B(t)| &= \frac{1}{\sqrt{2}}n^{1/4}\left| t\delta^{-2}(t)\left(\frac{I + r}{W + a} - \lambda\right) + \delta^{-2}(t)(I + r)Q_1(t)\right| \le \\ &\le \frac{1}{\sqrt{2}}n^{-1/4}\left(\sqrt{n}\left|\frac{I + r}{W + a} - \lambda\right| + \frac{(I + r)\sqrt{n}}{2(W + a)^2}\right) \xrightarrow[n\to\infty]{\mathrm{P}} 0, \end{aligned}$$

hence $\mathrm{P}[\sup_{t\in[0,\infty]} B^2(t) > \varepsilon] \to 0$ as $n \to \infty$.

Returning to (6) we have $\sup_{t\in[0,\infty]} \sqrt{n}\,\mathrm{e}^{-\lambda t}\, Q_2(t) \to 0$ in probability. Therefore both (5) and $-\sqrt{n}\,\mathrm{e}^{-\lambda t}\, U(t)$ converge to the process $-Z$ (according to Lemma), with the same distribution as process Z. □

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